Relaxation Methods

We use finite difference approximations to approximate the different forms of Laplace's equation. In 1D, the central difference approximation of the first and second order derivative is given by

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h},$$
$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

1.1 Cartesian Coordinates (2D)

We want to solve

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Using the above approximations, our equation becomes

$$\frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{u(x,y+h') - 2u(x,y) + u(x,y-h')}{h'^2} = 0$$
$$\implies u(x,y) = \frac{1}{2\left(\frac{1}{h^2} + \frac{1}{h'^2}\right)} \left[\frac{1}{h^2}\left(u(x+h,y) + u(x-h,y)\right) + \frac{1}{h'^2}\left(u(x,y+h') + u(x,y-h')\right)\right].$$

So interior points are given by

$$u_{i,j} = \frac{1}{2\left(\frac{1}{h^2} + \frac{1}{h'^2}\right)} \left[\frac{1}{h^2}\left(u_{i+1,j} + u_{i-1,j}\right) + \frac{1}{h'^2}\left(u_{i,j+1} + u_{i,j-1}\right)\right].$$

1.2 Cartesian Coordinates (3D)

Our 2D equation expands to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Introducing this term gives us

$$u_{i,j,k} = \frac{1}{2\left(\frac{1}{h^2} + \frac{1}{h'^2} + \frac{1}{h''^2}\right)} \left[\frac{1}{h^2}\left(u_{i+1,j,k} + u_{i-1,j,k}\right) + \frac{1}{h'^2}\left(u_{i,j+1,k} + u_{i,j-1,k}\right) + \frac{1}{h''^2}\left(u_{i,j,k+1} + u_{i,j-1,k-1}\right)\right]$$

1.3 Polar coordinates

We want to solve

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Using finite difference approximations, our equation becomes

$$\frac{u(r+h,\theta) - 2u(r,\theta) + u(r-h,\theta)}{h^2} + \frac{1}{r}\frac{u(r+h,\theta) - u(r-h,\theta)}{2h} + \frac{1}{r^2}\frac{u(r,\theta+h') - 2u(r,\theta) + u(r,\theta-h')}{h'^2} = 0.$$

Solving for $u(r,\theta)$ we get

$$\frac{1}{h^2}u(r+h,\theta)-\frac{2}{h^2}u(r,\theta)+\frac{1}{h^2}u(r-h,\theta)+\frac{1}{2hr}u(r+h,\theta)-\frac{1}{2hr}u(r-h,\theta)$$
$$+\frac{1}{h'^2r^2}u(r,\theta+h')-\frac{2}{h'^2r^2}u(r,\theta)+\frac{1}{h'^2r^2}u(r,\theta-h')=0.$$

$$\implies 2\Big(\frac{1}{h^2}+\frac{1}{h'^2r^2}\Big)u(r,\theta)=\Big(\frac{1}{h^2}+\frac{1}{2hr}\Big)u(r+h,\theta)+\Big(\frac{1}{h^2}-\frac{1}{2hr}\Big)u(r-h,\theta)+\frac{1}{h'^2r^2}\Big(u(r,\theta+h')+u(r,\theta-h')\Big)$$

$$\implies u(r,\theta)=\frac{1}{2\Big(\frac{1}{h^2}+\frac{1}{h'^2r^2}\Big)}\Bigg[\Big(\frac{1}{h^2}+\frac{1}{2hr}\Big)u(r+h,\theta)+\Big(\frac{1}{h^2}-\frac{1}{2hr}\Big)u(r-h,\theta)+\frac{1}{h'^2r^2}\Big(u(r,\theta+h')+u(r,\theta-h')\Big)\Bigg].$$

On our polar grid, we then have interior points given by

$$u_{r,\theta}=\frac{1}{2\Big(\frac{1}{h^2}+\frac{1}{h'^2r^2}\Big)}\Bigg[\Big(\frac{1}{h^2}+\frac{1}{2hr}\Big)u_{r+1,\theta}+\Big(\frac{1}{h^2}-\frac{1}{2hr}\Big)u_{r-1,\theta}+\frac{1}{h'^2r^2}\Big(u_{r,\theta+1}+u_{r,\theta-1}\Big)\Bigg]$$

1.4 Cylindrical coordinates

We just need to add one slight adjustment to our polar equation:

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+u_{zz}=0,$$

where we approximate u_{zz} as

$$u_{zz}=\frac{u(r,\theta,z+h'')-2u(r,\theta,z)+u(r,\theta,z-h'')}{h''^2}$$

On our cylindrical grid, we then have interior points given by

$$u_{r,\theta}=\frac{1}{2\Big(\frac{1}{h^2}+\frac{1}{h'^2r^2}+\frac{1}{h''^2}\Big)}\Bigg[\Big(\frac{1}{h^2}+\frac{1}{2hr}\Big)u_{r+1,\theta,z}+\Big(\frac{1}{h^2}-\frac{1}{2hr}\Big)u_{r-1,\theta,z}+\frac{1}{h'^2r^2}\Big(u_{r,\theta+1,z}+u_{r,\theta-1,z}\Big)$$
$$+\frac{1}{h''^2}\Big(u_{r,\theta,z+1}+u_{r,\theta,z-1}\Big)\Bigg]$$

1.5 Spherical coordinates

We want to solve

$$\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial f}{\partial r}\big)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial f}{\partial\theta}\big)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}=0$$
$$\implies \frac{1}{r^2}\big(2r\frac{\partial f}{\partial r}+r^2\frac{\partial^2 f}{\partial r^2}\big)+\frac{1}{r^2\sin\theta}\big(\cos\theta\frac{\partial f}{\partial\theta}+\sin\theta\frac{\partial^2 f}{\partial\theta^2}\big)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}=0.$$

Using finite differences, our equation becomes

$$\frac{1}{r^2}\Bigg(2r\frac{u(r+h,\theta,\phi)-u(r-h,\theta,\phi)}{2h}+r^2\frac{u(r+h,\theta,\phi)-2u(r,\theta,\phi)+u(r-h,\theta,\phi)}{h^2}\Bigg)$$
$$+\frac{1}{r^2\sin\theta}\Bigg(\cos\theta\frac{u(r,\theta+h',\phi)-u(r,\theta-h',\phi)}{2h'}+\sin\theta\frac{u(r,\theta+h',\phi)-2u(r,\theta,\phi)+u(r,\theta-h')}{h'^2}\Bigg)$$
$$+\frac{1}{r^2\sin^2\theta}\Bigg(\frac{u(r,\theta,\phi+h'')-2u(r,\theta,\phi)+u(r,\theta,\phi-h'')}{h''^2}\Bigg)=0.$$

This simplifies to:

$$
\begin{aligned}
&\left(\frac{2}{h^2}+\frac{2}{r^2h'^2}+\frac{2}{r^2\sin^2\theta h''^2}\right)u(r,\theta,\phi)\\
&=\frac{1}{r^2}\left(2r\frac{u(r+h,\theta,\phi)-u(r-h,\theta,\phi)}{2h}+r^2\frac{u(r+h,\theta,\phi)+u(r-h,\theta,\phi)}{h^2}\right)\\
&+\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{u(r,\theta+h',\phi)-u(r,\theta-h',\phi)}{2h'}+\sin\theta\frac{u(r,\theta+h',\phi)+u(r,\theta-h',\phi)}{h'^2}\right)\\
&+\frac{1}{r^2\sin^2\theta}\left(\frac{u(r,\theta,\phi+h'')+u(r,\theta,\phi-h'')}{h''^2}\right).
\end{aligned}
$$

So our interior points become

$$
\begin{aligned}
&\left(\frac{2}{h^2}+\frac{2}{h'^2r^2}+\frac{2}{r^2\sin^2\theta h''^2}\right)u_{r,\theta,\phi}\\
&=\frac{1}{r^2}\left(2r\frac{u_{r+1,\theta,\phi}-u_{r-1,\theta,\phi}}{2h}+r^2\frac{u_{r+1,\theta,\phi}+u_{r-1,\theta,\phi}}{h^2}\right)\\
&+\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{u_{r,\theta+1,\phi}-u_{r,\theta-1,\phi}}{2h'}+\sin\theta\frac{u_{r,\theta+1,\phi}+u_{r,\theta-1,\phi}}{h'^2}\right)\\
&+\frac{1}{r^2\sin^2\theta}\left(\frac{u_{r,\theta,\phi+1}+u_{r,\theta,\phi-1}}{h''^2}\right),
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
u_{r,\theta,\phi}=&\frac{1}{\left(\frac{2}{h^2}+\frac{2}{h'^2r^2}+\frac{2}{r^2\sin^2\theta h''^2}\right)}\Bigg[\frac{1}{hr}\left(u_{r+1,\theta,\phi}-u_{r-1,\theta,\phi}\right)+\frac{1}{h^2}\left(u_{r+1,\theta,\phi}+u_{r-1,\theta,\phi}\right)\\
&+\frac{1}{2h'r^2\tan\theta}\left(u_{r,\theta+1,\phi}-u_{r,\theta-1,\phi}\right)+\frac{1}{r^2h'^2}\left(u_{r,\theta+1,\phi}+u_{r,\theta-1,\phi}\right)\\
&+\frac{1}{r^2\sin^2\theta h''^2}\left(u_{r,\theta,\phi+1}+u_{r,\theta,\phi-1}\right)\Bigg],
\end{aligned}
$$

1.6 Vector Cartesian form

The vector form of the Laplacian is just given component-wise:

$$
\nabla^2\boldsymbol{A}=\begin{pmatrix}\nabla^2A_x\\ \nabla^2A_y\\ \nabla^2A_z\end{pmatrix}=\boldsymbol{0},
$$

so we can just apply our scalar function component-wise.

1.7 Vector Cylindrical form

We want to solve:

$$
\begin{pmatrix}
\frac{\partial^2A_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2A_r}{\partial\phi^2}+\frac{\partial^2A_r}{\partial z^2}+\frac{1}{r}\frac{\partial A_r}{\partial r}-\frac{2}{r^2}\frac{\partial A_\phi}{\partial\phi}-\frac{A_r}{r^2}\\
\frac{\partial^2A_\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2A_\phi}{\partial\phi^2}+\frac{\partial^2A_\phi}{\partial z^2}+\frac{1}{r}\frac{\partial A_\phi}{\partial r}+\frac{2}{r^2}\frac{\partial A_r}{\partial\phi}-\frac{A_\phi}{r^2}\\
\frac{\partial^2A_z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2A_z}{\partial\phi^2}+\frac{\partial^2A_z}{\partial z^2}+\frac{1}{r}\frac{\partial A_z}{\partial r}
\end{pmatrix}=\boldsymbol{0}.
$$

Expanding out with finite differences, we get the respective components given by

$$\frac{A_r(r+h,\phi,z)-2A_r(r,\phi,z)+A_r(r-h,\phi,z)}{h^2}+\frac{A_r(r,\phi+h',z)-2A_r(r,\phi,z)+A_r(r,\phi-h',z)}{r^2h'^2}$$
$$+\frac{A_r(r,\phi,z+h'')-2A_r(r,\phi,z)+A_r(r,\phi,z-h'')}{h''^2}+\frac{A_r(r+h,\phi,z)-A_r(r-h,\phi,z)}{2hr}$$
$$-\frac{A_\phi(r,\phi+h',z)-A_\phi(r,\phi-h',z)}{h'r^2}-\frac{A_r(r,\phi,z)}{r^2}=0,$$

$$\frac{A_\phi(r+h,\phi,z)-2A_\phi(r,\phi,z)+A_\phi(r-h,\phi,z)}{h^2}+\frac{A_\phi(r,\phi+h',z)-2A_\phi(r,\phi,z)+A_\phi(r,\phi-h',z)}{r^2h'^2}$$
$$+\frac{A_\phi(r,\phi,z+h'')-2A_\phi(r,\phi,z)+A_\phi(r,\phi,z-h'')}{h''^2}+\frac{A_\phi(r+h,\phi,z)-A_\phi(r-h,\phi,z)}{2hr}$$
$$+\frac{A_r(r,\phi+h',z)-A_r(r,\phi-h',z)}{h'r^2}-\frac{A_\phi(r,\phi,z)}{r^2}=0,$$

$$\frac{A_z(r+h,\phi,z)-2A_z(r,\phi,z)+A_z(r-h,\phi,z)}{h^2}+\frac{A_z(r,\phi+h',z)-2A_z(r,\phi,z)+A_z(r,\phi-h',z)}{r^2h'^2}$$
$$+\frac{A_z(r,\phi,z+h'')-2A_z(r,\phi,z)+A_z(r,\phi,z-h'')}{h''^2}+\frac{A_z(r+h,\phi,z)-A_z(r-h,\phi,z)}{2hr}=0.$$

NB: The z component can just be calculated by the scalar cylindrical function. With $A_r = R$, $A_\phi = \Phi$, the first component can be written as

$$\left(\frac{2}{h^2}+\frac{2}{r^2h'^2}+\frac{2}{h''^2}+\frac{1}{r^2}\right)R_{r,\phi,z}=\frac{1}{h^2}\left(R_{r+1,\phi,z}+R_{r-1,\phi,z}\right)+\frac{1}{r^2h'^2}\left(R_{r,\phi+1,z}+R_{r,\phi-1,z}\right)$$
$$+\frac{1}{h''^2}\left(R_{r,\phi,z+1}+R_{r,\phi,z-1}\right)+\frac{1}{2hr}\left(R_{r+1,\phi,z}+R_{r-1,\phi,z}\right)$$
$$-\frac{1}{h'r^2}\left(\Phi_{r,\phi+1,z}-\Phi_{r,\phi-1,z}\right),$$

and the second component can be written as

$$\left(\frac{2}{h^2}+\frac{2}{r^2h'^2}+\frac{2}{h''^2}+\frac{1}{r^2}\right)\Phi_{r,\phi,z}=\frac{1}{h^2}\left(\Phi_{r+1,\phi,z}+\Phi_{r-1,\phi,z}\right)+\frac{1}{r^2h'^2}\left(\Phi_{r,\phi+1,z}+\Phi_{r,\phi-1,z}\right)$$
$$+\frac{1}{h''^2}\left(\Phi_{r,\phi,z+1}+\Phi_{r,\phi,z-1}\right)+\frac{1}{2hr}\left(\Phi_{r+1,\phi,z}+\Phi_{r-1,\phi,z}\right)$$
$$+\frac{1}{h'r^2}\left(R_{r,\phi+1,z}-R_{r,\phi-1,z}\right).$$

1.8 Vector Spherical Form

Given a vector given in spherical coordinates by $\boldsymbol{v}=(v_r,v_\theta,v_\phi)=(R,\Theta,\Phi)$, the components of $\Delta\boldsymbol{v}$ are given in spherical coordinates by

$$\begin{bmatrix} \frac{2}{r}R_r+R_{rr}+\frac{1}{r^2}R_{\theta\theta}+\frac{1}{r^2\sin^2\theta}R_{\phi\phi}+\frac{\cot\theta}{r^2}R_\theta-\frac{2}{r^2}\Theta_\theta-\frac{2}{r^2\sin\theta}\Phi_\phi-\frac{2}{r^2}R-\frac{2\cot\theta}{r^2}\Theta \\ \frac{2}{r}\Theta_r+\Theta_{rr}+\frac{1}{r^2}\Theta_{\theta\theta}+\frac{1}{r^2\sin^2\theta}\Theta_{\phi\phi}+\frac{\cot\theta}{r^2}\Theta_\theta-\frac{2\cot\theta}{r^2\sin\theta}\Phi_\phi+\frac{2}{r^2}R_\theta-\frac{1}{r^2\sin^2\theta}\Theta \\ \frac{2}{r}\Phi_r+\Phi_{rr}+\frac{1}{r^2}\Phi_{\theta\theta}+\frac{1}{r^2\sin^2\theta}\Phi_{\phi\phi}+\frac{\cot\theta}{r^2}\Phi_\phi+\frac{2}{r^2\sin\theta}R_\phi+\frac{2\cot\theta}{r^2\sin\theta}\Theta_\phi-\frac{1}{r^2\sin^2\theta}\Phi \end{bmatrix}=\mathbf{0}.$$

We approximate using finite differences. The first component becomes:

$$
\begin{aligned}
2\left(\frac{1}{h^2}+\frac{1}{h'^2r^2}+\frac{1}{h''^2r^2\sin^2\theta}+\frac{1}{r^2}\right)R = {} & \frac{1}{rh}\left(R_{r+1,\theta,\phi}-R_{r-1,\theta\phi}\right)+\frac{1}{h^2}\left(R_{r+1,\theta,\phi}+R_{r-1,\theta,\phi}\right)\\
&+\frac{1}{h'^2r^2}\left(R_{r,\theta+1,\phi}+R_{r,\theta-1,\phi}\right)+\frac{1}{h''^2r^2\sin^2\theta}\left(R_{r,\theta,\phi+1}+R_{r,\theta,\phi-1}\right)\\
&+\frac{1}{2h'r^2\tan\theta}\left(R_{r,\theta+1,\phi}+R_{r,\theta-1,\phi}\right)-\frac{1}{h'r^2}\left(\Theta_{r,\theta+1,\phi}-\Theta_{r,\theta-1,\phi}\right)\\
&-\frac{1}{h''r^2\sin\theta}\left(\Phi_{r,\theta,\phi+1}-\Phi_{r,\theta,\phi-1}\right)-\frac{2}{r^2\tan\theta}\Theta.
\end{aligned}
$$

The second component is given by

$$
\begin{aligned}
\left(\frac{2}{h^2}+\frac{2}{h'^2r^2}+\frac{2}{h''^2r^2\sin^2\theta}+\frac{1}{r^2\sin^2\theta}\right)\Theta = {} & \frac{1}{rh}\left(\Theta_{r+1,\theta,\phi}-\Theta_{r-1,\theta,\phi}\right)+\frac{1}{h^2}\left(\Theta_{r+1,\theta,\phi}+\Theta_{r-1,\theta,\phi}\right)\\
&+\frac{1}{h'^2r^2}\left(\Theta_{r,\theta+1,\phi}+\Theta_{r,\theta-1,\phi}\right)+\frac{1}{h''^2r^2\sin^2\theta}\left(\Theta_{r,\theta,\phi+1}-\Theta_{r,\theta,\phi-1}\right)\\
&+\frac{1}{2h'r^2\tan\theta}\left(\Theta_{r,\theta+1,\phi}-\Theta_{r,\theta-1,\phi}\right)-\frac{\cot\theta}{h''r^2\sin\theta}\left(\Phi_{r,\theta,\phi+1}-\Phi_{r,\theta,\phi-1}\right)\\
&+\frac{1}{r^2h'}\left(R_{r,\theta+1,\phi}-R_{r,\theta-1,\phi}\right).
\end{aligned}
$$

Finally the third component is given by

$$
\begin{aligned}
\left(\frac{2}{h^2}+\frac{2}{h'^2r^2}+\frac{2}{h''^2r^2\sin^2\theta}+\frac{1}{r^2\sin^2\theta}\right)\Phi = {} & \frac{1}{rh}\left(\Phi_{r+1,\theta,\phi}-\Phi_{r-1,\theta,\phi}\right)+\frac{1}{h^2}\left(\Phi_{r+1,\theta,\phi}+\Phi_{r-1,\theta,\phi}\right)\\
&+\frac{1}{h'^2r^2}\left(\Phi_{r,\theta+1,\phi}+\Phi_{r,\theta-1,\phi}\right)+\frac{1}{h''^2r^2\sin^2\theta}\left(\Phi_{r,\theta,\phi+1}+\Phi_{r,\theta,\phi-1}\right)\\
&+\frac{1}{2h'r^2\tan\theta}\left(\Phi_{r,\theta+1,\phi}-\Phi_{r,\theta-1,\phi}\right)+\frac{1}{h''r^2\sin\theta}\left(R_{r,\theta,\phi+1}-R_{r-1,\theta,\phi-1}\right)\\
&+\frac{\cot\theta}{h''r^2\sin\theta}\left(\Theta_{r,\theta,\phi+1}-\Theta_{r-1,\theta,\phi-1}\right).
\end{aligned}
$$

Tests

1.9 Vector Cartesian form

Given a known vector field $\boldsymbol{\omega}$, suppose we want to find an unknown vector field $\boldsymbol{u}$ given $\nabla \times \boldsymbol{u} = \boldsymbol{\omega}$ and $\nabla \cdot \boldsymbol{u} = 0$. We transform this to the Laplace equation by taking the curl of each side and then using the common vector calculus identity: $\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}$ for any vector $\boldsymbol{v}$.

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{u}) &= \nabla \times \boldsymbol{\omega} \\ \implies -\nabla^2 \boldsymbol{u} + \nabla(\nabla \cdot \boldsymbol{u}) &= \nabla \times \boldsymbol{\omega} \\ \implies \nabla^2 \boldsymbol{u} &= -\nabla \times \boldsymbol{\omega}. \end{aligned}$$

In Cartesian coordinates, we set $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega} = (x^2 + y^2) e^{-x^2 - y^2} \boldsymbol{e}_z,$$

so we have

$$\begin{aligned} \nabla \times \boldsymbol{\omega} &= \left(2ye^{-x^2-y^2} - 2y(x^2+y^2)e^{-x^2-y^2}\right)\boldsymbol{e}_x + \left(2xe^{-x^2-y^2} - 2x(x^2+y^2)e^{-x^2-y^2}\right)\boldsymbol{e}_y \\ &= \left(1 - (x^2+y^2)\right) 2ye^{-x^2-y^2}\boldsymbol{e}_x + \left(1 - (x^2+y^2)\right) 2xe^{-x^2-y^2}\boldsymbol{e}_y. \end{aligned}$$

So in component form we have:

$$\begin{aligned} \nabla^2 u_x &= \left(x^2 + y^2 - 1\right) 2ye^{-x^2-y^2}, \\ \nabla^2 u_y &= \left(x^2 + y^2 - 1\right) 2xe^{-x^2-y^2}, \\ \nabla^2 u_z &= 0. \end{aligned}$$

We are turning our Laplace equation into Poission's equation so, for example, our x component equation becomes

$$u_{i,j,k} = \frac{1}{2\left(\frac{1}{h^2} + \frac{1}{h'^2} + \frac{1}{h''^2}\right)} \left[\frac{1}{h^2}\left(u_{i+1,j,k} + u_{i-1,j,k}\right) + \frac{1}{h'^2}\left(u_{i,j+1,k} + u_{i,j-1,k}\right) + \frac{1}{h''^2}\left(u_{i,j,k+1} + u_{i,j-1,k-1}\right) - h_x \right],$$

where $h_x = \left(x^2 + y^2 - 1\right) 2ye^{-x^2-y^2}$.