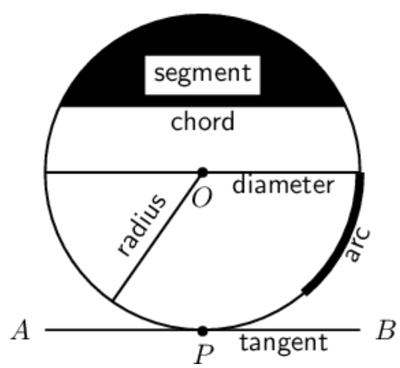
Circle geometry

Terminology

The following terms are regularly used when referring to circles:

- **Arc** — a portion of the circumference of a circle.
- **Chord** — a straight line joining the ends of an arc.
- **Circumference** — the perimeter or boundary line of a circle.
- **Radius (r)** — any straight line from the centre of the circle to a point on the circumference.
- **Diameter** — a special chord that passes through the centre of the circle. A diameter is a straight line segment from one point on the circumference to another point on the circumference that passes through the centre of the circle.
- **Segment** — part of the circle that is cut off by a chord. A chord divides a circle into two segments.
- **Tangent** — a straight line that makes contact with a circle at only one point on the circumference.

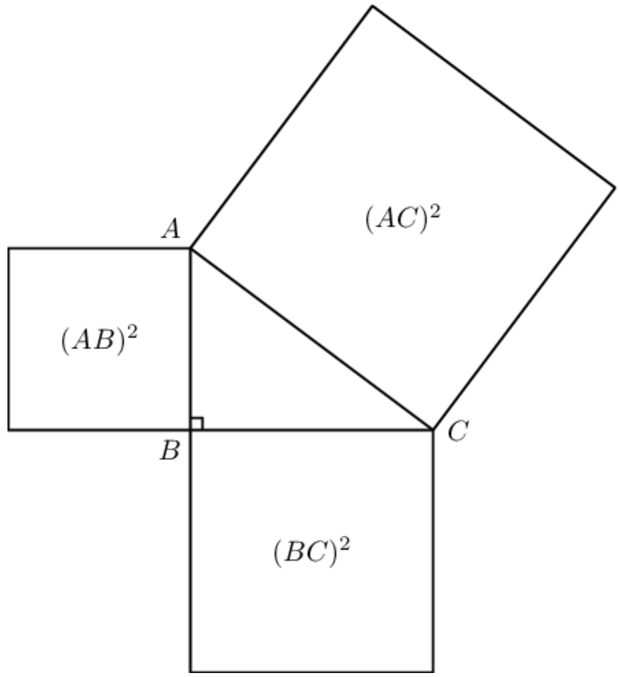


Axioms

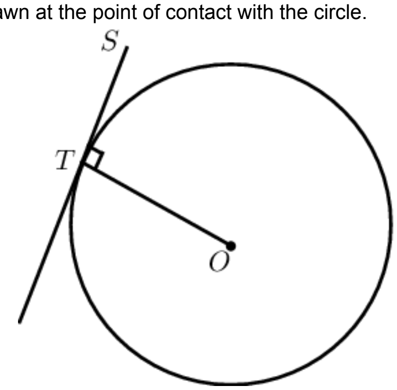
An axiom is an established or accepted principle. For this section, the following are accepted as axioms.

The theorem of Pythagoras states that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

$$(AC)^2 = (AB)^2 + (BC)^2$$



A tangent is perpendicular to the radius ($OT \perp ST$), drawn at the point of contact with the circle.



Theorems

A theorem is a hypothesis (proposition) that can be shown to be true by accepted mathematical operations and arguments. A proof is the process of showing a theorem to be correct.

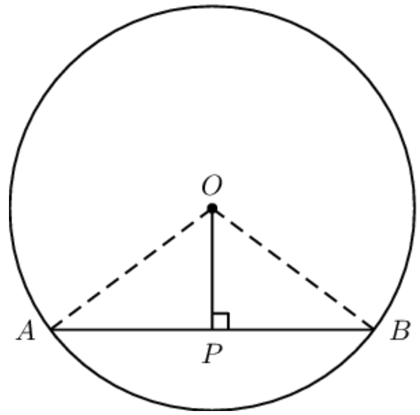
The converse of a theorem is the reverse of the hypothesis and the conclusion. For example, given the theorem “if A, then B”, the converse is “if B, then A”.

Perpendicular line from circle centre bisects chord

If a line is drawn from the centre of a circle perpendicular to a chord, then it bisects the chord.

(Reason: $\perp$ from centre bisects chord)

Circle with centre O and line OP perpendicular to chord AB.



$AP = PB$

Proof

Draw OA and OB.

In $\triangle OPA$ and in $\triangle OPB$,

$$\begin{aligned} OA^2 &= OP^2 + AP^2 \quad &(\text{Pythagoras}) \\ OB^2 &= OP^2 + BP^2 \quad &(\text{Pythagoras}) \end{aligned}$$

and

$$\begin{aligned} OA &= OB \quad &(\text{equal radii}) \\ \therefore AP^2 &= BP^2 & \\ \therefore AP &= BP & \end{aligned}$$

Therefore OP bisects AB.

Alternative proof:

In $\triangle OPA$ and in $\triangle OPB$,

$$\begin{aligned} O\hat{P}A &= O\hat{P}B \quad &(\text{given } OP \perp AB) \\ OA &= OB &(\text{equal radii}) \\ OP &= OP &(\text{common side}) \\ \therefore \triangle OPA &\equiv \triangle OPB &(\text{RHS}) \\ \therefore AP &= PB & \end{aligned}$$

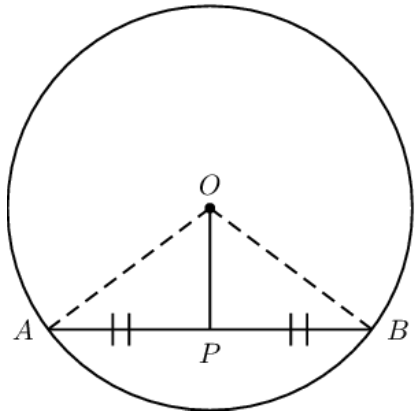
Therefore OP bisects AB.

(PROOF NOT FOR EXAMS) Converse **Line from circle centre to mid-point of chord is perpendicular**

If a line is drawn from the centre of a circle to the mid-point of a chord, then the line is perpendicular to the chord.

(Reason: line from centre to mid-point $\perp$)

Circle with centre O and line OP to mid-point P on chord AB.



$OP \perp AB$

Proof

Draw OA and OB.

In $\triangle OPA$ and in $\triangle OPB$,

$$\begin{aligned} OA &= OB && \text{(equal radii)} \\ AP &= PB && \text{(given)} \\ OP &= OP && \text{(common side)} \\ \therefore \triangle OPA &\equiv \triangle OPB && \text{(SSS)} \\ \therefore O\hat{P}A &= O\hat{P}B && \\ \text{and } O\hat{P}A + O\hat{P}B &= 180^\circ && (\angle \text{ on str. line}) \\ \therefore O\hat{P}A = O\hat{P}B &= 90^\circ && \end{aligned}$$

Therefore $OP \perp AB$.

Perpendicular bisector of chord passes through circle centre

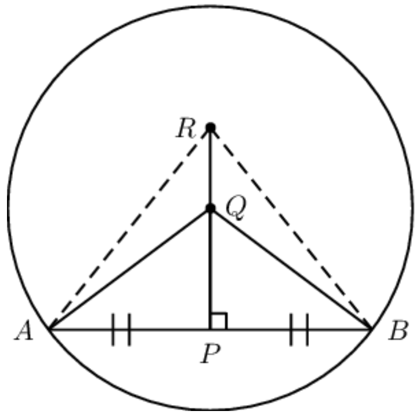
If the perpendicular bisector of a chord is drawn, then the line will pass through the centre of the circle.

(Reason: $\perp$ bisector through centre)

Circle with mid-point P on chord AB.

Line QP is drawn such that $Q\hat{P}A = Q\hat{P}B = 90^\circ$.

Line RP is drawn such that $R\hat{P}A = R\hat{P}B = 90^\circ$.



Circle centre O lies on the line PR

Proof

Draw lines QA and QB.

Draw lines RA and RB.

In $\triangle QPA$ and in $\triangle QPB$,

$$\begin{aligned} AP &= PB && \text{(given)} \\ QP &= QP && \text{(common side)} \\ Q\hat{P}A = Q\hat{P}B &= 90^\circ && \text{(given)} \\ \therefore \triangle QPA &\equiv \triangle QPB && \text{(SAS)} \\ \therefore QA &= QB \end{aligned}$$

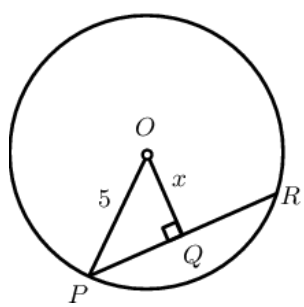
Similarly it can be shown that in $\triangle RPA$ and in $\triangle RPB$, $RA = RB$.

We conclude that all the points that are equidistant from A and B will lie on the line PR extended. Therefore the centre O, which is equidistant to all points on the circumference, must also lie on the line PR.

Example 1: Perpendicular line from circle centre bisects chord

Question

Given $OQ \perp PR$ and $PR = 8$ units, determine the value of x.



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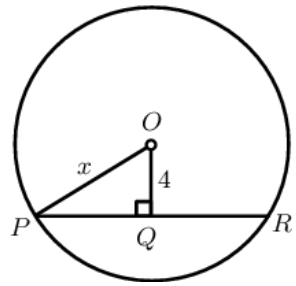
Exercise 1: Perpendicular line from center bisects chord

Problem 1:

In the circle with centre O, $OQ \perp PR$, $OQ = 4$ units and $PR = 10$. Determine x.



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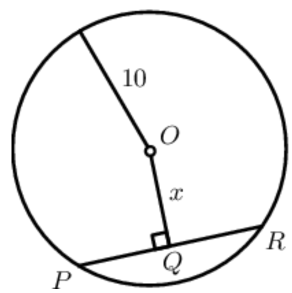
Problem 2:

In the circle with centre O and radius $= 10$ units, $OQ \perp PR$ and $PR = 8$. Determine x.



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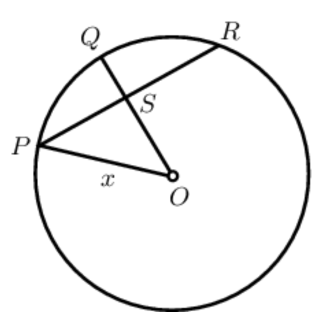
Problem 3:

In the circle with centre O, $OQ \perp PR$, $PR = 12$ units and $SQ = 2$ units. Determine x.



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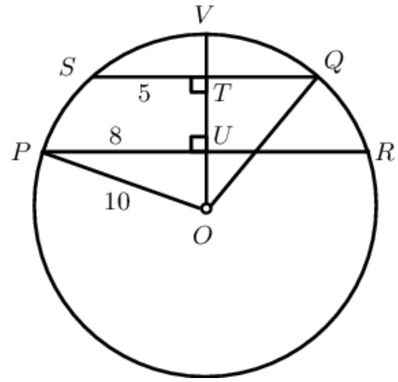
Problem 4:

In the circle with centre O, $OT \perp SQ$, $OT \perp PR$, $OP = 10$ units, $ST = 5$ units and $PU = 8$ units. Determine TU.



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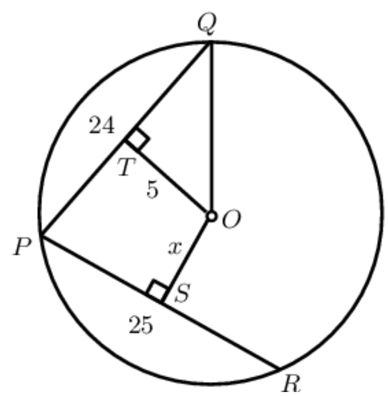
Problem 5:

In the circle with centre O, $OT \perp QP$, $OS \perp PR$, $OT = 5$ units, $PQ = 24$ units and $PR = 25$ units. Determine $OS = x$.



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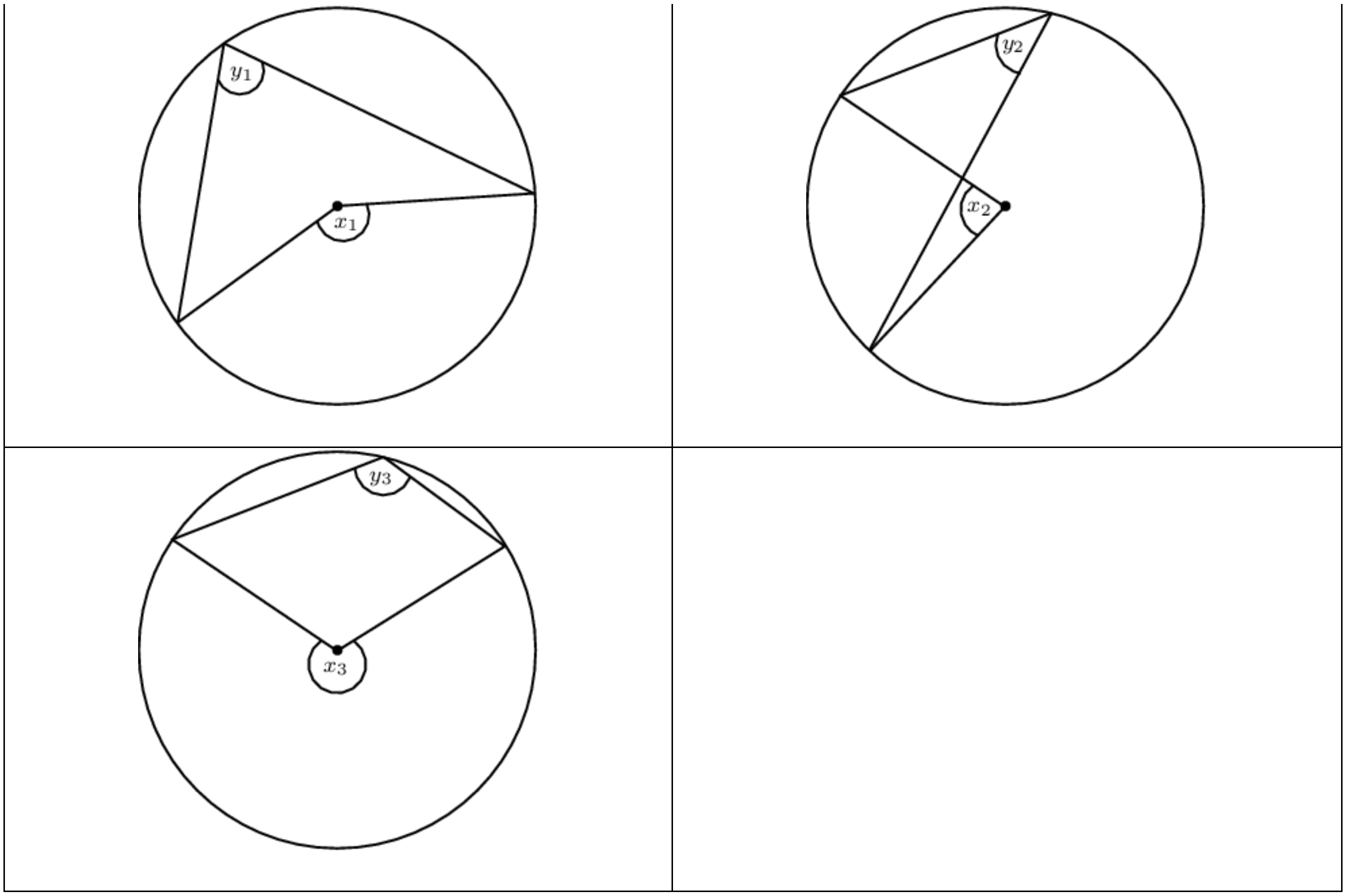
Activity 1: Investigation: Angles subtended by an arc at the centre and the circumference of a circle

Measure angles x and y in each of the following graphs:



Complete the table:

x	y

Use your results to make a conjecture about the relationship between angles subtended by an arc at the centre of a circle and angles at the circumference of a circle.

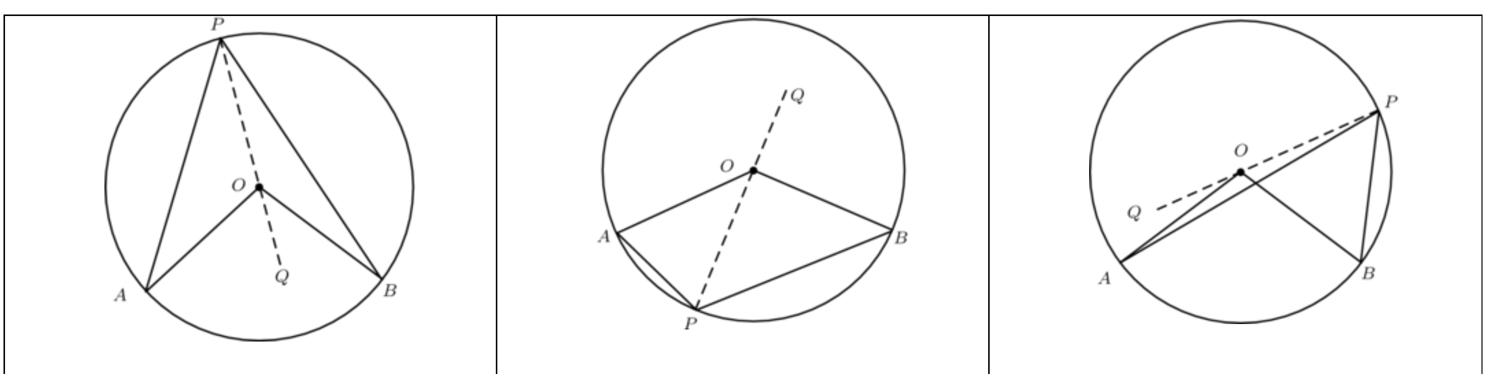
Now draw three of your own similar diagrams and measure the angles to check your conjecture.

Angle at the centre of a circle is twice the size of the angle at the circumference

If an arc subtends an angle at the centre of a circle and at the circumference, then the angle at the centre is twice the size of the angle at the circumference.

(Reason: $\angle \text{ at centre } = 2\angle \text{ at circum.}$)

Circle with centre O, arc AB subtending $A\hat{O}B$ at the centre of the circle, and $A\hat{P}B$ at the circumference.



$A\hat{O}B = 2A\hat{P}B$

Proof

Draw PO extended to Q and let $A\hat{O}Q = \hat{O}_1$ and $B\hat{O}Q = \hat{O}_2$.

Similarly, we can also show that $\hat{O}_2 = 2B\hat{P}O$.

For the first two diagrams shown above we have that:

$$\begin{aligned} A\hat{O}B &= \hat{O}_1 + \hat{O}_2 \\ &= 2A\hat{P}O + 2B\hat{P}O \\ &= 2(A\hat{P}O + B\hat{P}O) \\ \therefore A\hat{O}B &= 2(A\hat{P}B) \end{aligned}$$

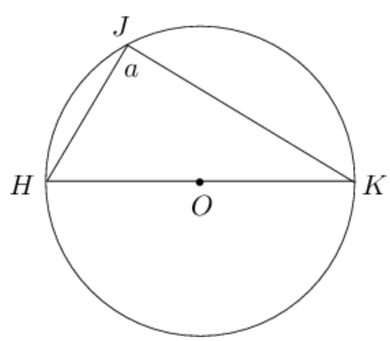
And for the last diagram:

$$\begin{aligned} A\hat{O}B &= \hat{O}_2 - \hat{O}_1 \\ &= 2B\hat{P}O - 2A\hat{P}O \\ &= 2(B\hat{P}O - A\hat{P}O) \\ \therefore A\hat{O}B &= 2(A\hat{P}B) \end{aligned}$$

Example 2: Angle at the centre of circle is twice angle at circumference

Question

Given HK, the diameter of the circle passing through centre O.



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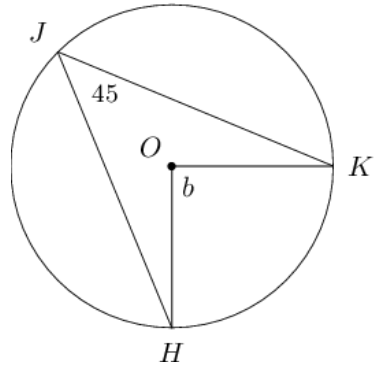
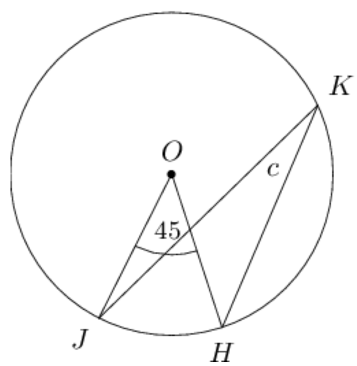
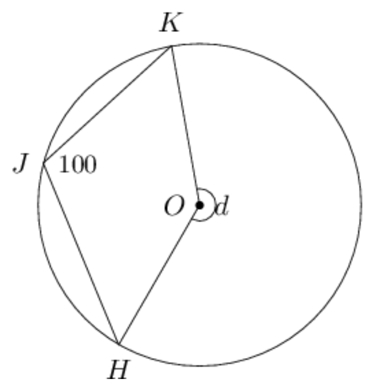
Exercise 2: Angle at the centre of circle is twice angle at circumference

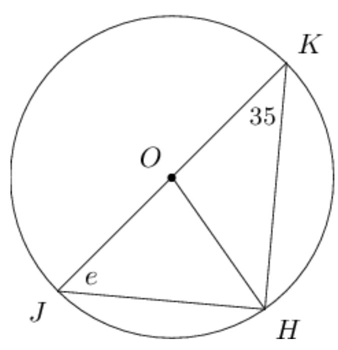
Problem 1:

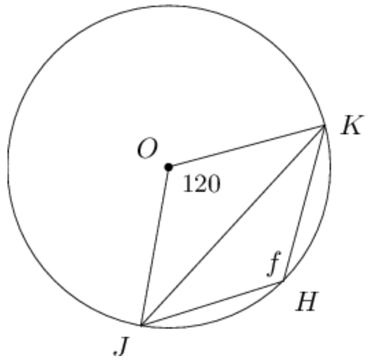
Given O is the centre of the circle, determine the unknown angle in each of the following diagrams:











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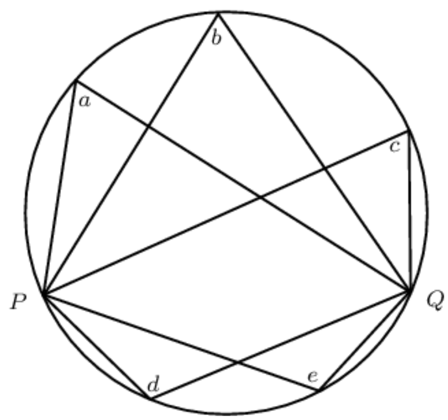
Activity 2: Investigation: Subtended angles in the same segment of a circle

Measure angles a, b, c, d and e in the diagram below:



Choose any two points on the circumference of the circle and label them A and B.

Draw AP and BP, and measure $A\hat{P}B$.

Draw AQ and BQ, and measure $A\hat{Q}B$.

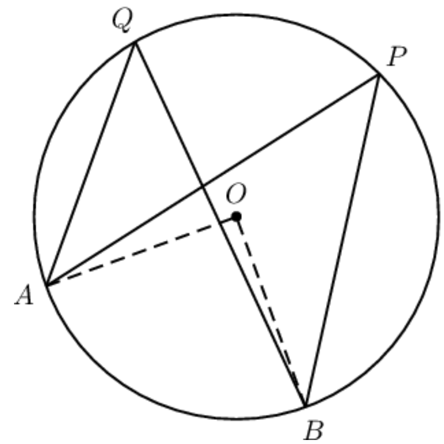
What do you observe? Make a conjecture about these types of angles.

Subtended angles in the same segment of a circle are equal

If the angles subtended by a chord of the circle are on the same side of the chord, then the angles are equal.

(Reason: $\angle$s in same seg.)

Circle with centre O, and points P and Q on the circumference of the circle. Arc AB subtends $A\hat{P}B$ and $A\hat{Q}B$ in the same segment of the circle.

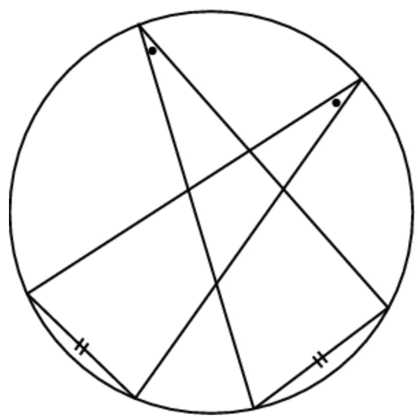


$A\hat{P}B = A\hat{Q}B$

Proof

Equal arcs subtend equal angles

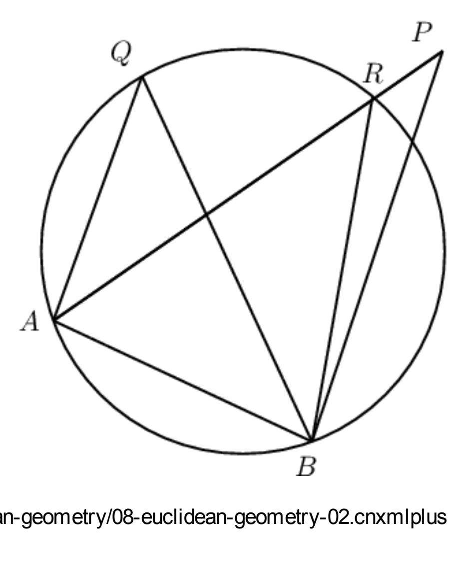
From the theorem above we can deduce that if angles at the circumference of a circle are subtended by arcs of equal length, then the angles are equal. In the figure below, notice that if we were to move the two chords with equal length closer to each other, until they overlap, we would have the same situation as with the theorem above. This shows that the angles subtended by arcs of equal length are also equal.



(PROOF NOT FOR EXAMS) Converse **Concyclic points**

If a line segment subtends equal angles at two other points on the same side of the line segment, then these four points are concyclic (lie on a circle).

Line segment AB subtending equal angles at points P and Q on the same side of the line segment AB.



A, B, P and Q lie on a circle.

Proof

Proof by contradiction:

Points on the circumference of a circle: we know that there are only two possible options regarding a given point — it either lies on circumference or it does not.

We will assume that point P does not lie on the circumference.

We draw a circle that cuts AP at R and passes through A, B and Q.

$$\begin{aligned} A\hat{Q}B &= A\hat{R}B && (\angle\text{s in same seg.}) \\ \text{but } A\hat{Q}B &= A\hat{P}B && (\text{given}) \\ \therefore A\hat{R}B &= A\hat{P}B && \\ \text{but } A\hat{R}B &= A\hat{P}B + R\hat{B}P && (\text{ext. } \angle\triangle = \text{sum int. opp.}) \\ \therefore R\hat{B}P &= 0^\circ && \end{aligned}$$

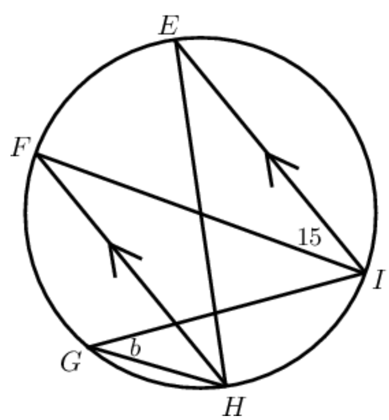
Therefore the assumption that the circle does not pass through P must be false.

We can conclude that A, B, Q and P lie on a circle (A, B, Q and P are concyclic).

Example 3: Concyclic points

Question

Given $FH \parallel EI$ and $E\hat{I}F = 15^\circ$, determine the value of b.



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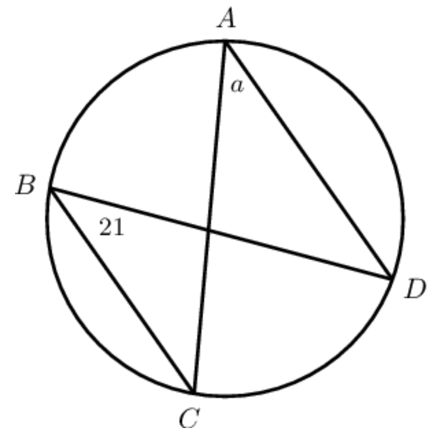
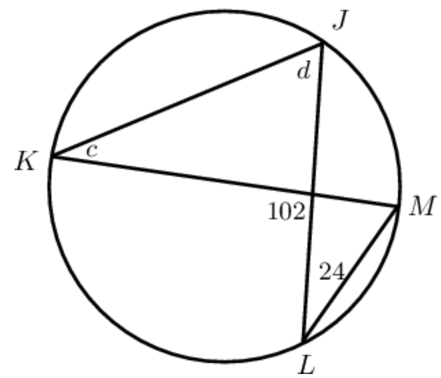
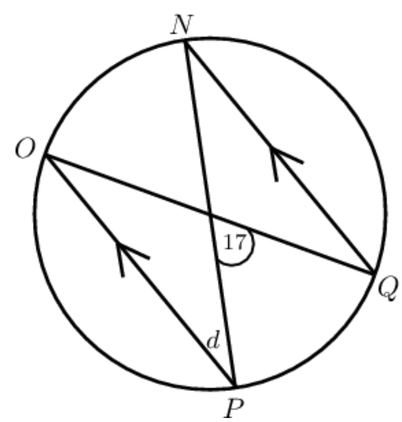
Exercise 3: Subtended angles in the same segment

Problem 1:

Find the values of the unknown angles.







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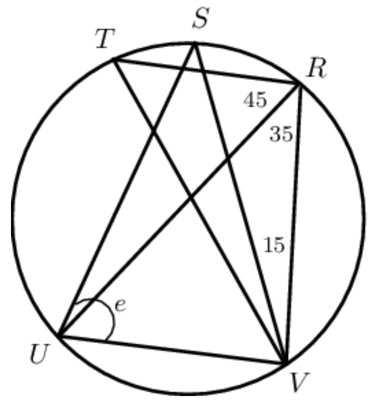
Problem 2:



Given $T\hat{V}S = S\hat{V}R$, determine the value of e.

Is TV a diameter of the circle? Explain your answer.

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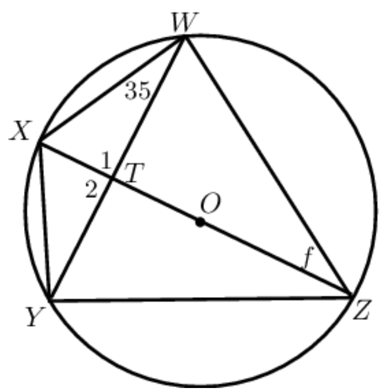
Problem 3:

Given circle with centre O, $WT = TY$ and $X\hat{W}T = 35^{\circ}$. Determine f.



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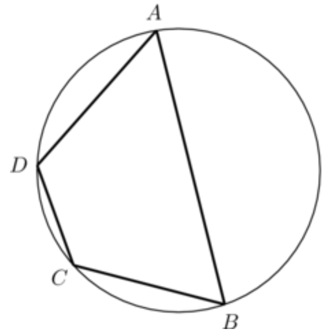
Cyclic quadrilaterals

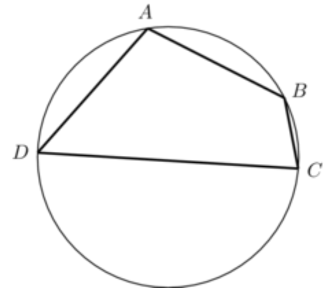
Cyclic quadrilaterals are quadrilaterals with all four vertices lying on the circumference of a circle (concyclic).

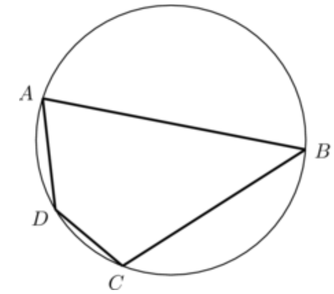
Activity 3: Investigation: Cyclic quadrilaterals

Consider the diagrams given below:

Circle 1	Circle 2	Circle 3







Complete the following: $ABCD$ is a cyclic quadrilateral because $\ldots\ldots$

Complete the table:

	Circle 1	Circle 2	Circle 3
$\hat{A} =$			
$\hat{B} =$			
$\hat{C} =$			
$\hat{D} =$			
$\hat{A} + \hat{C} =$			
$\hat{B} + \hat{D} =$			

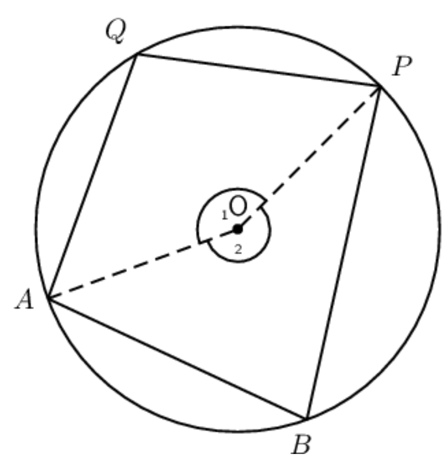
Use your results to make a conjecture about the relationship between angles of cyclic quadrilaterals.

Opposite angles of a cyclic quadrilateral

The opposite angles of a cyclic quadrilateral are supplementary.

(Reason: opp. $\angle$s cyclic quad.)

Circle with centre O with points A, B, P and Q on the circumference such that $ABPQ$ is a cyclic quadrilateral.



$A\hat{B}P + A\hat{Q}P = 180^{\circ}$ and $Q\hat{A}B + Q\hat{P}B = 180^{\circ}$

Proof

Draw AO and OP. Label $\hat{O}_1$ and $\hat{O}_2$.

$$\begin{aligned}
\hat{O}_1 &= 2A\hat{B}P \quad (\angle \text{ at centre} = 2\angle \text{ at circum.}) \\
\hat{O}_2 &= 2A\hat{Q}P \quad (\angle \text{ at centre} = 2\angle \text{ at circum.}) \\
\text{and } \hat{O}_1 + \hat{O}_2 &= 360^\circ \quad (\angle\text{s around a point}) \\
\therefore 2A\hat{B}P + 2A\hat{Q}P &= 360^\circ \\
A\hat{B}P + A\hat{Q}P &= 180^\circ
\end{aligned}$$

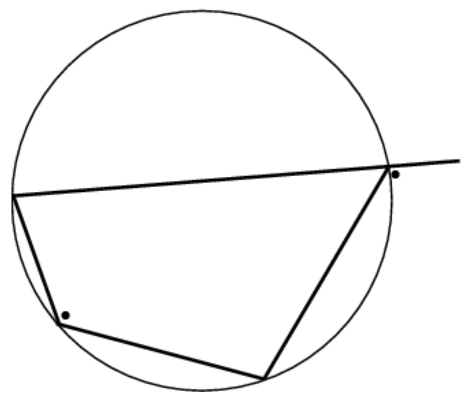
Similarly, we can show that $Q\hat{A}B + Q\hat{P}B = 180^\circ$.

Converse: interior opposite angles of a quadrilateral

If the interior opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Exterior angle of a cyclic quadrilateral

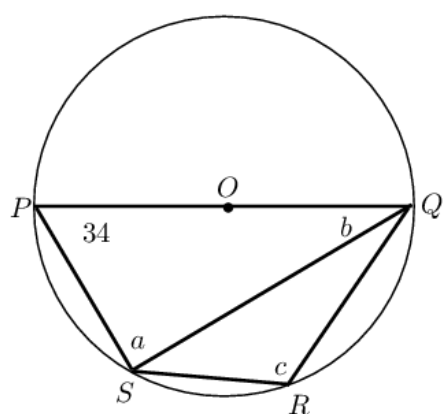
If a quadrilateral is cyclic, then the exterior angle is equal to the interior opposite angle.



Example 4: Opposite angles of a cyclic quadrilateral

Question

Given the circle with centre O and cyclic quadrilateral $PQRS$. SQ is drawn and $S\hat{P}Q = 34^\circ$. Determine the values of a, b and c.



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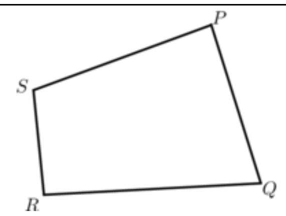
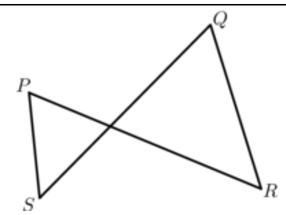
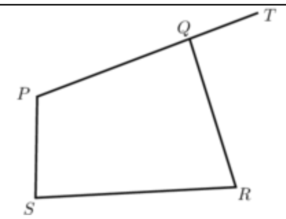
Methods for proving a quadrilateral is cyclic

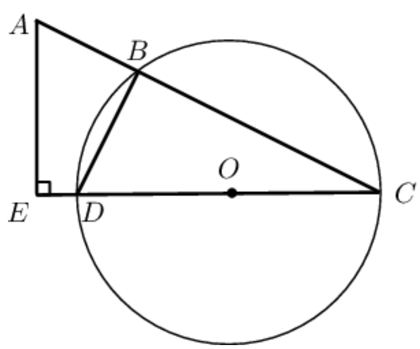
There are three ways to prove that a quadrilateral is a cyclic quadrilateral:

	Method of proof	Reason
	If $\hat{P} + \hat{R} = 180^{\circ}$ or $\hat{S} + \hat{Q} = 180^{\circ}$, then $PQRS$ is a cyclic quad.	opp. int. angles suppl.
	If $\hat{P} = \hat{Q}$ or $\hat{S} = \hat{R}$, then $PQRS$ is a cyclic quad.	angles in the same seg.
	If $T\hat{Q}R = \hat{S}$, then $PQRS$ is a cyclic quad.	ext. angle equal to int. opp. angle

Example 5: Proving a quadrilateral is a cyclic quadrilateral

Question

Prove that $ABDE$ is a cyclic quadrilateral.

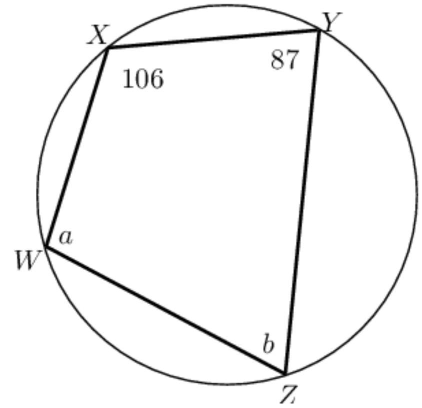


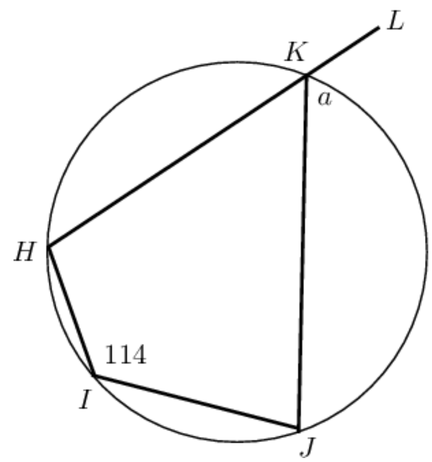
Show me this worked solution

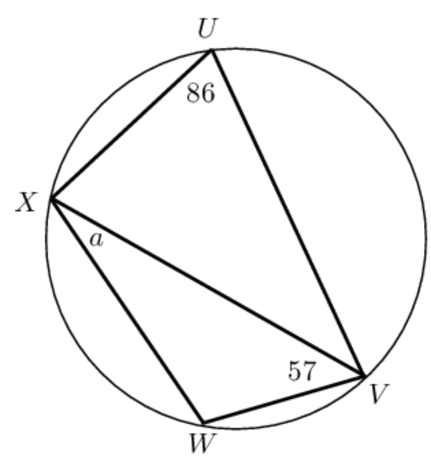
Exercise 4: Cyclic quadrilaterals

Problem 1:

Find the values of the unknown angles.







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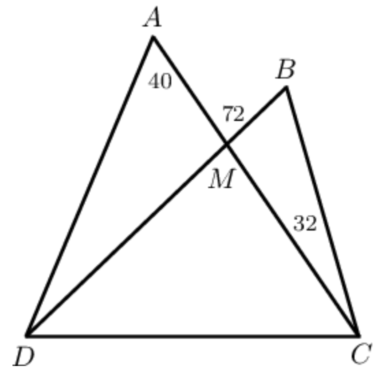
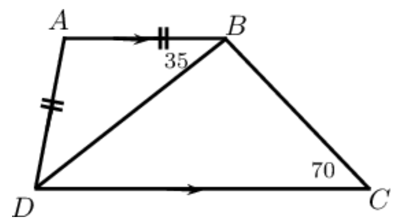
Problem 2:

Prove that $ABCD$ is a cyclic quadrilateral:





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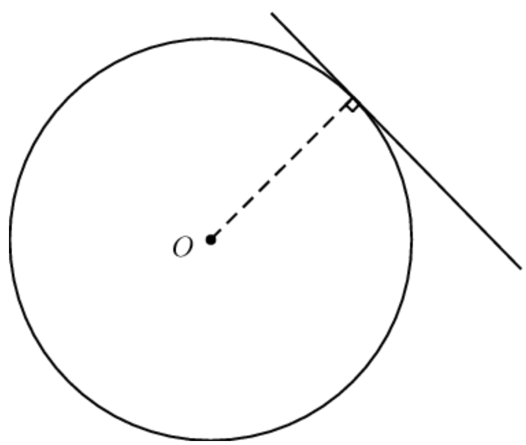
Tangent line to a circle

A tangent is a line that touches the circumference of a circle at only one place. The radius of a circle is perpendicular to the tangent at the point of contact.

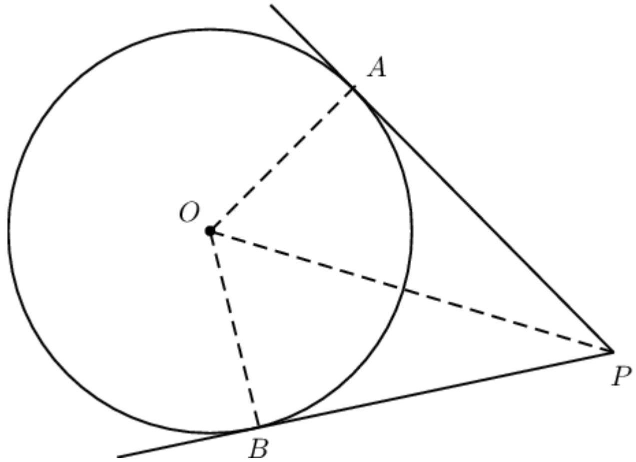


Two tangents drawn from the same point outside a circle

If two tangents are drawn from the same point outside a circle, then they are equal in length.

(Reason: tangents from same point equal)

Circle with centre O and tangents PA and PB, where A and B are the respective points of contact for the two lines.



$AP = BP$

Proof

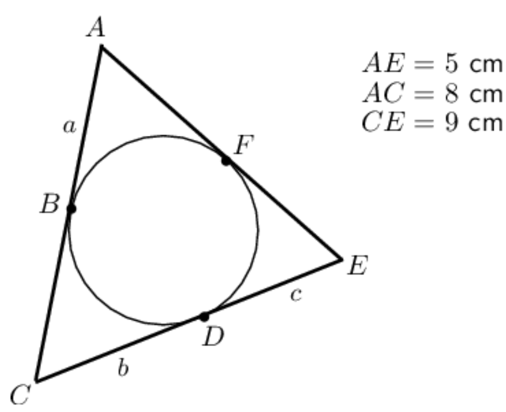
In $\triangle AOP$ and $\triangle BOP$,

$$\begin{aligned} O\hat{A}P = O\hat{B}P &= 90^\circ && \text{(tangent } \perp \text{ radius)} \\ AO &= BO && \text{(equal radii)} \\ OP &= OP && \text{(common side)} \\ \therefore \triangle AOP &\equiv \triangle BOP && \text{(RHS)} \\ \therefore AP &= BP \end{aligned}$$

Example 6: Tangents from the same point outside a circle

Question

In the diagram below $AE = 5\text{ cm}$, $AC = 8\text{ cm}$ and $CE = 9\text{ cm}$. Determine the values of a, b and c.



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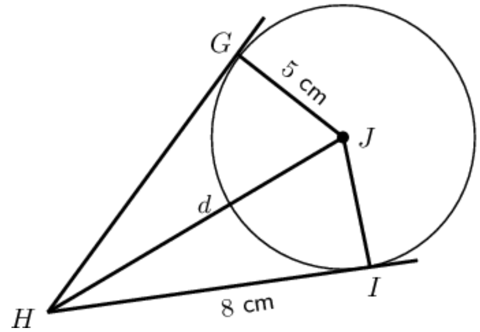
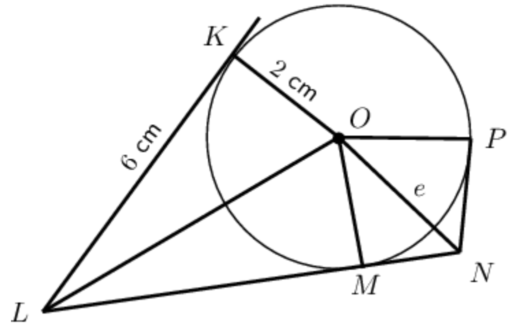
Exercise 5: Tangents to a circle

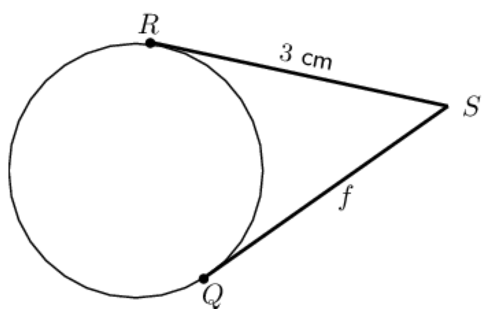
Problem 1:

Find the values of the unknown lengths.





$LN = 7{,}5$ cm



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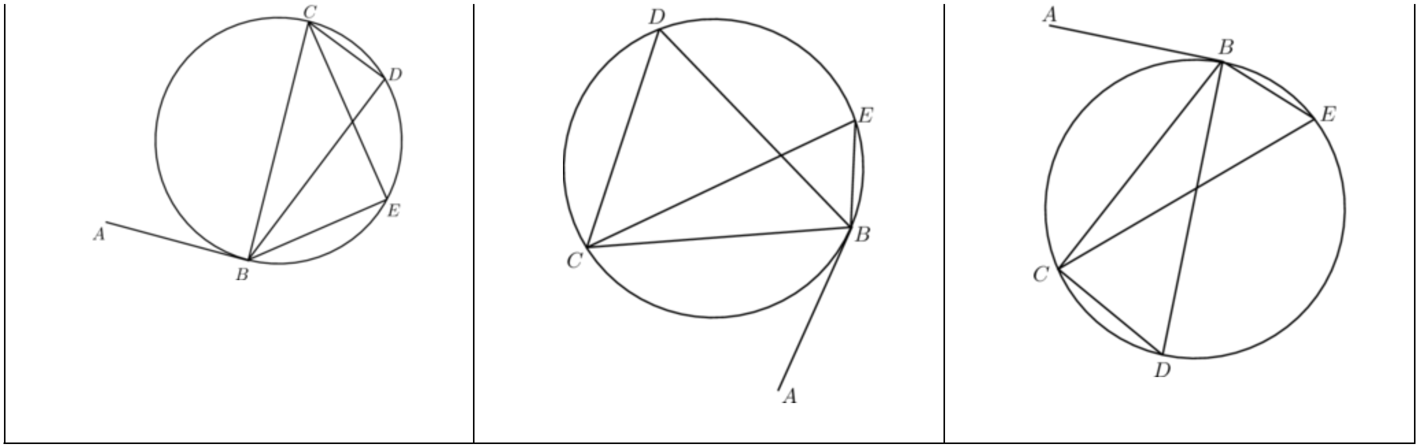
Activity 4: Investigation: Tangent-chord theorem

Consider the diagrams given below:

Diagram 1	Diagram 2	Diagram 3



Measure the following angles with a protractor and complete the table:

	Diagram 1	Diagram 2	Diagram 3
$A\hat{B}C =$			
$\hat{D} =$			
$\hat{E} =$			

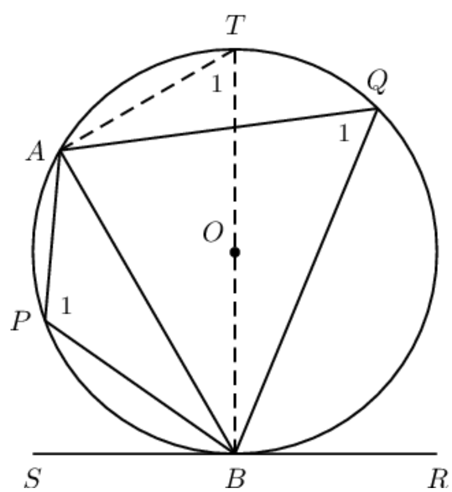
Use your results to complete the following: the angle between a tangent to a circle and a chord is $\ldots\ldots$ to the angle in the alternate segment.

Tangent-chord theorem

The angle between a tangent to a circle and a chord drawn at the point of contact, is equal to the angle which the chord subtends in the alternate segment.

(Reason: tan. chord theorem)

Circle with centre O and tangent SR touching the circle at B. Chord AB subtends $\hat{P}_1$ and $\hat{Q}_1$.



$$A\hat{B}R = A\hat{P}B$$

$$A\hat{B}S = A\hat{Q}B$$

Proof

Draw diameter BT and join T to A.

Let $A\hat{T}B = T_1$.

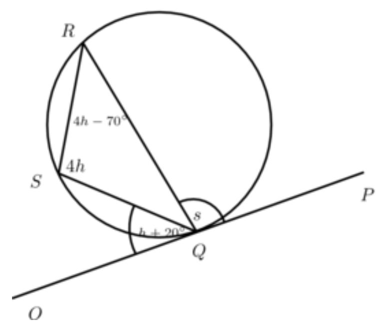
$$\begin{aligned}
A\hat{B}S + A\hat{B}T &= 90^\circ && (\text{tangent} \perp \text{radius}) \\
B\hat{A}T &= 90^\circ && (\angle \text{ in semi circle}) \\
\therefore A\hat{B}T + T_1 &= 90^\circ && (\angle \text{ sum of } \triangle BAT) \\
\therefore A\hat{B}S &= T_1 && \\
\text{but } Q_1 &= T_1 && (\angle\text{s in same segment}) \\
\therefore Q_1 &= A\hat{B}S &&
\end{aligned}$$

$$\begin{aligned}
A\hat{B}S + A\hat{B}R &= 180^\circ && (\angle\text{s on str. line}) \\
\hat{Q}_1 + \hat{P}_1 &= 180^\circ && (\text{opp. } \angle\text{s cyclic quad. supp.}) \\
\therefore A\hat{B}S + A\hat{B}R &= Q_1 + P_1 && \\
\text{and } A\hat{B}S &= Q_1 && \\
\therefore A\hat{B}R &= P_1 &&
\end{aligned}$$

Example 7: Tangent-chord theorem

Question

Determine the values of h and s.

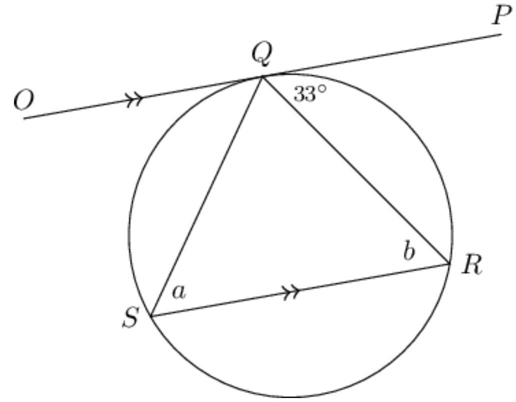


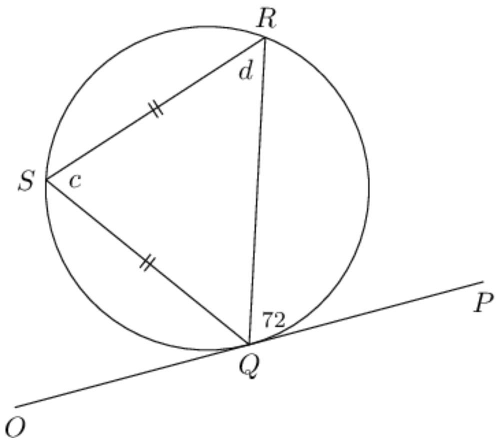
Show me this worked solution

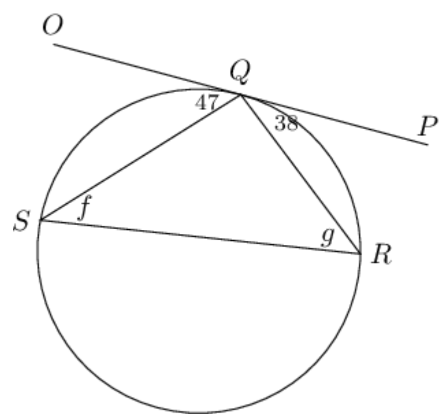
Exercise 6: Tangent-chord theorem

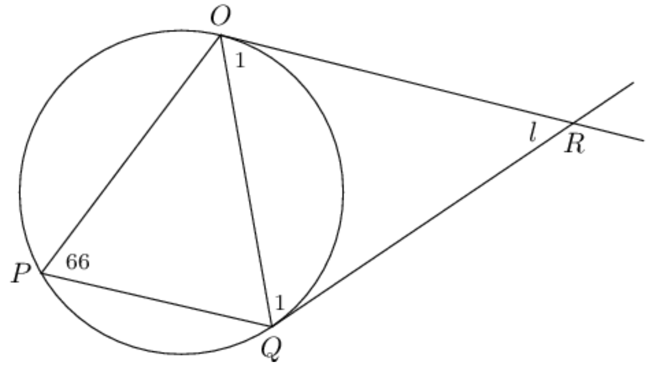
Problem 1:

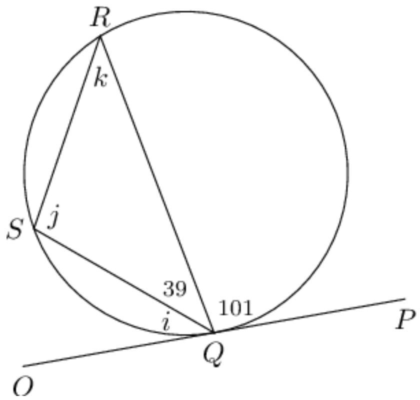
Find the values of the unknown letters, stating reasons.

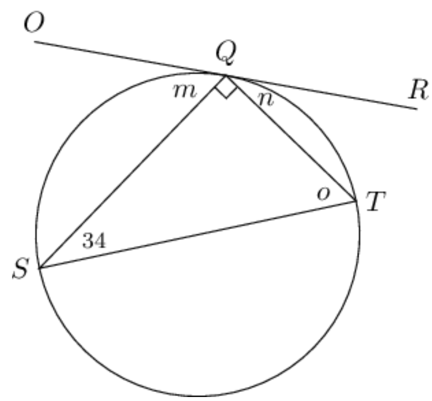

P
Q
O
33°
b
R
a
S

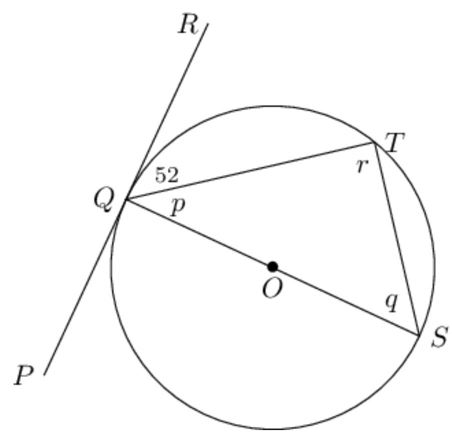

R
d
S
c
72
P
Q
O


O
Q
47
38
P
S
f
g
R


O
1
l
R
P
66
1
Q







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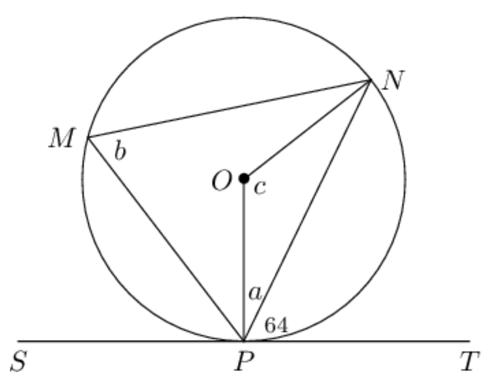
Problem 2:

O is the centre of the circle and SPT is a tangent, with $OP \perp ST$. Determine a, b and c, giving reasons.



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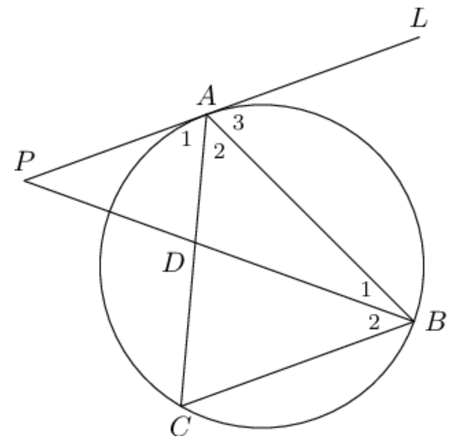
Problem 3:



Given $AB = AC$, $AP \parallel BC$ and $\hat{A}_2 = \hat{B}_2$. Prove:

PAL is a tangent to the circle ABC.

AB is a tangent to the circle ADP.

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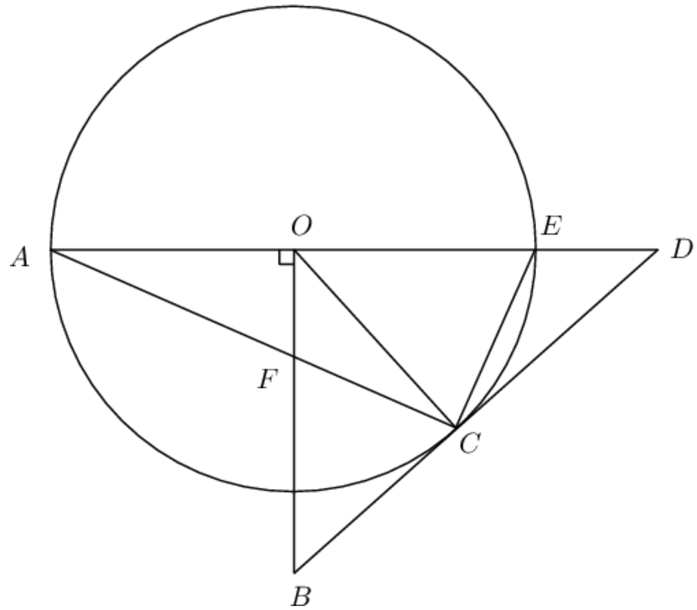
Converse: tangent-chord theorem

If a line drawn through the end point of a chord forms an angle equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

(Reason: $\angle$ between line and chord $= \angle$ in alt. seg.)

Example 8: Applying the theorems

Question



BD is a tangent to the circle with centre O, with $BO \perp AD$.

Prove that:

$CFOE$ is a cyclic quadrilateral

$FB = BC$

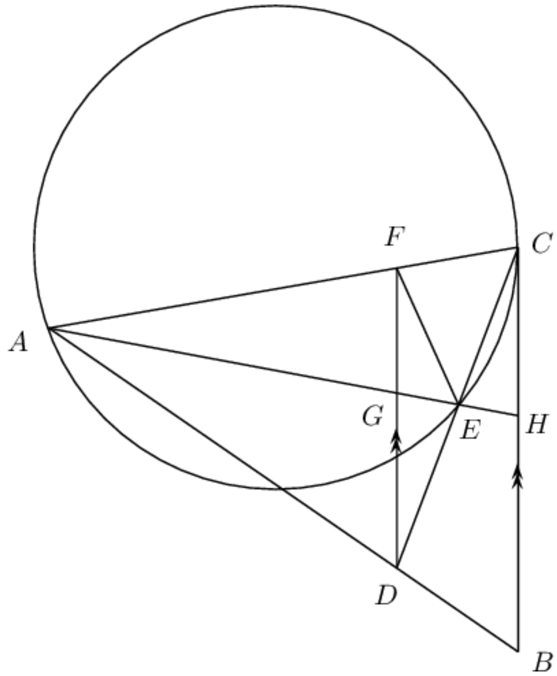
$\angle A\hat{O}C = 2B\hat{F}C$

Will DC be a tangent to the circle passing through C, F, O and E? Motivate your answer.

Show me this worked solution

Example 9: Applying the theorems

Question



FD is drawn parallel to the tangent CB

Prove that:

$FADE$ is a cyclic quadrilateral

$F\hat{E}A = \hat{B}$

Show me this worked solution