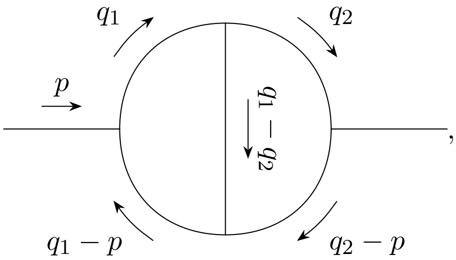
In this manuscript I would like to carry out a two-loop calculation that I did as homework at some point. The calculation will be a general example of loop calculations in quantum field theory, as well as a good demonstration of dimensional regularization and integration-by-parts identities (IBPs).

We are working in ϕ^3 theory, though we will be ignoring all its features and focus solely on the loop calculation. Consider the two-loop

$I(p^2) =$ 

, (1)

where for simplicity we assume that everything is massless. Remember that despite this, we cannot write $k^2 = 0$ for any momentum k since nothing here is on-shell. With the given momentum labeling we can write

$$I(p^2) = \int \frac{d^d q_1}{(2\pi)^d} \int \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_1^2 (q_1 - p)^2 (q_1 - q_2)^2 q_2^2 (q_2 - p)^2}. \tag{2}$$

To solve eq. (2), we will make use of the so-called Feynman parametrization without proof,

$$\frac{1}{P_1^{a_1} P_2^{a_2} \dots P_n^{a_n}} = \frac{\Gamma(a_1 + a_2 + \cdots + a_n)}{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_n)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-1}} dx_n \\ \times \delta(1 - x_1 - \cdots - x_n) \frac{x_1^{a_1-1} x_2^{a_2-1} \dots x_n^{a_n-1}}{(x_1 P_1 + x_2 P_2 + \cdots + x_n P_n)^{a_1+a_2+\cdots+a_n}}. \tag{3}$$

However, as you would imagine, solving an integral with 5 distinct propagators using eq. (3) can be quite taxing. A better way is to reduce eq. (2) into a linear combination of much more palpable integrals I_i such that I_i have up to two distinct propagators and therefore be computed using two Feynman parameters only. The reduction of complicated loop integrals into a linear combination of a much smaller number of simpler integrals (so-called **master integrals**) is done with the help of the so-called **Integration-by-Parts** identities (or IBPs),

$$\int \frac{d^d q_1}{(2\pi)^d} \int \frac{d^d q_2}{(2\pi)^d} \frac{\partial}{\partial q_i^\mu} \frac{v_\mu}{q_1^2 (q_1 - p)^2 (q_1 - q_2)^2 q_2^2 (q_2 - p)^2} = 0, \tag{4}$$

where v_μ can be any of the involved external or loop momenta, or a linear combination thereof. We won't be proving this identity here, but to convince ourselves that this is true, let's look at the case with one integration variable. Schematically, using Gauss' theorem,

$$\int d^d q \partial_\mu \left(\frac{v^\mu}{D} \right) = \lim_{R \to \infty} \int_{|q|=R} d^{d-1} q S(q),$$

where D is some propagator and S is some $d-1$ dimensional surface term which is evaluated from the origin to infinity. It either converges to zero at infinity, or diverges. However, in dimensional regularization, all such divergences are regulated by a $1/\varepsilon$ term giving $1/\varepsilon_{UV} - 1/\varepsilon_{IR} = 0$. Thus, in dimensional regularization, all surface terms are set to zero.

Next we want to use eq. (4) to reduce eq. (2) to a more familiar massless bubble. But we before we start, let's do a Wick rotation on the momenta.

$$q_1^0 =: -ik^0, \;\; q_2^0 =: -il^0, \;\; p^0 =: -ip_E^0,$$

where any subscript E stands for the Euclidean version. Then

$$\int d^d q_1 d^d q_2 = -\int d^d k \int d^d l.$$

Since $f(k^2) = f(-k_E^2)$ and

$$\begin{aligned}
(q_1 - p)^2 &= q_1^2 + p^2 - 2q_1 p \\
&= -k^2 - p_E^2 - 2\left((-ik^0)(-ip_E^0) - \underbrace{\vec{q_1}\cdot\vec{p}}_{=\vec{k}\cdot\vec{p}_E}\right) \\
&= -k^2 - p_E^2 - 2(-k^0 p_E^0 - \vec{k}\cdot\vec{p}_E) \\
&= -(k^2 + p_E^2 - 2kp_E) \\
&= -(k - p_E)^2,
\end{aligned}$$

we obtain the Euclidean version of eq. (2),

$$\begin{aligned}
I(p^2) &= \int \frac{d^d k}{(2\pi)^d}\frac{d^d l}{(2\pi)^d}\frac{1}{k^2(k-p_E)^2(k-l)^2 l^2 (l-p_E)^2} \\
&=: I_E(p_E^2).
\end{aligned} \tag{5}$$

Let us now use eq. (4). For ease of writing, define

$$\int := \int \frac{d^d k}{(2\pi)^d}\frac{d^d l}{(2\pi)^d}.$$

Then we pick $v_\mu = k_\mu - l_\mu$ and get

$$\begin{aligned}
\int \frac{\partial}{\partial k_\mu}\frac{k_\mu - l_\mu}{k^2(k-p_E)^2(k-l)^2 l^2(l-p_E)^2} &= \int \underbrace{\left(\frac{\partial}{\partial k_\mu}(k_\mu - l_\mu)\right)}_{=\frac{\partial k_\mu}{\partial k_\mu}=\delta^\mu_\mu=d} \frac{1}{k^2(k-p_E)^2(k-l)^2 l^2(l-p_E)^2} \\
&\quad - 2\int (k_\mu - l_\mu)\frac{1}{l^2}\frac{1}{(l-p_E)^2}\left\{\frac{1}{(k-p_E)^2}\frac{1}{(k-l)^2}\frac{1}{k^4}(k-l)^\mu\right. \\
&\quad + \frac{1}{k^2}\frac{(k-p_E)^\mu}{(k-p_E)^4}\frac{1}{(k-l)^2} \\
&\quad \left. + \frac{1}{k^2}\frac{1}{(k-p_E)^2}\frac{(k-l)^\mu}{(k-l)^2}\right\} \\
&= dI_E(p_E^2) - 2(A + B + C),
\end{aligned}$$

where

$$\begin{aligned}
A &:= \int \frac{k^2 - lk}{l^2(l-p_E)^2(k-p_E)^2(k-l)^2 k^4}, \\
B &:= \int \frac{k^2 - kp_E - lk - lp_E}{l^2(l-p_E)^2(k-p_E)^4(k-l)^2 k^2}, \\
C &:= \int \frac{1}{l^2(l-p_E)^2 k^2(k-p_E)^2(k-l)^2}.
\end{aligned}$$

For A we can complete the square of the numerator using $k^2 - lk = k^2 - 2lk + lk + l^2 - l^2$ which gives

$$A = \int \left(\frac{(k-l)^2}{l^2(l-p_E)^2(k-p_E)^2(k-l)^2k^4} - \frac{l^2}{l^2(l-p_E)^2(k-p_E)^2(k-l)^2k^4} \right)$$
$$= \underbrace{\int \frac{1}{l^2(l-p_E)^2(k-p_E)^2k^4}}_{:=I_1(p_E^2)} - \underbrace{\int \frac{1}{(l-p_E)^2(k-p_E)^2(k-l)^2k^4}}_{:=I_2(p_E^2)}.$$

A similar calculation for B and C gives $B = C = I_E(p_E^2)$. Thus, the IBP eq. (4) enforces

$$dI_E - 2A - 2(B+C) = dI_E - 2(I_1 - I_2) - 4I_E \stackrel{!}{=} 0,$$

which immediately gives

$$I_E(p_E^2) = \frac{2}{d-4}\Big(I_1(p_E^2) - I_2(p_E^2)\Big). \tag{6}$$

$I_{1,2}$ are our master integrals. Indeed we see that both only feature up to two distinct propagators. In fact, in I_1, the two loop momenta have completely decoupled and we can write

$$I_1(p_E^2) = \int \frac{1}{l^2(l-p_E)^2(k-p_E)^2k^4}$$
$$= \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^4(k-p_E)^2} \int \frac{d^dl}{(2\pi)^d} \frac{1}{l^2(l-p_E)^2}.$$

In I_2 the two loop momenta are still coupled via the $(k-l)^2$ term. Nevertheless we can solve for one of the loop momenta separately since

$$I_2(p_E^2) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^4(k-p_E)^2} \int \frac{d^dl}{(2\pi)^d} \frac{1}{(l-p_E)^2(l-k)^2}$$
$$\stackrel{l\to l+k}{=} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^4(k-p_E)^2} \int \frac{d^dl}{(2\pi)^d} \frac{1}{l^2(l+k-p_E)^2}.$$

Therefore we have to solve the general integral

$$B(q_E^2; a, b) := \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^a((k-q_E)^2)^b}$$

once and then just apply the result. Here we can finally employ eq. (3). Defining $[dk] := d^dk/(2\pi)^d$,

$$B(q_E^2; a, b) = \int [dk] \int_0^1 dx \left(k^2 - xk^2 + (k-q_E)^2\right)^{-a-b} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$$
$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx x^{a-1}(1-x)^{b-1} \int [dk] \left(k^2 - xk^2 + (k-q_E)^2\right)^{-a-b}.$$

When shifting the momentum $k \to k + xq_E$, the expression in the last parentheses becomes

$$(k+xq_E)^2 - 2x(k+xq_E)q_E + xq_E^2 = k^2 - x^2q_E^2 + xq_E^2$$
$$= k^2 + \underbrace{q_E^2(x-x^2)}_{=:\Delta(q_E^2,x)},$$

so

$$B(q_E^2;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx x^{a-1}(1-x)^{b-1} \underbrace{\int [dk] \frac{1}{(k^2+\Delta(q_E^2,x))^{a+b}}}_{=:J(q_E^2,x;a,b)}. \tag{7}$$

To solve $J(q_E^2,x;a,b)$ notice that it only depends on k^2 and not k. This means that we can switch to d dimensional spherical coordinates where angular integration is trivial. Using

$$d^d k = d|k||k|^{d-1} \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

and

$$k^2 = |k|^2 =: l \Longrightarrow \frac{dl}{d|k|} = 2|k| \Longrightarrow d|k| = \frac{dl}{2|k|},$$

we get

$$\begin{aligned} J(q_E^2,x;a,b) &= \frac{1}{(2\pi)^d} \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{1}{2} \int dl \underbrace{|k|^{d-2}}_{=l^{\frac{d-2}{2}}} \frac{1}{(l+\Delta(q_E^2,x))^{a+b}} \\ &= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma(\frac{d}{2})} \underbrace{\int_0^\infty dl \frac{l^{\frac{d-2}{2}}}{(l+\Delta(q_E^2,x))^{a+b}}}_{=:L(q_E^2,x;a,b)}. \end{aligned} \tag{8}$$

We are now solving

$$L = \int_0^\infty dl \frac{l^{\frac{d-2}{2}}}{(l+\Delta)^{a+b}}.$$

We want to bring this to the form of the **beta function**,

$$\int_0^1 dz z^{\alpha-1}(1-z)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

To do this, use the substitution

$$z := \frac{\Delta}{l+\Delta} \Longrightarrow l = \Delta \frac{1-z}{z} \Longrightarrow dl = -\Delta \frac{1}{z^2} dz$$

to get

$$\begin{aligned} L &= \int_0^1 dz \Delta^{\frac{d-2}{2}} \left(\frac{1-z}{z}\right)^{\frac{d-2}{2}} \left(\Delta \frac{1-z}{z} + \Delta\right)^{-a-b} (-\Delta) \frac{1}{z^2} \\ &= -\Delta^{\frac{d-2}{2}-a-b} \int_0^1 dz \left(\frac{1-z}{z}\right)^{\frac{d-2}{2}} \left(\frac{1}{z}\right)^{-a-b-2} \\ &= -\Delta^{\frac{d-2}{2}-a-b+1} \underbrace{\int_0^1 dz (1-z)^{\frac{d-2}{2}} z^{-\frac{d}{2}+a+b-1}}_{\text{beta function for } \alpha=-\frac{d}{2}+a+b,\ \beta=\frac{d}{2}} \\ &= -\Delta^{\frac{d-2}{2}-a-b+1} \frac{\Gamma(-\frac{d}{2}+a+b)\Gamma(\frac{d}{2})}{\Gamma(a+b)}. \end{aligned}$$

Putting this into eq. (8) and then inserting that into eq. (7) we obtain

$$B(q_E^2;a,b) = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Gamma(\frac{d}{2})}\frac{\Gamma(-\frac{d}{2}+a+b)\Gamma(\frac{d}{2})}{\Gamma(a)\Gamma(b)}(q_E^2)^{\frac{d-2}{2}-a-b+1}(-1)\underbrace{\int_0^1 dx x^{a-1}(1-x)^{b-1}(x-x^2)^{\frac{d-2}{2}-a-b+1}}_{=:F(a,b)}.$$

$F(a,b)$ can also be expressed in terms of the beta function,

$$\begin{aligned}F(a,b) &= \int_0^1 dx x^{a-1+\frac{d-2}{2}-a-b+1}(1-x)^{b-1+\frac{d-2}{2}-a-b+1} \\ &= \int_0^1 dx x^{\frac{d}{2}-b-1}(1-x)^{\frac{d}{2}-a-1} \\ &= \frac{\Gamma(\frac{d}{2}-b)\Gamma(\frac{d}{2}-a)}{\Gamma(d-b-a)}.\end{aligned}$$

Next let us define $d =: 4-2\varepsilon$ and instead of taking $d \to 4$ we will investigate what happens for $\varepsilon \to 0$. This will allows us to better isolate the divergent terms. Then

$$\begin{aligned}(4\pi)^{\frac{4-2\varepsilon}{2}} &= 16\pi^2(4\pi)^{-\varepsilon}, \\ \Gamma(-\frac{d}{2}+a+b) &= \Gamma(-2+\varepsilon+a+b), \\ \frac{\Gamma(\frac{d}{2}-b)\Gamma(\frac{d}{2}-a)}{\Gamma(d-b-a)} &= \frac{\Gamma(2-\varepsilon-b)\Gamma(2-\varepsilon-a)}{\Gamma(4-2\varepsilon-a-b)}.\end{aligned}$$

Finally we obtain

$$B(q_E^2;a,b) = \frac{(4\pi)^\varepsilon}{16\pi^2}\frac{\Gamma(2-\varepsilon-a)\Gamma(2-\varepsilon-b)\Gamma(-2+\varepsilon+a+b)}{\Gamma(a)\Gamma(b)\Gamma(4-2\varepsilon-a-b)}(q_E^2)^{2-\varepsilon-a-b}. \tag{9}$$

Let us now look at our two master integrals and express them in terms of B.

$$\begin{aligned}I_1 &= \underbrace{\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4(k-p_E)^2}}_{=B(p_E^2,2,1)}\underbrace{\int \frac{d^dl}{(2\pi)^d}\frac{1}{l^2(l-p_E)^2}}_{=B(p_E^2,1,1)} \\ &= \left(\frac{1}{16\pi^2}\right)^2(4\pi)^{2\varepsilon}\frac{\Gamma(-\varepsilon)\Gamma(\varepsilon)\Gamma(1-\varepsilon)^3\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(2-2\varepsilon)}(p_E^2)^{1-2\varepsilon}.\end{aligned}$$

Now notice that for $\varepsilon \to 0$, some of the terms are perfectly finite since $\Gamma(x+\varepsilon) = \Gamma(x)(1+\mathcal{O}(\varepsilon))$ for $x > 0$, while for others the gamma functions give simple poles since $\Gamma(-x+\varepsilon) \propto 1/\varepsilon + \mathcal{O}(1)$. Our goal is to isolate the divergent terms from the finite ones. Using $\Gamma(x+1) = x\Gamma(x)$ we can write

$$\Gamma(-\varepsilon) = \frac{1}{\varepsilon}\Gamma(1-\varepsilon)$$

to obtain

$$\begin{aligned}I_1 &= \left(\frac{1}{16\pi^2}\right)^2\underbrace{\left(\frac{(4\pi)^\varepsilon\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\right)^2}_{=:S_\varepsilon^2}\left(-\frac{1}{\varepsilon^2}\frac{1}{1-2\varepsilon}\right)(p_E^2)^{1-2\varepsilon} \\ &= \left(\frac{S_\varepsilon}{16\pi^2}\right)^2\left(-\frac{1}{\varepsilon^2}\frac{1}{1-2\varepsilon}\right)(p_E^2)^{1-2\varepsilon}.\end{aligned}$$

Similarly, for I_2 we get

$$
\begin{aligned}
I_2 &= \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4(k-p_E)^2}\underbrace{\int \frac{d^dl}{(2\pi)^d}\frac{1}{l^2(l+k-p_E)^2}}_{B((k-p_E)^2,1,1)} \\
&= \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4(k-p_E)^2}\frac{(4\pi)^\varepsilon}{16\pi^2}\frac{\Gamma(1-\varepsilon)^2\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)})((k-p_E)^2)^{-\varepsilon} \\
&= \frac{S_\varepsilon}{16\pi^2}\frac{1}{\varepsilon}\frac{1}{1-2\varepsilon}\underbrace{\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4(k-p_E)^2}}_{B(p_E^2,2,1+\varepsilon)} \\
&= \left(\frac{S_\varepsilon}{16\pi^2}\right)^2\left(-\frac{1}{\varepsilon^2}\frac{1}{1-2\varepsilon}\right)\frac{\Gamma(1-2\varepsilon)^2\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)^2\Gamma(1-\varepsilon)\Gamma(1-3\varepsilon)}(p_E^2)^{-1-2\varepsilon}. \qquad (10)
\end{aligned}
$$

Our next and final step is to disentangle the gamma function mess by expanding in powers of ε. We will use

$$
\Gamma(1+n\varepsilon)e^{n\gamma_E\varepsilon} = 1 + \frac{\pi^2}{12}n^2\varepsilon^2 - \frac{\zeta_3}{3}n^3\varepsilon^3 + \mathcal{O}(\varepsilon^4), \qquad (11)
$$

where γ_E is the Euler-Mascheroni constant and $\zeta_3 = \zeta(3)$ is the Riemann-Zeta function. Expanding all the Gamma functions up to some order m and the multiplying out the brackets is straight-forward but extremely tedious, especially since one does not know a priori what m is (here I have already spoiled the answer in eq. (11)). This is extremely important since if we have some factor $1/\varepsilon^m$ and accidentally only expand up to ε^{m-1} we risk losing track of all the infinities and/or finite parts. A nice way to mitigate this is to expand all Gammas without computing the numerical values as follows:

$$
\begin{aligned}
\Gamma(1+n\varepsilon) &= A_0 + A_1\varepsilon + \dots, \\
\Gamma(-1+n) &= B_{-1}\frac{1}{\varepsilon} + B_0 + B_1\varepsilon + \dots,
\end{aligned}
$$

then multiply out the brackets and see which powers of ε remain relevant in the limit $\varepsilon \to 0$ and only then attempt to calculate the finite coefficients A_i and B_i. This is especially handy in an exam situation or when one is asked to only discuss the *types* of divergences that arise in the theory, rather than, for example, the exact values of the counterterms. The standard way of calculating this, however, is using Mathematica. For the Gamma function factor in eq. (10) we obtain

$$
\frac{\Gamma(1-2\varepsilon)^2\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)^2\Gamma(1-\varepsilon)\Gamma(1-3\varepsilon)} = 1 - 6\zeta_3\varepsilon^3 + \mathcal{O}(\varepsilon^4).
$$

Finally, putting it all together,

$$
\begin{aligned}
I_E(p_E^2) &= \frac{2}{d-4}\Big(I_1(p_E^2) - I_2(p_E^2)\Big) \\
&= \frac{1}{\varepsilon^3}\frac{1}{1-2\varepsilon}\left(\frac{S_\varepsilon}{16\pi^2}\right)^2(p_E^2)^{-1-2\varepsilon}(6\zeta_3\varepsilon^3 + \mathcal{O}(\varepsilon^4)) \\
&= (1+2\varepsilon+\mathcal{O}(\varepsilon^2))\left(6\zeta_3 + \frac{\mathcal{O}(\varepsilon^4)}{\varepsilon^3}\right)\left(\frac{S_\varepsilon}{16\pi^2}\right)^2(p_E^2)^{-1-2\varepsilon} \\
&= (p_E^2)^{-1}\left(\frac{S_\varepsilon}{16\pi^2}\right)^2 6\zeta_3 + \mathcal{O}(\varepsilon^4).
\end{aligned}
$$

We take $\varepsilon \to 0$ and use $S_{\varepsilon\to 0} = 1$, as well as $p_E^2 = -p^2$ to obtain the final answer

$$I(p^2) = -\frac{1}{(4\pi)^4}\frac{6\zeta_3}{p^2}. \tag{12}$$

Remarkably, the two-loop eq. (2) is perfectly finite. That is is UV-finite is not so surprising since this is exactly the result we would expect from power counting. However, the diagram is also IR-finite despite it being massless and having 2 powers of momentum in each loop variable.