$f: E \subseteq \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \text{int}(E)$ derivate parziali, differenziabilità di f in x_0

PROPOSIZIONE Se f è differenziabile in $x_0 \in \text{int}(E)$, $f: E \to \mathbb{R}$, allora f è continua in x_0

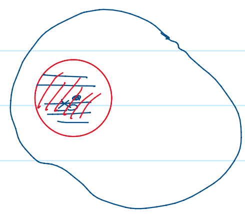
DIM $f(\underline{x_0}+\underline{h}) = f(\underline{x_0}) + \nabla f(\underline{x_0}) \cdot \underline{h} + \mathcal{E}(\underline{h})$ $\qquad \lim\limits_{\underline{h}\to \underline{0}} \frac{\mathcal{E}(\underline{h})}{|\underline{h}|} = 0$

$$\lim_{\underline{h}\to \underline{0}} f(\underline{x_0}+\underline{h}) = \lim_{\underline{h}\to \underline{0}} \Big(f(\underline{x_0}) + \underbrace{\nabla f(\underline{x_0}) \cdot \underline{h}}_{\to 0 \ \ \underline{h}\to \underline{0}} + \underbrace{\mathcal{E}(\underline{h})}_{\to 0} \Big) = f(\underline{x_0})$$

DEF La funzione $df(\underline{x_0}): \underline{h} \in \mathbb{R}^n \to \nabla f(\underline{x_0}) \cdot \underline{h} \in \mathbb{R}$ è una funzione lineare su $\mathbb{R}^n$, si chiama DIFFERENZIALE DI f IN $\underline{x_0}$

$$\lim_{\underline{h}\to \underline{0}} \frac{f(\underline{x_0}+\underline{h}) - f(\underline{x_0}) - \nabla f(\underline{x_0}) \cdot \underline{h}}{|\underline{h}|} \overset{?}{=} 0$$

TEOREMA (Condizione sufficiente per la differenziabilità)
Sia $A \subseteq \mathbb{R}^n$ aperto, sia $f: A \to \mathbb{R}$ e sia $x_0 \in A$
Supponiamo che le derivate parziali f esistano in almeno un intorno di x_0 e che siano continue nel pto x_0
Allora f è differenziabile in x_0



DIM $n=2$ $\qquad \underline{h} = (h,k)$

$$f(x_0+h, y_0+k) - f(x_0,y_0) = \big(f(x_0+h, y_0+k) - f(x_0+h, y_0)\big) + \big(f(x_0+h, y_0) - f(x_0,y_0)\big)$$

$$f(x_0+h, y_0+k) - f(x_0+h, y_0) = \frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k)\,(y_0+k-y_0) \qquad \theta_1 \in (0,1)$$

$y_0+\theta_1 k \quad \theta_1 \in (0,1)$

$y_0 \qquad y_0+k \qquad \mathbb{R}$

$$= k \frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k)$$

$$f(x_0+h, y_0) - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0)\,(x_0+h-x_0) \qquad \theta_2 \in (0,1)$$

$x_0+\theta_2 h \quad \theta_2 \in (0,1)$

$x_0 \qquad x_0+h \qquad \mathbb{R}$

$$= h \frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0)$$

$$f(x_0+h, y_0+k) - f(x_0,y_0) = h \frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0) + k \frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) =$$

$$\frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0) = \frac{\partial f}{\partial x}(x_0,y_0) + \mathcal{E}_2(h) \qquad \lim_{h\to 0} \mathcal{E}_2(h) = 0$$

$$\frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) = \frac{\partial f}{\partial y}(x_0,y_0) + \mathcal{E}_1(h,k) \qquad \lim \mathcal{E}_1(h,k) = 0$$

$$\frac{\partial f}{\partial x}(x_0+\theta_2 h, y_0) = \frac{\partial f}{\partial x}(x_0,y_0) + \varepsilon_2(h) \qquad \lim_{h\to 0} \varepsilon_2(h) = 0$$

$$\frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) = \frac{\partial f}{\partial y}(x_0,y_0) + \varepsilon_1(h,k) \qquad \lim_{(h,k)\to(0,0)} \varepsilon_1(h,k) = 0$$

$$f(x_0+h, y_0+k) - f(x_0,y_0) = \underbrace{h\frac{\partial f}{\partial x}(x_0,y_0)} + h\,\varepsilon_2(h) + \underbrace{k\frac{\partial f}{\partial y}(x_0,y_0)} + k\,\varepsilon_1(h,k)$$

$$\left|\frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \nabla f(x_0,y_0)\cdot(h,k)}{\sqrt{h^2+k^2}}\right| = \left|\frac{h\varepsilon_2(h)+k\varepsilon_1(h,k)}{\sqrt{h^2+k^2}}\right| \leq$$

$$\leq \underbrace{\frac{|h|}{\sqrt{h^2+k^2}}}_{\leq 1}|\varepsilon_2(h)| + \underbrace{\frac{|k|}{\sqrt{h^2+k^2}}}_{\leq 1}|\varepsilon_1(h,k)| \leq \underbrace{|\varepsilon_2(h)|+|\varepsilon_1(h,k)|} \longrightarrow 0$$

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \nabla f(x_0,y_0)\cdot(h,k)}{\|(h,k)\|} = 0$$

DERIVATE DIREZIONALI

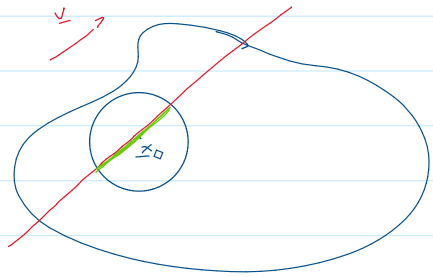
Le derivate parziale di f in un pto $\underline{x}_0$ rispetto alla i-esima variabile

$$\frac{d}{dt} f(\underline{x}_0 + t\underline{e}_i)\Big|_{t=0} \qquad \underline{e}_i = (0, \_\_, 0, 1, 0, \_\_\_ 0)$$

$\hookrightarrow$ i-esima componente

$f: E \subseteq \mathbb{R}^n \to \mathbb{R}$ $\quad \underline{x}_0 \in \operatorname{int}(E)$

Sia $\underline{v} \in \mathbb{R}^n : \|\underline{v}\| = 1$



$g(t) = f(\underline{x}_0 + t\underline{v}) \qquad t \in (-r, r)$

Se $\exists\, g'(0)$, dico che il suo valore è la derivata direzionale di f nel pto $\underline{x}_0$ e nella direzione $\underline{v}$ e la indico $D_{\underline{v}} f(\underline{x}_0)$ cioè

$$D_{\underline{v}} f(\underline{x}_0) = \lim_{t\to 0} \frac{f(\underline{x}_0 + t\underline{v}) - f(\underline{x}_0)}{t} \quad \text{se questo limite esiste finito}$$

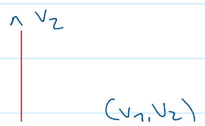
N.B. $\dfrac{\partial f}{\partial x_i}(\underline{x}_0) = D_{\underline{e}_i} f(\underline{x}_0)$

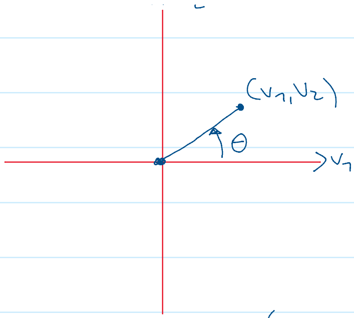
ESEMPIO $\quad f(x,y) = x^2\cos(y-x) \qquad D_{\underline{v}} f(x_0,y_0)$

$\underline{v} = (v_1, v_2)$

$v_1^2 + v_2^2 = 1$

$$g(t) = f\big((x_0,y_0) + t(\cos\theta, \sin\theta)\big) = f(x_0 + t\cos\theta,\ y_0 + t\sin\theta)$$





$f(x_0 + t\cos\theta,\ y_0 + t\sin\theta)$

$(v_1, v_2) = (\cos\theta, \sin\theta)$

$g(t) = (x_0 + t\cos\theta)^2 \cos(y_0 + t\sin\theta - x_0 - t\cos\theta)$

$g(t) = (x_0 + t\cos\theta)^2 \cos(y_0 - x_0 + t(\sin\theta - \cos\theta))$

$g'(t) = 2(x_0 + t\cos\theta)\cos(\theta)\cos(y_0 - x_0 + t(\sin\theta - \cos\theta)) +$
$-(x_0 + t\cos\theta)^2 \cdot \sin(y_0 - x_0 + t(\sin\theta - \cos\theta)) \cdot (\sin\theta - \cos\theta)$

$D_{\underline{v}} f(x_0, y_0) = g'(0) = 2\cos\theta\cos(y_0 - x_0) - x_0^2 \sin(y_0 - x_0)(\sin\theta - \cos\theta)$

derivate direzionali con gradiente

TEO (FORMULA DEL GRADIENTE)

Sia $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$, $\underline{x}_0 \in \operatorname{int}(E)$, Supponiamo che f sia differenziabile in x_0.

Allora $\forall \underline{v}$ direzione, $D_{\underline{v}} f(\underline{x}_0) = \nabla f(\underline{x}_0) \cdot \underline{v} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{x}_0) v_i$

$\underline{v} = (v_1, v_2, \ldots v_n)$

DIM $f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + \nabla f(\underline{x}_0) \cdot \underline{h} + o(\|\underline{h}\|)$ $\qquad \lim_{\underline{h} \to 0} \frac{o(\|\underline{h}\|)}{\|\underline{h}\|} = 0$

Scelgo $\underline{h} = t\underline{v} \Rightarrow \|\underline{h}\| = |t|$

$f(\underline{x}_0 + t\underline{v}) - f(\underline{x}_0) = t\nabla f(\underline{x}_0) \cdot \underline{v} + o(|t|)$

$$\frac{f(\underline{x}_0 + t\underline{v}) - f(\underline{x}_0)}{t} = \nabla f(\underline{x}_0) \cdot \underline{v} + \underbrace{\boxed{\frac{o(|t|)}{t}}}_{\to 0} \xrightarrow{t \to 0} \nabla f(\underline{x}_0) \cdot \underline{v}$$

$$\exists \lim_{t \to 0} \frac{f(\underline{x}_0 + t\underline{v}) - f(\underline{x}_0)}{t} = \nabla f(\underline{x}_0) \cdot \underline{v}$$

cioè $D_{\underline{v}} f(\underline{x}_0) = \nabla f(\underline{x}_0) \cdot \underline{v}$

Algebra

CALCOLO DELLE DERIVATE tramite derivate e gradiente

f e g derivabili in un pto x_0 $\quad \alpha, \beta \in \mathbb{R} \Rightarrow \alpha f + \beta g$ è derivabile in x_0

e
$$\frac{\partial}{\partial x_i}(\alpha f + \beta g)(\underline{x}_0) = \alpha \frac{\partial f}{\partial x_i}(\underline{x}_0) + \beta \frac{\partial g}{\partial x_i}(\underline{x}_0) \qquad \forall i = 1 \ldots n$$

$$\nabla(\alpha f + \beta g)(\underline{x}_0) = \alpha \nabla f(\underline{x}_0) + \beta \nabla g(\underline{x}_0)$$

$$\frac{\partial}{\partial x_i}(fg)(\underline{x}_0) = f(\underline{x}_0)\frac{\partial g}{\partial x_i}(\underline{x}_0) + g(\underline{x}_0)\frac{\partial f}{\partial x_i}(\underline{x}_0) \qquad \forall i = 1 \ldots n$$

$$\nabla(fg)(\underline{x}_0) = f(\underline{x}_0)\nabla g(\underline{x}_0) + g(\underline{x}_0)\nabla f(\underline{x}_0)$$

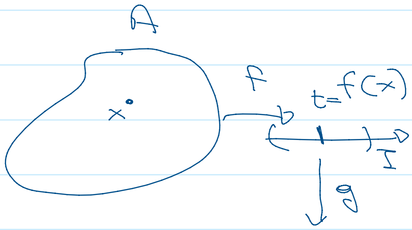
Se $g(\underline{x}_0) \neq 0$ $\quad \dfrac{\partial}{\partial x_i}\left(\dfrac{f}{g}\right)(\underline{x}_0) = \dfrac{g(\underline{x}_0)\frac{\partial f}{\partial x_i}(\underline{x}_0) - f(\underline{x}_0)\frac{\partial g}{\partial x_i}(\underline{x}_0)}{g^2(\underline{x}_0)}$

$$\nabla\left(\frac{f}{g}\right)(\underline{x}_0) = \frac{1}{g^2(\underline{x}_0)}\left(g(\underline{x}_0)\nabla f(\underline{x}_0) - f(\underline{x}_0)\nabla g(\underline{x}_0)\right)$$

DERIVABILITÀ DI FUNZIONI COMPOSTE

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, A aperto,

$g: I \subseteq \mathbb{R} \to \mathbb{R}$ t.c. $f(A) \subseteq I$



$h(x) := g(f(x))$ $\qquad h: x \in A \mapsto g(f(x)) \in \mathbb{R}$

$g(t) = g(f(x))$

PROPRIETÀ Se f è DIFFERENZIABILE in un pto $\underline{x}_0 \in A$ e g è derivabile in $f(\underline{x}_0)$, allora h è DIFFERENZIABILE in $\underline{x}_0$ e $\nabla h(\underline{x}_0) = g'(f(\underline{x}_0))\nabla f(\underline{x}_0)$

DIM $h(\underline{x}) = g(f(\underline{x}))$

$h(x_1 \ldots x_n) = g(f(x_1 \ldots x_n))$

$$\frac{\partial h}{\partial x_i}(x_1 \ldots x_n) = g'(f(x_1 \ldots x_n))\frac{\partial f}{\partial x_i}(x_1 \ldots x_n) \qquad \forall i = 1 \ldots n$$

$$\nabla h(\underline{x}_0) = g'(f(\underline{x}_0))\nabla f(\underline{x}_0)$$

$k \in \mathbb{R}$ $\quad g(\underline{\underline{f(\underline{x}_0)}} + k) = g(f(\underline{x}_0)) + k\,g'(f(\underline{x}_0)) + o(k) =$

$= g(f(\underline{x}_0)) + k\,g'(f(\underline{x}_0)) + k\,\theta(k) \qquad \lim_{k \to 0}\theta(k) = 0$

Scelgo $k = f(\underline{x}_0 + \underline{w}) - f(\underline{x}_0) \qquad \underline{w} \in \mathbb{R}^n$

$f(\underline{x}_0) + k = f(\underline{x}_0 + \underline{w})$

$$g(f(\underline{x}_0 + \underline{w})) = g(f(\underline{x}_0)) + \left(\underline{f(\underline{x}_0+\underline{w}) - f(\underline{x}_0)}\right)g'(f(\underline{x}_0)) +$$

$$+ \left(\underline{f(\underline{x}_0+\underline{w}) - f(\underline{x}_0)}\right)\theta\left(f(\underline{x}_0+\underline{w}) - f(\underline{x}_0)\right)$$

f differenziabile in $\underline{x}_0$ $\Rightarrow$ $f(\underline{x}_0+\underline{w}) - f(\underline{x}_0) = \nabla f(\underline{x}_0)\cdot\underline{w} + o(\|\underline{w}\|)$

$$\underbrace{g(f(\underline{x}_0+\underline{w})) - g(f(\underline{x}_0))}_{h(\underline{x}_0+\underline{w}) - h(\underline{x}_0)} = g'(f(\underline{x}_0))\nabla f(\underline{x}_0)\cdot\underline{w} + g'(f(\underline{x}_0))\,o(\|\underline{w}\|)$$

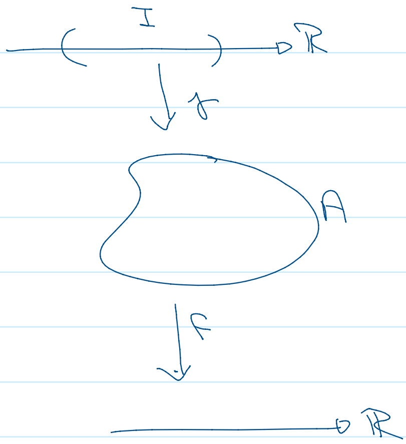
$$+ \left(\underline{f(\underline{x}_0+\underline{w}) - f(\underline{x}_0)}\right)\theta\left(\underline{f(\underline{x}_0+\underline{w}) - f(\underline{x}_0)}\right)$$

$$h(\underline{x}_0+\underline{w})-h(\underline{x}_0)=g'(f(\underline{x}_0))\,\nabla f(\underline{x}_0)\cdot\underline{w}+o(\|\underline{w}\|)+\underbrace{(\nabla f(\underline{x}_0)\cdot\underline{w}+o(\|\underline{w}\|))\,\vartheta(\nabla f(\underline{x}_0)\cdot\underline{w}+o(\|\underline{w}\|))}_{o(\|\underline{w}\|)}$$

$$h(\underline{x}_0+\underline{w})-h(\underline{x}_0)=g'(f(\underline{x}_0))\,\nabla f(\underline{x}_0)\cdot\underline{w}+o(\|\underline{w}\|)$$

I intervallo aperto $\quad \gamma: I\to\mathbb{R}^n$

$A\subseteq\mathbb{R}^n$ aperto $\quad f: A\to\mathbb{R}$



Se $\gamma(I)\subseteq A$ posso considerare

$$f(\gamma(t)) \quad \forall t\in I$$

$$g:=f\circ\gamma: t\in I \longmapsto f(\gamma(t))\in\mathbb{R}$$

PROPRIETÀ Se γ è derivabile in t_0 e se f è differenziabile in $\gamma(t_0)$, allora g è derivabile in t_0 e

$$g'(t_0)=\nabla f(\gamma(t_0))\cdot\gamma'(t_0)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\gamma(t_0))\,\gamma_i'(t_0)$$

DIM

So che f è differenziabile in $\gamma(t_0)$

$$f(\gamma(t_0)+\underline{k})-f(\gamma(t_0))=\nabla f(\gamma(t_0))\cdot\underline{k}+\|\underline{k}\|\varepsilon(\underline{k}) \qquad \lim_{\underline{k}\to 0}\varepsilon(\underline{k})=0$$

Scelgo $\underline{k}=\gamma(t_0+h)-\gamma(t_0) \qquad h\in\mathbb{R}$

$\gamma(t_0)+k=\gamma(t_0+h)$

$$f(\gamma(t_0+h))-f(\gamma(t_0))=\nabla f(\gamma(t_0))\cdot(\gamma(t_0+h)-\gamma(t_0))+$$
$$+\|\gamma(t_0+h)-\gamma(t_0)\|\,\varepsilon(\underbrace{\gamma(t_0+h)-\gamma(t_0)}_{\to 0 \text{ per } h\to 0})$$

$\gamma(t_0+h)=\gamma(t_0)+h\gamma'(t_0)+o(h)$ perché γ è derivabile in t_0

$$f(\gamma(t_0+h))-f(\gamma(t_0))=\nabla f(\gamma(t_0))\cdot(h\gamma'(t_0)+o(h))+o(h)$$

$$g(t_0+h)-g(t_0)=h\,\nabla f(\gamma(t_0))\cdot\gamma'(t_0)+o(h)$$

$$\exists\, g'(t_0)=\nabla f(\gamma(t_0))\cdot\gamma'(t_0)$$

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ differenziabile in $\underline{x}_0$

$\underline{x}_0 = (x_1^0, x_2^0, \ldots, x_n^0)$

$Gr(f) \subset \mathbb{R}^{n+1}$ $\qquad (\underline{x}, x_{n+1}) = (x_1 \ldots x_n, x_{n+1})$

$$x_{n+1} = f(\underline{x}_0) + Df(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0)$$

$$x_{n+1} = f(x_1^0 \ldots x_n^0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1^0 \ldots x_n^0)(x_i - x_i^0)$$

$n=2$ $\qquad Gr(f) \subset \mathbb{R}^3$ $\qquad (x,y,z)$ $\qquad z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

(x_0, y_0)

$\gamma : t \in I \mapsto \gamma(t) \in A$ $\qquad \gamma(t_0) = (x_0, y_0)$ $\qquad f(\gamma(t))$

$\gamma(t) = (\gamma_1(t), \gamma_2(t))$

$\eta : t \in I \mapsto \eta(t) \in \mathbb{R}^3$ $\qquad \eta(t) = \begin{cases} x = \gamma_1(t) \\ y = \gamma_2(t) \\ z = \underbrace{f(\gamma_1(t), \gamma_2(t))}_{(f \circ \gamma)(t)} \end{cases}$ $\qquad t \in I$

$$\eta'(t_0) = \Big(\gamma_1'(t_0), \gamma_2'(t_0), Df(\underbrace{\gamma(t_0)}_{(x_0,y_0)}) \cdot \gamma'(t_0)\Big) =$$

$$= \Big(\gamma_1'(t_0), \gamma_2'(t_0), f_x(x_0,y_0)\gamma_1'(t_0) + f_y(x_0,y_0)\gamma_2'(t_0)\Big)$$

$$\eta(t_0) = (x_0, y_0, f(x_0,y_0))$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \eta(t_0) = \lambda \eta'(t_0) \qquad \lambda \in \mathbb{R}$$

$$\to \begin{cases} x - x_0 = \lambda \gamma_1'(t_0) \leftarrow \\ y - y_0 = \lambda \gamma_2'(t_0) \leftarrow \\ z - f(x_0,y_0) = \underline{\lambda}\,(f_x(x_0,y_0)\gamma_1'(t_0) + f_y(x_0,y_0)\underline{\gamma_2'(t_0)}) \end{cases} \qquad \lambda \in \mathbb{R}$$

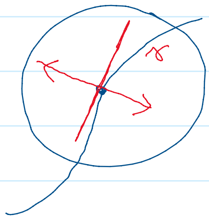
$$\to z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$\frac{y-y_0}{x-x_0} = \frac{\gamma_2'(t_0)}{\gamma_1'(t_0)}$$

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ $\qquad A$ aperto $\qquad \underline{x}_0 \in A$ $\quad$ T.c. f è differenziabile in A

$c := f(\underline{x}_0)$

$$L_c = \{\underline{x} \in A : f(\underline{x}) = c\}$$



Suppongo che $\exists r>0$ t.c. $L_c \cap B_r(x_0)$ sia una curva derivabile

$\gamma: t \in (-r,r) \mapsto \gamma(t) \in \mathbb{R}^n$ derivabile

$\gamma(0) = x_0$

$$g(t) = f(\gamma(t)) = c \qquad g'(t) = 0$$

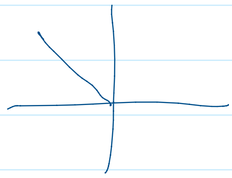
$$0 = g'(0) = \nabla f(\gamma(0)) \cdot \gamma'(0) = \nabla f(x_0) \cdot \gamma'(0)$$

ESERCIZIO $f(x,y) = \ln(xy) + \cos(x+y)$ $\qquad P\left(\frac{\pi}{2}, \frac{\pi}{4}, f\left(\frac{\pi}{2}, \frac{\pi}{4}\right)\right)$

$$f\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \ln\left(\frac{\pi^2}{8}\right) + \cos\left(\frac{3}{4}\pi\right) = \ln\left(\frac{\pi^2}{8}\right) - \frac{\sqrt{2}}{2}$$

$$f_x(x,y) = \frac{1}{xy} y - \sin(x+y) = \frac{1}{x} - \sin(x+y)$$

$$f_x\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \frac{2}{\pi} - \sin\left(\frac{3}{4}\pi\right) = \frac{2}{\pi} - \frac{\sqrt{2}}{2}$$



$$f_y(x,y) = \frac{1}{xy} x - \sin(x+y) = \frac{1}{y} - \sin(x+y)$$

$$f_y\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \frac{4}{\pi} - \sin\left(\frac{3}{4}\pi\right) = \frac{4}{\pi} - \frac{\sqrt{2}}{2}$$

$$z = \ln\left(\frac{\pi^2}{8}\right) - \frac{\sqrt{2}}{2} + \left(\frac{2}{\pi} - \frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{2}\right) + \left(\frac{4}{\pi} - \frac{\sqrt{2}}{2}\right)\left(y - \frac{\pi}{4}\right)$$

ESERCIZIO $\alpha: t \in \mathbb{R} \mapsto (t-1, t+1) \in \mathbb{R}$

$\beta: t \in \mathbb{R} \mapsto (t^2-1, 2) \in \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}$ differenziabile

calcolate in $t=1$

$$\frac{d}{dt}(f\circ\alpha)(t)\Big|_{t=1} = 2 \qquad \frac{d}{dt}(f\circ\beta)(t)\Big|_{t=1} = 3$$

sostituisco $t=1$

$\nabla f(0,2) = ?$

$\alpha(1) = (0,2) \qquad \alpha'(t) = (1,1)$

$\beta(1) = (0,2) \qquad \beta'(t) = (2t, 0)$

$\beta'(1) = (2,0)$

$$2 = \frac{d}{dt}(f\circ\alpha)(t)\Big|_{t=1} = \nabla f(\alpha(1)) \cdot \alpha'(1) = (f_x(0,2), f_y(0,2)) \cdot (1,1) = f_x(0,2) + f_y(0,2)$$

$$\boxed{f_x(0,2) + f_y(0,2) = 2}$$

$$\boxed{f_x(0,2)+f_y(0,2)=2}$$

$$3=\frac{d}{dt}(f\circ\beta)(t)\Big|_{t=1}=\nabla f(\beta(1))\cdot\beta'(1)=\left(f_x(0,2),\,f_y(0,2)\right)\cdot(2,0)=2f_x(0,2)$$

$$\boxed{2f_x(0,2)=3}$$

$$\begin{cases}2f_x(0,2)=3\\ f_x(0,2)+f_y(0,2)=2\end{cases}\qquad f_x(0,2)=3/2$$
$$f_y(0,2)=2-f_x(0,2)=2-\frac{3}{2}=\frac{1}{2}$$

$$\nabla f(0,2)=\left(\frac{3}{2},\frac{1}{2}\right)$$

ESERCIZIO $\rightarrow f(x,y)=x^4e^{xy^2}$ $\quad P(2,1)$ $\quad \underline{v}$ direzione individuata da $\underline{w}=(1,3)$

$$\underline{v}=\frac{\underline{w}}{\|\underline{w}\|}\qquad \|\underline{w}\|=\sqrt{1^2+3^2}=\sqrt{10}$$

$$\underline{v}=\left(\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\right)$$

$f(x,y)=x^4e^{xy^2}$
$P(2,1)$

$$f_x=4x^3\cdot e^{xy^2}+x^4\cdot e^{xy^2}y^2=e^{xy^2}\cdot x^3(4+xy^2)$$
$$f_y=x^4e^{xy^2}2xy=2x^5ye^{xy^2}$$

$$f_x(2,1)=e^2\cdot 8(4+2)=48e^2\qquad \nabla f(2,1)=(48e^2,64e^2)$$
$$f_y(2,1)=64e^2$$

$$D_{\underline{v}}f(2,1)=\nabla f(2,1)\cdot\underline{v}=(48e^2,64e^2)\cdot\left(\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\right)$$
$$=\frac{1}{\sqrt{10}}\left(48e^2+192e^2\right)=\frac{e^2}{\sqrt{10}}240=\frac{e^2}{\cancel{\sqrt{10}}}24\cdot\cancel{10}^{\sqrt{10}}$$
$$=24e^2\sqrt{10}$$

$f: A\subseteq\mathbb{R}^n\to\mathbb{R}$, A aperto - Supponiamo che f è derivabile rispetto ad x_i in ogni pto di A $\quad\exists f_{x_i}(\underline{x})\quad\forall\underline{x}\in A$

$$f_{x_i}:\underline{x}\in A\longmapsto f_{x_i}(\underline{x})\in\mathbb{R}$$

$n=2$ $\quad f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$, A aperto $\quad (x,y)$

$\forall (x,y) \in A \quad f_x(x,y) \qquad f_x : (x,y) \in A \mapsto f_x(x,y) \in \mathbb{R}$

$$\frac{\partial}{\partial x}(f_x) \; ? \qquad \frac{\partial}{\partial y}(f_x) \; ?$$

Se $(x_0,y_0) \in A$ e f_x è derivabile in (x_0,y_0) rispetto a x

cioè se

$$\exists \frac{\partial}{\partial x}(f_x)(x_0,y_0) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right)(x_0,y_0), \text{ la indico}$$

$$\frac{\partial^2 f}{\partial x^2}(x_0\, y_0) \quad \text{o anche} \quad f_{xx}(x_0,y_0)$$

e dico che f è derivabile due volte rispetto a x nel pto (x_0,y_0)

Se esiste $\frac{\partial}{\partial y}(f_x)(x_0,y_0) \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0,y_0)$, la indico

$$\frac{\partial^2}{\partial y\, \partial x} f(x_0,y_0)$$ derivata mista di f rispetto a x e poi rispetto a y

Se $f_y(x,y)$ è definita $\forall (x,y) \in A$, mi posso chiedere se

$$\exists \frac{\partial}{\partial x}(f_y(x,y)) = \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x,y)$$

$$\exists \frac{\partial}{\partial y}(f_y(x,y)) = \frac{\partial}{\partial y} \frac{\partial f}{\partial y}(x,y)$$

Se $\frac{\partial}{\partial x} f_y(x_0,y_0)$ esiste, dico che f è derivabile prima rispetto a y e poi rispetto a x nel pto (x_0,y_0)

$$\frac{\partial^2}{\partial x\, \partial y} f(x_0,y_0)$$

Se $\frac{\partial}{\partial y} f_y(x_0,y_0)$ esiste, dico che f è derivabile due volte rispetto a y nel pto (x_0,y_0) e la indico

$$\frac{\partial^2}{\partial y^2} f(x_0,y_0) \qquad \frac{\partial^2 f}{\partial y^2}(x_0,y_0)$$

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ derivabile in ogni pto di A (A aperto)

$\forall \underline{x} = (x_1 \ldots x_n) \in A$ $\qquad \dfrac{\partial f}{\partial x_i}(\underline{x}) \quad \forall i = 1 \ldots n$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\underline{x}) \qquad j = 1 \ldots n$$

Se $\forall i, j = 1 \ldots n$ esiste la derivata $\dfrac{\partial}{\partial x_j}\left(\dfrac{\partial f}{\partial x_i}\right)(\underline{x}_0)$, $\underline{x}_0 \in A$, dico che f è derivabile due volte nel pto $\underline{x}_0$

ESEMPIO $\quad f(x,y,z) = \underline{x^2 z}\, e^{y-z}$

$$f_x(x,y,z) = 2x\, z e^{y-z}$$
$$f_y(x,y,z) = x^2 z e^{y-z} \Leftarrow$$
$$f_z(x,y,z) = x^2 \cdot e^{y-z} + x^2 z\, e^{y-z} \cdot (-1) = e^{y-z}\left(x^2 - x^2 z\right) =$$
$$= x^2(1-z)e^{y-z}$$

$$\frac{\partial^2 f}{\partial x^2} = 2z e^{y-z}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = 2xze^{y-z} \Leftarrow$$

$$\frac{\partial}{\partial z}\frac{\partial f}{\partial x} = 2x e^{y-z} + 2xz\, e^{y-z}(-1) = 2xe^{y-z}(1-z) \Leftarrow$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}\left(x^2 z e^{y-z}\right) = 2xze^{y-z} \Leftarrow$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \frac{\partial}{\partial x}\left(x^2(1-z)e^{y-z}\right) = 2x(1-z)e^{y-z} \Leftarrow$$

$\dfrac{\partial^2 f}{\partial x^2} \quad f_{xx}$

$\dfrac{\partial^2 f}{\partial x \partial y} = f_{xy}$

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

CALCOLARE LE DERIVATE SECONDE MANCANTI