Série d'exercices 2

Mars 2020

Unconstrained and constrained optimization

Exercice 1

Find the maximum and minimum values of f in the following cases:

1. $f(x,y) = 81\,x^2 + y^2$ s.t. $4x^2 + y^2 = 9$.
2. $f(x,y) = 8x^2 - 2y$ s.t. $x^2 + y^2 = 1$.
3. $f(x,y,z) = y^2 - 10z$ s.t. $x^2 + y^2 + z^2 = 36$.
4. $f(x,y,z) = xyz$ s.t. $x + 9y^2 + z^2 = 4$. Assume that $x \geqslant$ for this problem. Why is this assumption needed?
5. $f(x,y,z) = 3x^2 + y$ s.t. $4x - 3y = 9$ and $x^2 + z^2 = 9$.
6. $f(x,y,z) = 4y - 2z$ s.t. $2x - y - z = 2$ and $x^2 + y^2 = 1$.

Exercice 2

Find the maximum and minimum values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$.

Exercice 3

Given $(n_i)_{i=1,\ldots,N} \in \mathbb{N}^*$ and $P > 0$, solve the following problem

$$\begin{aligned} \underset{(p_1,\ldots,p_n)}{\text{maximize}} \quad & \sum_{i=1}^{N} \ln\left(1 + \frac{p_i}{n_i}\right) \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i \leq P, \\ & p_i > 0, \forall\, i = 1,..N \end{aligned}$$

Exercice 4

Given A, b, and m, use the SVD algorithm to find a vector x with $\|x\|_2 < m$ minimizing $\|Ax - b\|_2$.

Exercice 5

We want to find a normal $n \times 1$ vector h satisfying $Ah = 0$, where A is $m \times n$ matrix, $m \geqslant n$, and $rank(A) = n$. We consider the following problem:

$$\begin{aligned} \underset{h \in \mathbb{R}^n}{\text{minimize}} \quad & \|Ah\| \\ \text{subject to} \quad & 1 - h^T h = 0, \end{aligned} \tag{II.12}$$

1. Using Lagrange multipliers, derive a characteristic equation.Derive h in terms of eigenvector of $(A^T A)$.

2. Let $A = USV^T$ the SVD decomposition.
   a. Show that $\|USV^T h\| = \|SV^T h\|$ and $\|Vh\| = \|h\|$.
   b. Express the minimization problem in terms of $y = V^T h$.
   c. Deduce that $y = [0, 0, ..., 1]^T$ and the corresponding vector h.

Exercice 6

Find the solution x to the least squares problem:

$$\underset{x}{\operatorname{argmin}} \|\mathbf{y} - A\mathbf{x}\|_2^2 + \lambda \|\mathbf{b} - \mathbf{x}\|_2^2$$

Exercice 7 Problem

Find a circle that approximates given m points $\mathbf{a}_1, \cdots, \mathbf{a}_m$ of $\mathbb{R}^n$.

Exercice 8 Problem

Consider a noisy image $Y \in \mathbb{R}^{n\times n}$ and its associated vectorial represntation $\boldsymbol{y}$. As in the previous section, we express the denoising of $\boldsymbol{y}$ as the following least squares problem:

$$\min_{\boldsymbol{x}\in\mathbb{R}^N} \|\boldsymbol{x} - \boldsymbol{y}\|^2 \ + \ \lambda \ R(\boldsymbol{x}), \tag{II.13}$$

where λ is a given regularization para- meter, $N = n^2$, and $R(\boldsymbol{x})$ is a regularization term.

1. Express R in terms of the Frobenius nor mof the gradient norm matrix.
2. Write the denoising problem as a least squres problem.
3. Grive the explicit solution of the minimization problem.