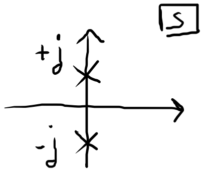
Esercizi sull'analisi modale

1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\varphi(s) = \det(sI - A) = \det \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = s^2 + 1$$

$$\varphi(s) = 0 \iff s^2 = -1 \iff s = \pm j$$



$\lambda_1 = j \qquad \mu_1 = 1$

$\lambda_2 = -j \qquad \mu_2 = 1$

$$\varphi(s) = (s-j)(s+j)$$

$$1 \le m_i \le \mu_i \quad \Rightarrow \quad m_1 = m_2 = 1$$

molteplicità in $m(s)$

molteplicità in $\varphi(s)$

$$m(s) = \varphi(s) = s^2 + 1$$

$\Rightarrow$ modi naturali $\sin(t)$, $\cos(t)$

limitati oscillanti

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$t \geq 0 \qquad e^{At} = \mathcal{L}^{-1}\left\{ (sI-A)^{-1} \right\}$$

$$(sI-A)^{-1} = \frac{1}{\varphi(s)} \, \mathrm{Adj}\,(sI-A)$$

$$\varphi(s) = s^2+1 \qquad \mathrm{Adj}\,(sI-A) = \mathrm{Adj}\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$$

$$t \geq 0 \qquad e^{At} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} & \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ \mathcal{L}^{-1}\left\{\frac{-1}{s^2+1}\right\} & \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

NOTA: gli elementi di e^{At} sono combinazione lineare dei modi naturali $\sin(t)$, $\cos(t)$

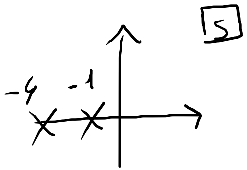
2) $A = \begin{bmatrix} -3 & 2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\varphi(s) = \det(sI - A) = \begin{bmatrix} s+3 & -2 & 0 \\ -1 & s+2 & 0 \\ 0 & 0 & \boxed{s+1} \end{bmatrix} = (s+1)\left[(s+3)(s+2) - 2\right]$$

$$= (s+1)(s^2+5s+6-2) = (s+1)(s^2+5s+4) = (s+1)(s+1)(s+4) = (s+1)^2(s+4)$$

$\lambda_1 = -1 \qquad \mu_1 = 2$

$\lambda_2 = -4 \qquad \mu_2 = 1$



$1 \leq m_i \leq \mu_i \quad \Rightarrow \quad 1 \leq m_1 \leq 2$

$m_2 = 1$

Per calcolare m_1 devo calcolare $(sI - A)^{-1}$

$$(sI-A)^{-1} = \begin{bmatrix} s+3 & -2 & 0 \\ -1 & s+2 & 0 \\ 0 & 0 & \boxed{s+1} \end{bmatrix}^{-1} = \begin{bmatrix} \begin{bmatrix} s+3 & -2 \\ -1 & s+2 \end{bmatrix}^{-1} & & 0 \\ & & 0 \\ 0 & 0 & \boxed{(s+1)^{-1}} \end{bmatrix}$$

$$\begin{bmatrix} s+3 & -2 \\ -1 & s+2 \end{bmatrix} = \frac{1}{\det \begin{bmatrix} s+3 & -2 \\ -1 & s+2 \end{bmatrix}} \operatorname{Adj} \begin{bmatrix} s+3 & -2 \\ -1 & s+2 \end{bmatrix} = \frac{1}{s^2+5s+4} \begin{bmatrix} s+2 & 2 \\ 1 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s+2}{s^2+5s+4} & \frac{2}{s^2+5s+4} & 0 \\ \frac{1}{s^2+5s+4} & \frac{s+3}{s^2+5s+4} & 0 \\ 0 & 0 & \boxed{\frac{1}{s+1}} \end{bmatrix}$$

$m(s)$ minimo comune multiplo dei denominatori degli elementi di $sI-A^{-1}$

$s^2+5s+4 = (s+1)(s+4)$, $s+1$

$\Rightarrow \quad m(s) = (s+1)(s+4)$

$\lambda_1 = -1 \qquad m_1 = 1$

$\lambda_2 = -4 \qquad m_2 = 1$

$\Rightarrow$ modi naturali $e^{\lambda_1 t} = e^{-t}$, $e^{\lambda_2 t} = e^{-4t}$

convergenti non oscillanti

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s+2}{(s+1)(s+4)} & \frac{2}{(s+1)(s+4)} & 0 \\ \frac{1}{(s+1)(s+4)} & \frac{s+3}{(s+1)(s+4)} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{bmatrix}\right\}$$

$$\frac{s+2}{(s+1)(s+4)} = \frac{k_1}{s+1} + \frac{k_2}{s+4} = \frac{1}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{1}{s+4}$$

$$k_1 = \lim_{s \to -1} \cancel{(s+1)} \frac{(s+2)}{\cancel{(s+1)}(s+4)} = \frac{1}{3}$$

$$k_2 = \lim_{s \to -4} \cancel{(s+4)} \frac{(s+2)}{(s+1)\cancel{(s+4)}} = \frac{-2}{-3} = \frac{2}{3}$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)(s+4)}\right\} = \frac{1}{3}\cdot e^{-t} + \frac{2}{3} e^{-4t}$$

$$\boxed{e^{\lambda t} \leftrightarrow \frac{1}{s-\lambda}}$$

$$e^{At} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s+4} & \frac{2}{3}\frac{1}{s+1} - \frac{2}{3}\frac{1}{s+4} & 0 \\ \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4} & \frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4} & 0 \\ 0 & 0 & \frac{1}{s+1} \end{bmatrix}\right\} = \begin{bmatrix} \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t} & 0 \\ \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$