Exercise 1. *Find the general solution of the following equations:*

1. $y'' - 8y' + 16y = 0$.
2. $y'' + k^2 y = 0$, *for a constant* $k > 0$.
3. $y'' - 6y' + 13y = 0, y(0) = 0, y'(0) = 10$.

Exercise 2. *Which of the following operators are linear?*

- $\mathcal{L}_1 u = u_x + x u_y$; $\quad \mathcal{L}_2 u = u_x + u u_y$; $\quad \mathcal{L}_3 u = u_x + u_y^2$; $\quad \mathcal{L}_4 u = u_x + u_y + 1$
- $\mathcal{L}_5 u = \sqrt{1+x^2}(\cos y) u_x + u_{yxy} - [\arctan(x/y)]u$

Exercise 3. *For each of the following equations, state the order and whether it is linear/nonlinear/semilinear, quasilinear, homogeneous; provide reasons.*

1. $u_t - u_{xx} + 1 = 0$; $\ u_t - u_{xx} + xu = 0$; $\ u_t - u_{xxt} + uu_x = 0$; $\ u_{tt} - u_{xx} + x^2 = 0$
2. $iu_t - u_{xx} + u/x = 0$; $\ u_x + e^y u_y = 0$; $\ xu_x + yu_y = u$; $\ xu_x + yu_y = u^2$
3. $u_x + (x+y)u_y = xy$; $\ uu_x + u_y = 0$; $\ xu_x^2 + yu_y^2 = 2$.
4. *Shock wave:* $u_x + uu_y = 0$; *Wave with interaction:* $u_{tt} - u_{xx} + u^3 = 0$; *Dispersive wave:* $u_t + uu_x + u_{xxx} = 0$

Exercise 4.

- *Verify by direct substitution that* $u_n(x,y) = \sin nx \sinh ny$ *is a solution of* $u_{xx} + u_{yy} = 0$ *for every* $n > 0$.
- *Verify that* $u(x,y) = f(x)g(y)$ *is a solution of the PDE* $uu_{xy} = u_x u_y$ *for all pairs of (differentiable) functions* f *and* g *of one variable.*
- *Show that* $u_1(x,y) = x$ *and* $u_2(x,y) = x^2 - y^2$ *are solutions to Laplace's equation. How can you combine them to create a new solution?*
- *Show that the soliton*
$$h(x,t) = 2\alpha^2 \mathrm{sech}\left(\alpha(x - 4\alpha^2 t)\right)$$
satisfies the the Korteweg-deVries equation,
$$h_t + 6hh_x = h_{xxx}$$
- *The PDE*
$$v_t - 6v^2 v_x + v_{xxx} = 0$$
is known as the modified Korteweg de Vries (mKdV) equation. Show that if v *is a solution of the mKdV, then*
$$u = v_x - v^2$$
is a solution of the KdV
$$u_t + 6uu_x + u_{xxx} = 0.$$

Exercise 5. *Consider Laplace's equation $u_{xx} + u_{yy} = 0$ in $\mathbb{R}^2$ with the boundary conditions $u(x, 0) = 0$.*

1. *Show that*
$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny$$
are solutions of the problem.
2. *Compute the limit of u_n when $n \to \infty$.*
3. *Is the problem defined by the PDE and the boundary condition stable?*

Exercise 6. *Consider the traffic flow in a highway as shown on the next figure.*

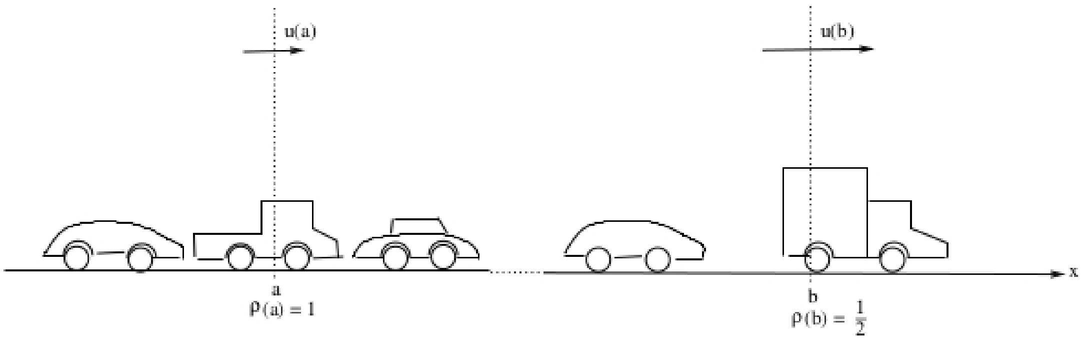

Figure 1: Traffic flow in a highway

Let $\rho(x, t)$ be the density of cars and $u(x, t)$ their velocity. We would like the express a PDE that describes the density function ρ.

1. *How can you express the quantity of vehicles between a and b and its variation over time?*
2. *Express the difference between the traffic inflow (at $x = a$) and outflow ($x = b$) in terms of ρu.*
3. *Combine the two previous relations to deduce the equation $\rho_t + (u\rho)_x = 0$.*
4. *Assume that $u = 1 - \rho$. Motivate this choice then deduce a nonlinear convection equation of ρ.*
5. *Assume that $u = c - \epsilon(\rho'/\rho)$. Motivate this choice then deduce a convection-diffusion equation of ρ.*

Exercise 7. *Consider a smooth surface in $\mathbb{R}^{n+1}$ representing the graph of a function $x_{n+1} = u(x_1, ..., x_n)$ defined on a bounded open set Ω in $\mathbb{R}^n$. Assuming that u is sufficiently smooth, the area of the surface is given by the nonlinear functional*

$$\mathcal{A}(u) = \int\limits_{\Omega} \left(1 + |\nabla u|^2\right)^{1/2} dx_1 ... dx_n.$$

The minimal surface problem is the problem of minimizing $\mathcal{A}(u)$ subject to a prescribed boundary condition $u = g$ on the boundary of Ω. A classical result from the calculus of variations asserts that if u is a minimiser of $\mathcal{A}(u)$, then it satisfies the Euler-Lagrange equation:

$$\nabla \cdot \left(\nabla u / \left(1 + |\nabla u|^2\right)^{1/2} \right) = 0.$$

This PDE is known as the minimal surface equation.

1. *Write down the previous PDE in the case $n = 2$.*
2. *Show that the plane $u(x, y) = Ax + By + C$ is a (trivial) solution to this equation.*
3. *Show that the following are non-trivial solutions*

   $$u_1(x, y) = \tan^{-1}(y/x); \quad u_2(x, y) = \frac{1}{a} \cosh^{-1} \left(a\sqrt{x^2 + y^2} \right); \quad u_3(x, y) = \frac{1}{a} \log \frac{\cos ay}{\cos ax}, \tag{1}$$

   where a is a real constant.
4. *Show that the helicoid surface (u_1) is a harmonic function. In fact, it is the is the only non-trivial solution that is a harmonic function while the catenoid (u_2) is the only non-trivial solution that is a surface of revolution and the Scherk surface (u_3) ant, is the only nontrivial solution that can be written in the form $u(x, y) = f(x) + g(y)$.*