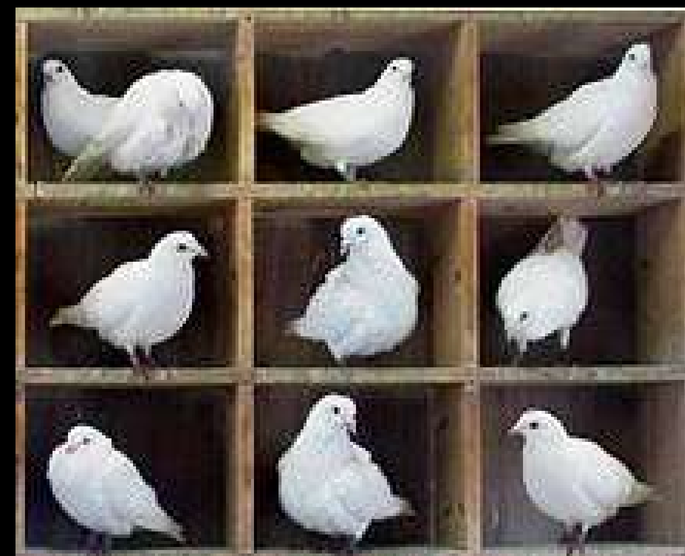
Pigeonhole Principle

Principle I

When $m + 1$ pigeons enter m pigeonholes (m is positive integer), there must be at least one hole having more than 1 pigeon.



Q. In the morning, John draws socks randomly from his drawer. If there are 12 pairs of socks each pair a different colour, in the drawer, how many socks does John have to draw at most in order to get a matched pair?

Solution:

There are 12 different colours (pigeonholes) and in order to ensure at least two socks (pigeons) have the same colour,

No. of socks drawn > 12.

So at most $12 + 1 = 13$ socks have to be drawn to get a matched pair.

Q. How many students must be in a class to guarantee that at least two student receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

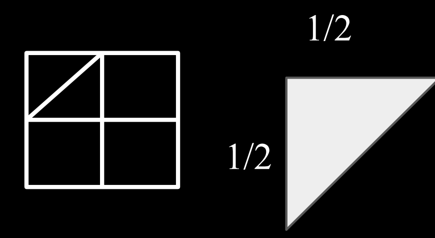
Solution:

There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Q. Five points are drawn randomly inside a unit square, show that there is at least two points with a distance less than $\frac{\sqrt{2}}{2}$.

Solution:

Consider a unit square and cut it into 4 equal smaller square, as shown below on the left.



There are 4 equal smaller squares (pigeonholes) and 5 points (pigeons). By the pigeonhole principle, at least two points are in the same smaller square. The distance between these two points < length of the diagonal of the smaller square.

Length of the diagonal =

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}.$$

So there is at least two points with a distance less than

$\frac{\sqrt{2}}{2}$.

Q. Prove that for any given 50 positive integers, it is always possible to select out four numbers a_1; a_2; a_3 and a_4 from them, such that $(a_2 - a_1)(a_4 - a_3)$ is a multiple of 2009.

Solution:

First of all, note that $2009 = 49 \times 41$. Consider the 50 remainders of the 50 given integers modulo 49, by the pigeonhole principle, there must be two numbers selected from the 50 integers, denoted by a_1 and a_2, such that a_1 and a_2 are congruent modulo 49, so $a_2 - a_1$ is divisible by 49.

Next, by same reason, it must be possible that two numbers a_3 and a_4 can be selected from the remaining 48 numbers such that $a_4 - a_3$ is divisible by 41.

Thus, $49.41 \mid (a_2 - a_1)(a_4 - a_3)$, i.e. $2009 \mid (a_2 - a_1)(a_4 - a_3)$.

Principle II

Let k and n be any two positive integers. If at least $kn + 1$ objects are distributed among n boxes, then one of the boxes must contain at least $k + 1$ objects.

In particular, if at least $n + 1$ objects are to be put into n boxes, then one of the boxes must contain at least two objects.

Q. Show that if 9 colours are used to paint 100 houses, then at least 12 houses will be of the same colour.

Solution:

We have 9 colours (pigeonholes) and 100 houses (pigeons). By the generalized pigeonhole principle, there is at least one colour that will be used to paint at least $\left\lceil \frac{100}{9} \right\rceil = 12$ houses.

So there is at least 12 houses painted with the same colour.

Q. How many cards must be selected from a deck of 52 cards to make sure that at least 3 cards of the same suit are selected?

Solution:

We have 4 suits (pigeonholes) and if all of them does not appear with at least 3 card (pigeons), i.e. each suit has at most 2 cards, then the maximum number of cards that can possibly be drawn is (4). (2) = 8.

So the number of cards that must be drawn to ensure at least 3 cards are of the same suit is 8 + 1 = 9

Alternatively, let there be n cards drawn, then according to the generalized pigeon principle,

$$\left\lceil \frac{n}{4} \right\rceil = 3$$

$$\Rightarrow \frac{n}{4} > 3 - 1 = 2$$

$$\Rightarrow n > 1(4).(2) = 8.$$

So at least 8 + 1 = 9 cards to be drawn.

CHINA-2003

Q. In a bag, there are some balls of the same size that are colored by 7 colors, and for each color the number of balls is 77. At least how many balls are needed to be picked out at random to ensure that one can obtain 7 groups of 7 balls each such that in each group the balls are homochromatic?

Solution:

For this problem, it is natural to let each color be one pigeonhole, and a ball drawn be a pigeon. At the first step, for getting a group of 7 balls with the same color, at least 43 balls are needed to be picked out from the bag at random, since if only 42 balls are picked out, there may be exactly 6 for each color.

By pigeonhole principle, there must be one color such that at least b42=7c+1 = 7 drawn balls have this color.

Next, after getting the first group, it is sufficient to pick out from the bag another 7 balls for getting 43 balls once again. Then, by the same reason, the second group of 7 homochromatic drawn balls can be obtained. Repeating this process for 6 times, the 7 groups of 7 homochromatic balls are obtained.

Thus, the least number of drawn balls is $43 + 6 \times 7 = 85$.

SSSMO - 2001

Q. A bag contain 200 marbles. There are 60 red ones, 60 blue ones, 60 green ones and the remaining 20 consist of yellow and white ones. If marbles are chosen from the bag without looking, what is the smallest number one must pick in order to ensure that, among the chosen marbles, at least 20 are of the same colour?

Solution:

When 77 marbles are chosen, there may be 19 red, 19 blue, 19 green and 20 yellow and white.

If 78 marbles are chosen at random, the number of yellow and white ones among them is at most 20. Therefore there are at least 58 marbles of red, blue or green colours. According to the Pigeonhole Principle, the number of drawn marbles of some color is not less that

$$\left\lfloor \frac{57}{3} \right\rfloor + 1 = 20,$$

i.e. are least 20. Thus, the smallest number of marble to be picked is 78.

In this problem, a colour is taken as a pigeonhole, and then a drawn marble is taken as a pigeon.

Q. Prove that in a set containing n positive integers there must be a subset such that the sum of all numbers in it is divisible by n.

Solution:

Let the n positive integers be $a_1, a_2, \ldots, a_n$. Consider n new positive integers:

$b_1 = a_1, b_2 = a_1 + a_2, \ldots, b_n = a_1 + a_2 + \ldots + a_n$.

Then all the n values are distinct. When some of $b_1, b_2, \ldots b_n$ is divisible by n, the conclusion is proven. Otherwise, if all b_i are not divisible by n, then their remainders are all not zero, i.e. at most they can take $n - 1$ different values. By the pigeonhole principle, there must be b_i and b_j with $i < j$ such that $b_j - b_i \neq 0$ is divisible by n. Since $b_j - b_i = a_{i+1} + a_{i+2} + \ldots + a_j$ is a sum of some given numbers, the conclusion is proven..