Week 3 & Week 4: The Three Body Problem

Accelerate

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1 Introduction

The three-body problem is one of the most famous problems in classical mechanics. It asks: given three point masses with initial positions and velocities, interacting only through Newtonian gravitation, what are their subsequent motions? While the two-body problem has a closed-form analytical solution, the three-body problem is generally non-integrable and exhibits chaotic behavior.

2 Mathematical Formulation

Consider three bodies with masses m_1, m_2, and m_3 at positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ respectively. The gravitational force on body i due to body j is given by Newton's law of universal gravitation:

$$\mathbf{F}_{ij} = G\frac{m_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|^3}(\mathbf{r}_j - \mathbf{r}_i) \tag{1}$$

where G is the gravitational constant.

2.1 Equations of Motion

The acceleration of each body is determined by the net gravitational force acting upon it:

$$\ddot{\mathbf{r}}_1 = Gm_2\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + Gm_3\frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} \tag{2}$$

$$\ddot{\mathbf{r}}_2 = Gm_1\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + Gm_3\frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3} \tag{3}$$

$$\ddot{\mathbf{r}}_3 = Gm_1\frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} + Gm_2\frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \tag{4}$$

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This system of nine coupled second-order differential equations (three spatial dimensions for each of three bodies) generally has no closed-form solution.

2.2 Conservation Laws

Despite the complexity, the three-body system conserves several quantities:

- **Total Energy:**
$$E = \frac{1}{2}\sum_{i=1}^{3} m_i|\dot{\mathbf{r}}_i|^2 - G\sum_{i<j}\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{5}$$

- **Linear Momentum:**
$$\mathbf{P} = \sum_{i=1}^{3} m_i\dot{\mathbf{r}}_i \tag{6}$$

- **Angular Momentum:**
$$\mathbf{L} = \sum_{i=1}^{3} m_i(\mathbf{r}_i \times \dot{\mathbf{r}}_i) \tag{7}$$

These ten conserved quantities (energy, three components each for linear and angular momentum) reduce the effective dimensionality of the problem but are insufficient to make it integrable.

3 Computational Modeling

Since analytical solutions are generally unavailable, numerical integration is essential for studying three-body systems.

3.1 State Space Representation

We reformulate the second-order system as a first-order system by introducing velocities $\mathbf{v}_i = \dot{\mathbf{r}}_i$:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\\ \mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{pmatrix} = \begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\\ \mathbf{a}_1(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\\ \mathbf{a}_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\\ \mathbf{a}_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\end{pmatrix} \tag{8}$$

where $\mathbf{a}_i$ represents the acceleration of body i as computed from equations (2)-(4).

3.2 Numerical Integration Methods

Several numerical methods can integrate this system:

Euler Method: The simplest approach updates positions and velocities using:

$$\mathbf{r}_i^{n+1} = \mathbf{r}_i^n + \Delta t\, \mathbf{v}_i^n \tag{9}$$

$$\mathbf{v}_i^{n+1} = \mathbf{v}_i^n + \Delta t\, \mathbf{a}_i^n \tag{10}$$

However, this approach gets more inaccurate as n gets large.

3.3 Implementation Considerations

- **Time Step:** Choose Δt small enough to resolve the shortest orbital period and closest approaches
- **Collisions:** As we divide by ($|\mathbf{r}_i - \mathbf{r}_j|$) as ($|\mathbf{r}_i - \mathbf{r}_j| \to 0$) the accleration gets large, this results in large inaccuracies using numerical integration techniques
- **Conservation Monitoring:** You could try to track total energy and momentum to detect numerical errors
- **Adapt your Methods:** Use adaptive time-stepping (e.g., Runge-Kutta-Fehlberg) to balance accuracy and efficiency

3.4 Basic Computational Algorithm

A simple computational approach follows this structure:

1. Initialize positions $\mathbf{r}_i(0)$ and velocities $\mathbf{v}_i(0)$
2. For each time step $n = 0, 1, 2, \ldots$:
   (a) Compute all pairwise separations $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$
   (b) Calculate accelerations $\mathbf{a}_i$ using equations (2)-(4)
   (c) Update velocities and positions using chosen integrator
   (d) Check conservation laws and detect close encounters
3. Output trajectory data for analysis and visualization

4 Conclusion

The three-body problem demonstrates the transition from integrable to chaotic dynamics. While no general analytical solution exists, modern numerical methods allow precise simulation of three-body systems. The key is choosing appropriate integration schemes that preserve the physical properties of the system over long timescales, particularly energy and momentum conservation inherent to Hamiltonian mechanics.

References

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