Task 3

1. $k < i \leq j$, then S_k is an ancestor of both S_i and S_j if S_k has the largest priority among the $j - k + 1$ keys between S_k and S_j. In this case, $P[A^k_{i,j} = 1] = \frac{1}{j-k+1}$.
2. $i \leq k \leq j$, then S_k is an ancestor of both S_i and S_j if S_k has the largest priority among the $j - i + 1$ keys between S_i and S_j. In this case, $P[A^k_{i,j} = 1] = \frac{1}{j-i+1}$.
3. $i \leq j < k$, then S_k is an ancestor of both S_i and S_j if S_k has the largest priority among the $k - i + 1$ keys between S_i and S_k. In this case, $P[A^k_{i,j} = 1] = \frac{1}{k-i+1}$.

$$\therefore E[A^k_{i,j}] = \begin{cases} \frac{1}{j-k+1} & k < i \leq j \\ \frac{1}{j-i+1} & i \leq k \leq j \\ \frac{1}{k-i+1} & i \leq j < k \end{cases}$$

Task 4 To find the path from S_i and S_j, we go up from S_i, visit each of its ancestors bottom-up until we reach the one which is both an ancestor of S_i and S_j, then we go down from that node, visit each of A_j's ancestors top-down from there until we reach A_j.

Therefore, the number of nodes on the path from S_i to S_j can be written as (number of S_i's ancestors - number of common ancestors) + 1 + (number of S_j's ancestors - number of common ancestors), since we don't want to count any of the common ancestors except for the lowest (closest to leaves) one, and we count that common ancestor only once (that's where the 1 comes from).

$$\therefore l_{i,j} = 1 + \sum_{k=0}^{n-1}(A_i^k + A_j^k - 2A_{i,j}^k)$$

Task 5

$$
\begin{aligned}
E[l_{i,j}] &= E[1 + \sum_{k=0}^{n-1}(A_i^k + A_j^k - 2A_{i,j}^k)] \\
&= 1 + \sum_{k=0}^{n-1} E[A_i^k] + \sum_{k=0}^{n-1} E[A_j^k] - 2\sum_{k=0}^{n-1} E[A_{i,j}^k] \\
&= 1 + \sum_{k=0}^{i} \frac{1}{i-k+1} + \sum_{k=i+1}^{n-1} \frac{1}{k-i+1} + \sum_{k=0}^{j} \frac{1}{j-k+1} + \sum_{k=j+1}^{n-1} \frac{1}{k-j+1} \\
&- 2\sum_{k=0}^{i-1} \frac{1}{j-k+1} - 2\sum_{k=i}^{j} \frac{1}{j-i+1} - 2\sum_{k=j+1}^{n-1} \frac{1}{k-i+1} \\
&= 1 + H_{i+1} + (H_{n-i} - 1) + H_{j+1} + (H_{n-j} - 1) \\
&- 2(H_{j+1} - H_{j-i+1}) - \frac{2(j-i+1)}{j-i+1} - 2(H_{n-i} - H_{j-i+1}) \\
&= 4H_{j-i+1} + (H_{i+1} - H_{j+1}) + (H_{n-j} - H_{n-i}) - 3
\end{aligned} \tag{1}
$$

Task 6

$$\begin{aligned}
\because i \leq j, &\therefore H_{i+1} < H_{j+1}, H_{n-j} - H_{n-i} \\
&\Rightarrow H_{i+1} - H_{j+1} < 0, H_{n-j} - H_{n-i} < 0 \\
\therefore E[cost] &= O(E[l_{i,j}]) \\
&= O(4H_{j-i+1} + (H_{i+1} - H_{j+1}) + (H_{n-j} - H_{n-i}) - 3) \\
&\leq O(4H_{j-i+1}) \\
&\leq O(4 + 4\ln(j - i + 1)) \\
&\in O(\log(j - i + 1))
\end{aligned} \tag{2}$$

Task 7 Let Max_1 denotes the event where the maximum priority among T_1, T_2 is in T_1, and Max_2 denotes its complement, which is when the maximum priority is in T_2. Max_1 and Max_2 partition the event space. Thus, using the law of total probability, we have that

$$P[p_1 > p_2] = P[p_1 > p_2 | Max_1] \cdot P[Max_1] + P[p_1 > p_2 | Max_2] \cdot P[Max_2]$$

Note that given Max_2, i.e. the maximum priority among T_1, T_2 is in T_2, then since p_2 is the priority at the root of T_2, p_2 must be that maximum priority, so p_1 must be smaller than p_2. Otherwise, given Max_1, similar to above p_1 must be that maximum priority, thus bigger than p_2.

$$\therefore P[p_1 > p_2] = 1 \cdot P[Max_1] + 0 \cdot P[Max_2] = P[Max_1] = \frac{|T1|}{|T1 + T2|}$$