QNO 1:

ANSWER: a) $a_{ii} = a_{11} + a_{22} + a_{33}$

$= 1 + 4 + 1 = 6$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1+1+1+0+16+4+0+1+1 = 25$ (scalar)

$$a_{ij}a_{ik} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ (vector)}$$

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 +$
$a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1 + 0 + 2 + 0 + 0 + 0 + 0 + 0 + 4 = 7$ (scalar)

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \text{ (matrix)}$$

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1 + 2 + 2 = 5$ (scalar)

b) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 2 + 2 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} +$
$a_{32}a_{32} + a_{33}a_{33}$

$= 1+4+0+0+2+1+0+4+2 = 17$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} \text{ (vector)}$$

$$a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + a_{13} b_1 b_3 + a_{21} b_2 b_1 + a_{22} b_2 b_2 + a_{23} b_2 b_3 + a_{31} b_3 b_1 + a_{32} b_3 b_2 + a_{33} b_3 b_3$$

$$= 4 + 4 + 0 + 0 + 2 + 1 + 0 + 4 + 2 = 17 \text{ (scalar)}$$

$$b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{ (matrix)}$$

$$b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3 = 4 + 1 + 1 = 6 \text{ (scalar)}$$

c) $$a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 0 + 4 = 5 \text{ (scalar)}$$

$$a_{ij} a_{ij} = a_{11} a_{11} + a_{12} a_{12} + a_{13} a_{13} + a_{21} a_{21} + a_{22} a_{22} + a_{23} a_{23} + a_{31} a_{31} + a_{32} a_{32} + a_{33} a_{33}$$

$$= 1 + 1 + 1 + 1 + 0 + 4 + 0 + 1 + 16 = 25 \text{ (scalar)}$$

$$a_{ij} a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ (vector)}$$

$$a_{ij} b_i b_j = a_{11} b_1 b_1 + a_{12} b_1 b_2 + a_{13} b_1 b_3 + a_{21} b_2 b_1 + a_{22} b_2 b_2 + a_{23} b_2 b_3 \; a_{31} b_3 b_1 + a_{32} b_3 b_2 + a_{33} b_3 b_3$$

$$= 1 + 1 + 0 + 1 + 0 + 0 + 0 + 0 + 0 = 3 \text{ (scalar)}$$

$$b_i b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (matrix)}$$

$$b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3 = 1 + 1 + 0 = 2 \text{ (scalar)}$$

Q NO 2:

ANSWER: a) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

b) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

c) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions.

QNO 3:

ANSWER: $a_{ij}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij} \Rightarrow 2a_{ij}b_{ij} = 0 \Rightarrow a_{ij}b_{ij} = 0$

From Exercise 1-2 **(a)** $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

From Exercise 1-2 **(b)** $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}^T\right) = 0$

From Exercise 1-2 **(c)** $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

QNO 4:

ANSWER: $\delta_{ij}a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3 = \begin{bmatrix} \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 \\ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 \\ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$

$$\delta_{ij}a_{jk} = \begin{bmatrix} \delta_{11}a_{11} + \delta_{12}a_{21} + \delta_{13}a_{31} & \delta_{11}a_{12} + \delta_{12}a_{22} + \delta_{13}a_{32} & \delta_{11}a_{13} + \delta_{12}a_{23} + \delta_{13}a_{33} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

QNO 5:

ANSWER: $\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{231} a_{12} a_{23} a_{31} +$

$\varepsilon_{312} a_{13} a_{21} a_{32} + \varepsilon_{321} a_{13} a_{22} a_{31} + \varepsilon_{132} a_{11} a_{23} a_{32} + \varepsilon_{231} a_{12} a_{21} a_{33}$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$

$$= a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} a_{31}) + a_{13}(a_{21} a_{32} - a_{22} a_{31})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

QNO 6:

ANSWER: 45° rotation about x_1-axis $\Rightarrow$ $Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

From Exercise 1-1 **(a)**: $b'_i = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

From Exercise 1-1 **(b)**

$$b'_i = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

From Exercise 1-1 **(c)**

$$b'_i = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

QNO 7:

ANSWER:

$$Q_{ij} = \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \cos(90^\circ - \theta) \\ \cos(90^\circ + \theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b'_i = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \cos\theta + b_2 \sin\theta \\ -b_1 \sin\theta + b_2 \cos\theta \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

QNO 8:

ANSWER: $a'\delta'_{ij} = Q_{ip} Q_{jp} a \delta_{pq} = a Q_{ip} Q_{jp} = a \delta_{ij}$

QNO 9:

ANSWER: $\alpha'\delta'_{ij}\delta'_{kl} + \beta'\delta'_{ik}\delta'_{jl} + \gamma'\delta'_{il}\delta'_{jk} = Q_{im} Q_{jn} Q_{kp} Q_{lq} (\alpha \delta_{mn} \delta_{pq} + \beta \delta_{mp} \delta_{nq} + \gamma \delta_{mq} \delta_{np})$

$$= \alpha Q_{im} Q_{jm} Q_{kp} Q_{lp} + \beta Q_{im} Q_{jn} Q_{km} Q_{ln} + \gamma Q_{im} Q_{jn} Q_{kn} Q_{lm}$$

$$= \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

QNO 10:

ANSWER: $C_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$

$$= \alpha \delta_{ij} \delta_{kl} + \beta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$= \alpha \delta_{kl} \delta_{ij} + \beta (\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li}) = C_{klij}$$

QNO 11:

ANSWER: If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3$$

QNO 12:

a) $a_{ij} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I_a = -1,\ II_a = -2,\ III_a = 0$

Characteristics Eqn is $-\lambda^3 - \lambda^2 + 2\lambda = 0$

$$\lambda(\lambda^2 + \lambda - 2) = 0 \Rightarrow \lambda(\lambda+2)(\lambda-1) = 0$$

Roots $\Rightarrow \lambda_1 = -2,\ \lambda_2 = 0,\ \lambda_3 = 1$

$\lambda_1 = -2$ Case:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1 \end{array}$$

$$\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2,\ n^{(1)} = \pm(\sqrt{2}/2)(-1, 1, 0)$$

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$\lambda_2 = 0$ **Case :**

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \\ n_1^{(2)^2} + n_2^{(2)^2} + n_3^{(2)^2} = 1 \end{matrix}$$

$$\Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$$

$\lambda_3 = 1$ **Case :**

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ n_1^{(3)^2} + n_2^{(3)^2} + n_3^{(3)^2} = 1 \end{matrix}$$

$$\Rightarrow n_1 = n_2 = 0,\ n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm(0,0,1)$$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$ and

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $a_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -4,\ II_a = 3,\ III_a = 0$

Characteristic Eqn is $-\lambda^3 - 4\lambda^2 - 3\lambda = 0$

$$\lambda(\lambda^2 + 4\lambda + 3) = 0 \Rightarrow \lambda(\lambda+3)(\lambda+1) = 0$$

Roots $\Rightarrow \lambda_1 = -3,\ \lambda_2 = -1,\ \lambda_3 = 0$

$\lambda_1 = -3$ **Case:**

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{array}{c} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} = 1 \end{array}$$

$$\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2, \; n^{(1)} = \pm(\sqrt{2}/2)(-1,1,0)$$

$\lambda_2 = -1$ **Case:**

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{c} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \\ n_1^{(2)^2} + n_2^{(2)^2} + n_3^{(2)^2} = 1 \end{array}$$

$$\Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$$

$\lambda_3 = 0$ **Case:**

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{c} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ n_1^{(3)^2} + n_2^{(3)^2} + n_3^{(3)^2} = 1 \end{array}$$

$$\Rightarrow n_1 = n_2 = 0, \; n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm(0,0,1)$$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$ and

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) $a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -2, \; II_a = 0, \; III_a = 0$

$\therefore$ Characteristic Eqn is $-\lambda^3 - 2\lambda^2 = 0$ or $\lambda^2(\lambda+2) = 0$

Roots $\Rightarrow \lambda_1 = -2, \; \lambda_2 = \lambda_3 = 0$

$\lambda_1 = -2$ Case:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{matrix} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1 \end{matrix}$$

$\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2, \; n^{(1)} = \pm\sqrt{2}/2\,(-1, 1, 0)$

$\lambda_2 = \lambda_3 = 0$ Case:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -n_1 + n_2 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{matrix}$$

$\Rightarrow n_1 = n_2, \; n_3^2 = 1 - 2n_1^2 \Rightarrow n \pm (k, k, \sqrt{1-2k^2})$

for arbitary k, and thus directions are not uniquely determined. For convenience we may choose $k = \sqrt{2}/2$ and 0 to get $n^{(2)} = \pm\sqrt{2}/2\,(1, 1, 0)$ and $n^{(3)} = \pm(0, 0, 1)$

The rotation matrix is given by $Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$ and

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

QNO 14:

ANSWER: a) $u = x_1 e_1 + x_1 x_2 e_2 + 2x_1 x_2 x_3 e_3$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1 x_2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1 x_2 & 2x_1 x_2 x_3 \end{vmatrix} = 2x_1 x_3 e_1 - 2x_2 x_3 e_2 + x_2 e_3$$

$$\nabla^2 u = 0e_1 + 0e_2 + 0e_3 = 0$$

$$\nabla u = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2 x_3 & 2x_1 x_3 & 2x_1 x_2 \end{bmatrix}, \; tr(\nabla u) = 1 + x_1 + 2x_1 x_2$$

b) $u = x_1^2 e_1 + 2x_1x_2 e_2 + x_3^3 e_3$

$$\nabla \cdot u = U_{1,1} + U_{2,2} + U_{3,3} = 2x_1 + 2x_1 + 3x_3^2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^3 \end{vmatrix} = 0e_1 - 0e_2 + 2x_2 e_3$$

$$\nabla^2 u = 2e_1 + 0e_2 + 6x_3 e_3 = 0$$

$$\nabla u = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}, \quad tr(\nabla u) = 4x_1 + 3x_3^2$$

c) $u = x_2^2 e_1 + 2x_2x_3 e_2 + 4x_1^2 e_3$

$$\nabla \cdot u = U_{1,1} + U_{2,2} + U_{3,3} = 0 + 2x_3 + 0$$

$$\nabla \times u \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix} = -2x_2 e_1 - 8x_1 e_2 - 2x_2 e_3$$

$$\nabla^2 u = 2e_1 + 0e_2 + 8e_3 = 0$$

$$\nabla u = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}, \quad tr(\nabla u) = 3x_3$$

QNO 15:

ANSWER: $a_i = -\frac{1}{2} \varepsilon_{ijk} a_{jk}$

$$\varepsilon_{imn} a_i = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{imn} a_{jk} = -\frac{1}{2} \begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

$$= -\frac{1}{2} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_{jk}$$

$$= -\frac{1}{2}(a_{mn} - a_{nm}) \Rightarrow -\frac{1}{2}(a_{mn} + a_{mn}) = -a_{mn}$$

$$\therefore \; a_{jk} = -\varepsilon_{ijk} a_i$$

QNO 16:

ANSWER: a) $\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k} = \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$

$$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k = \nabla\phi\cdot u + \phi(\nabla\cdot u)$$

b) $\nabla\times(\phi u) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k) = \varepsilon_{ijk}\phi_{,j}u_k + \phi\varepsilon_{ijk}$

$$= \nabla\phi\times u + \phi(\nabla\times u)$$

$$\nabla\cdot(u\times v) = (\varepsilon_{ijk}u_j v_k)_{,i} = \varepsilon_{ijk}(u_j v_{k,i} + u_{j,i}v_k) = v_k\varepsilon_{ijk}u_{j,i} + u_j\varepsilon_{ijk}$$

$$= v\cdot(\nabla\times u) - u\cdot(\nabla\times v)$$

$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0$ bcz of symmetry & antisymmetry

$$\nabla\times\nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$$

c) $\nabla\cdot(\nabla\times u) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0$ bcz of symmetry & antisymmetry

$$\nabla\times(\nabla\times u) = \varepsilon_{mni}(\varepsilon_{ijk}u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn} = (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_k$$

$$= u_{n,nm} - u_{m,nn}$$

$$= \nabla(\nabla\cdot u) - \nabla^2 u$$

$$u\times(\nabla\times u) = \varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}u_j u_{n,m} = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})$$

$$= u_n u_{n,i} - u_m u_{i,m} \qquad = \frac{1}{2}\nabla(u\cdot u) - u\cdot\nabla u$$

QNO 17:

ANSWER: Cylindrical coordinates : $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = z$

$$(ds)^2 = (dr)^2 + (r d\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1$$

$$\hat{e}_r = \cos\theta e_1 + \sin\theta e_2,\ \hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2,\ \hat{e}_z = e_3$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta,\ \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r,\ \frac{\partial \hat{e}}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\partial \hat{e}_z}{\partial z} = 0$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_z \frac{\partial f}{\partial z}$$

$$\nabla \cdot u = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times u = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\hat{e}$$

QNO 18:

ANSWER: Spherical coordinates : $\xi^1 = R$, $\xi^2 = \phi$, $\xi^3 = \theta$

$$x^1 = \xi^1 \sin\xi^2 \cos\xi^3,\ x^2 = \xi^1 \sin\xi^2 \sin\xi^3,\ x^3 = \xi^1 \cos\xi^2$$

Scale factors:

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1 \Rightarrow h_1 =$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

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$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2 \sin^2\phi \Rightarrow h_3 = R\sin\phi$$

Unit Vectors:

$$\hat{e}_R = \cos\theta \sin\phi e_1 + \sin\theta \sin\phi e_2 + \cos\phi e_3$$

$$\hat{e}_\phi = \cos\theta \cos\phi e_1 + \sin\theta \cos\phi e_2 - \sin\phi e_3$$

$$\hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2$$

$$\frac{\partial \hat{e}_R}{\partial R} = 0, \quad \frac{\partial \hat{e}_R}{\partial \phi} = \hat{e}_\phi, \quad \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0, \quad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_r, \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = 0, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi \hat{e}_\phi$$

Using (1.9.12)–(1.9.16) $\Rightarrow$

$$\nabla = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}$$

$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\nabla \cdot u = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2\sin\phi u_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi u_\phi) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}(R u_\theta)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi u_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}(u_\theta)$$

$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

$$\nabla\times u = \left(\frac{1}{R^2\sin\phi}\left[\frac{\partial}{\partial\phi}(R\sin\phi u_\theta) - \frac{\partial}{\partial\theta}(Ru_\phi)\right]\right)\hat{e}_R + \left(\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial\theta}(u_R) - \frac{\partial}{\partial R}(R\sin\phi u_\theta)\right]\right)\hat{e}_\phi$$

$$+\left(\frac{1}{R}\frac{\partial}{\partial R}\left[(Ru_\phi) - \frac{\partial}{\partial\phi}(u_R)\right)\hat{e}_\theta$$

$$= \left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial\phi}(\sin\phi u_\theta) - \frac{\partial u_\phi}{\partial\theta}\right)\right]\hat{e}_R + \left[\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial\theta} - \frac{1}{R}\frac{\partial}{\partial R}(Ru_\theta)\right]\hat{e}_\phi$$

$$+\left[\frac{1}{R}\left(\frac{\partial}{\partial R}(Ru_\phi - \frac{\partial u_R}{\partial\phi}\right)\right]\hat{e}_\theta$$

Question no. 19:

Transform strain- Displacement relation from cartesian to cylinderical and spherical coordinates.

Answer:

Cylinderical coordinates

∴ Strain Displacement relation

$$e = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

where e = strain, ∇u = displacement

Writing

$$u_x = u_r\cos\theta + (-u_\theta\sin\theta)$$

$$u_y = u_r\sin\theta + u_\theta\cos\theta$$

$$u_z = u_z$$

as derivative of $x = r\cos\theta$; $y = r\sin\theta$; $z = z$ where $r = \sqrt{x^2+y^2}$; $\theta = \arctan(y/x)$ is given by

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

It follows that

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \left(\cos\theta\frac{\partial}{\partial r}\right)\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \cos\theta\sin\theta\left[-\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right] + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r}$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \sin^2\theta\left[\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] + 2\sin\theta\cos\theta\left[\frac{1}{2r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right]$$

Likewise,

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \frac{\partial^2}{\partial r^2} + \cos^2\theta\left[\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] - 2\sin\theta\cos\theta\left[\frac{1}{2r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right]$$

Finally we can determine the expression.

$$e_{xx} = \frac{\partial u_x}{\partial x} = \cos\theta\frac{\partial}{\partial r}(u_r\cos\theta - u_\theta\sin\theta) - \frac{\sin\theta}{r}\left(\frac{\partial}{\partial\theta}(u_r\cos\theta - u_\theta\sin\theta)\right)$$

$$= \frac{\partial}{\partial r}u_r\cos^2\theta - \frac{\partial u_\theta}{\partial r}\sin\theta\cos\theta - \frac{\partial u_r}{\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{u_r\sin^2\theta}{r} + \frac{\partial u_\theta}{\partial\theta}\frac{\sin^2\theta}{r} + \frac{u_\theta}{r}\sin\theta\cos\theta$$

$$= \frac{\partial u_r}{\partial r}\cos^2\theta + \left(\frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\sin\theta\cos\theta + \left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\sin^2\theta$$

$$e_{yy} = \frac{\partial u_y}{\partial y} = \sin\theta\frac{\partial}{\partial r}(u_r\sin\theta + u_\theta\cos\theta) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(u_r\sin\theta + u_\theta\cos\theta)$$

By using same method we obtain expression of e_{xy}.

$$e_{xy} = 2\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\right)$$

Hence we obtain

$$e_{rr} = \frac{\partial u_r}{\partial r}\ ;\ e_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right);\ e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{u_r} - \frac{u_\theta}{r}\right)$$

and $e_{zz} = \dfrac{\partial u_z}{\partial z}$

Question no. 20:

Verify that alternator e_{ijk} has property that

$e_{ijk} = Q_{ip}Q_{jq}Q_{kr}e_{pqr}$ for all proper orthogonal matrices $[Q]$.

but more generally

$e_{ijk} \neq Q_{ip}Q_{jq}Q_{kr}e_{pqr}$ for all orthogonal matrices $[Q]$.

For this the alternator is not isotropic