Lab08

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Q1 Part a

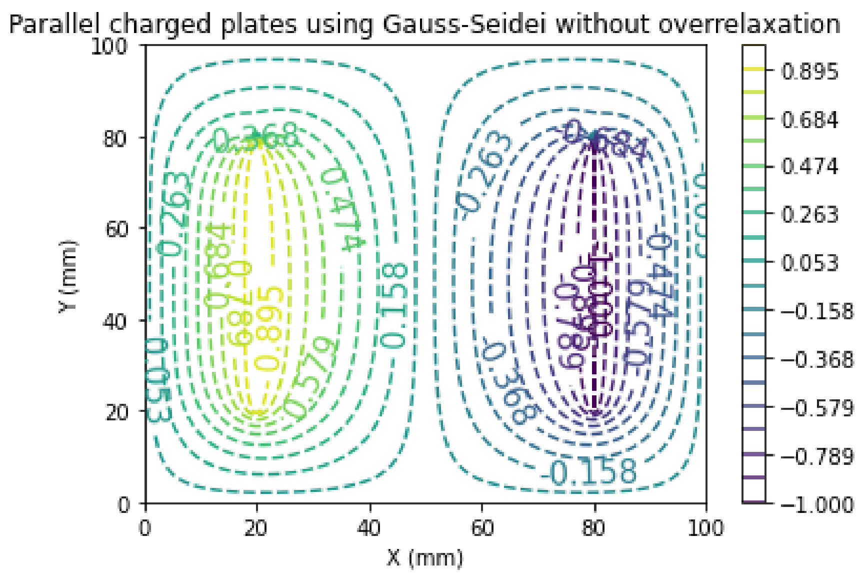

Figure 1: Model of a 2D electronic capacitor where 2 metal plates are in a square metal box that is 100mmx100mm. This model is by the use of the Gauss-Seidel method without over-relaxation

Q1 part b

By looking at figures 1,2, and 3, we can see that there is no significant difference between the Gauss-Seidel method without over-relaxation and with over-relaxation. Essentially, the use of over-relaxation gives a lower convergence time meaning its much faster when we use ω to change ϕ to values that are closer to what it will converge to, rather than keeping phi to what is calculated in the iterations. Below is a table outlining the convergence times with different values of ω.

ω	**Convergence time**
0 (Gauss-Seidel)	37.379369258880615 seconds
0.1	49.7877562046051 seconds
0.5	24.73794198036194 seconds
0.9	5.944880962371826 seconds
0.99	31.79275393486023 seconds

Table 1: Notice that as ω gets larger, the convergence time gets smaller but if we set it too high (close to 1) or too low (close to 0) the time increases again. By using this table we can conclude that the optimum value for ω is 0.9

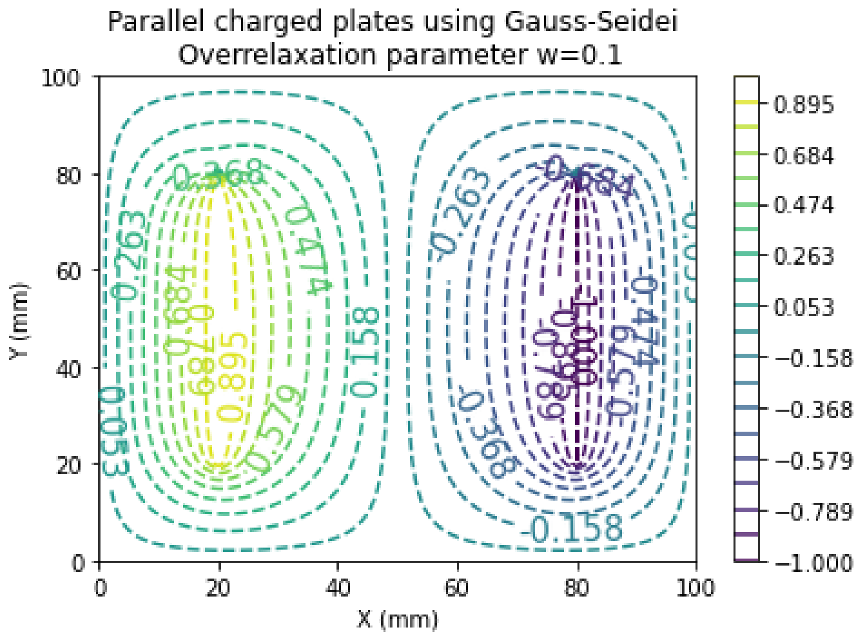

Figure 2: This contour plot shows the field of the two metal plates with voltages $\pm 1V$. It was done using the Gauss-Seidel method with over-relaxation using $\omega = 0.1$

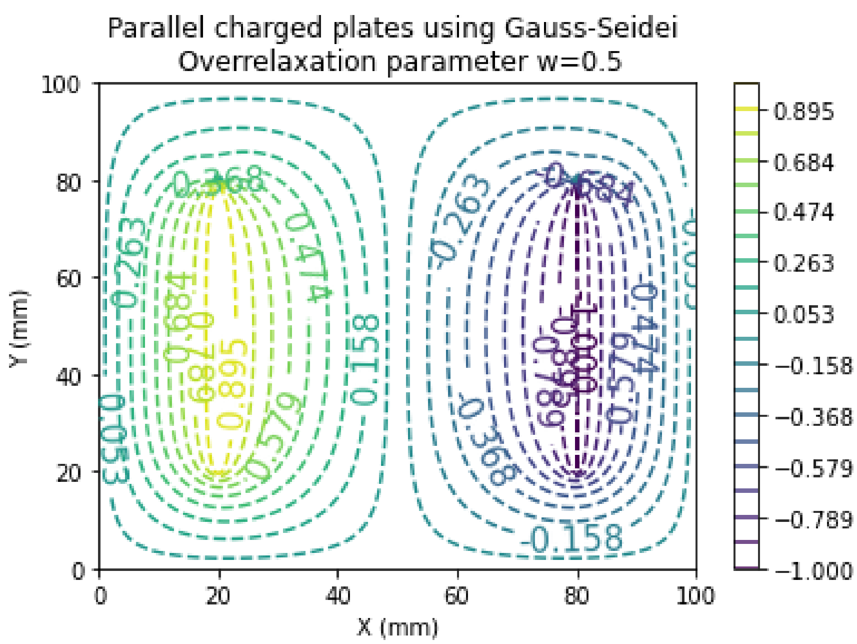

Figure 3: Potential fields of two metal plates in a box where the same method as part 1a was used, however this time we added an over-relaxation parameter $\omega = 0.5$

Q2 Part a

Starting from Equation (6) in the lab document we have:

$$\frac{du}{dt} + u\frac{du}{dx} = -g\frac{d\eta}{dx}$$

We can then rearrange to give:

$$\frac{du}{dt} = -(u\frac{du}{dx} + g\frac{d\eta}{dx})$$

Then because the integral of u is $\frac{u^2}{2}$, we can write:

$$\frac{du}{dt} = \frac{d}{dx}[\frac{1}{2}u^2 + g\eta] \qquad (1)$$

From just below Equation (6) in the lab document we have an Equation for eta:

$$\frac{d\eta}{dt} + \frac{d(uh)}{dx} = 0$$

Rearranging, and substituting h for $(\eta - \eta_b$ we have:

$$\frac{d\eta}{dt} = -\frac{d}{dx}[u(\eta - \eta_b)] \qquad (2)$$

Therefore we can write Equations (1) and (2) in the Flux Conservative Form:

$$-\frac{dF(u,\eta)}{dx} = [\frac{du}{dt}, \frac{d\eta}{dt}]$$

Or:

$$F(u,\eta) = [\frac{1}{2}u^2 + g\eta, (\eta - \eta_b)u]$$

Finally, we use the FTCS Scheme:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}[\frac{(u_{j+1}^n)^2}{2} - \frac{(u_{j-1}^n)^2}{2} + g\eta_{j+1}^n - g\eta_{j-1}^n]$$

$$\eta_j^{n+1} = \eta_j^n - \frac{\Delta t}{2\Delta x}[u_{j+1}^n(\eta_{j+1}^n - \eta_{b,j+1}) - u_{j-1}^n(\eta_{j-1}^n - \eta_{b,j+1})]$$

Q2 Part b

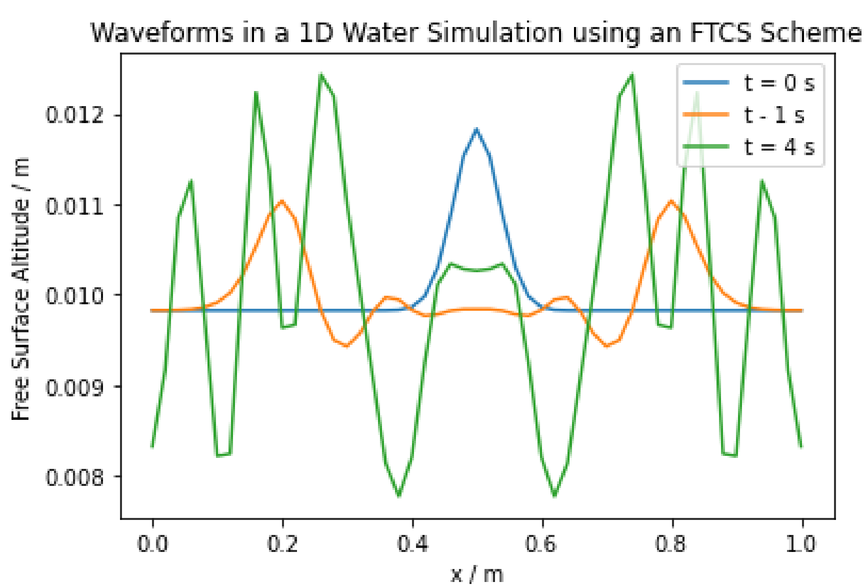

Figure 4:

Q2 Part c

To begin the stability analysis we can start from Equation (6) in the lab document, about $(u, \eta) = (0, H)$:

$$\frac{du}{dt} + u\frac{du}{dx} = -g\frac{d\eta}{dx}$$

Goes to:

$$\frac{du}{dt} = -g\frac{d\eta}{dx}$$

Similarly, with $\eta_b = 0$:

$$\frac{d\eta}{dt} + \frac{d(uh)}{dx} = 0$$

Goes to:

$$\frac{d\eta}{dt} = -\frac{d}{dx}[u\eta]$$

Via the Chain Rule, we can write:

$$\frac{d\eta}{dt} = \eta\frac{du}{dx} - u\frac{d\eta}{dx}$$

With u = 0, η = H, this becomes:

$$\frac{d\eta}{dt} = -H\frac{du}{dx}$$

This gives us the set of Equations (10) in the lab document. The next step is to perform the FTCS scheme on these equations:

$$u(t + \Delta t) = u(t) - g\frac{\Delta t}{2\Delta x}[\eta(x + \Delta x) - \eta(x - \Delta x)]$$

$$\eta(t + \Delta t) = \eta(t) - H\frac{\Delta t}{2\Delta x}[u(x + \Delta x) - u(x - \Delta x)]$$

We can now do a Fourier Transform on these equations, giving:

$$c_u(t + \Delta t)e^{ikx} = c_u(t)e^{ikx} - g\frac{\Delta t}{2\Delta x}[]e^{ikx+ik\Delta x} - e^{ikx-ik\Delta x}]c_\eta(t)$$

$$c_\eta(t + \Delta t)e^{ikx} = c_\eta(t)e^{ikx} - H\frac{\Delta t}{2\Delta x}[e^{ikx+ik\Delta x} - e^{ikx-ik\Delta x}]c_u(t)$$

With the substitution $e^{ik\Delta x} - e^{-ik\Delta x} = 2sin(k\Delta x)$, and cancelling terms when available, we get:

$$c_u(t + \Delta t) = c_u(t) - g\frac{\Delta t}{\Delta x}[sin(k\Delta x)]c_\eta(t)$$

$$c_\eta(t + \Delta t) = c_\eta(t) - H\frac{\Delta t}{\Delta x}[sin(k\Delta x)]c_u(t)$$

Taking $h = \frac{\Delta t}{\Delta x}sin(k\Delta x)$, we can write these equations in matrix form:

$$\begin{pmatrix} c_u(t + \Delta t) \\ c_\eta(t + \Delta t) \end{pmatrix} = \begin{pmatrix} 1 & -gh \\ Hh & 1 \end{pmatrix} \begin{pmatrix} c_u(t) \\ c_\eta(t) \end{pmatrix}$$

Now we must find the eigenvalues of matrix A where:

$$A = \begin{pmatrix} 1 & -gh \\ Hh & 1 \end{pmatrix}$$

So we must find the determinant of this matrix equation:

$$\begin{pmatrix} 1 & -gh \\ Hh & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Which goes to:

$$\begin{pmatrix} 1-\lambda & -gh \\ Hh & 1-\lambda \end{pmatrix}$$

The determinant of this matrix is:

$$(1-\lambda)^2 + gHh^2 = 0$$

So the eigenvalues are:

$$(1-\lambda) = \sqrt{-gHh^2}$$

$$(1-\lambda) = \pm ih\sqrt{gH}$$

$$\lambda = 1 \pm ih\sqrt{gH}$$

$$\lambda = 1 \pm i\frac{\Delta t}{\Delta x}\sqrt{gH} sin(k\Delta x)$$

Therefore, we can write Equation (11) from the lab document:

$$|\lambda| = \sqrt{1 + (\frac{\Delta t}{\Delta x})^2 gH sin^2(k\Delta x)}$$

To analyse whether this FTCS scheme is stable, we must see if the eigenvalues (λ) are always greater than the value of 1. We can see from Equation (11) of the lab document (derived above) that that eigenvalues in this case will always be larger than 1. This is because the sine function will always return a value in the range of [0,1]. Similarly, the $\frac{\Delta t}{\Delta x}^2$ term will always be positive because it is a squared term. Finally, because g and H are always positive, the entire second term in the square root will therefore always be positive. Hence, adding a positive number to 1 will always yield a number larger than 1. Square rooting this number will always yield a number larger than 1, therefore in this case, the FTCS scheme is unstable.