Série d'exercices 4

2020

First order methods

Exercice 1 Subgradients

Compute the subgradients of the following functions.

$$f(x) = \|x\|; \quad g(x) = \begin{cases} 0 & \text{if } \; x \in C \\ \infty & \text{if } \; x \notin C \end{cases}$$

where C is a nonempty convex subset of $\mathbb{R}^n$.

Exercice 2

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that a vector $d \in \mathbb{R}^n$ is a subgradient of f at x if and only if the function $d'y - f(y)$ attains its maximum at $y = x$.

Exercice 3

Let $C \subset \mathbb{R}^n$ be nonempty closed and convex set. Consider the function $\varphi_C : \mathbb{R}^n \to \mathbb{R}$ given by $\varphi_C(x) = \frac{1}{2}d_C^2(x) = \frac{1}{2}\|x - P_C(x)\|_2$, where P_C is the orthogonal projection mapping.

- Show that: $\bigtriangledown\varphi_C(x) = x - P_C(x)$.
- Compute $\partial d_C(x)$

Exercice 4 Proximal operators

Compute the prox of the following functions.

- Negative sum of Logs:

$$f(\boldsymbol{x}) = \begin{cases} -\lambda \sum\limits_{j=1}^{n} \log x_j, \, \boldsymbol{x} > 0 \\ \infty \qquad\qquad\qquad \text{else,} \end{cases}$$

  where $\lambda > 0$.

- ℓ_0-norm: $g(x) = \lambda\|\boldsymbol{x}\|_0$ where $\lambda > 0$.
- Norms: $h_1(\boldsymbol{x}) = \frac{\tau}{2}\|\boldsymbol{x}\|^2$, $h_2(\boldsymbol{x}) = \tau\|\boldsymbol{x}\|_1$.
- Loss function: $h_4(\boldsymbol{x}) = \frac{\tau}{2}\|A \cdot - y\|^2$, for $A \in \mathbb{R}^{p \times n}$.