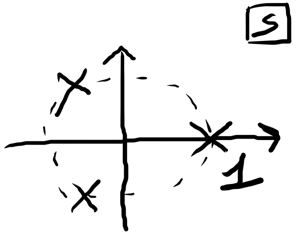
Esercizi sulla retroazione algebrica sull'uscita

$$\begin{cases} \dot{x}_1 = x_3 + \alpha^2 u \\ \dot{x}_2 = x_1 + 2\alpha u \\ \dot{x}_3 = x_2 + u \\ y = x_3 \end{cases} \qquad A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \alpha^2 \\ 2\alpha \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

a) $$\varphi(s) = \det(sI - A) = \det \begin{bmatrix} s & 0 & -1 \\ -1 & s & 0 \\ 0 & -1 & s \end{bmatrix} = s \cdot s^2 - 1 \cdot 1 = s^3 - 1$$

$$\varphi(s) = 0 \iff s^3 - 1 = 0 \iff s^3 = 1$$



NOTA: gli autovalori sono le 3 radici cubiche dell'unità nel campo dei numeri complessi

$$\varphi(s) = (s-1)(s^2 + as + b) = s^3 - 1$$

$$s^3 + as^2 + bs - s^2 - as - b = s^3 - 1$$

$$s^3 + (a-1)s^2 + (b-a)s - b = s^3 - 1 \iff \begin{matrix} a = 1 \\ b = 1 \end{matrix}$$

$$\varphi(s) = (s-1)(s^2 + s + 1)$$

$$G(s) = C(sI-A)^{-1}B = \frac{1}{\varphi(s)} C \, \mathrm{Adj}(sI-A) B =$$

$$= \frac{1}{(s-1)(s^2+s+1)} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 1 & s \\ s & s^2 & 1 \\ 1 & s & s^2 \end{bmatrix} \begin{bmatrix} \alpha^2 \\ 2\alpha \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s-1)(s^2+s+1)} \begin{bmatrix} 1 & s & s^2 \end{bmatrix} \begin{bmatrix} \alpha^2 \\ 2\alpha \\ 1 \end{bmatrix} = \frac{1}{(s-1)(s^2+s+1)} (\alpha^2 + s \cdot 2\alpha + s^2)$$

$$= \frac{s^2 + 2\alpha s + \alpha^2}{(s-1)(s^2+s+1)} = \frac{(s+\alpha)^2}{(s-1)(s^2+s+1)}$$

Devo vedere per quali α ho semplificazioni

$\alpha = -1$ $\qquad G(s) = \frac{(s-1)^{\cancel{2}}}{\cancel{(s-1)}(s^2+s+1)} = \frac{s-1}{s^2+s+1}$ $\qquad a(s) = s^2+s+1$

$\varphi_R(s) = s-1$

$\alpha \neq -1$ $\qquad$ non ci sono semplificazioni $\qquad a(s) = \varphi(s)$

$\varphi_R(s) = 1$

b) problema ben posto $\Leftrightarrow$ $\varphi_R(s)$ ha tutte radici con Re < 0

$\alpha = -1$ $\quad \varphi_R(s) = s - 1$ autovalore $\lambda_1 = 1$ nascosto

$\Rightarrow$ problema mal posto

$\alpha \neq -1$ $\quad \varphi_R(s) = 1 \Rightarrow$ problema ben posto

c) $\alpha = 2$ studiare stabilità in ciclo chiuso al variare di K

$$\varphi^*(s) = \varphi_R(s)\,[a(s) + K\,b(s)] = s^3 - 1 + K(s+2)^2$$

$$= s^3 - 1 + K(s^2 + 4s + 4) = s^3 + Ks^2 + 4Ks + 4K - 1$$

studio la stabilità costruendo la tabella di Routh

$$\begin{array}{c|ccc}
3 & 1 & 4K & 0 \\
2 & K & 4K-1 & 0 \\
1 & E_{11} & 0 & \\
0 & 4K-1 & &
\end{array}$$

$$E_{11} = -\frac{1}{K} \det\begin{bmatrix} 1 & 4K \\ K & 4K-1 \end{bmatrix} = -\frac{1}{K}\left(4K - 1 - 4K^2\right)$$

$$= \frac{4K^2 - 4K + 1}{K}$$

$$
\begin{array}{c|cc}
3 & 1 & 4k \\
2 & k & 4k-1 \\
1 & \frac{4k^2-4k+1}{k} & \\
0 & 4k-1 &
\end{array}
$$

tutte radici con Re < 0

$\Updownarrow$

tutti elementi della prima colonna $\neq 0$
e con segno concorde

$\Updownarrow$

$$
\begin{cases} k > 0 \\ \dfrac{4k^2 - 4k + 1}{k} > 0 \\ 4k - 1 > 0 \end{cases}
\qquad
\begin{cases} k > 0 \\ 4k^2 - 4k + 1 > 0 \\ k > \frac{1}{4} \end{cases}
$$

$\Updownarrow$

$$
\begin{cases} k > \frac{1}{4} \\ (2k-1)^2 > 0 \end{cases}
\qquad
\begin{cases} k > \frac{1}{4} \\ k \neq \frac{1}{2} \end{cases}
$$

stabilità esintotica in ciclo chiuso $\Leftrightarrow$ $k > \frac{1}{4}$ e $k \neq \frac{1}{2}$

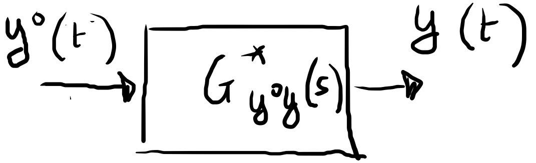
d) Prendo ad esempio $\boxed{k = \frac{1}{3}}$

$$G^*_{y^\circ y}(s) = \frac{b(s)}{a(s)+k\,b(s)}\,H = \frac{s^2+4s+4}{s^3+ks^2+4ks+4k-1}\,H = \frac{s^2+4s+4}{s^3+\frac{s^2}{3}+\frac{4}{3}s+\frac{1}{3}}\,H$$

$$G^*_{y^\circ y}(0) = \frac{4}{1/3}\,H = 12\,H$$

$$G^*_{y^\circ y}(0) = 1 \quad \Longleftrightarrow \quad \boxed{H = \frac{1}{12}}$$

e) $y^\circ(t) = 5 \cdot \sin(t) \cdot 1(t)$ regime permanente?



$G^*_{y^\circ y}(s)$ stabile $\Rightarrow$ posso applicare il teorema della risposta in frequenza

$$y^{Y^\circ}(t) = 5\left[\mathrm{Re}\left\{G^*_{y^\circ y}(j\omega_0)\right\}\sin(\omega_0 t) + \mathrm{Im}\left\{G^*_{y^\circ y}(j\omega_0)\right\}\cos(\omega_0 t)\right] 1(t)$$

$$y^{Y^0}(t) = 5\left[\mathrm{Re}\left\{G^{*}_{y^0y}(j\omega_0)\right\}\sin(\omega_0 t) + \mathrm{Im}\left\{G^{*}_{y^0y}(j\omega_0)\right\}\cos(\omega_0 t)\right]1(t)$$

$$\omega_0 = 1$$

$$G^{*}_{y^0y}(s) = \frac{s^2+4s+4}{s^3+\frac{s^2}{3}+\frac{4}{3}s+\frac{1}{3}} \quad \frac{1}{12}$$

$$G^{*}_{y^0y}(j) = \frac{(j)^2+4j+4}{(j)^3+\frac{j^2}{3}+\frac{4}{3}j+\frac{1}{3}}\frac{1}{12} = \frac{-1+4j+4}{-j-\cancel{\frac{1}{3}}+\frac{4}{3}j+\cancel{\frac{1}{3}}}\frac{1}{12} = \frac{3+4j}{\frac{1}{3}j}\cdot\frac{1}{12}$$

$$= \frac{9j-12}{-1}\cdot\frac{1}{12} = \frac{12-9j}{12} = 1-\frac{3}{4}j$$

$$y^{Y^0}(t) = 5\left[\sin(t) - \frac{3}{4}\cos(t)\right]1(t)$$