Doubt Clearing Session

Course on Discrete Mathematics for GATE 2023

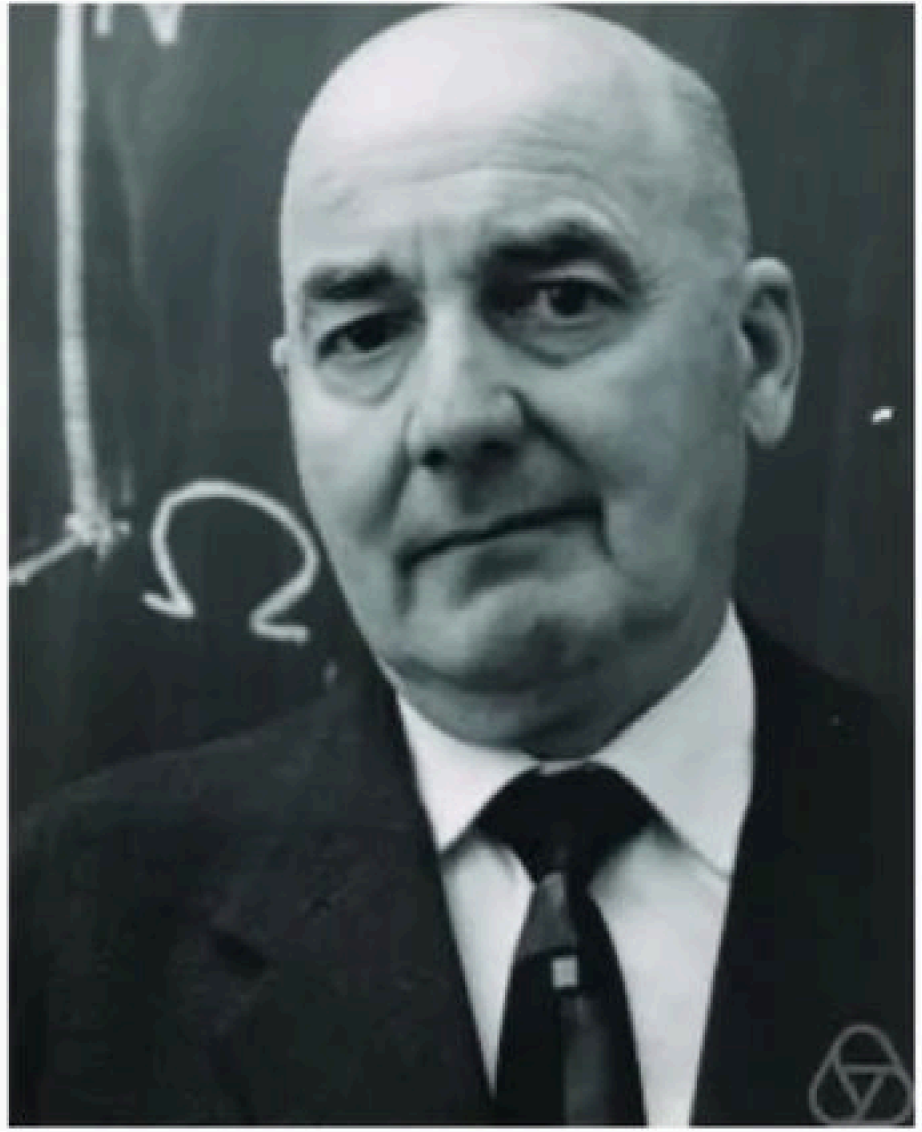
Sanchit Jain • Lesson 12 • Sept 14, 2022

Conversion of POSET into a Hasse Diagram

- If we want to study Partial order relation further then it will be better to convert it into more convenient notation so that it can be studied easily.
- This graphical representation is called Hasse Diagram

- In order theory, a Hasse diagram is a type of mathematical diagram used to represent a finite partially ordered set, in the form of a drawing of its transitive reduction.

- The diagrams are named after Helmut Hasse (1898–1979)



Steps to convert partial order relation into hasse diagram

1- Draw a vertex for each element in the Set

2- If (a, b) ϵ R then draw an edge from a to b

3- Remove all Reflexive and Transitive edges

4- Remove the direction of edges and arrange them in the increasing order of heights.

Q Consider a Partial order relation and convert it into hasse diagram?

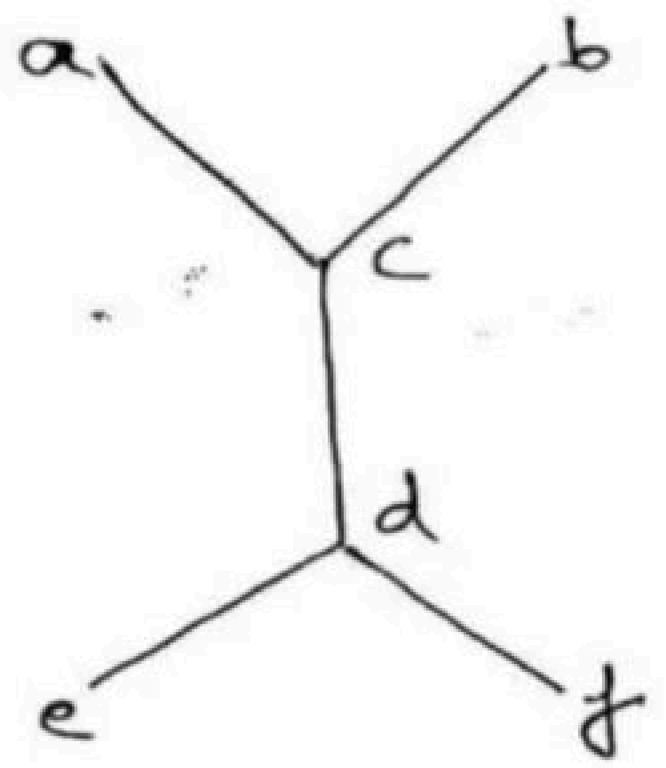
R = {(1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (4,4), (4,8), (8,8)}

Q Consider a Partial order relation and convert it into hasse diagram?

R = {(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)}

Break

Q Study the following hasse diagram and find which of the following are valid?



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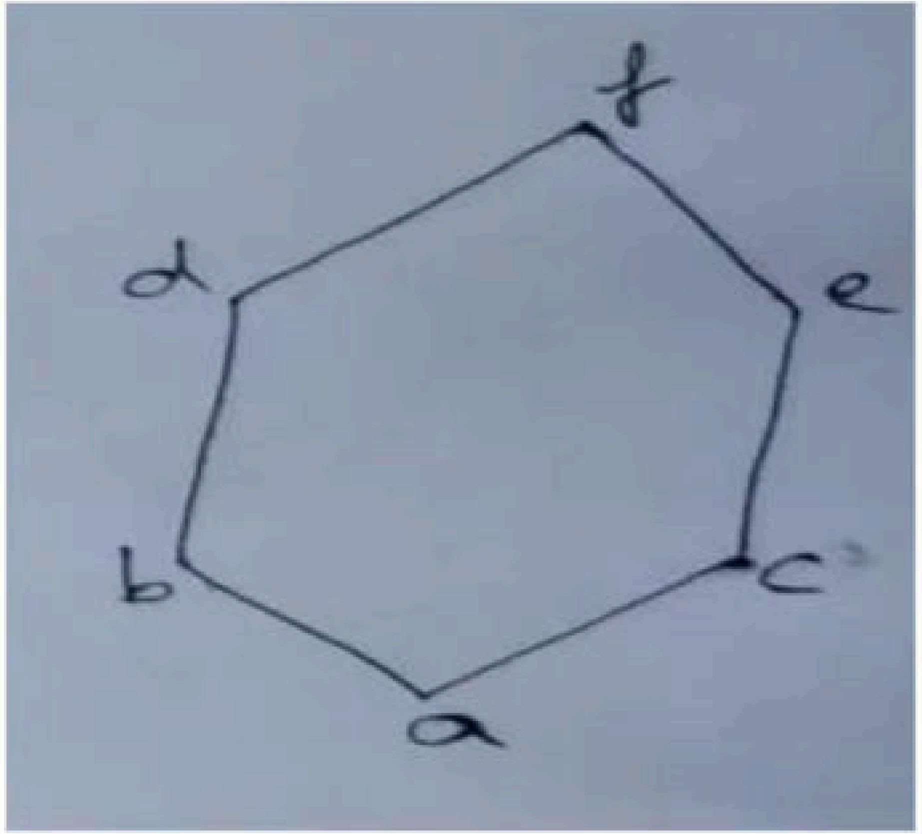

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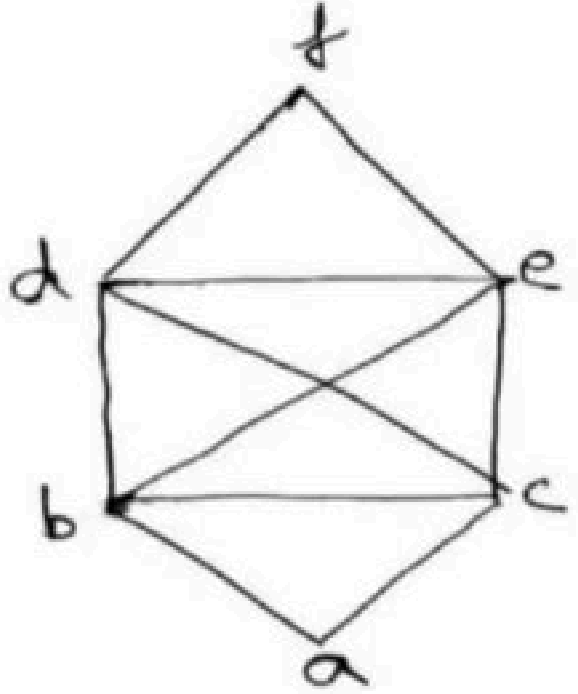

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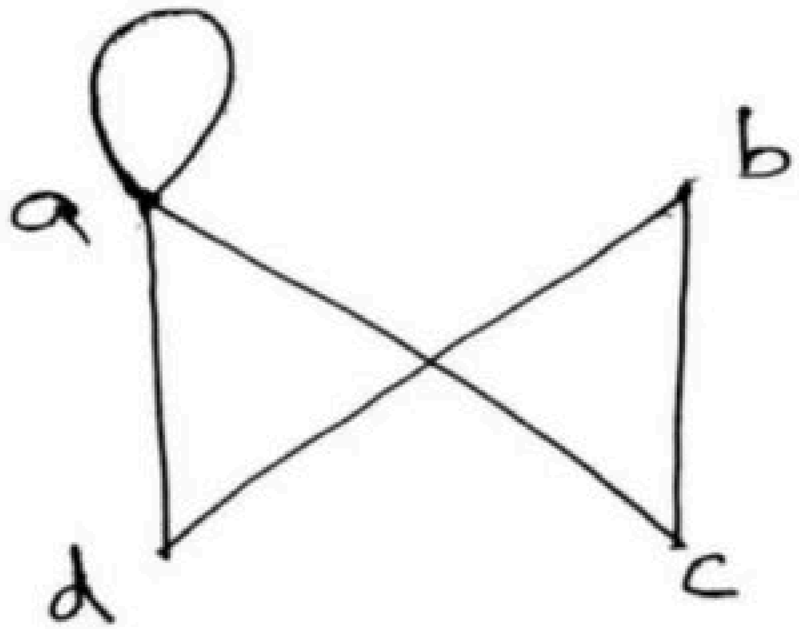

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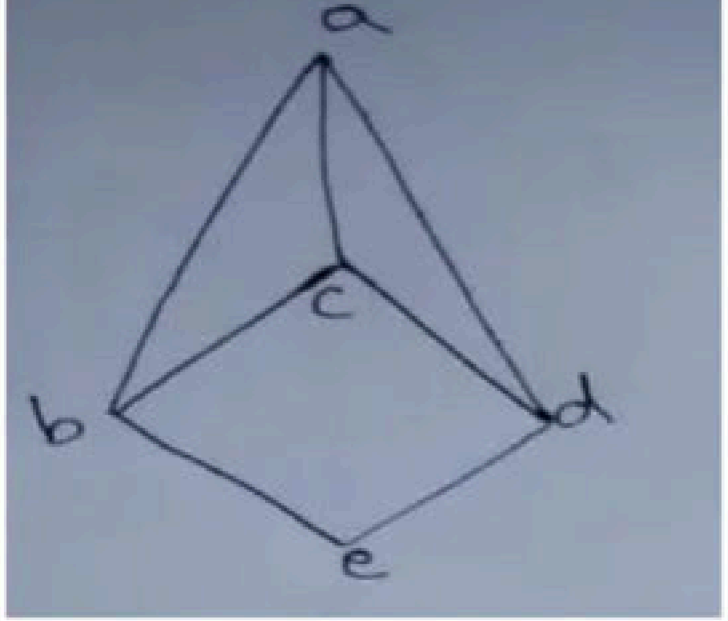

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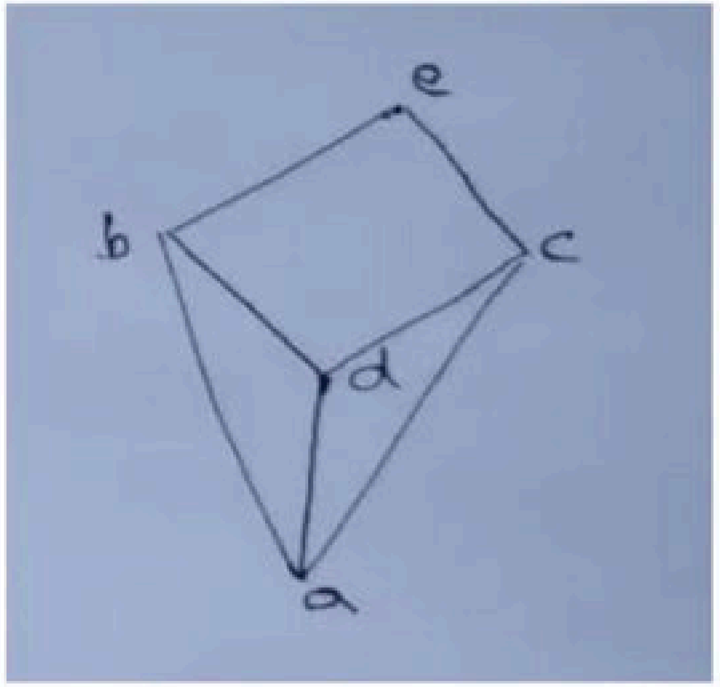

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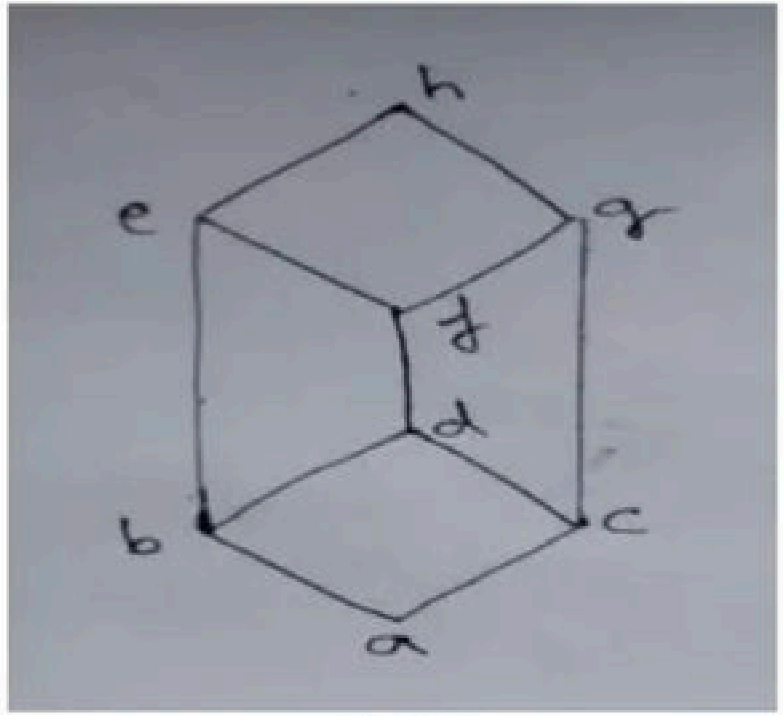

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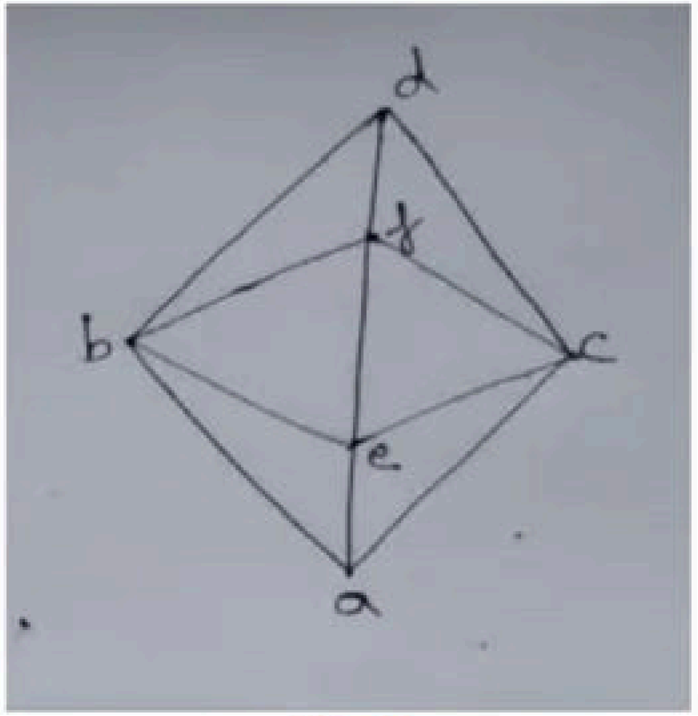

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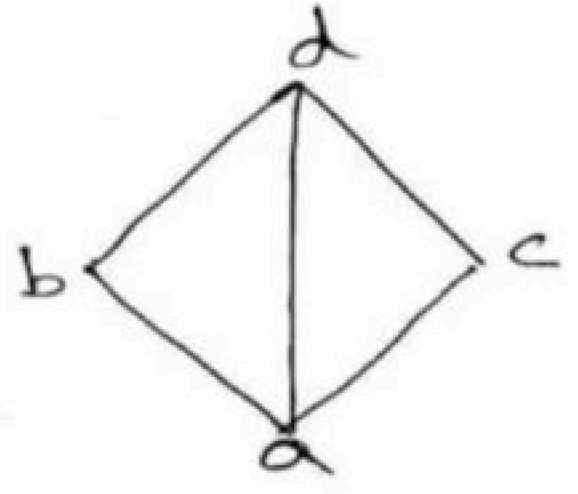

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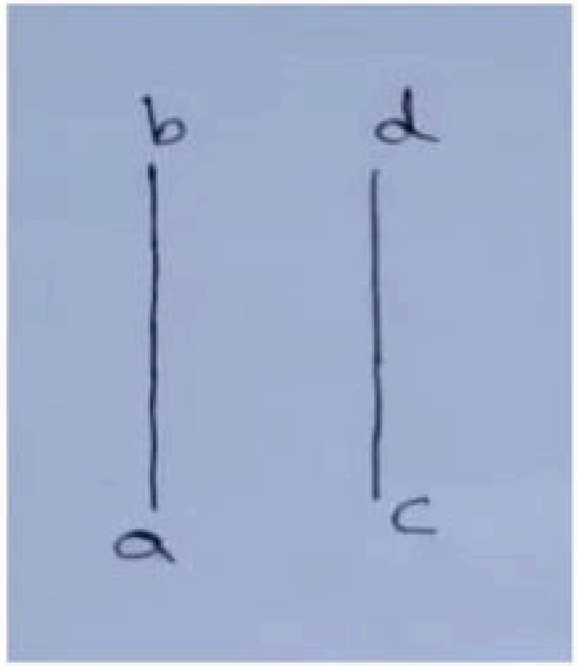

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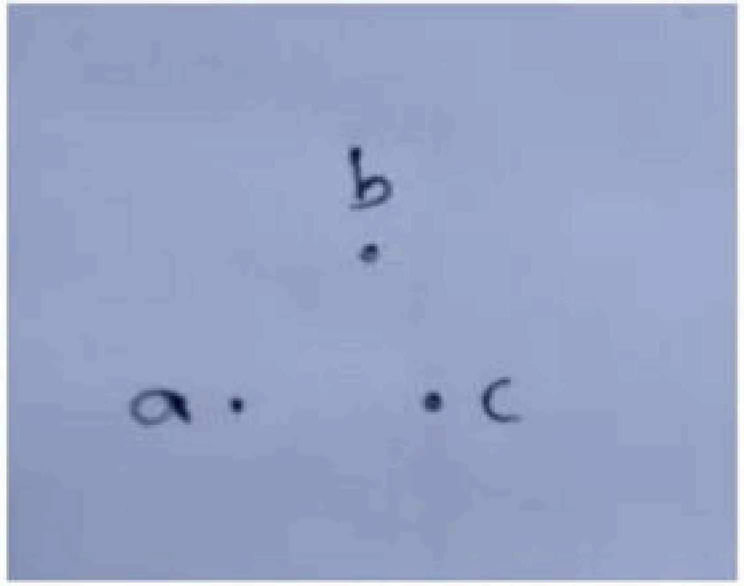


(10)

Conclusion

- We can not have a horizontal edge in a hasse diagram
- We can not have a reflexive and transitive edge in Hasse Diagram

Q Let X = {2,3,6,12,24}, Let ≤ be the partial order defined by X ≤ Y if x divides y. Number of edges as in the Hasse diagram of (X, ≤) is. **(GATE-1996) (1 Marks)**

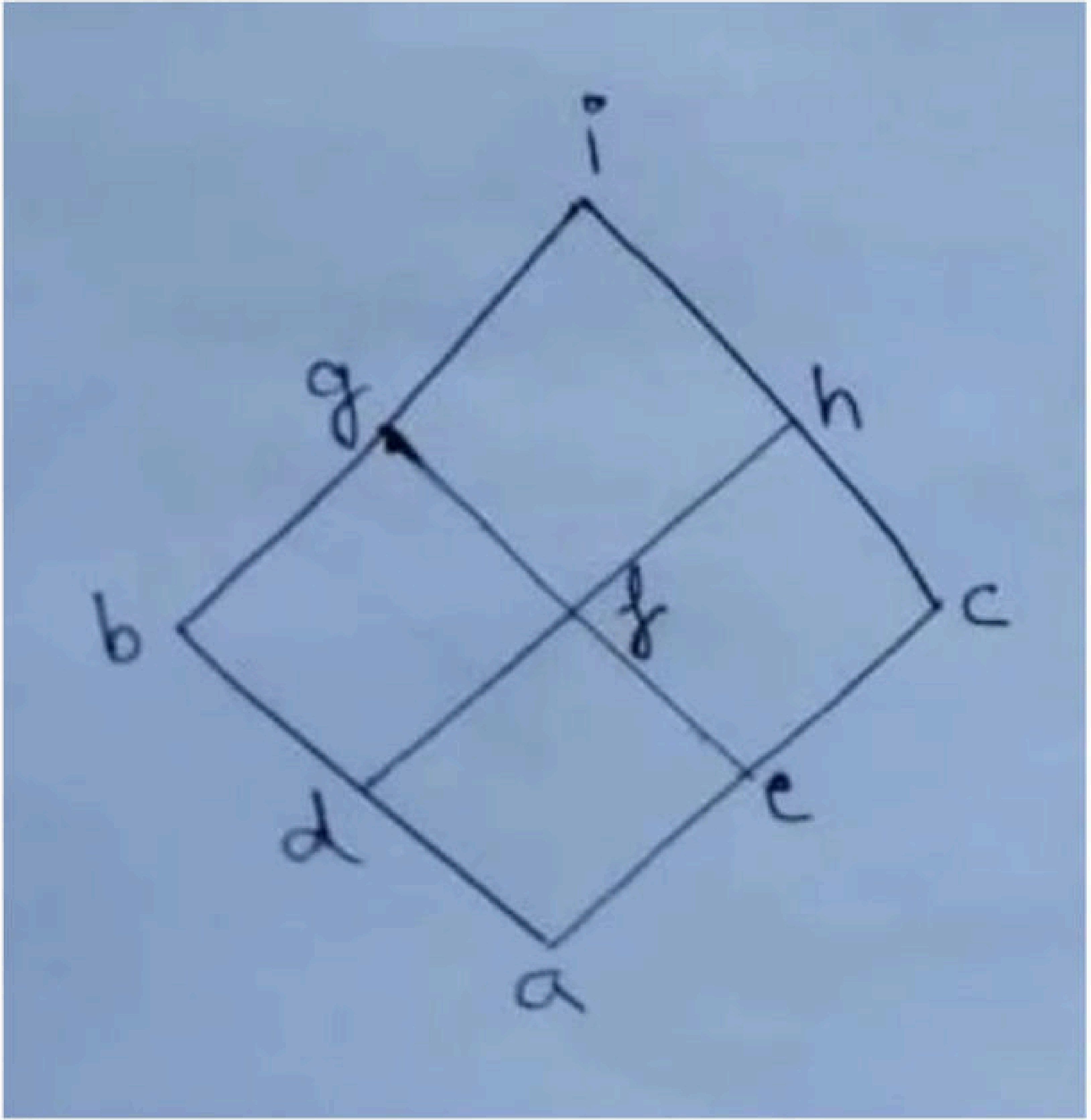
(a) 3 **(b)** 4 **(c)** 9 **(d)** None of the above

Break

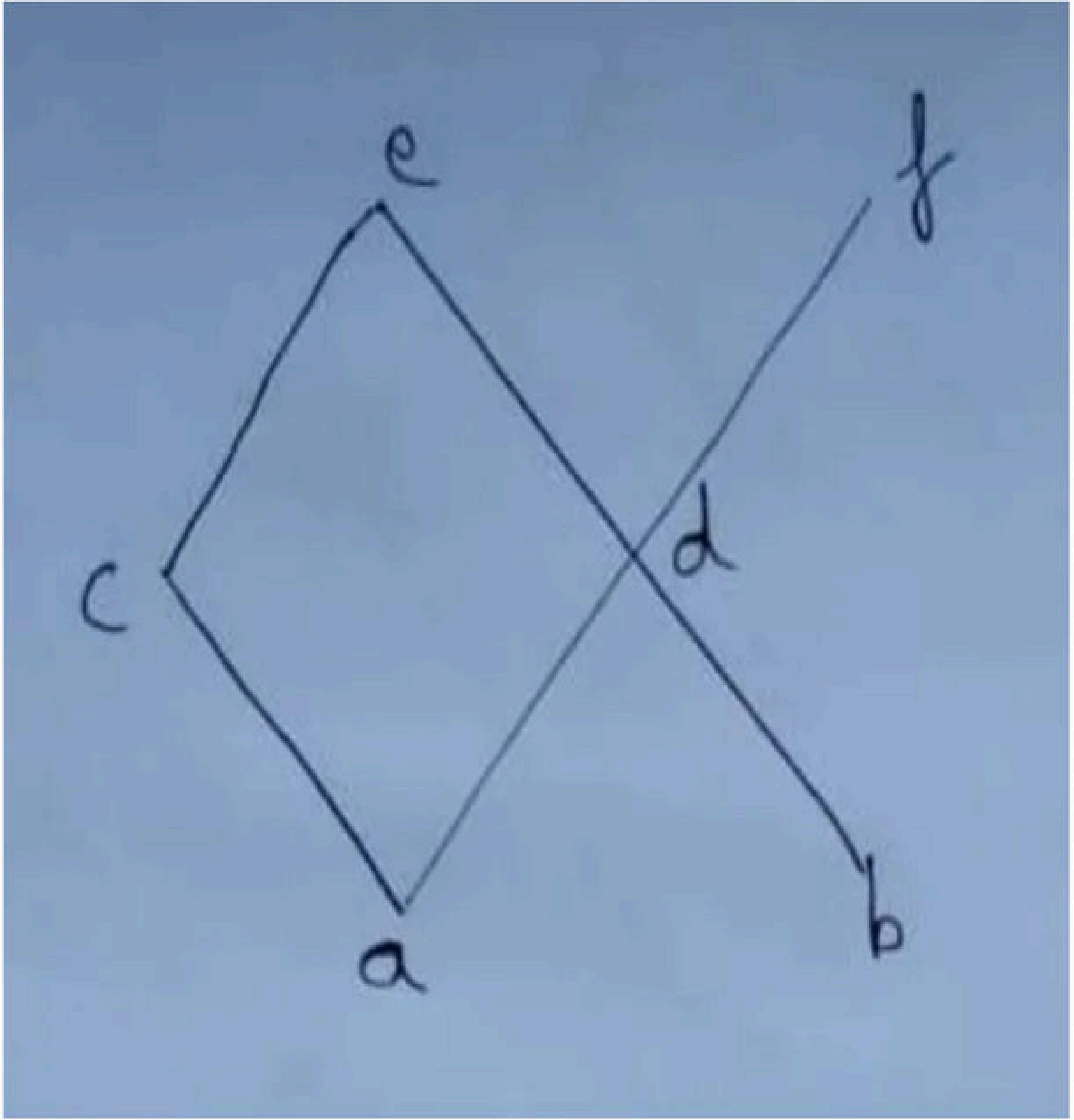
Elements of a Poset

1. **Maximal Element: -** An element is said to be maximal if it is not related to any other element in the Partial order relation.

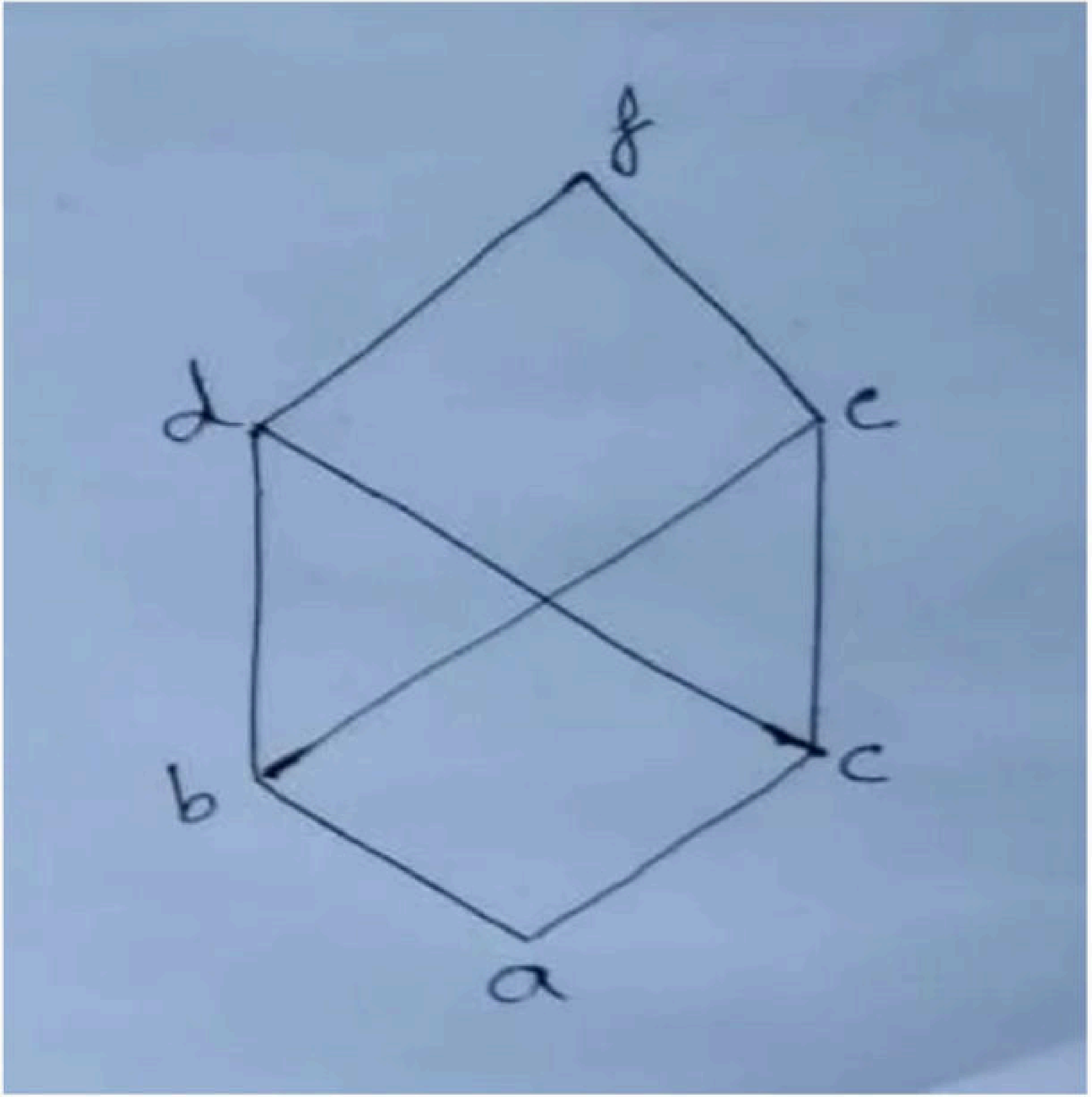
2. **Minimal Element: -** An element is said to be minimal if no other element is related to it in the Partial order relation.



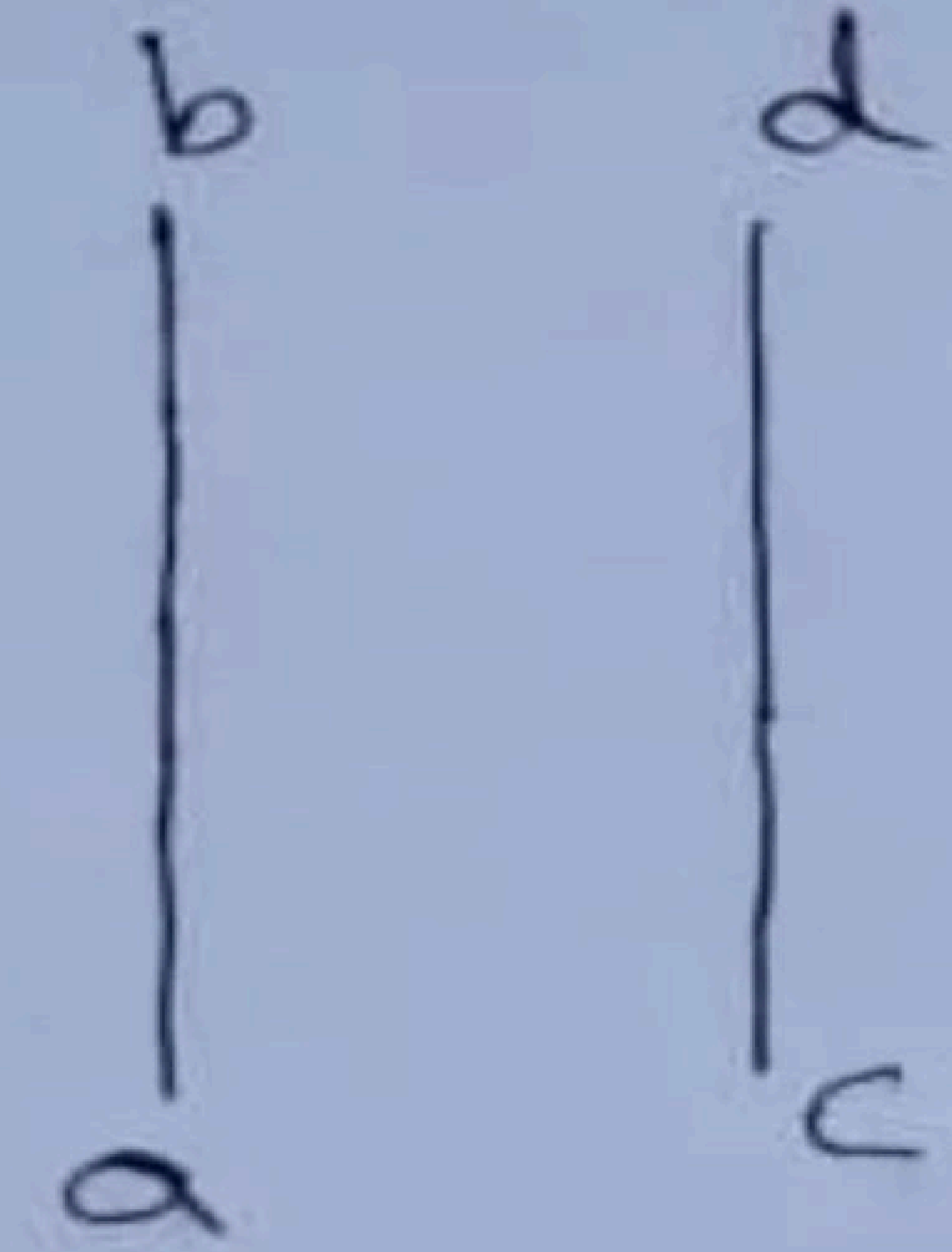
	Elements
Minimal	a
Least	a
Maximal	i
Greatest	i



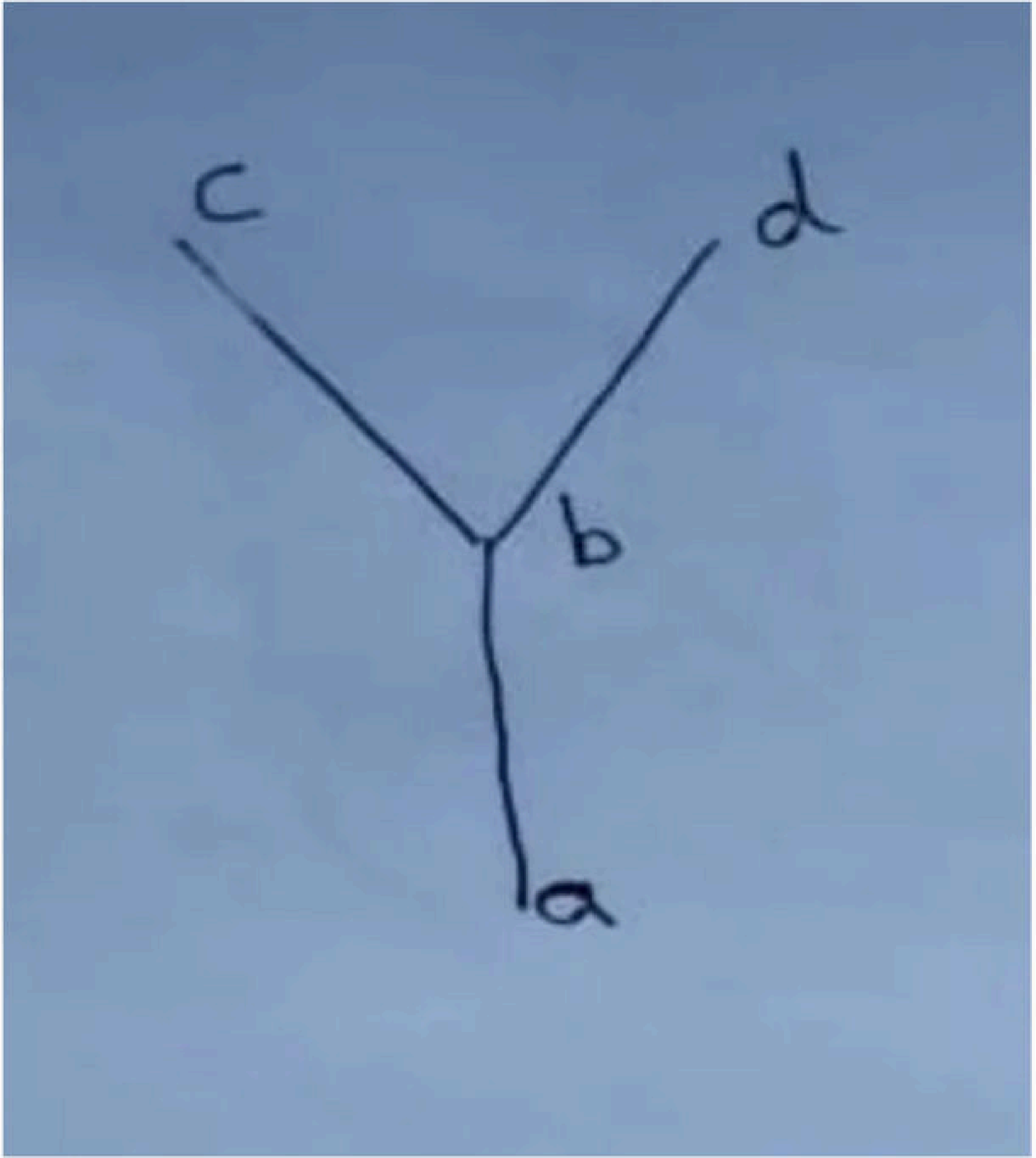
	Elements
Minimal	a , b
Least	ϕ
Maximal	e , f
Greatest	ϕ



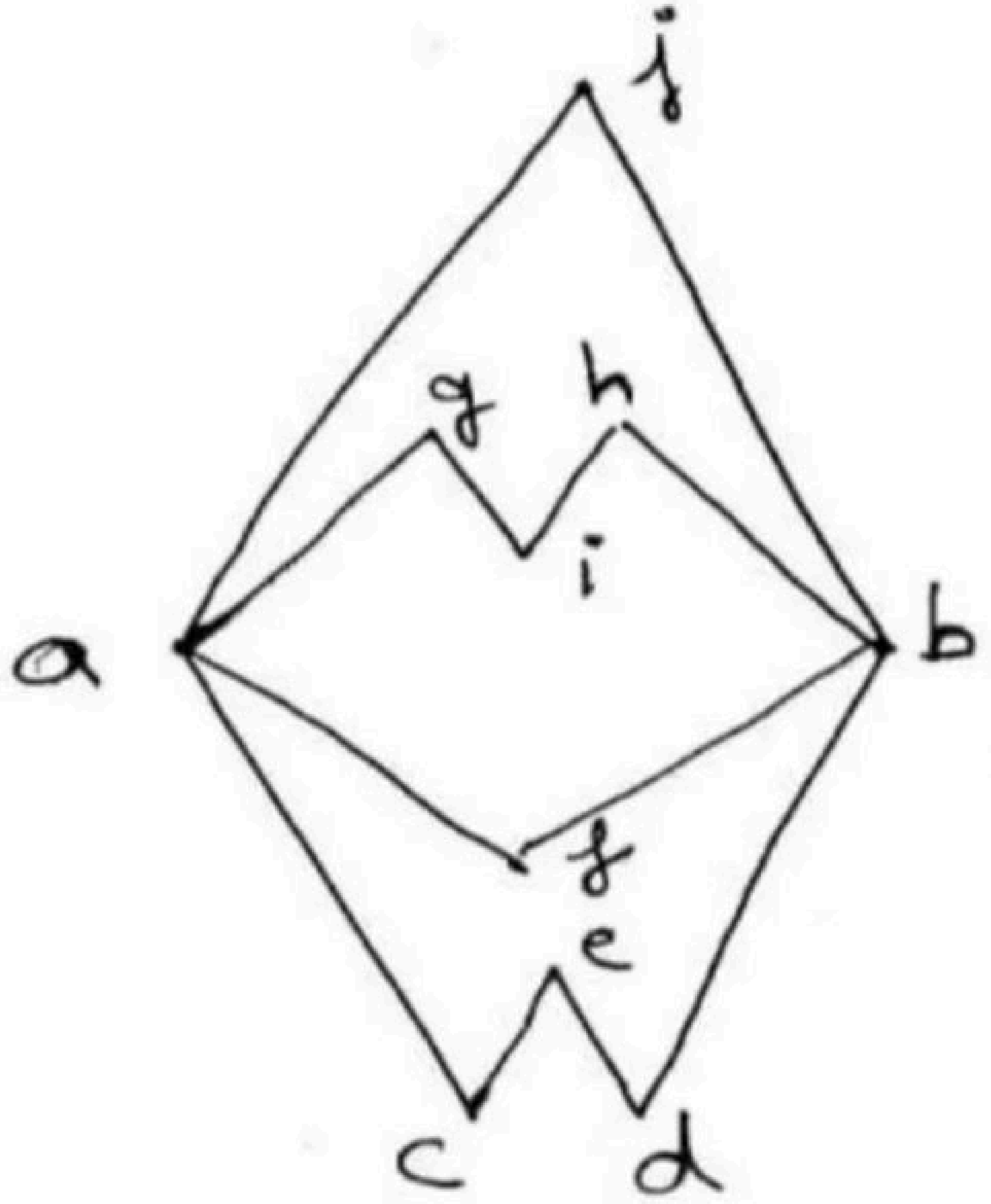
	Elements
Minimal	a
Least	a
Maximal	f
Greatest	f



	Elements
Minimal	a, c
Least	ϕ
Maximal	b, d
Greatest	ϕ



	Elements
Minimal	a
Least	a
Maximal	c, d
Greatest	ϕ



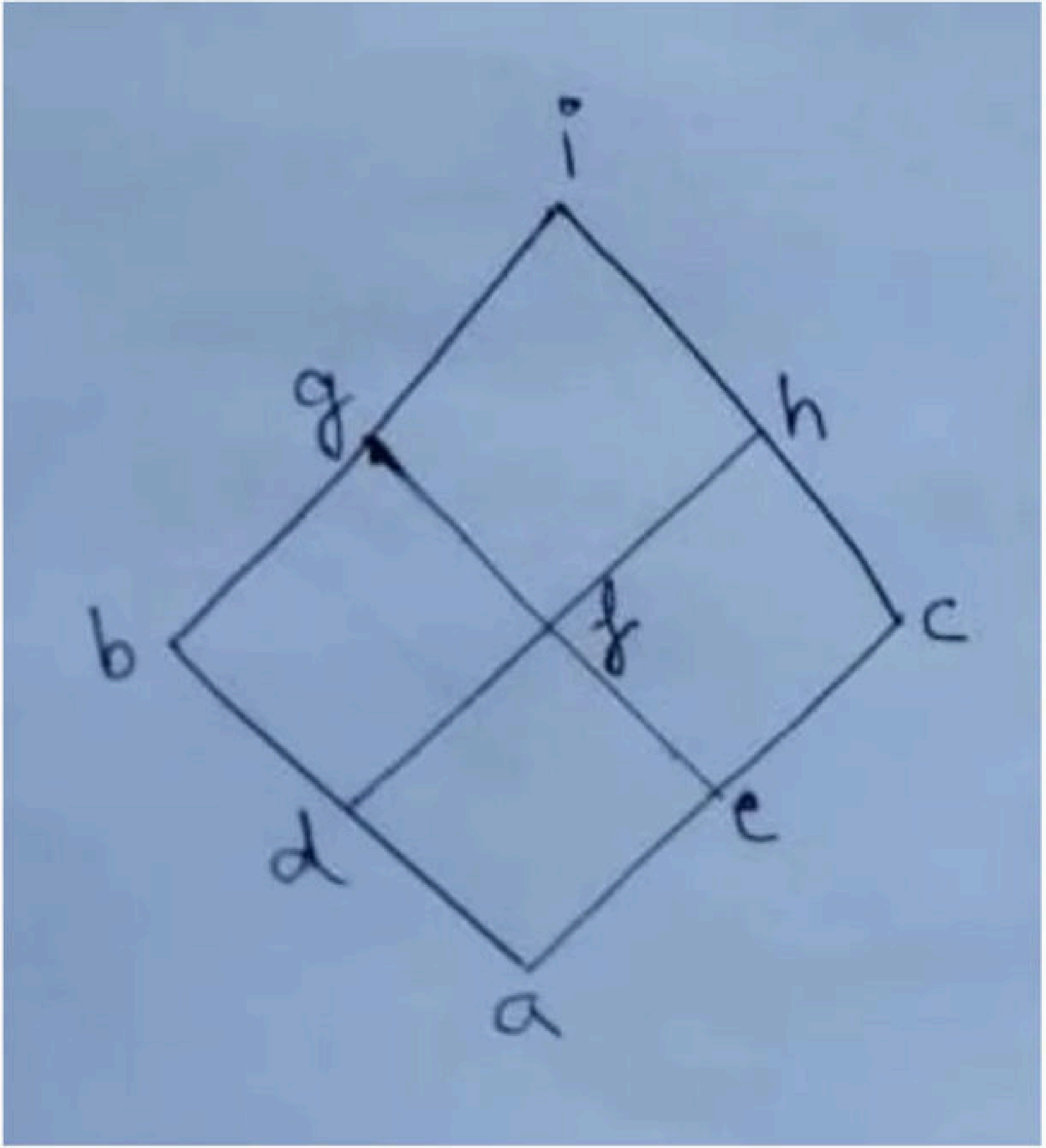
	Elements
Minimal	c, d, f, i
Least	ϕ
Maximal	j, g, h, e
Greatest	ϕ

- Every hasse diagram will have at least one Maximal and Minimal element(one or more).

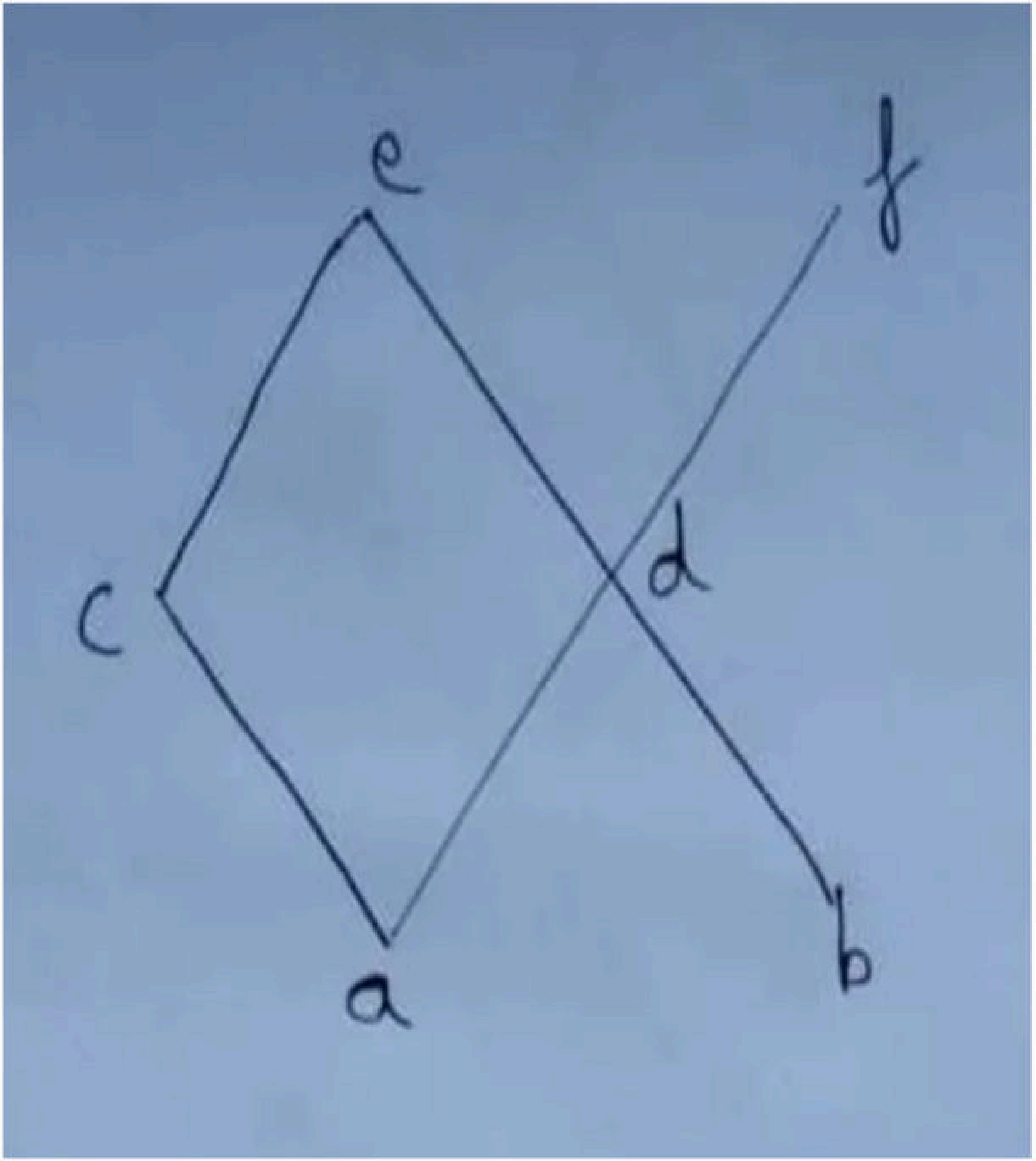
Break

- **Greatest Element: -** An element is said to be Maximum/Greatest if it is not related to any other element but every element is related to it in Partial order relation. Or if a hasse diagram has only one Maximal element then it will also be Maximum/Greatest element.

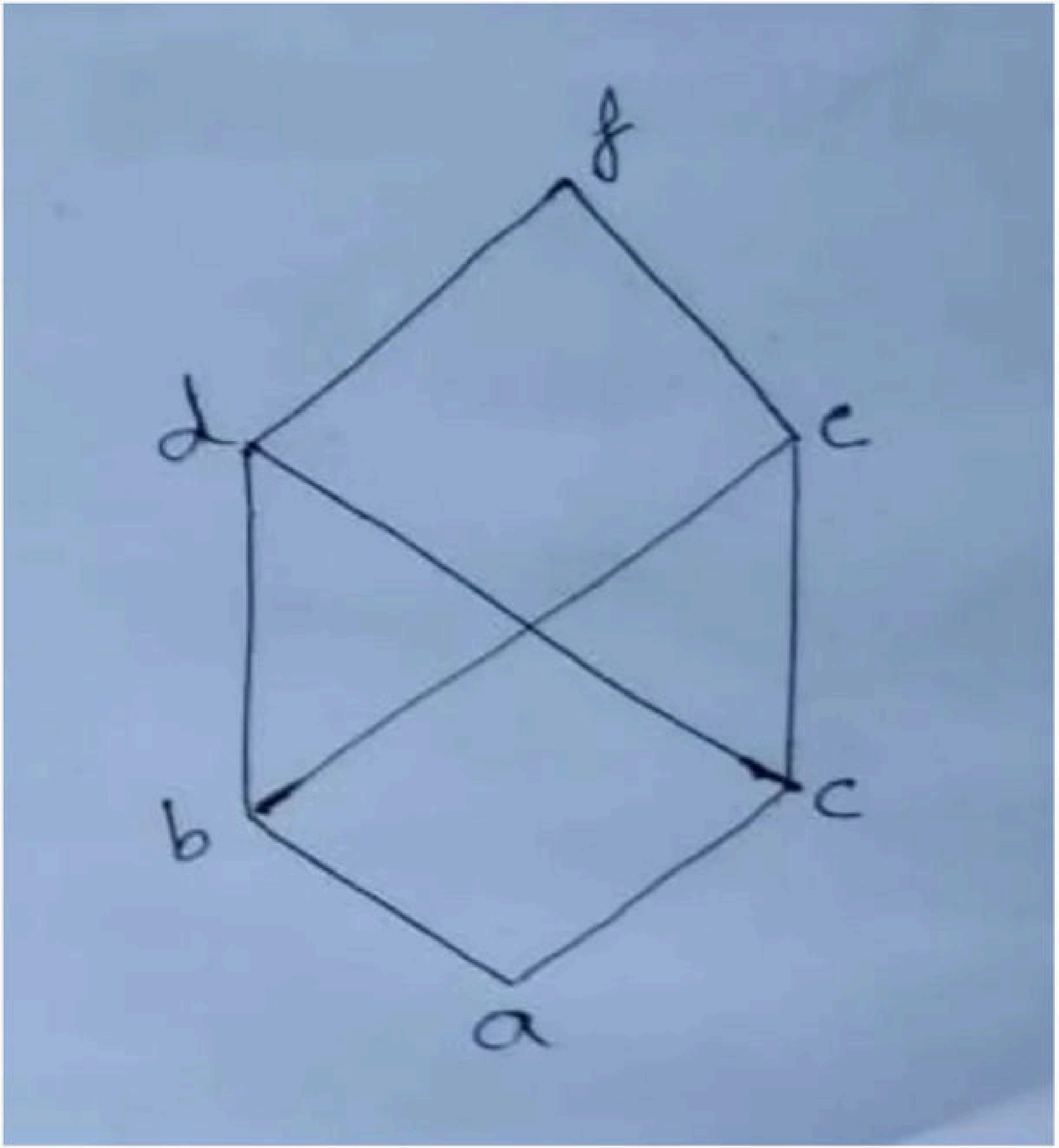
- **Least Element: -** An element is said to be Minimum/Least if no other element is related to it but it is related to every element Partial order relation. Or if a hasse diagram has only one Minimal element then it will also be Minimum/Least element.



	Elements
Minimal	a
Least	
Maximal	i
Greatest	



	Elements
Minimal	a, b
Least	
Maximal	e, f
Greatest	



	Elements
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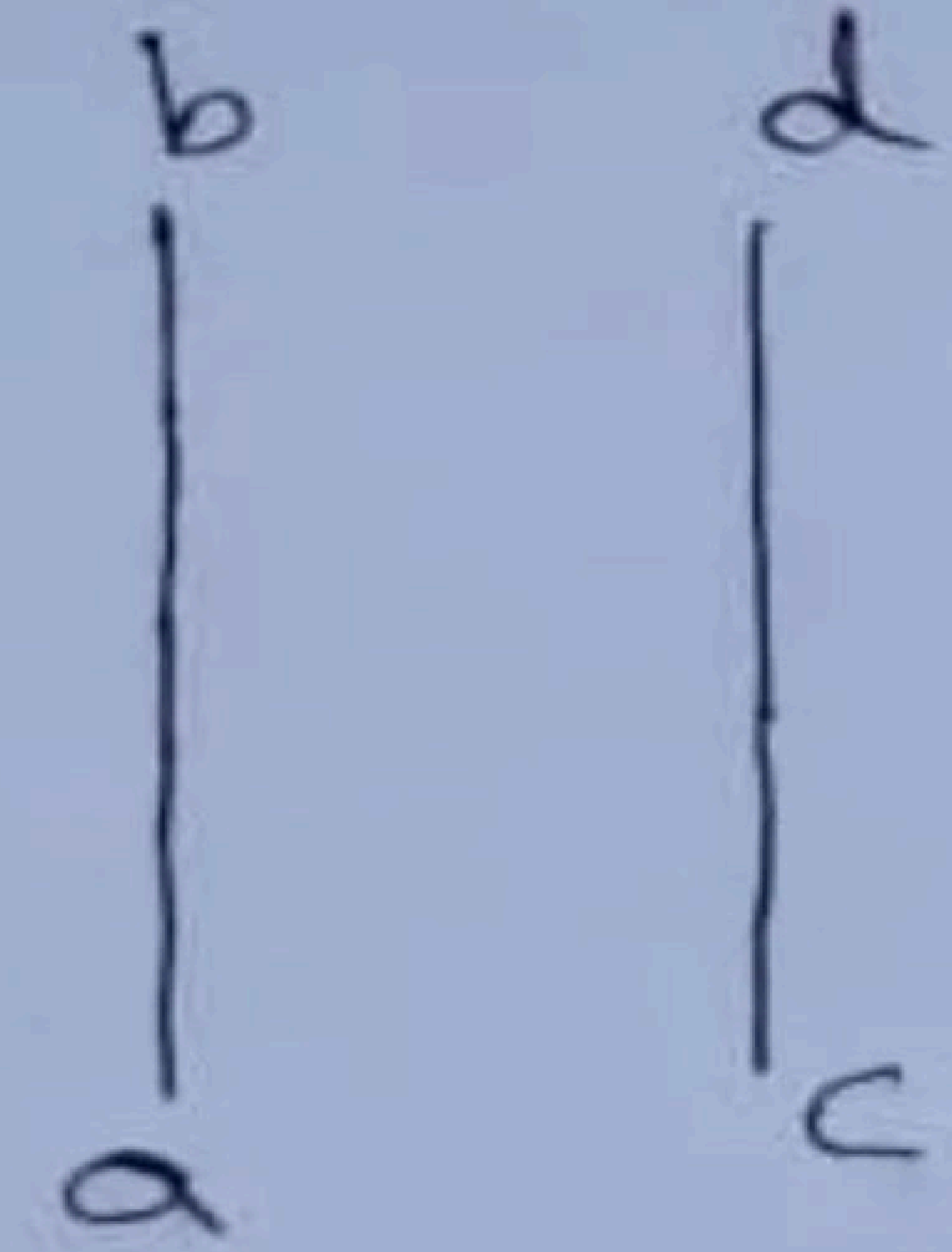
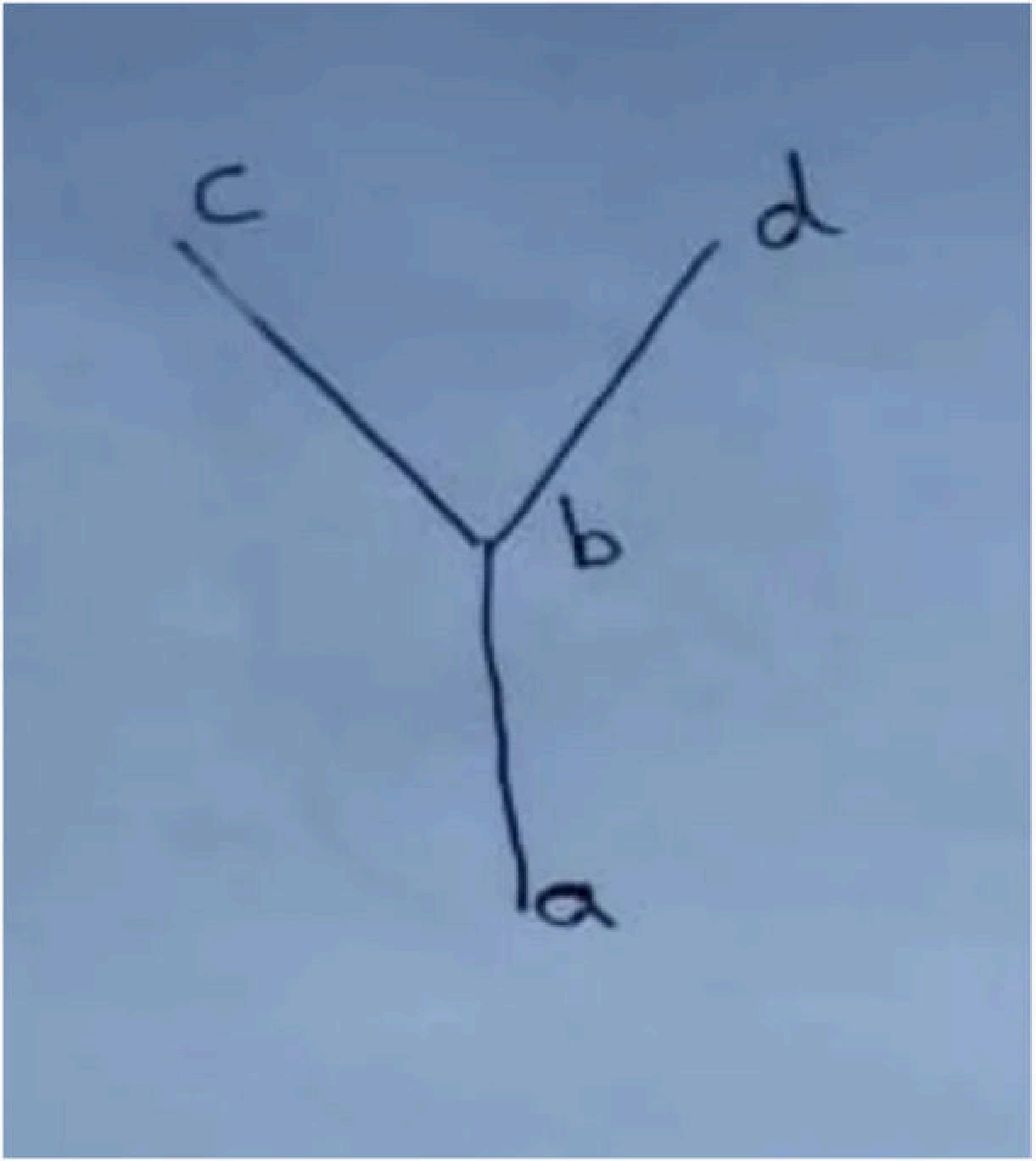
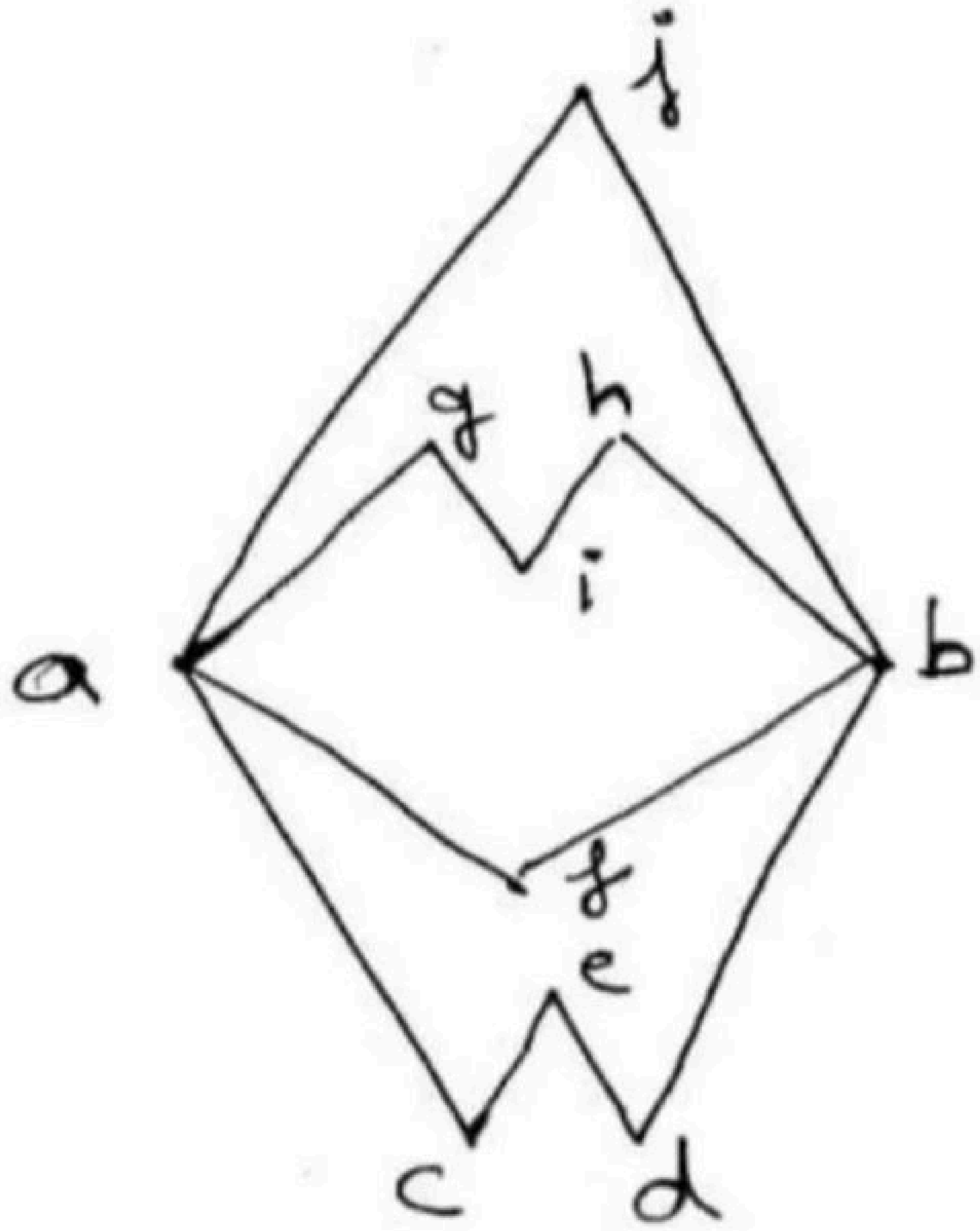
Maximal	f
Greatest	



	Elements
Minimal	a, c
Least	
Maximal	b, d
Greatest	



	Elements
Minimal	a
Least	
Maximal	c, d
Greatest	



	Elements
Minimal	c, d, f, i
Least	
Maximal	j, g, h, e
Greatest	

1. Every hasse diagram will have at most one Greatest and Least element(zero or one) (T/F) T

2. Every Greatest element is also Maximal (T/F)

3. Every Least element is also Minimal (T/F)

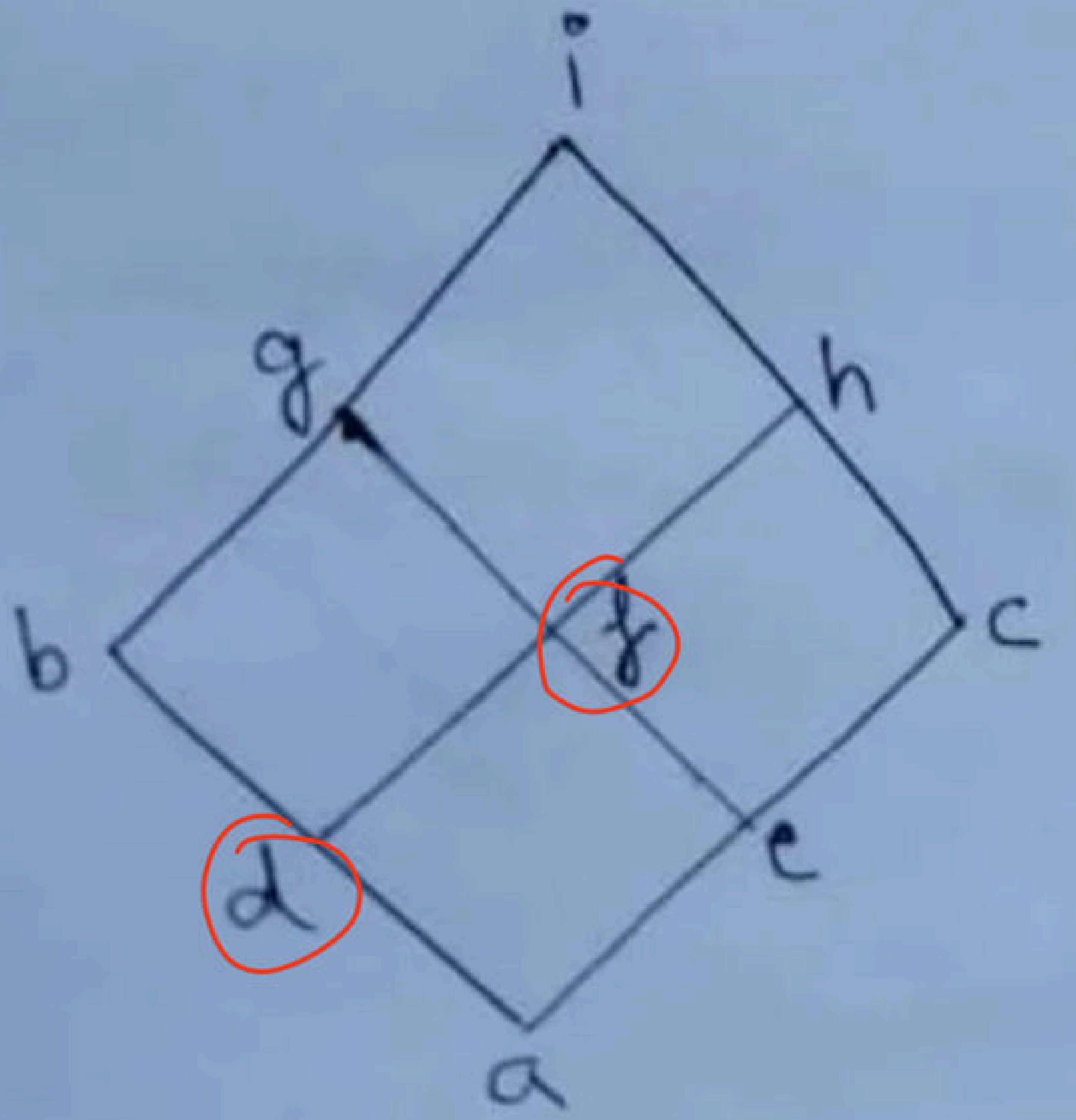
4. If there is only one Maximal element then it is called Greatest (T/F)

5. If there is only one Minimal element then it is called Least (T/F)

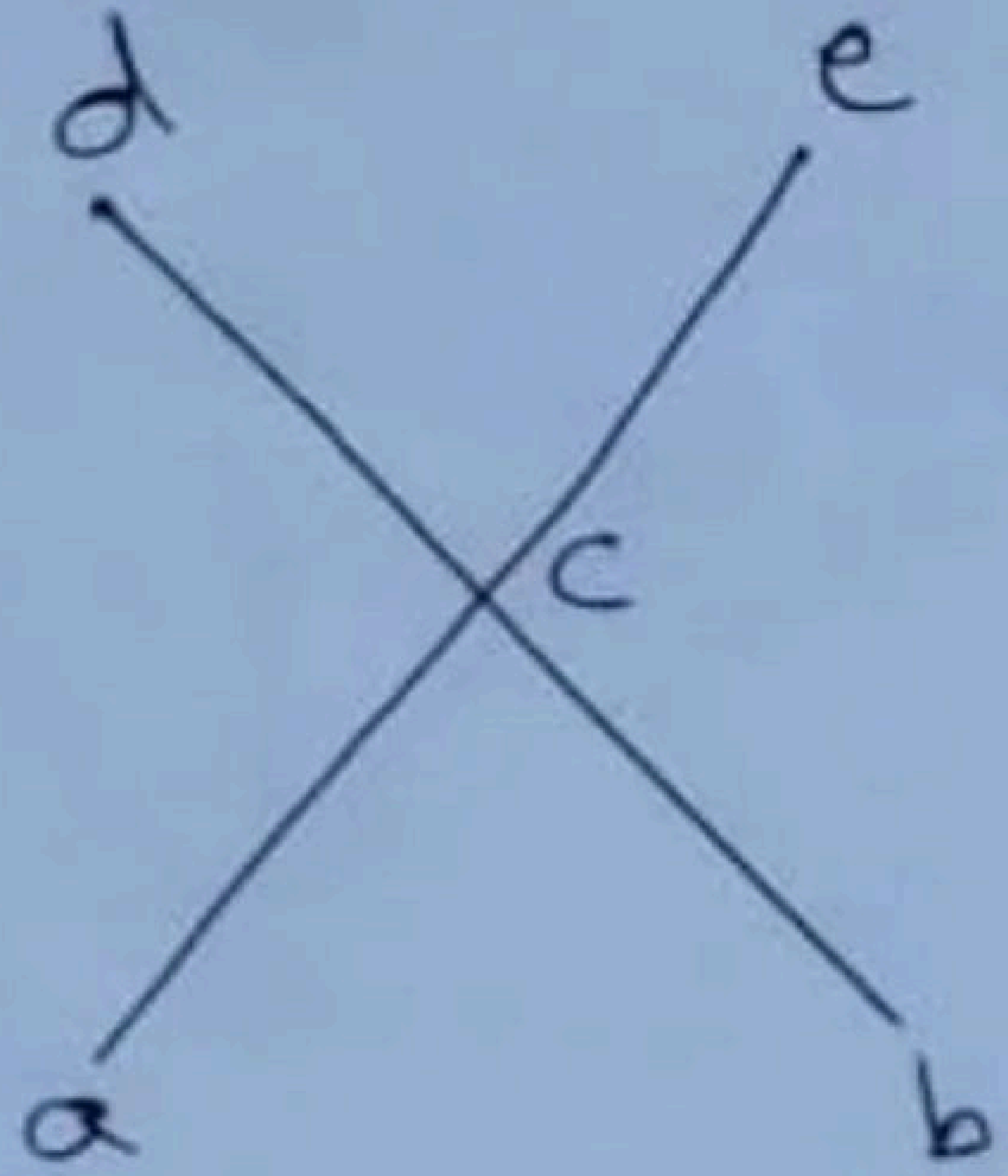
Break

- **Upper Bound**: - Upper bound of a subset B with respect to set A, will contain all those element to which all the elements of B is related.

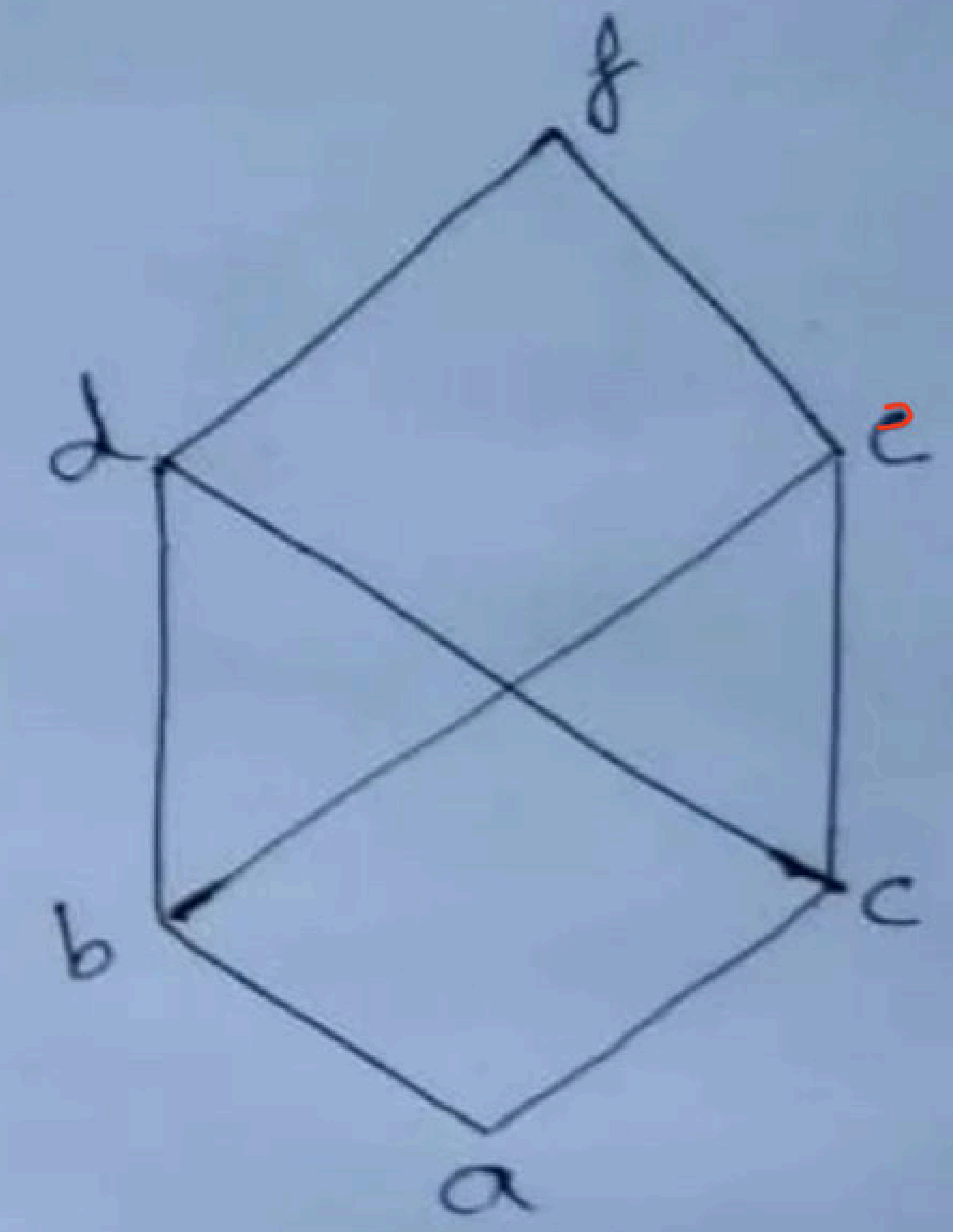
- **Lower Bound**: - lower bound of a subset B with respect to A, will contain all those elements which are related to every element of B.



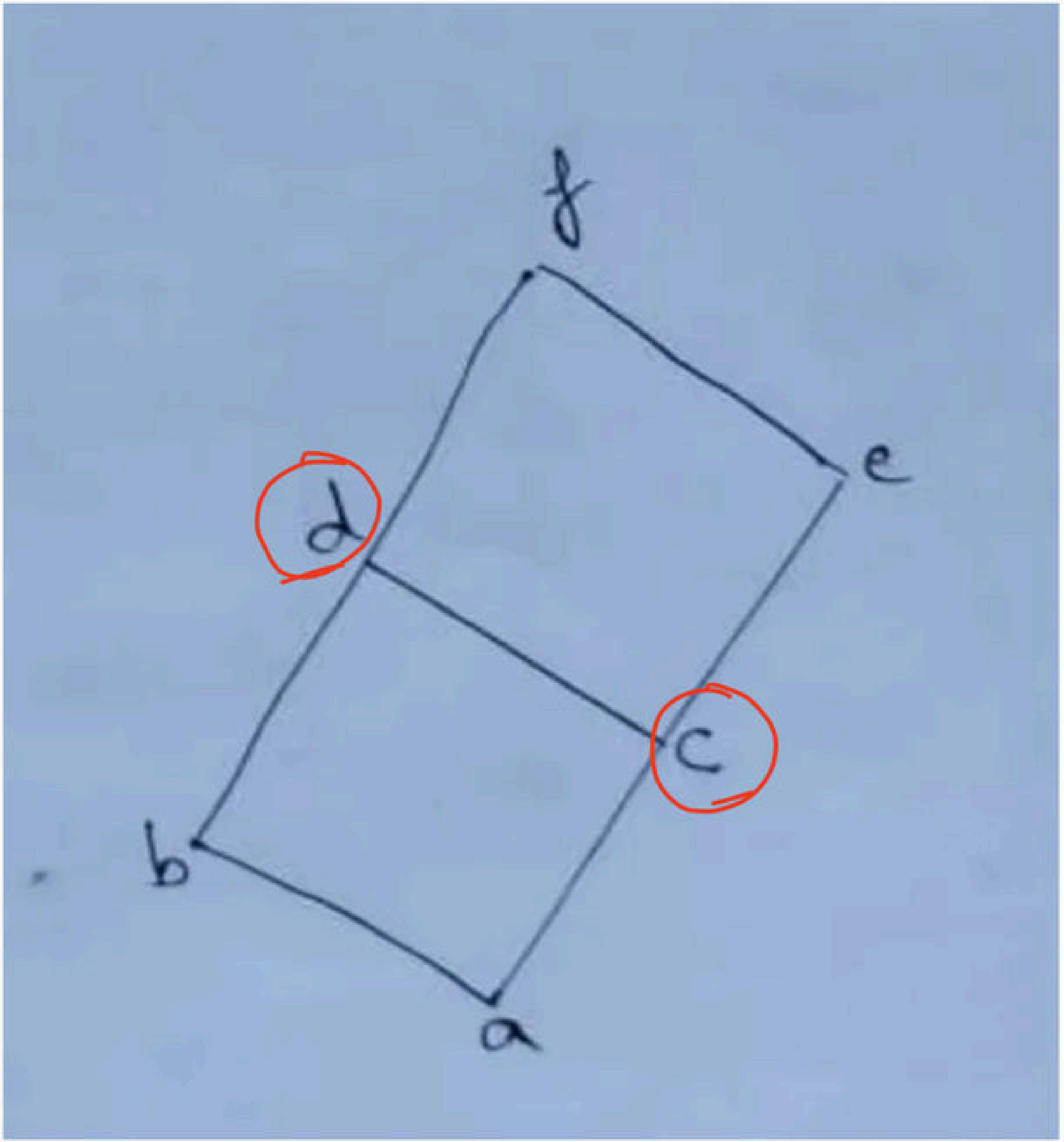
Elements	B = {g, f, e, c}	B = {d, f}
Upper Bound	i	i, g, h, f
Lower Bound	a, e,	a, d



Elements	B = {d, e}	B = {c}
Upper Bound	ϕ	d, e, c
Lower Bound	a, b, c	a, b, c



Elements	B = {d, e}	B = {b, c}
Upper Bound	f	d, f, e
Lower Bound	a, b, c	a



Elements	B = {d, c}	B = {b, e}
Upper Bound	f, d	f
Lower Bound	a, c	a

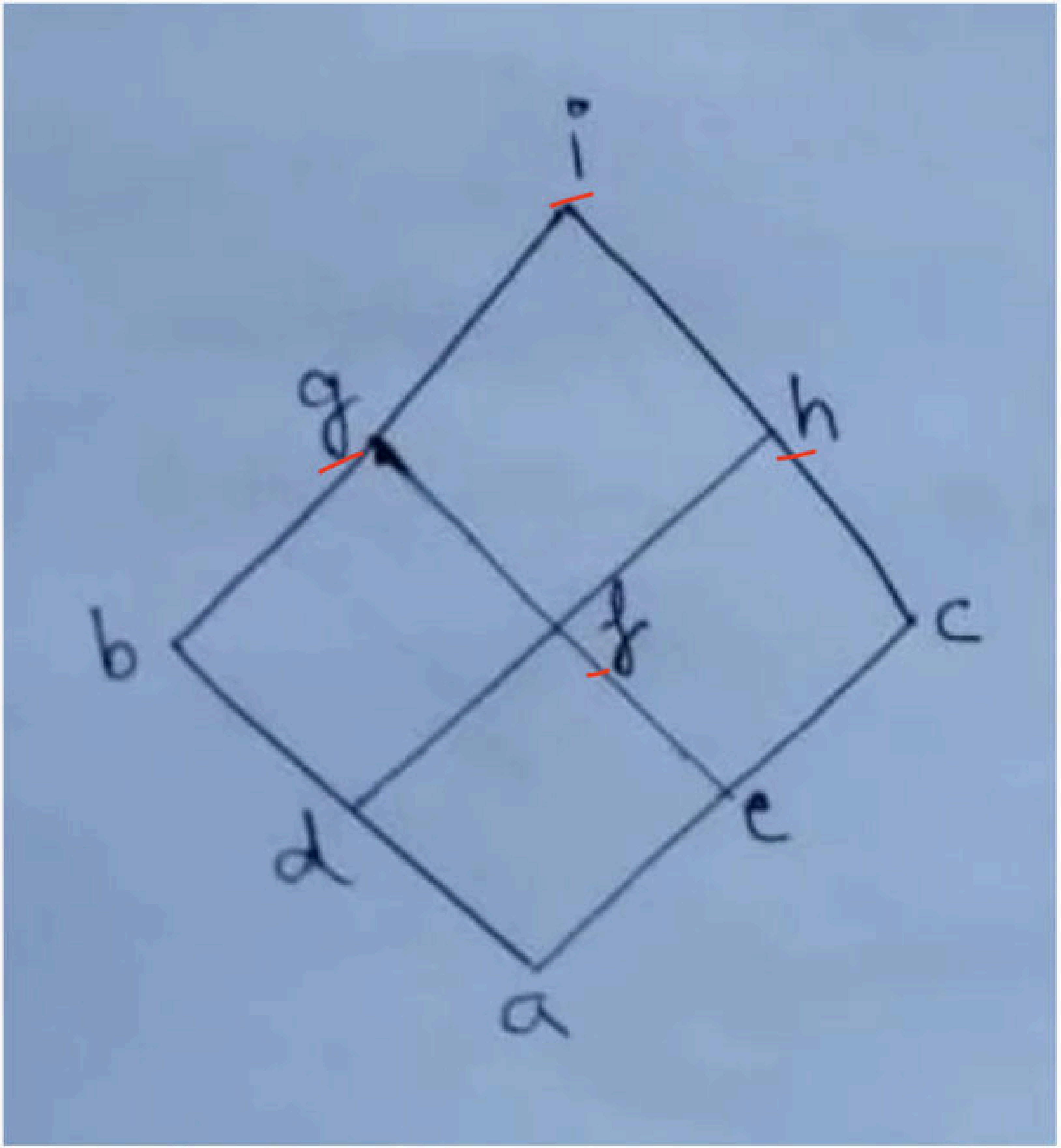
Break

Least Upper Bound / LUB / Join / Supremum / $\vee$

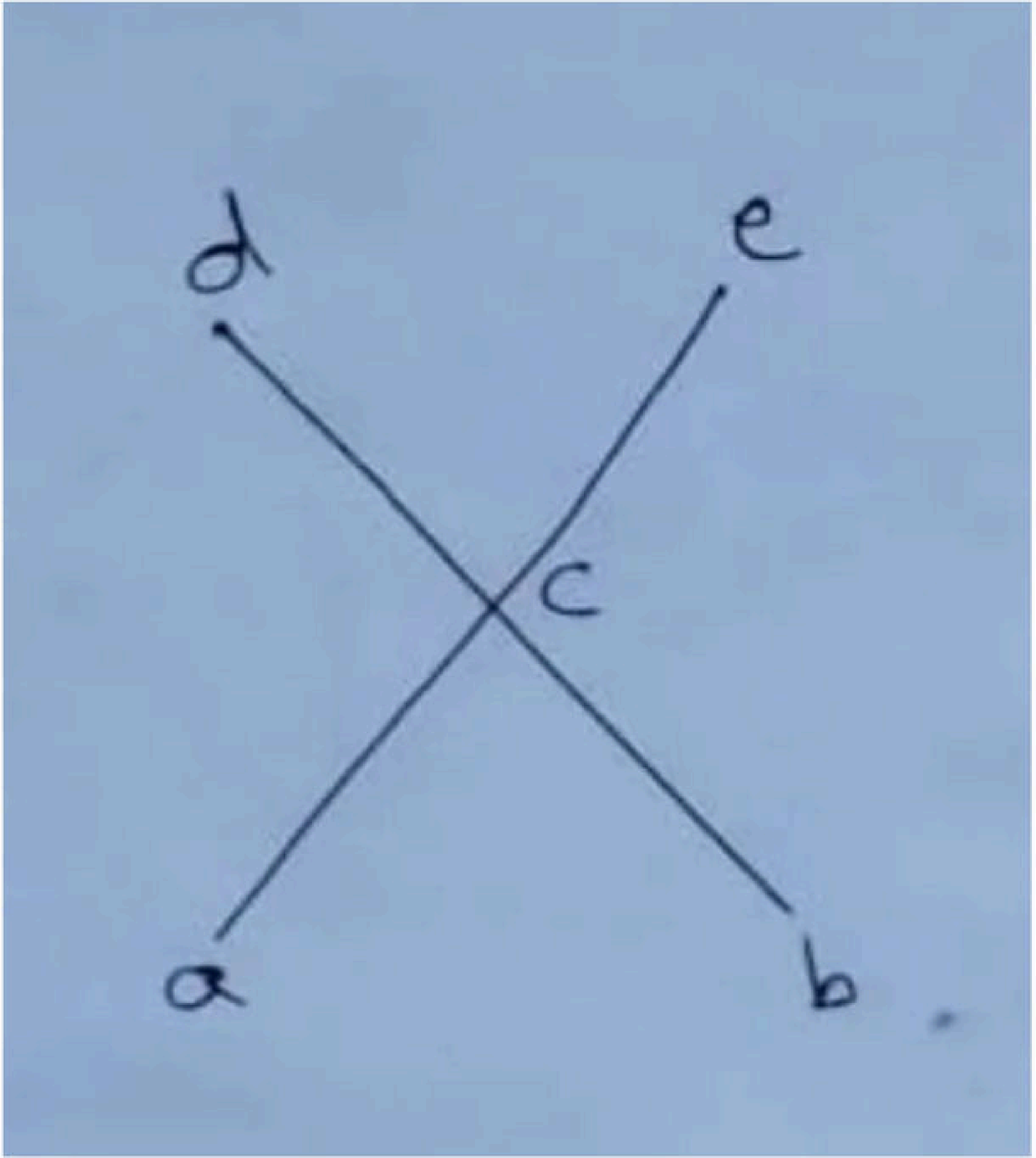
Least value in the upper bound

Greatest Lower Bound / GLB / Meet / Infimum / ∧

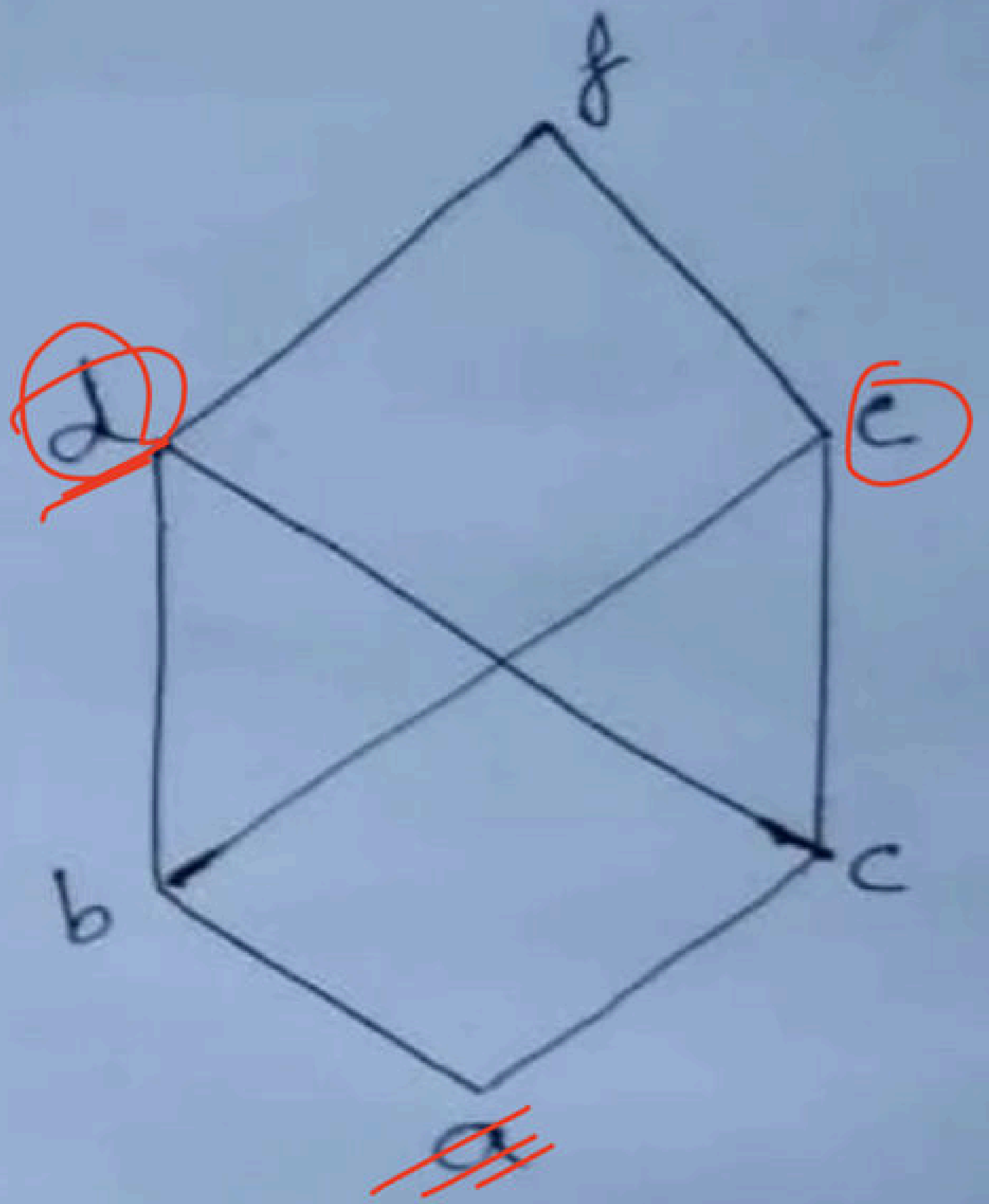
Greatest value in the lower bound



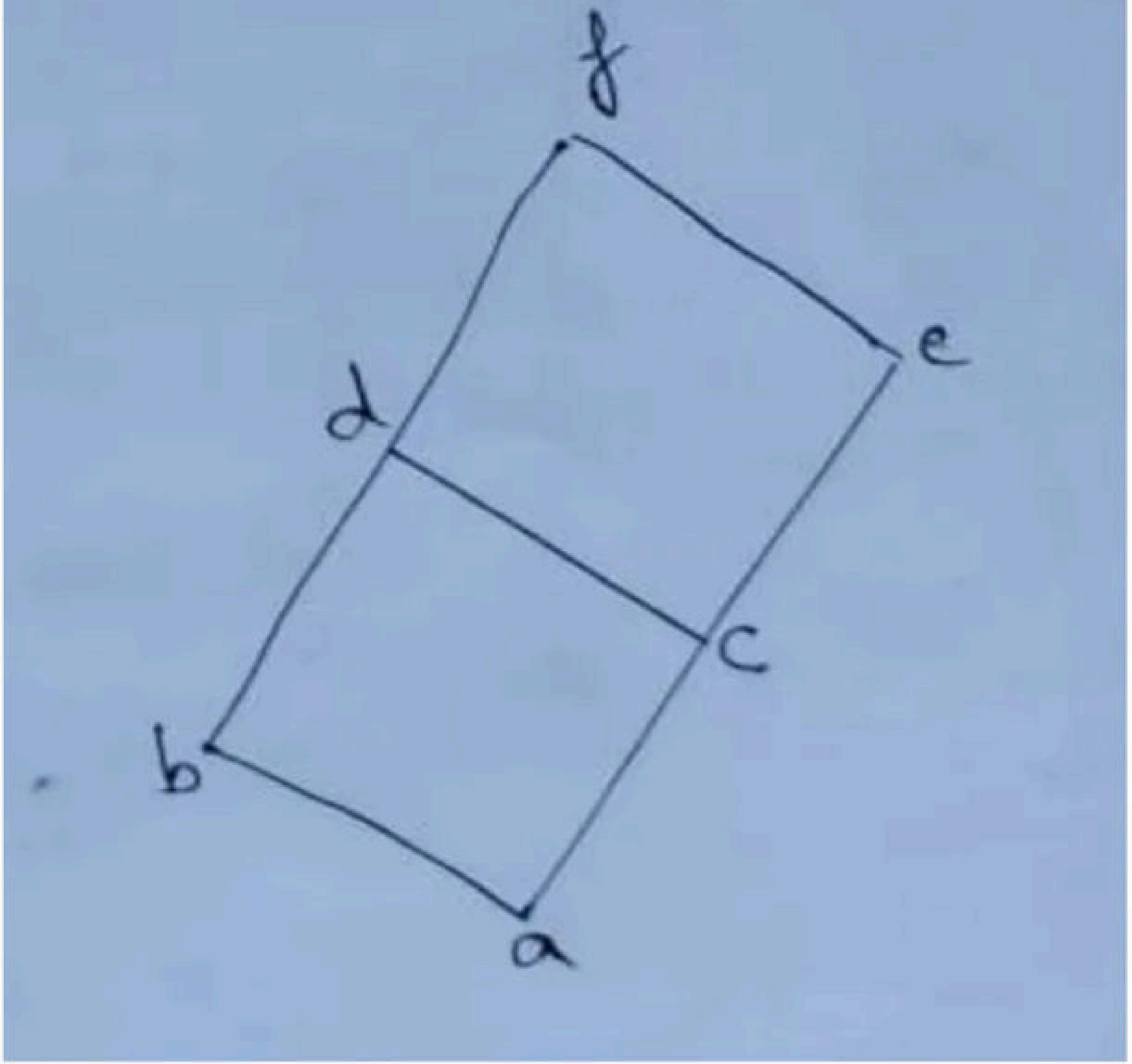
Elements	B = {g, f, e, c}	B = {d, f}
Upper Bound	{i}	{i, g, h, f}
Least Upper Bound	i	f
Lower Bound	{a, e}	{a, d}
Greatest Lower Bound	e	d



Elements	B = {d, e}	B = {c}
Upper Bound	{}	{d, e, c}
Least Upper Bound	ϕ	c
Lower Bound	{a, b, c}	{a, b, c}
Greatest Lower Bound	c	c



Elements	B = {d, e}	B = {b, c}
Upper Bound	{f}	{d, e, f}
Least Upper Bound	f	ϕ
Lower Bound	{a, b, c}	{a}
Greatest Lower Bound	ϕ	a



Elements	B = {d, c}	B = {b, e}
Upper Bound	{f, d}	{f}
Least Upper Bound	d	f
Lower Bound	{a, c}	{a}
Greatest Lower Bound	c	a

Q Consider the Poset ({3,5,9,15,24,45}, /). Which of the following is correct for the given Poset? **(NET-JUNE-2019)**

a) There exist a greatest element and a least element — 13

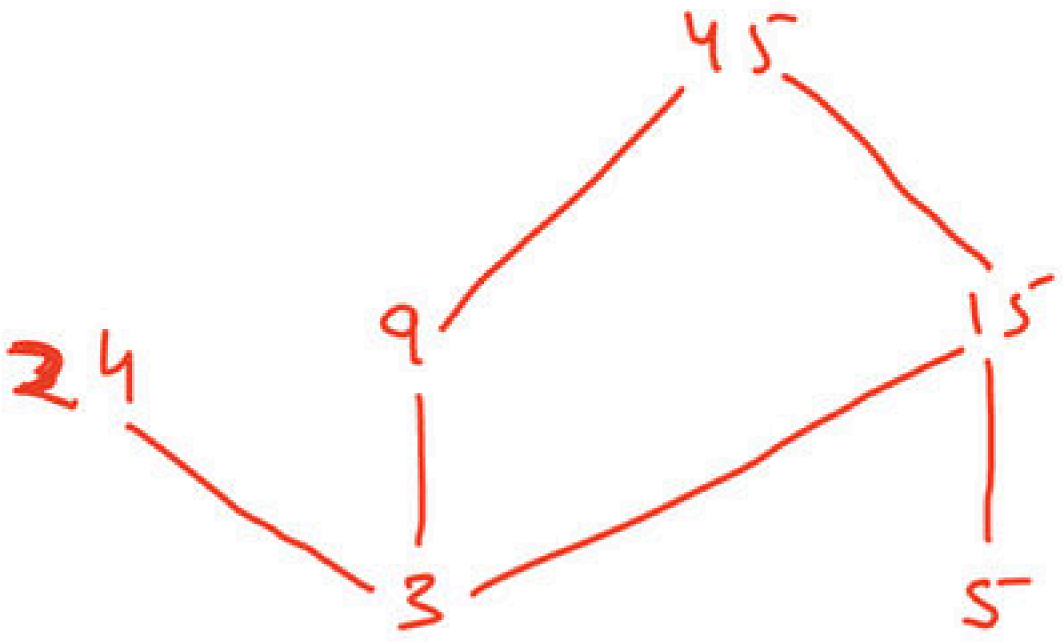
b) There exist a greatest element but not a least element — 23

Max - 24, 45

c) There exist a least element but not a greatest element — 9

Min - 3, 5

d) There does not exist a greatest element and a least element — 55

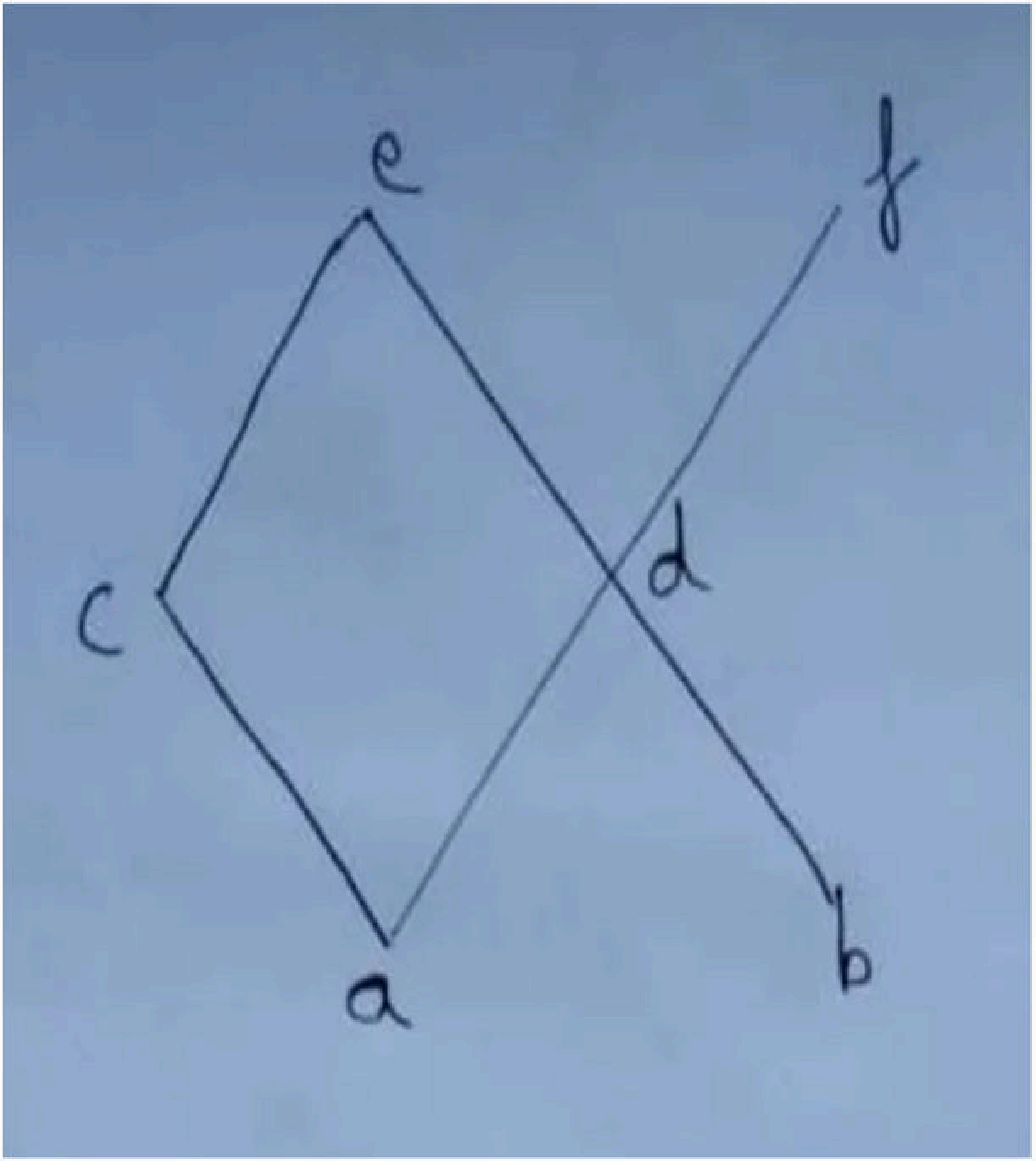


Break

Join Semi Lattice :- A hasse diagram/Partial order relation is called Join Semi Lattice if for every elements their exists a Join.

Meet Semi Lattice :- A hasse diagram/Partial order relation is called Meet Semi Lattice if for every elements their exists a Meet.

Lattice :- A hasse diagram/Partial order relation is called Lattice if their exist a Join and Meet for every pair of element. Or A hasse diagram/Partial order relation is called Lattice if it is both Join Semi Lattice and Meet Semi Lattice.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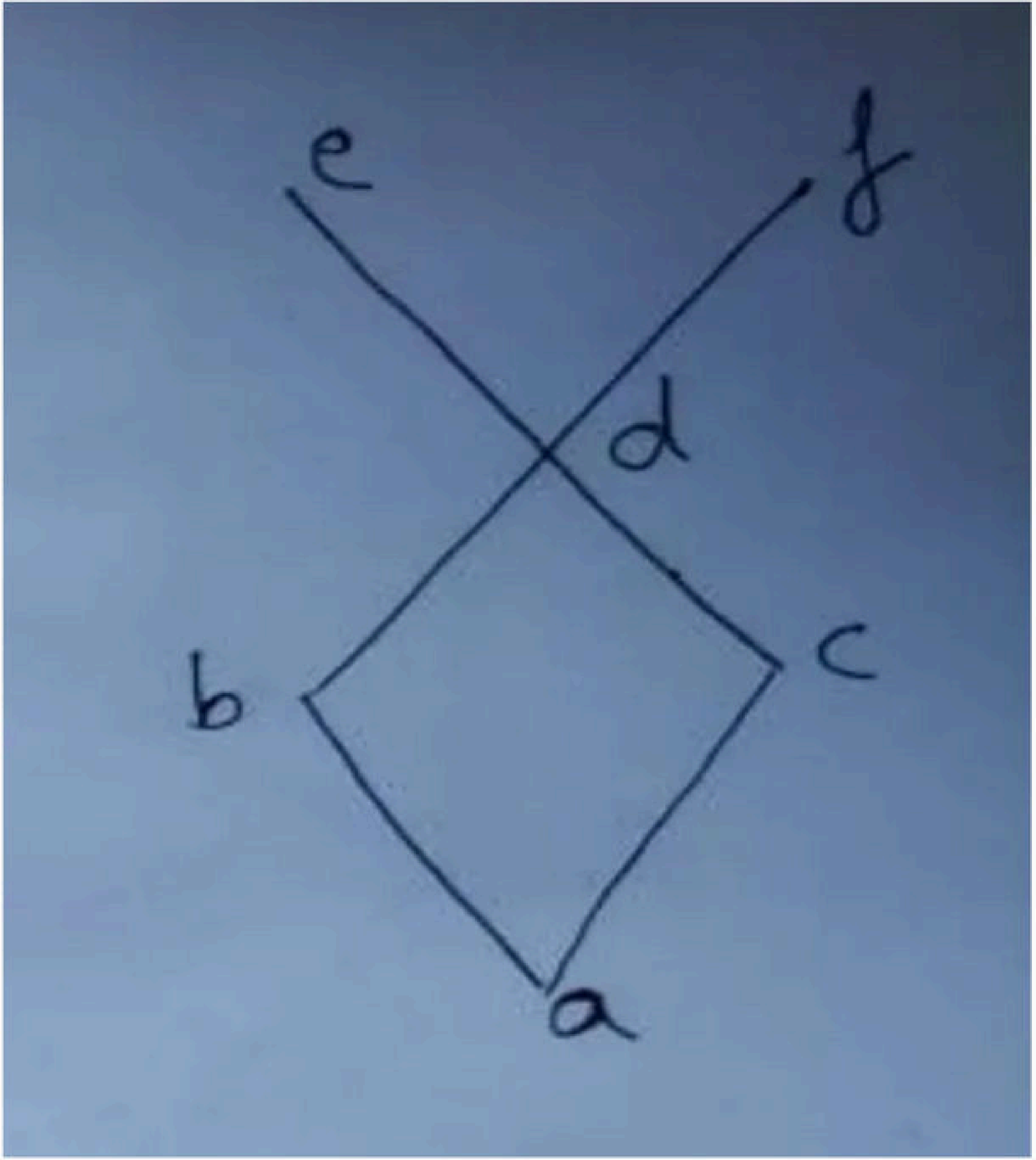
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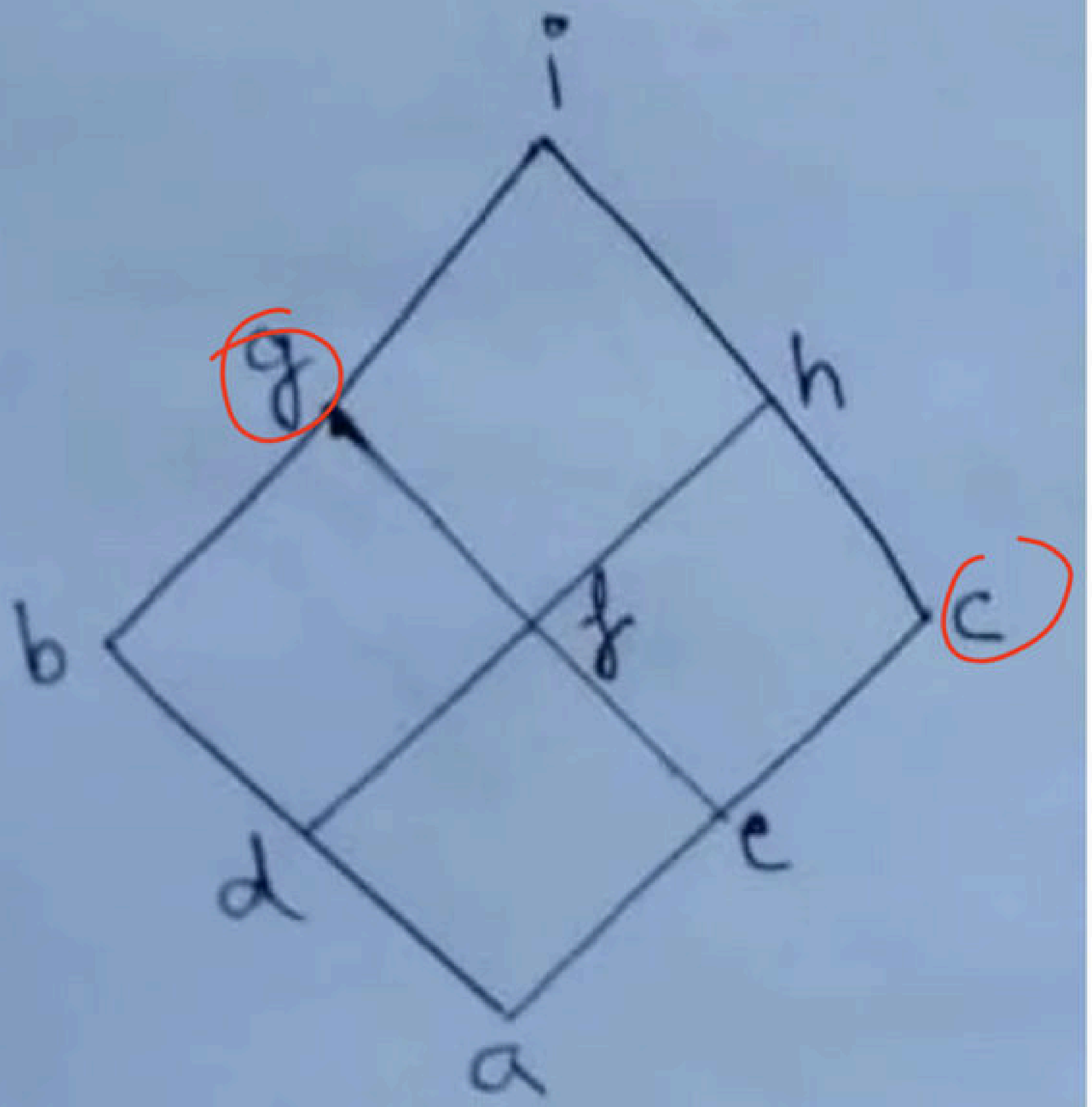

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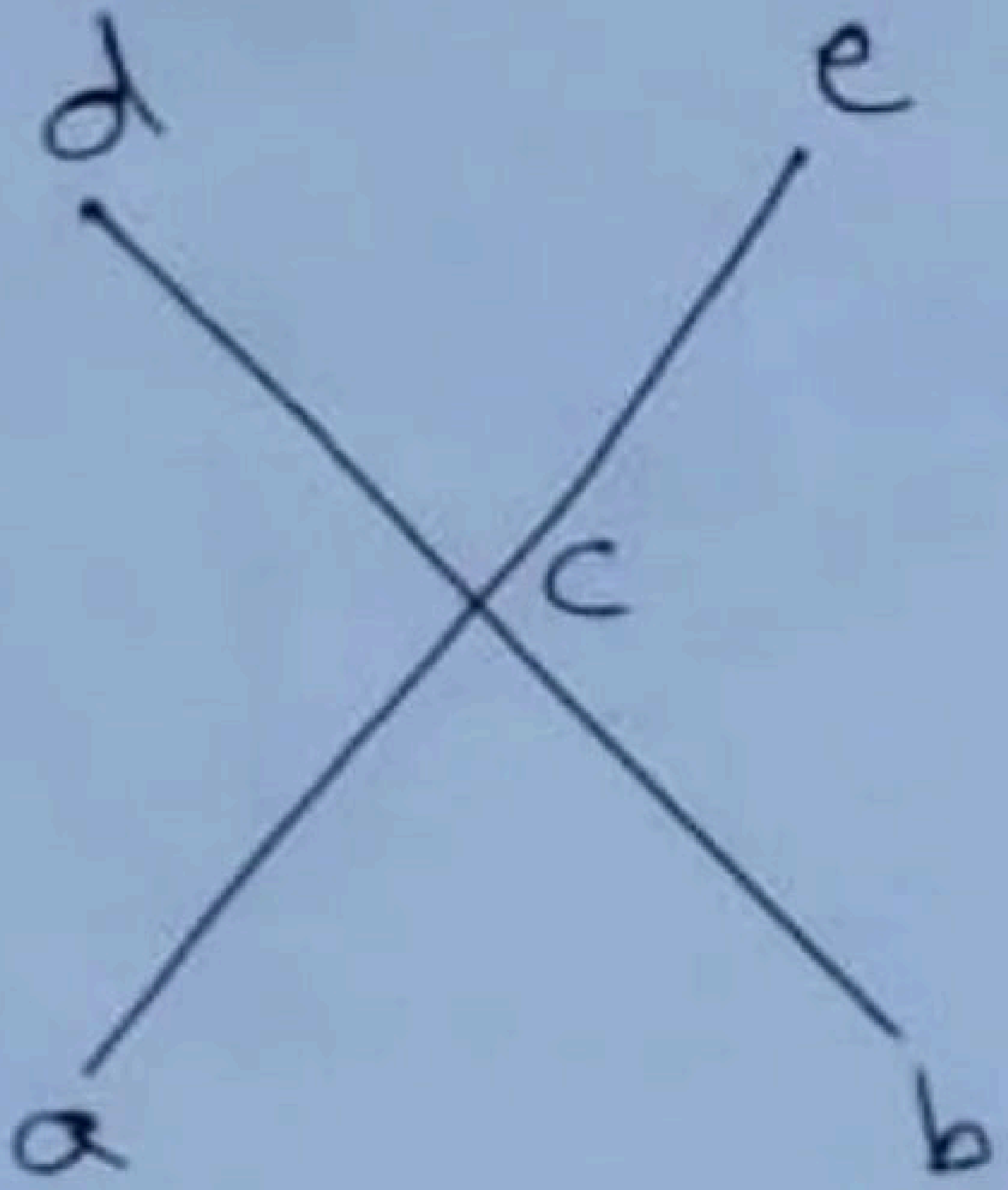
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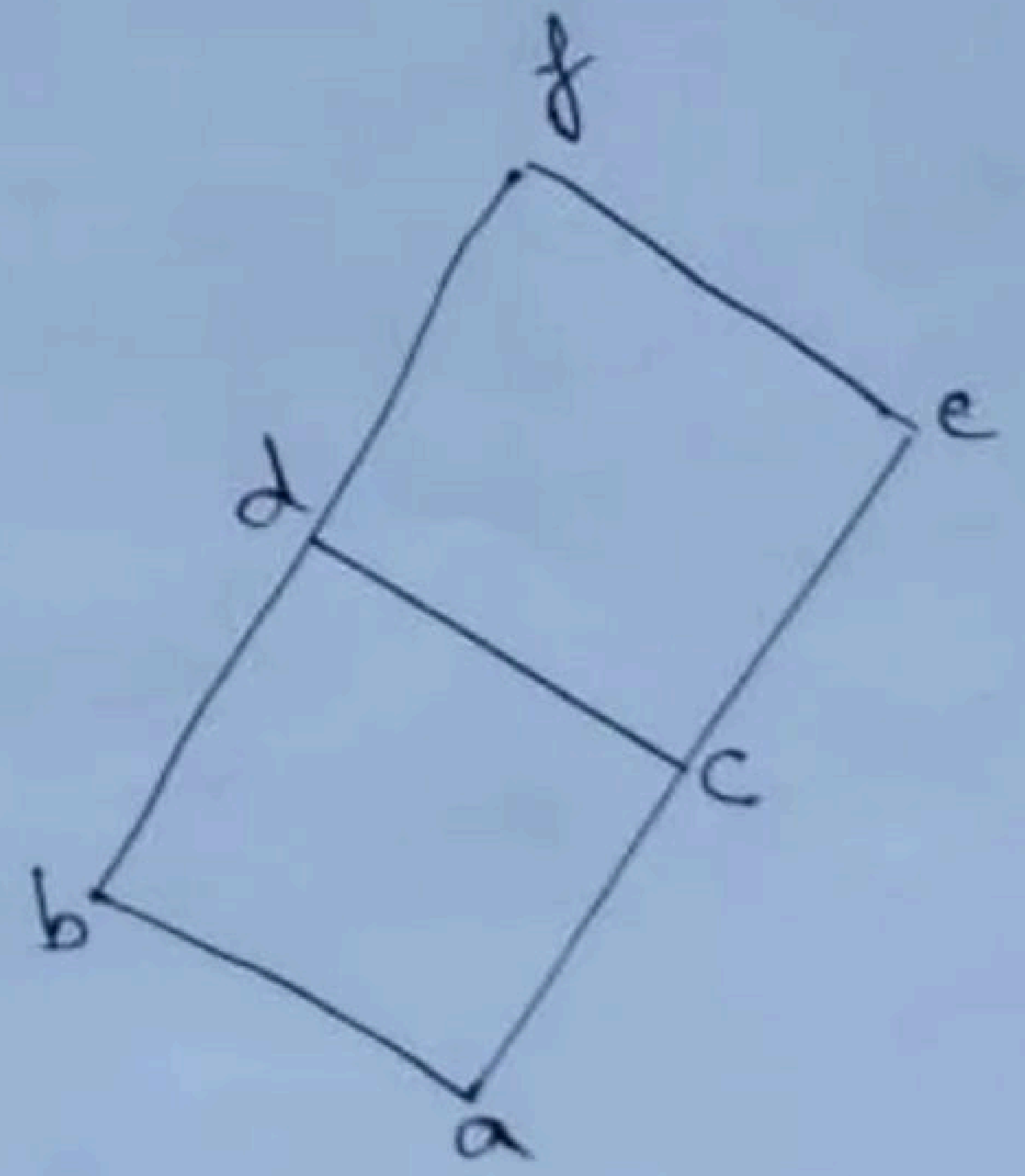
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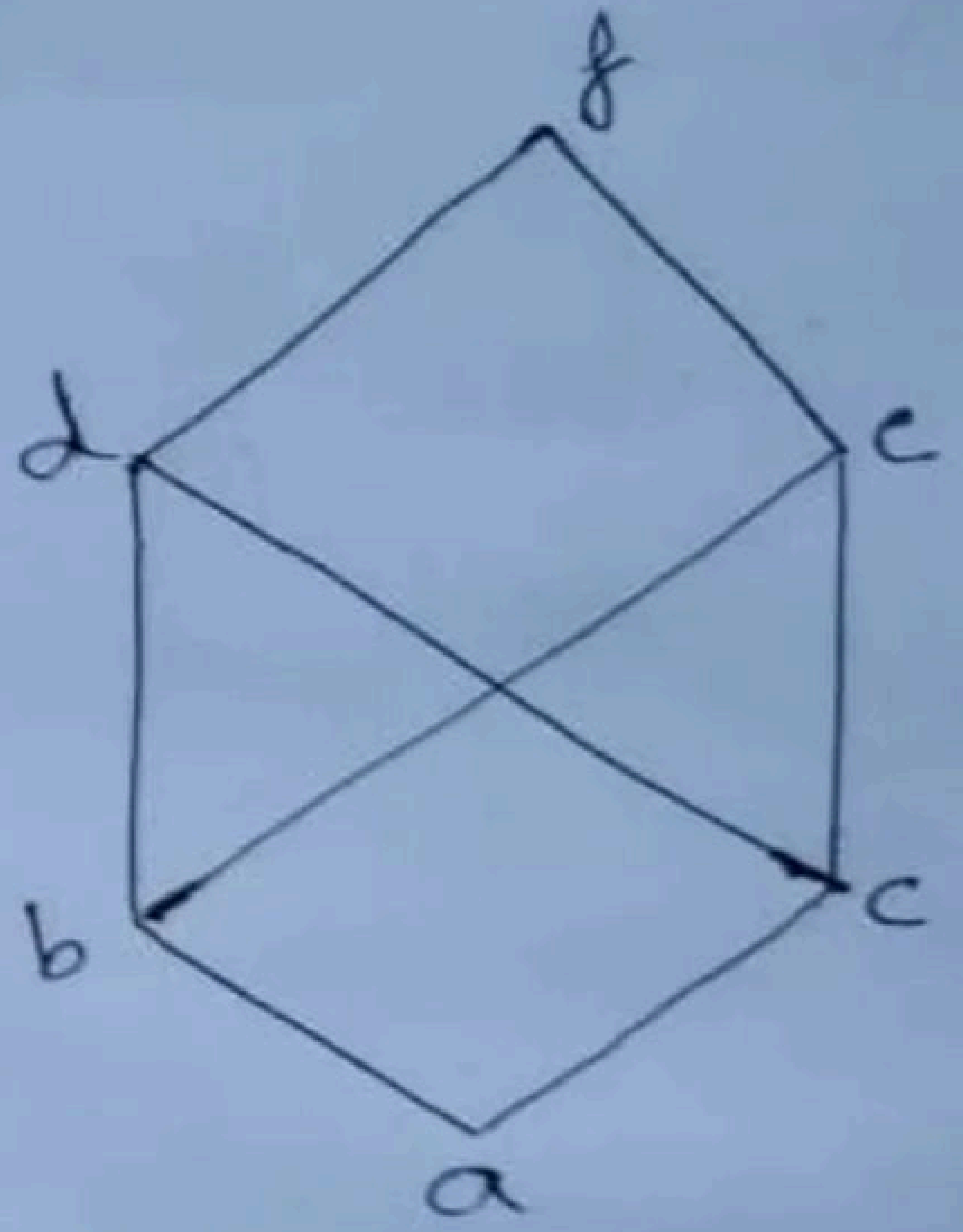
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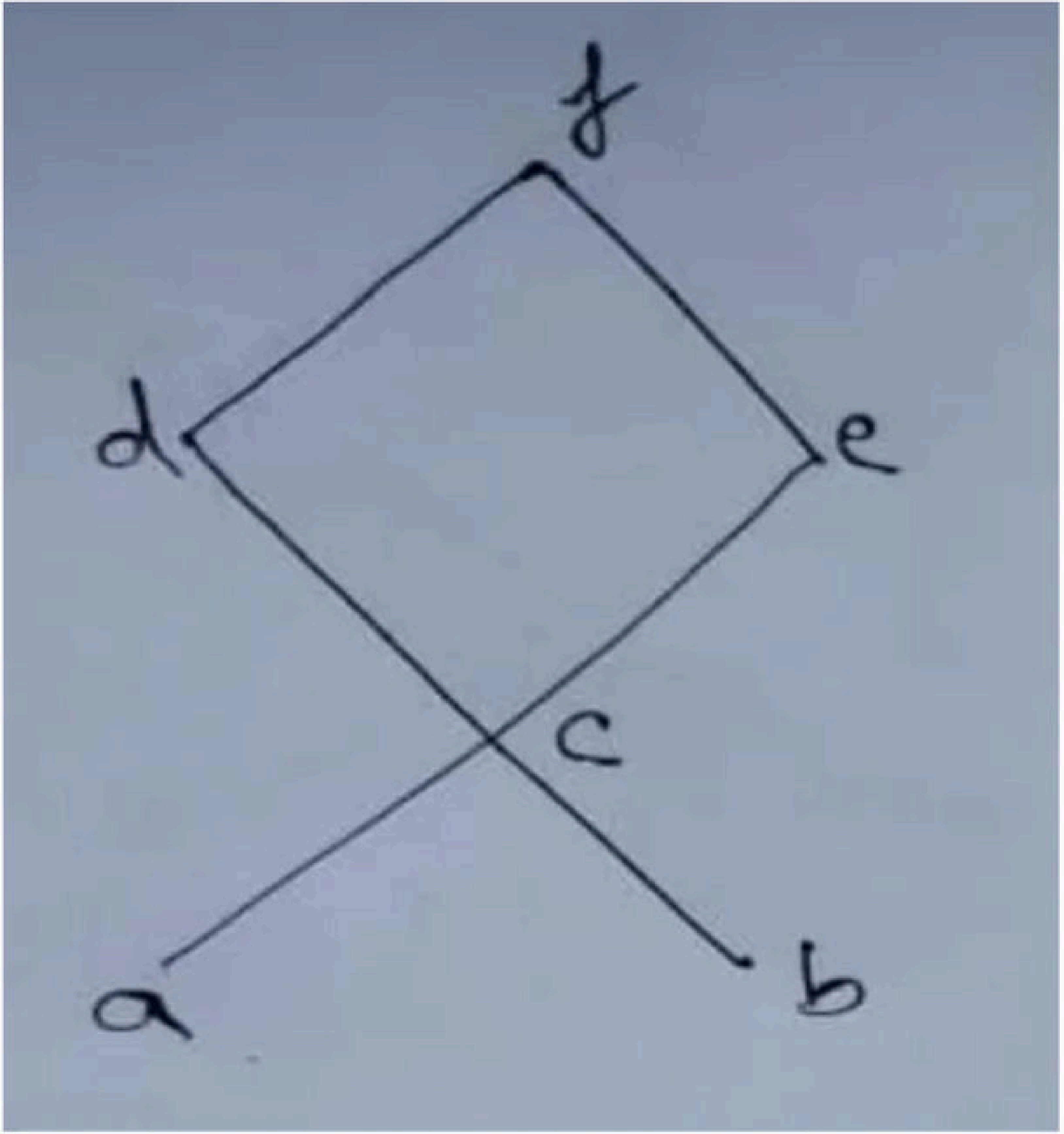
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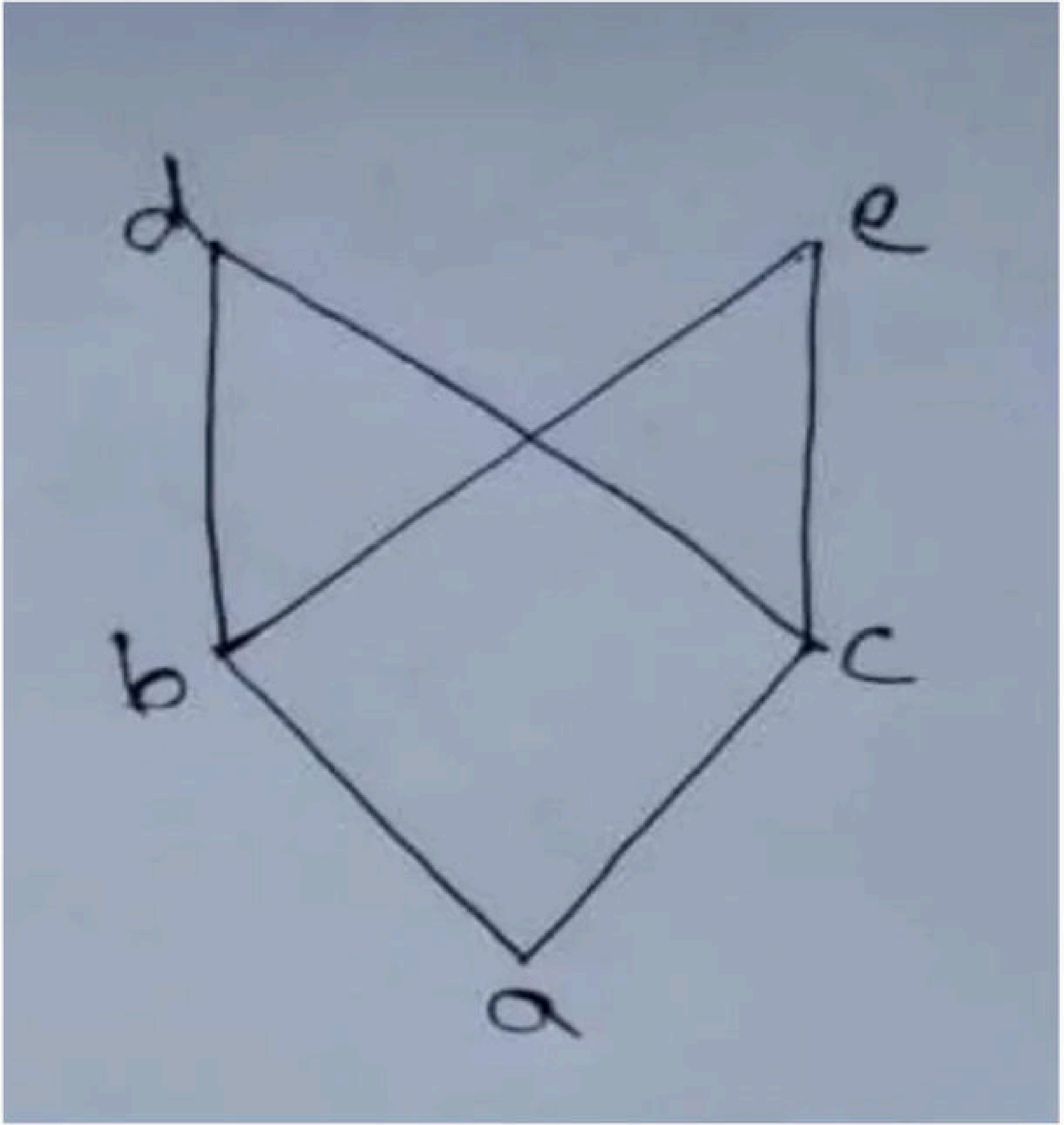
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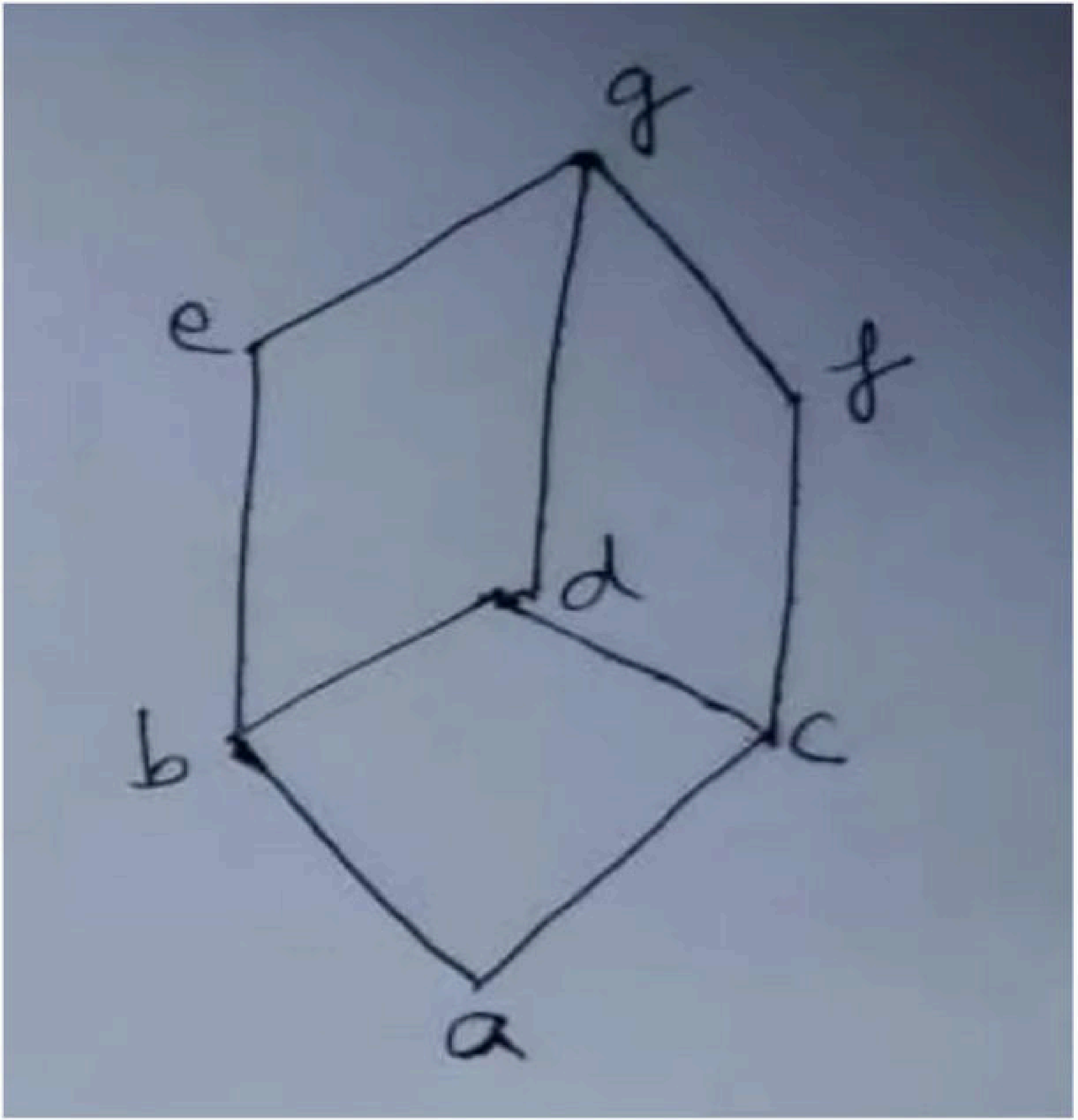

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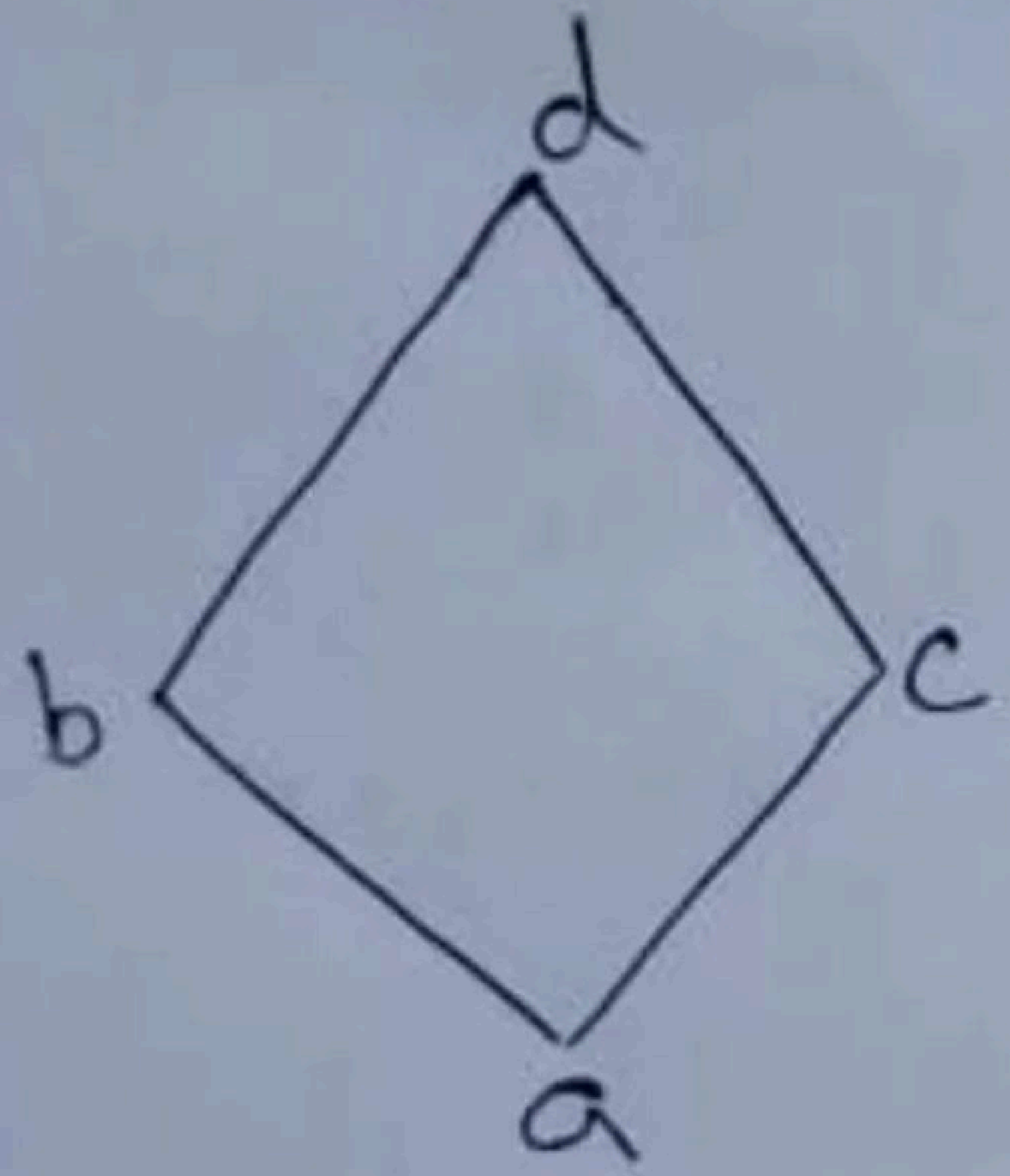
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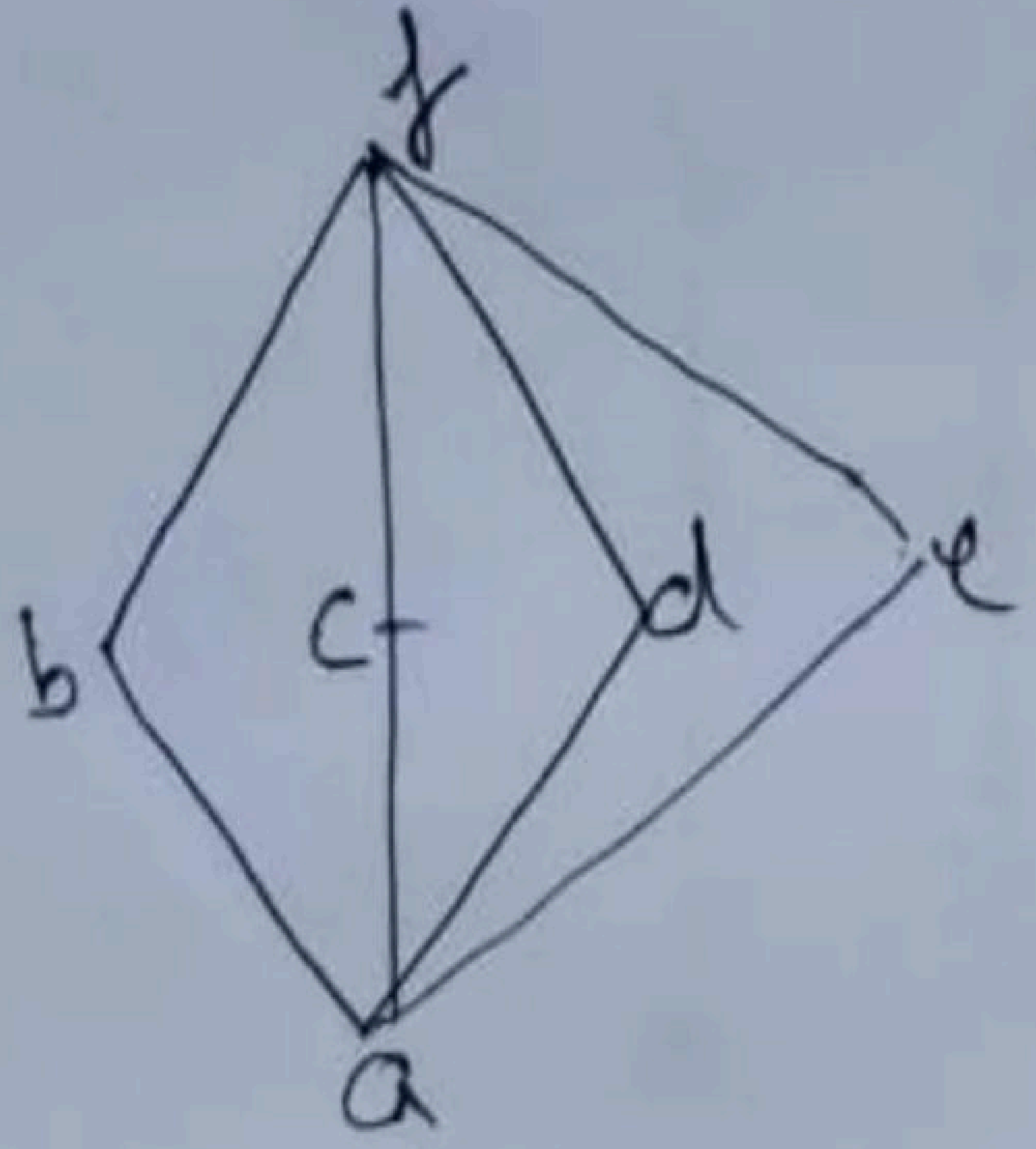
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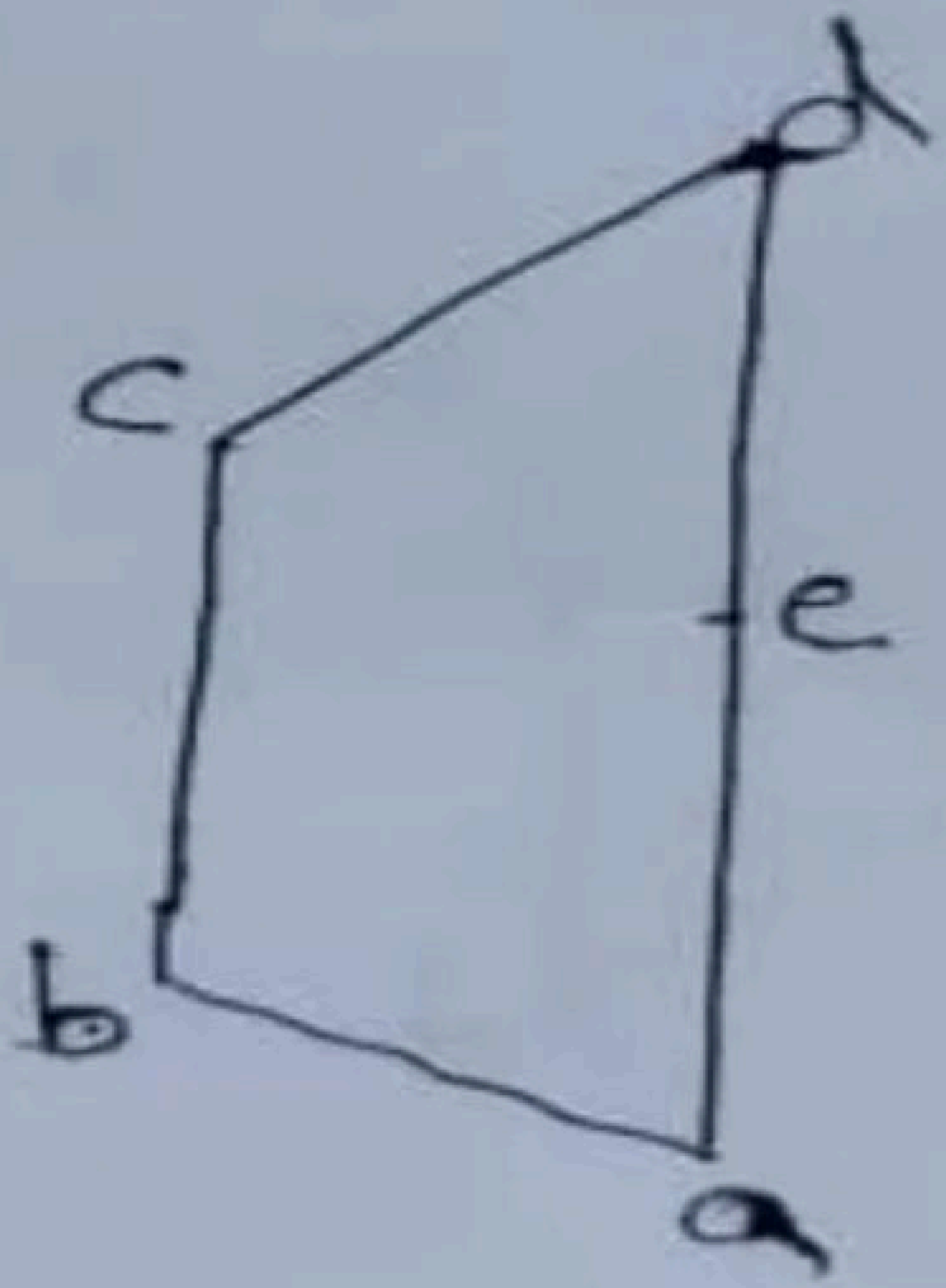
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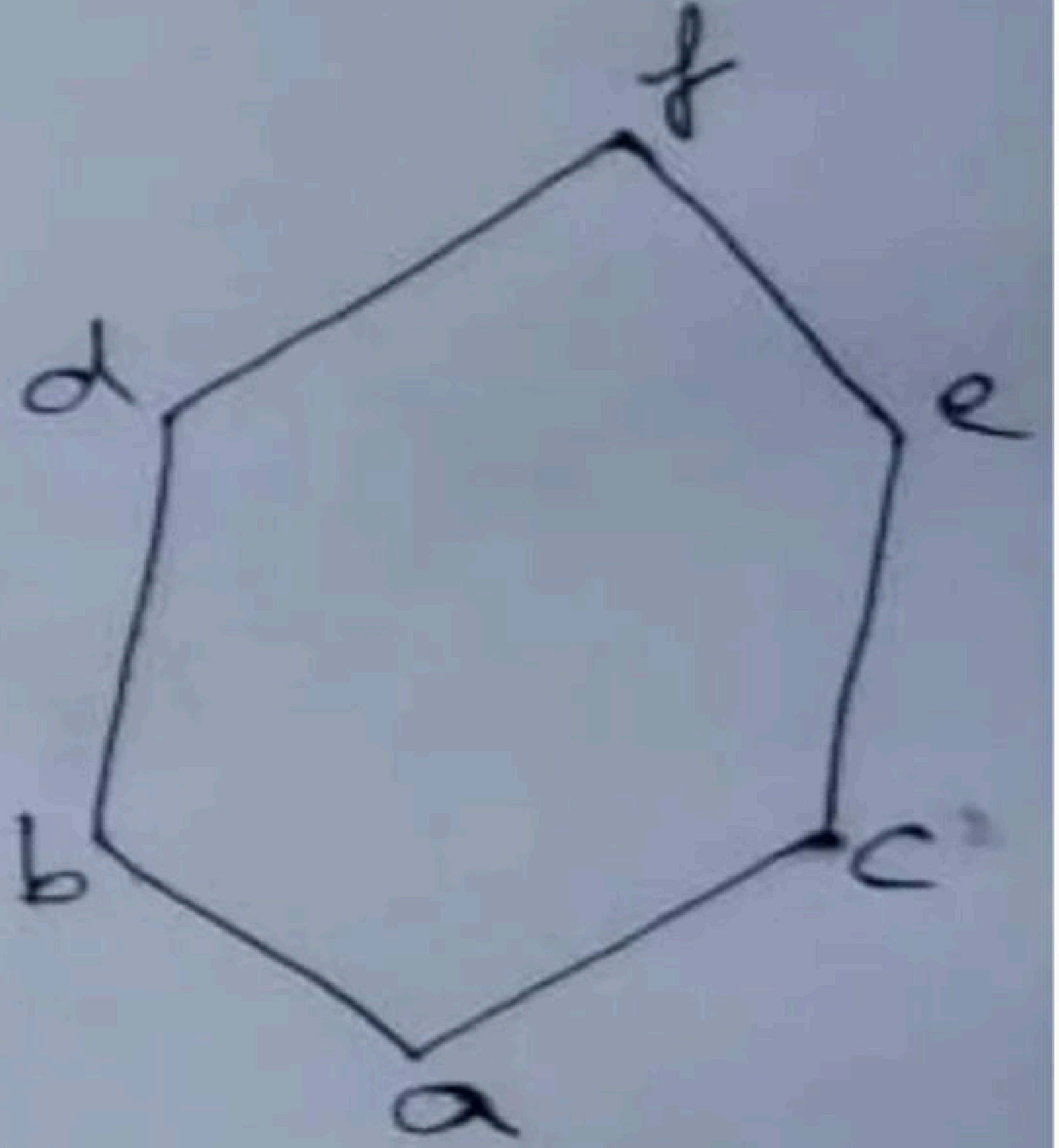
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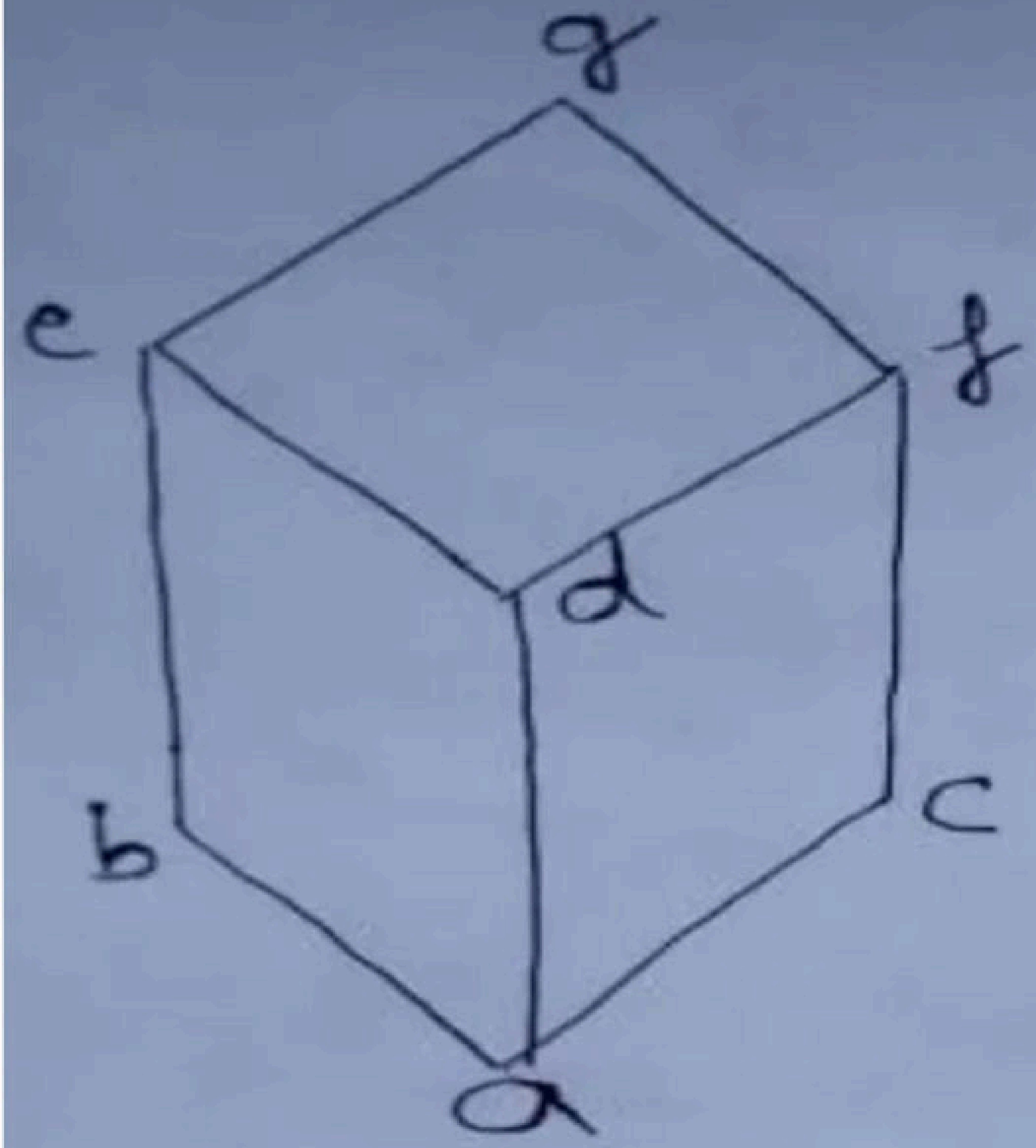
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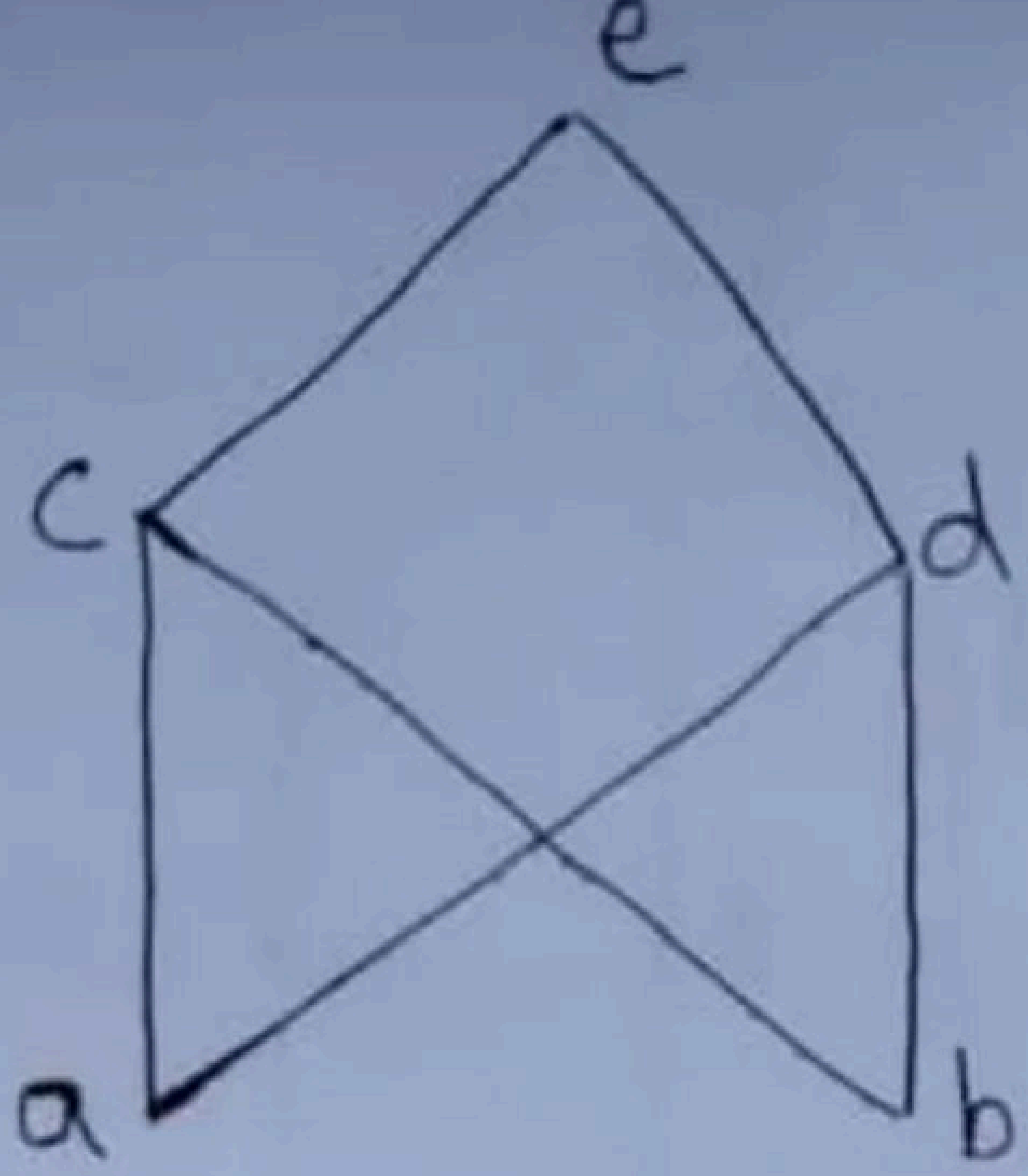

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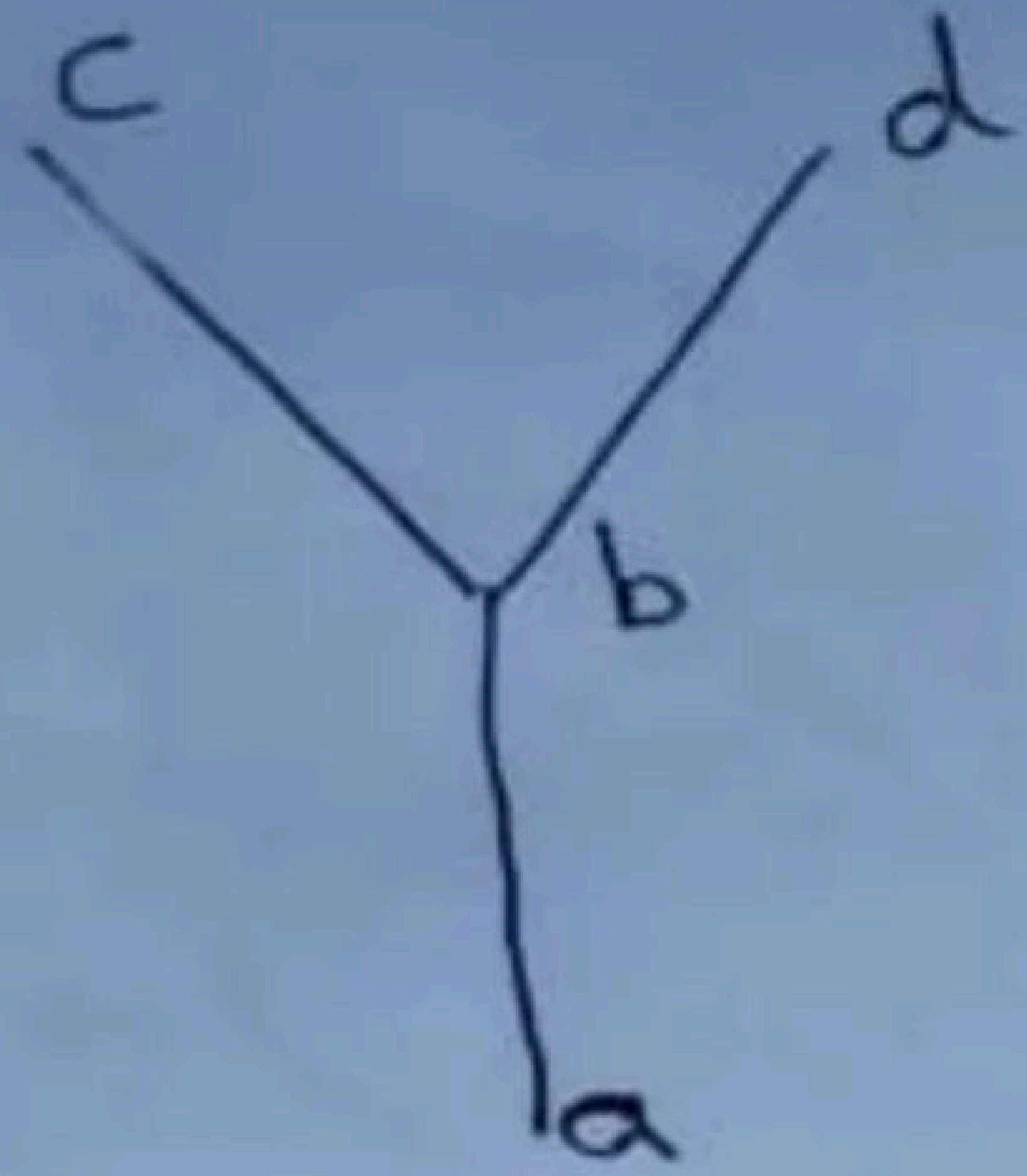
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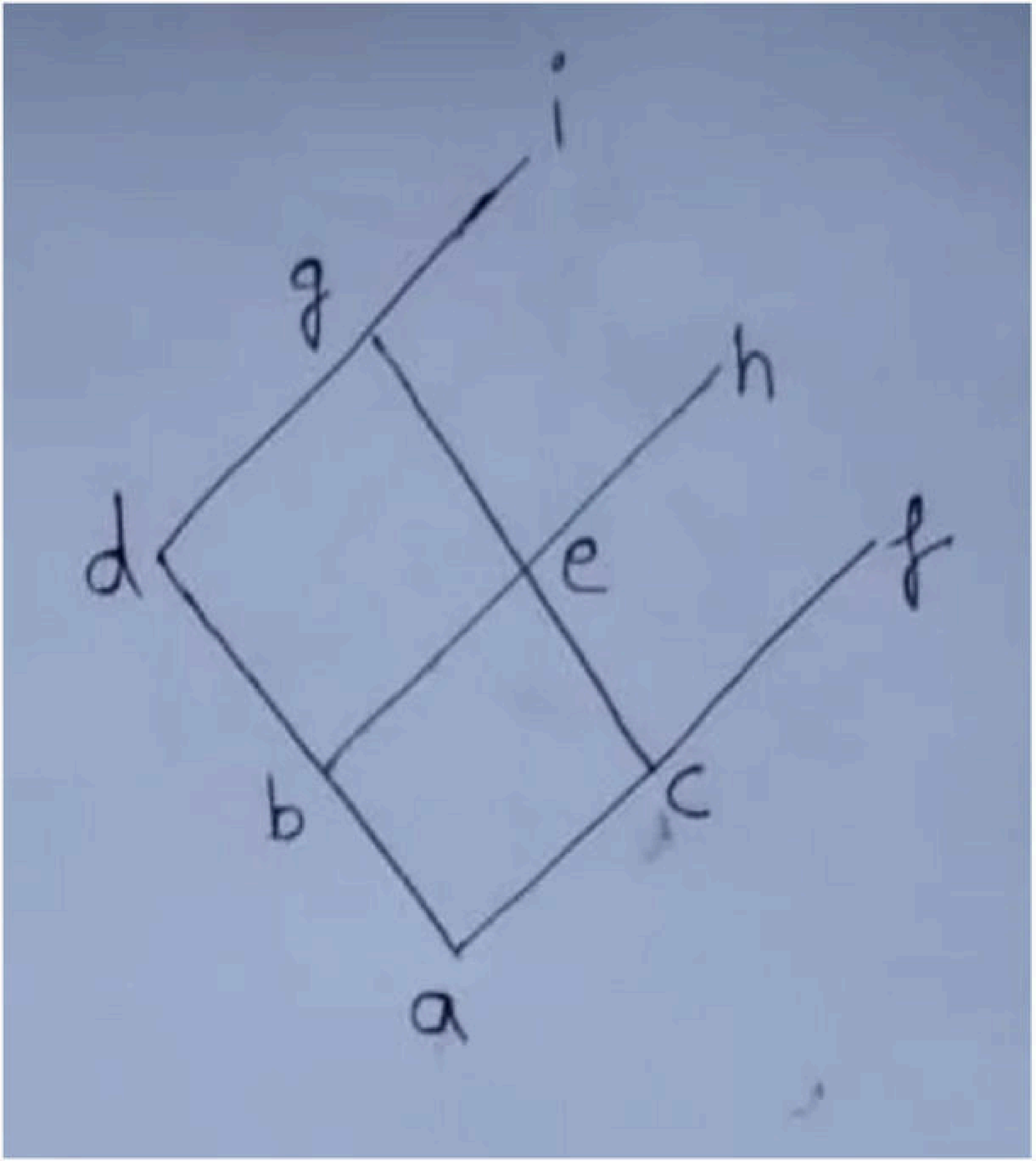
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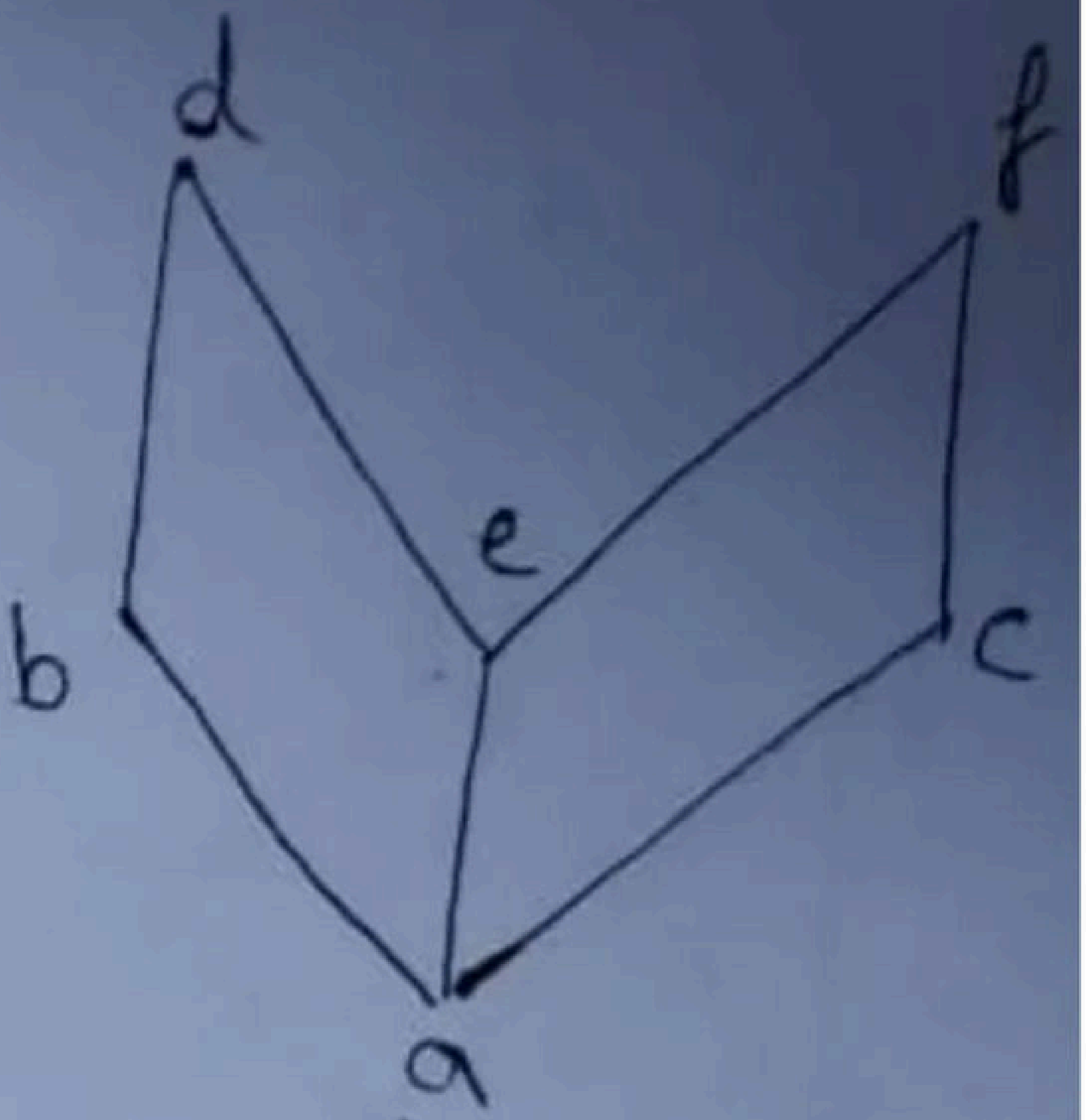
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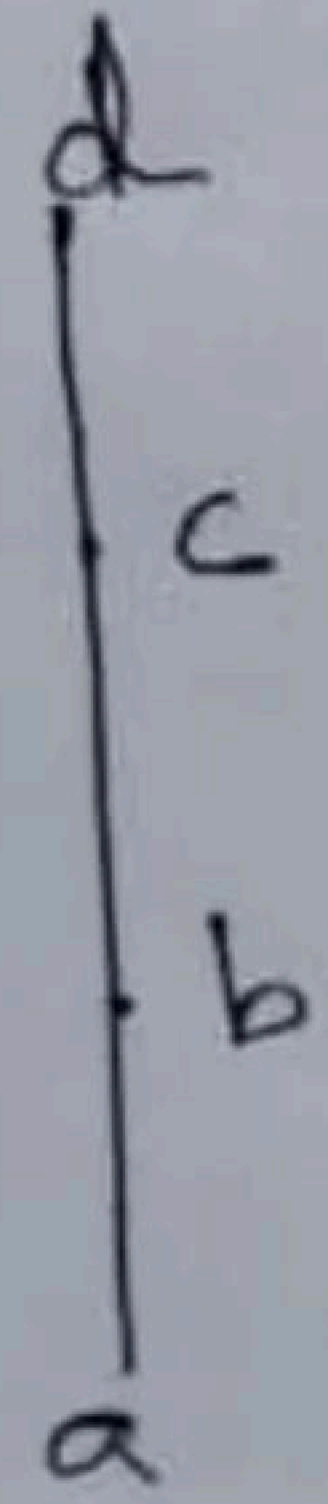
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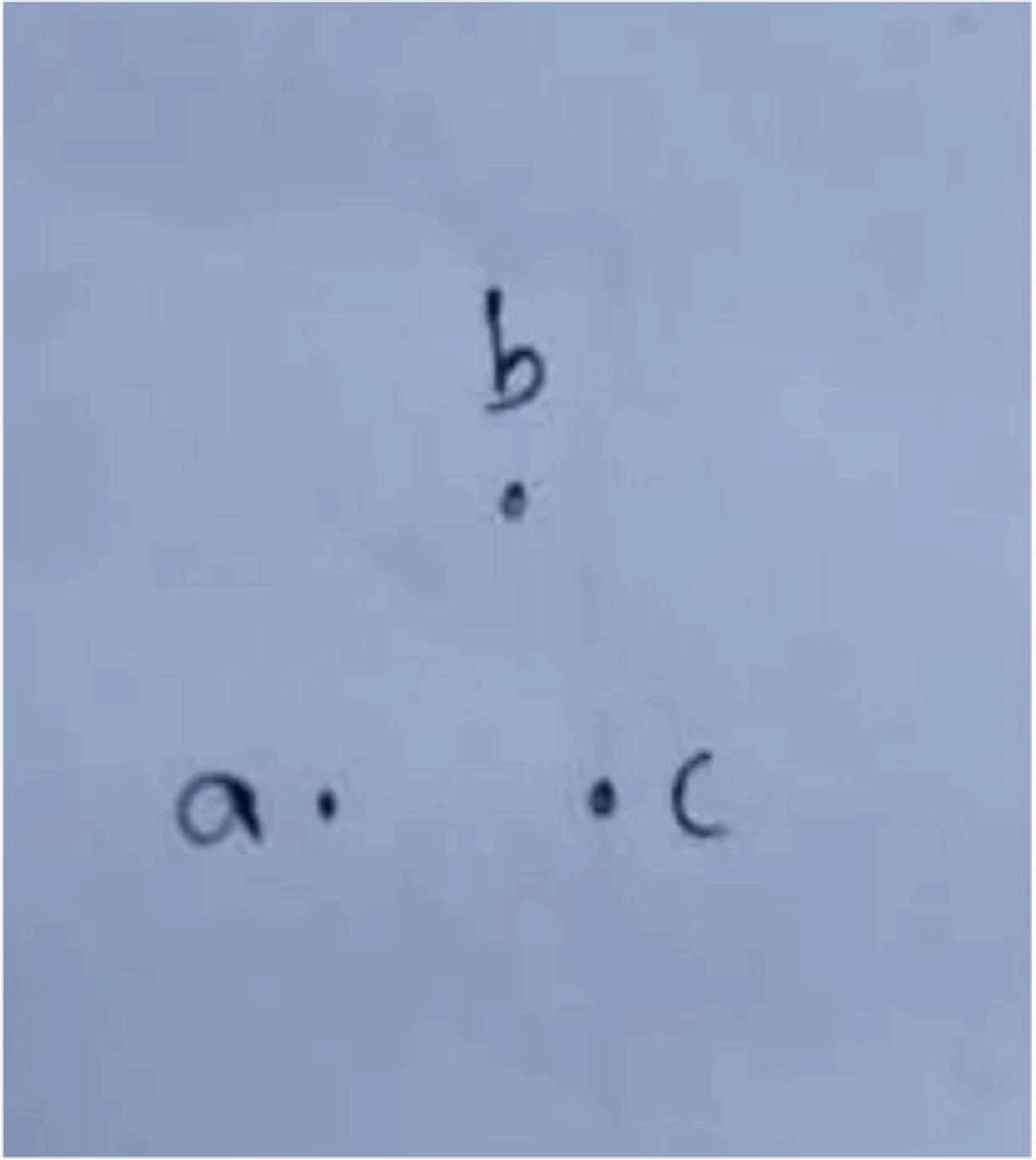
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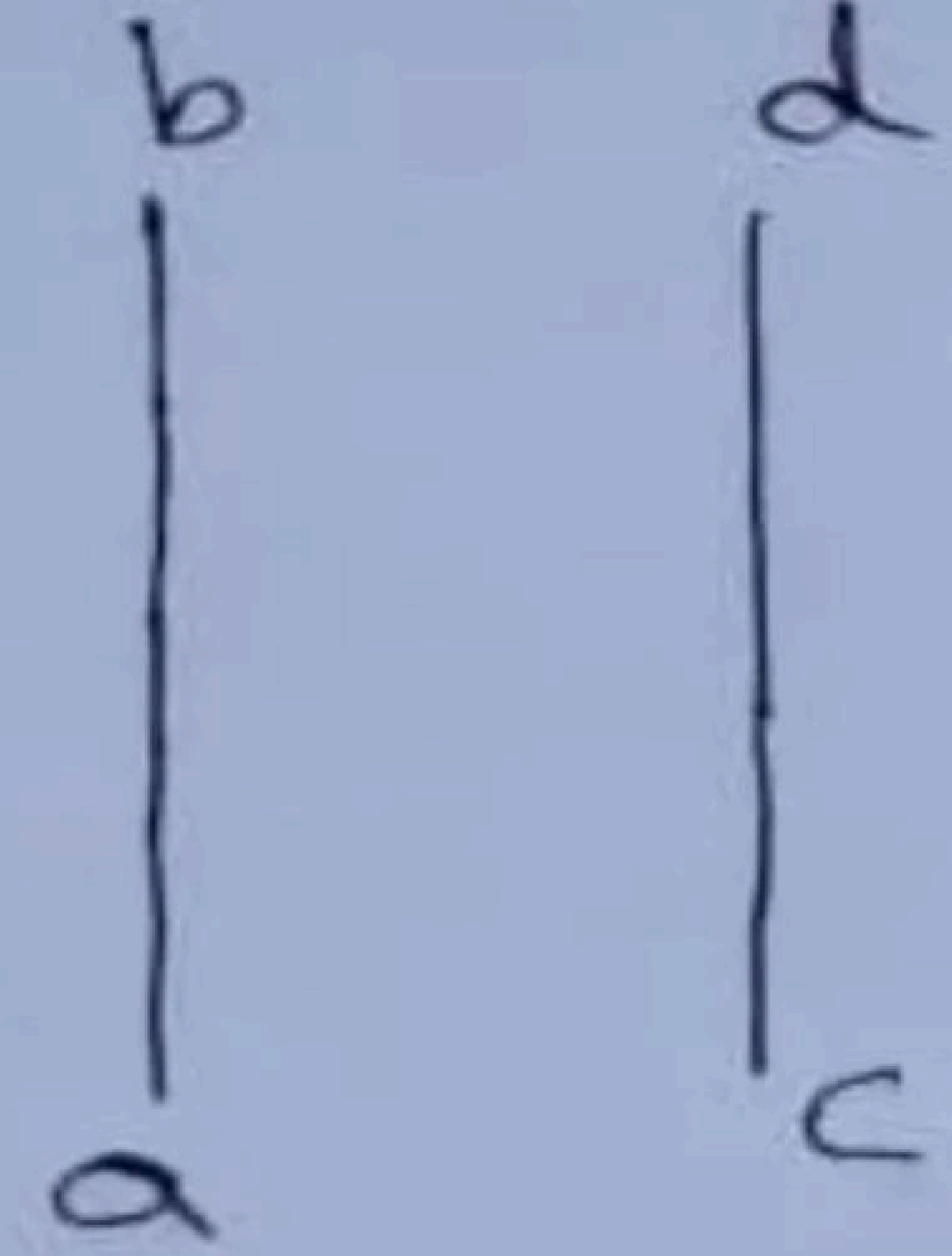
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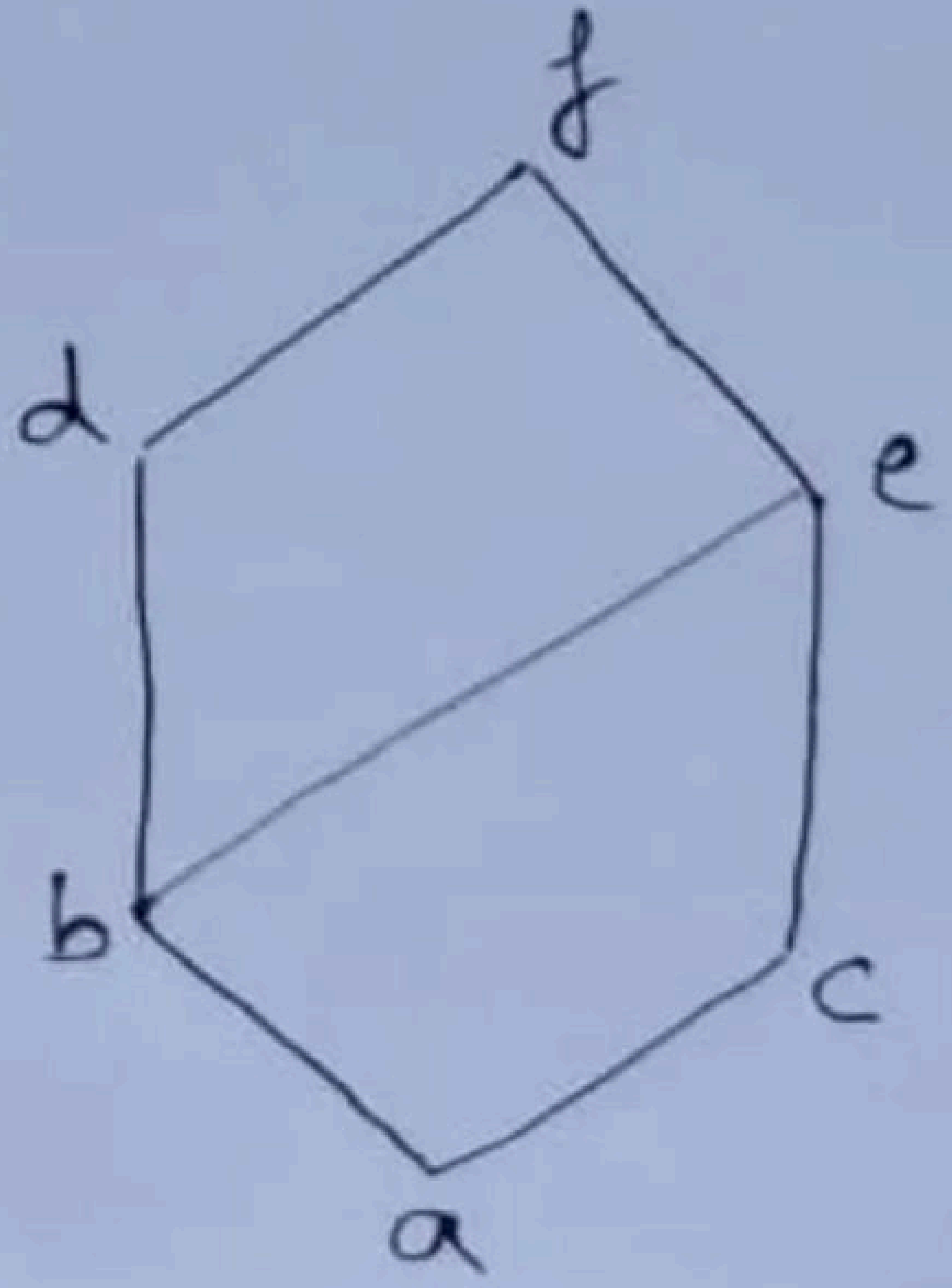
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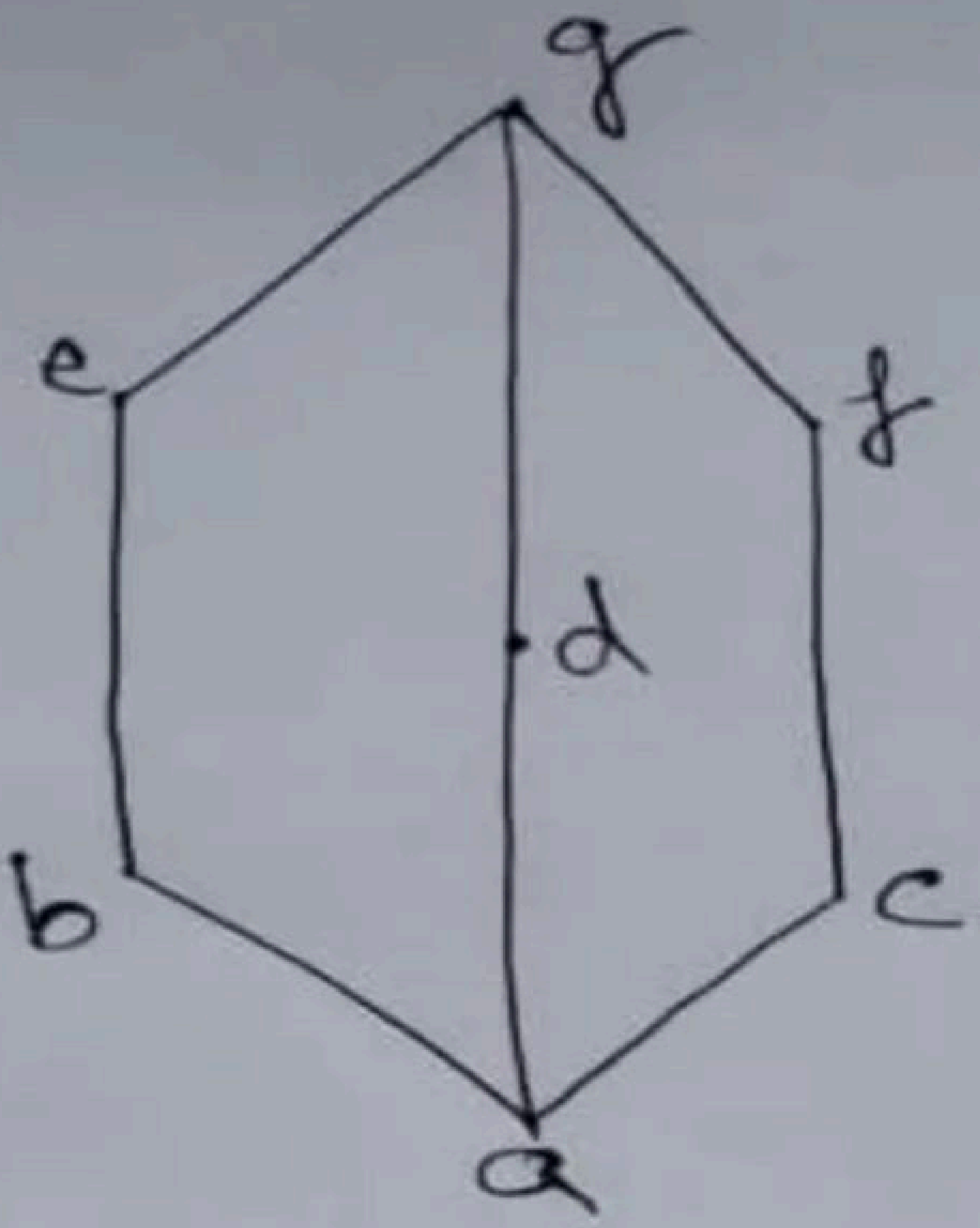
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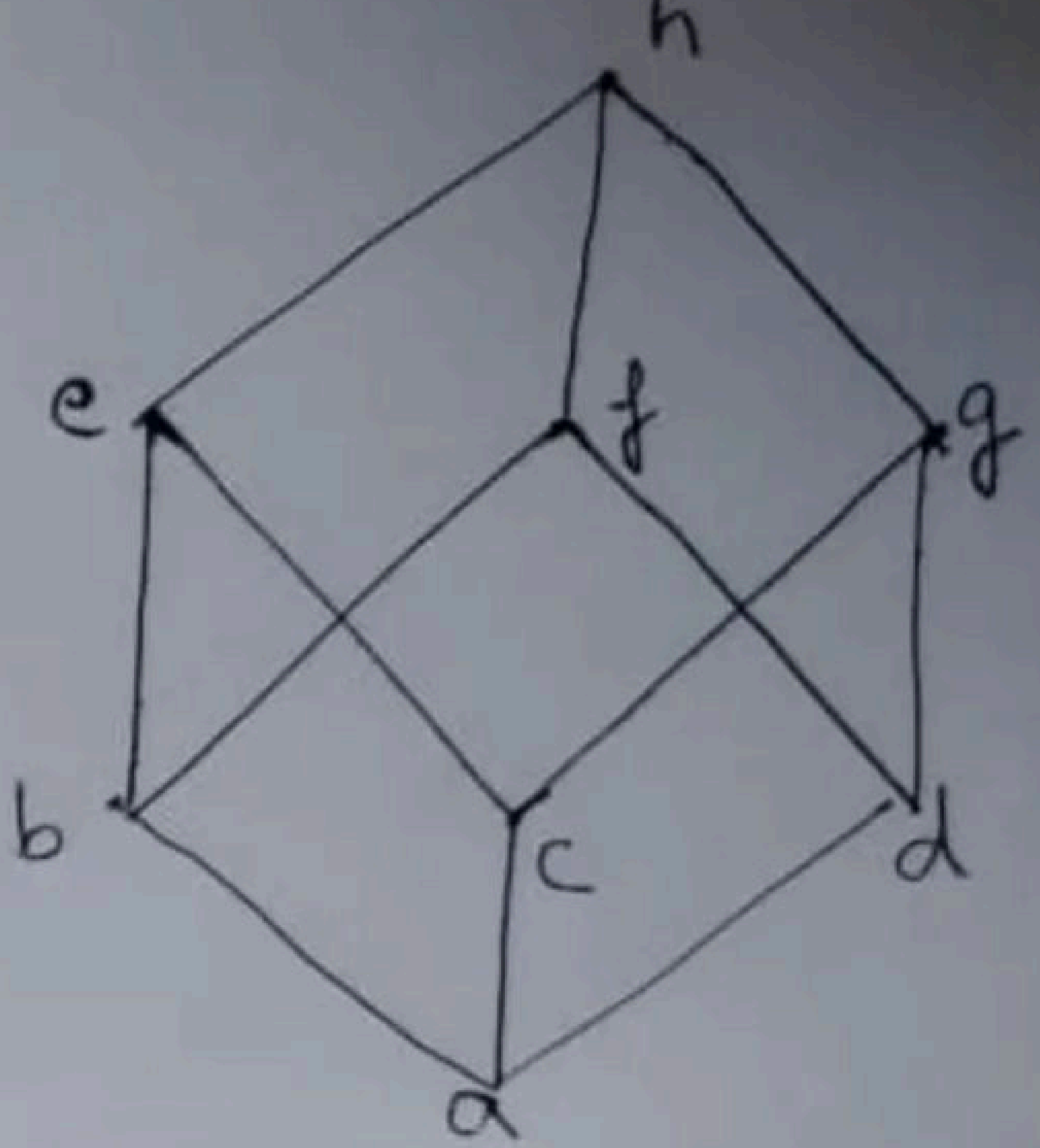
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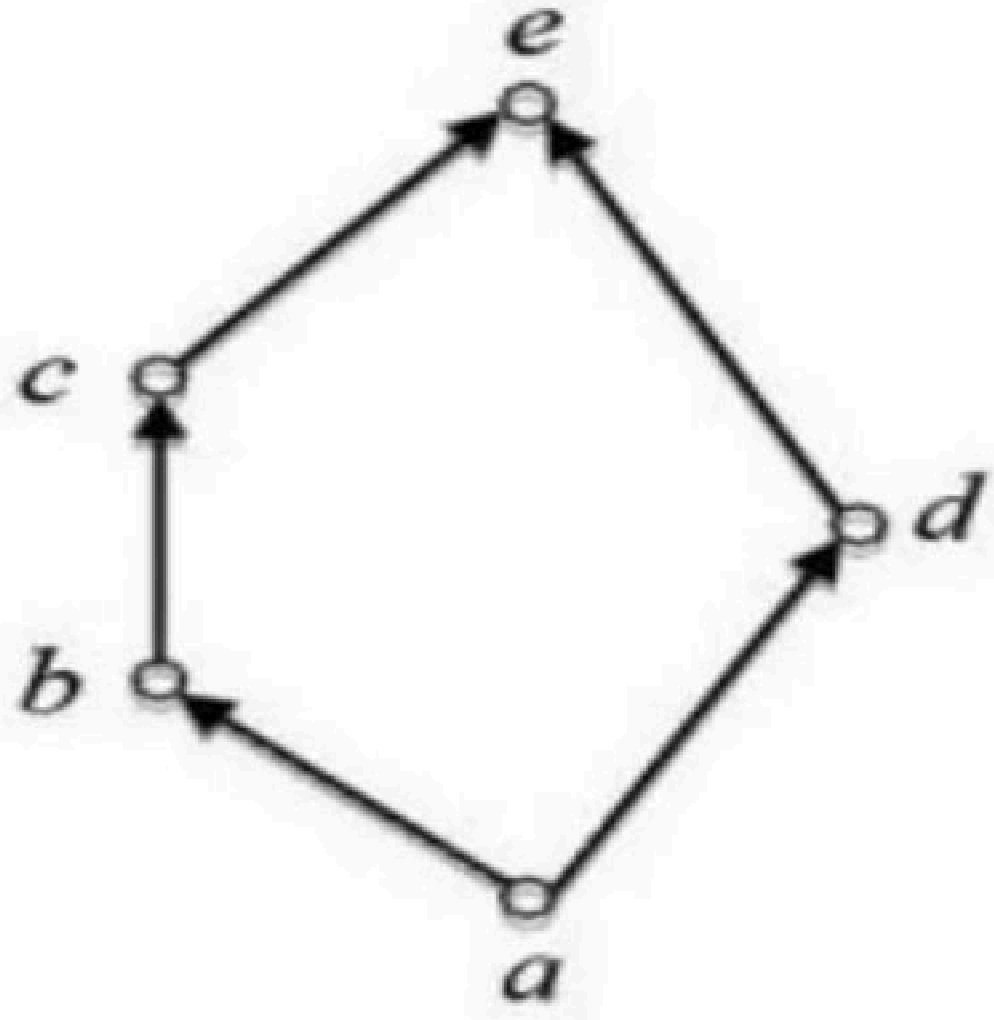
Break

Q Consider the set **X**={a, b, c, d, e} under partial ordering **R**={(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (b, c), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)}. The Hasse diagram of the partial order (X, R) is shown below. The minimum number of ordered pairs that need to be added to R to make (X, R) a lattice is ______ **(GATE-2017) (1 Marks)**



Q A partially ordered set is said to be a lattice if every two elements in the set have **(NET-Dec-2010)**

a) a unique least upper bound

b) a unique greatest lower bound

c) both (A) and (B)

d) none of the above

Q Consider the following Hasse diagrams

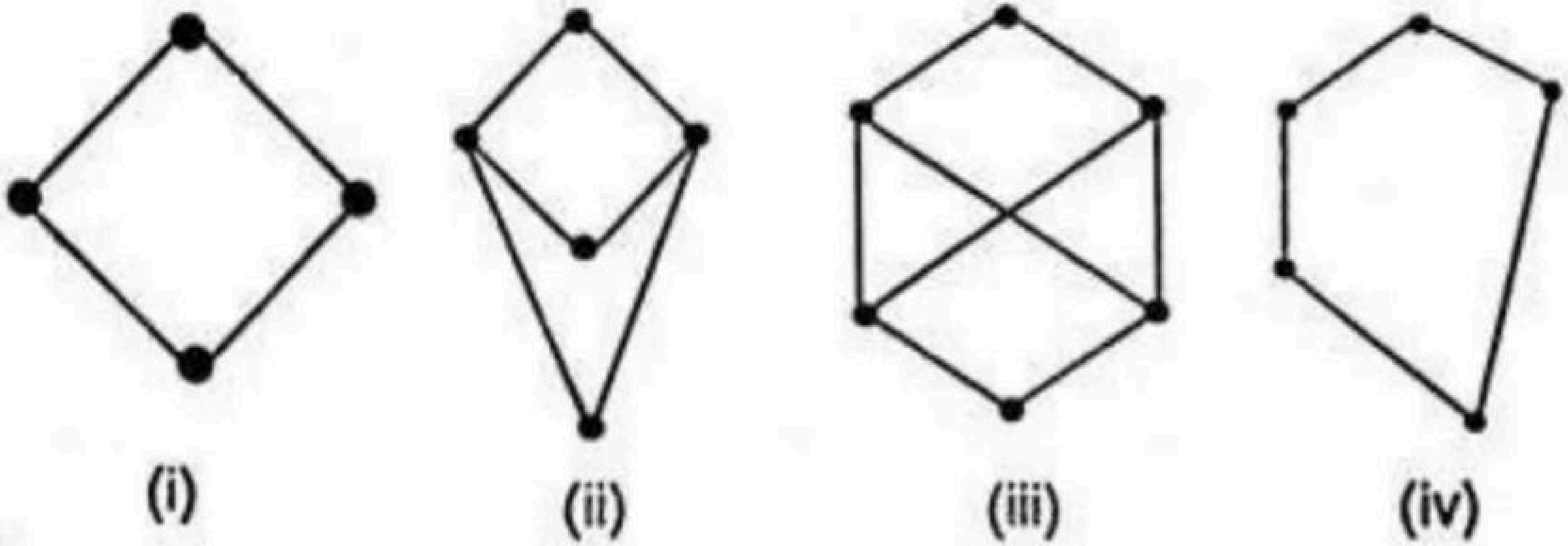
Which all of the above represent a lattice? **(GATE-2008) (2 Marks)**

(A) (i) and (iv) only

(B) (ii) and (iii) only

(C) (iii) only

(D) (i), (ii) and (iv) only



Q the inclusion of which of the following set into S = {{1,2}, {1,2,3}, {1,3,5}, {1,2,4}, {1,2,3,4,5}} is necessary and sufficient to make S a complete lattice under the partial order defined by set containment?
(GATE-2004) (2 Marks)

a) {1}

b) {1}, {2,3}

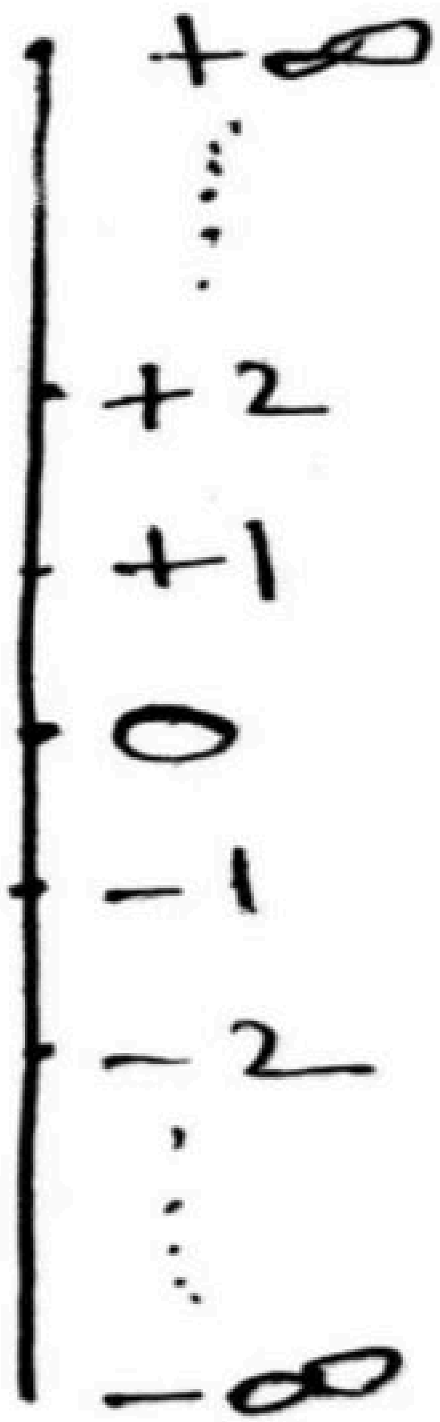
c) {1}, {1,3}

d) {1}, {1,3}, {1,2,3,4}, {1,2,3,5}

Break

Boolean algebra

- **Unbounded Lattice :-** If a lattice has infinite of elements then it is called Unbounded Lattice.



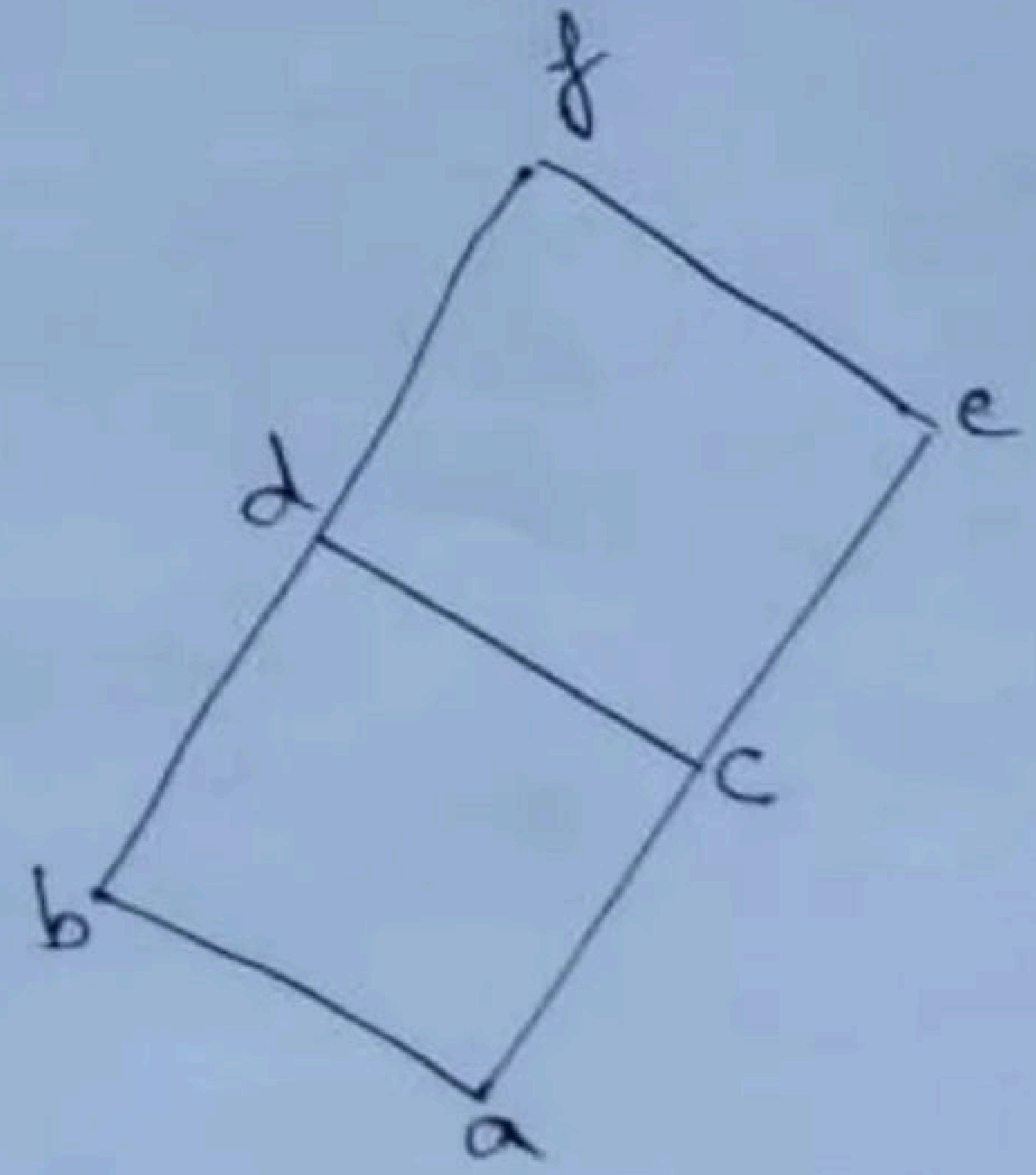
- **Bounded Lattice :-** If a lattice has finite number of elements then it is called Bounded lattice, there will be upper and lower bound in lattice.

- **<u>Complement of an element in a Lattice</u> :-** If two elements a and a^c, are complement of each other, then the following equations must always holds good.
  $a \vee a^c$ = Upper bound of lattice
  $a \wedge a^c$ = Lower bound of lattice

- **<u>Distributive Lattice</u> :-** A lattice is said to be distributed lattice. if for every element their exist at most one completemt(zero or one).

- **Complement Lattice :-** A Lattice is said to be Complement lattice. if for every element their exist at least one complement(one or more).

- **Boolean Algebra :-** A Lattice is said to be Boolean Algebra, if for every element their exist exactly one complement. Or if a lattice is both complemented and distributed then it is called Boolean Algebra.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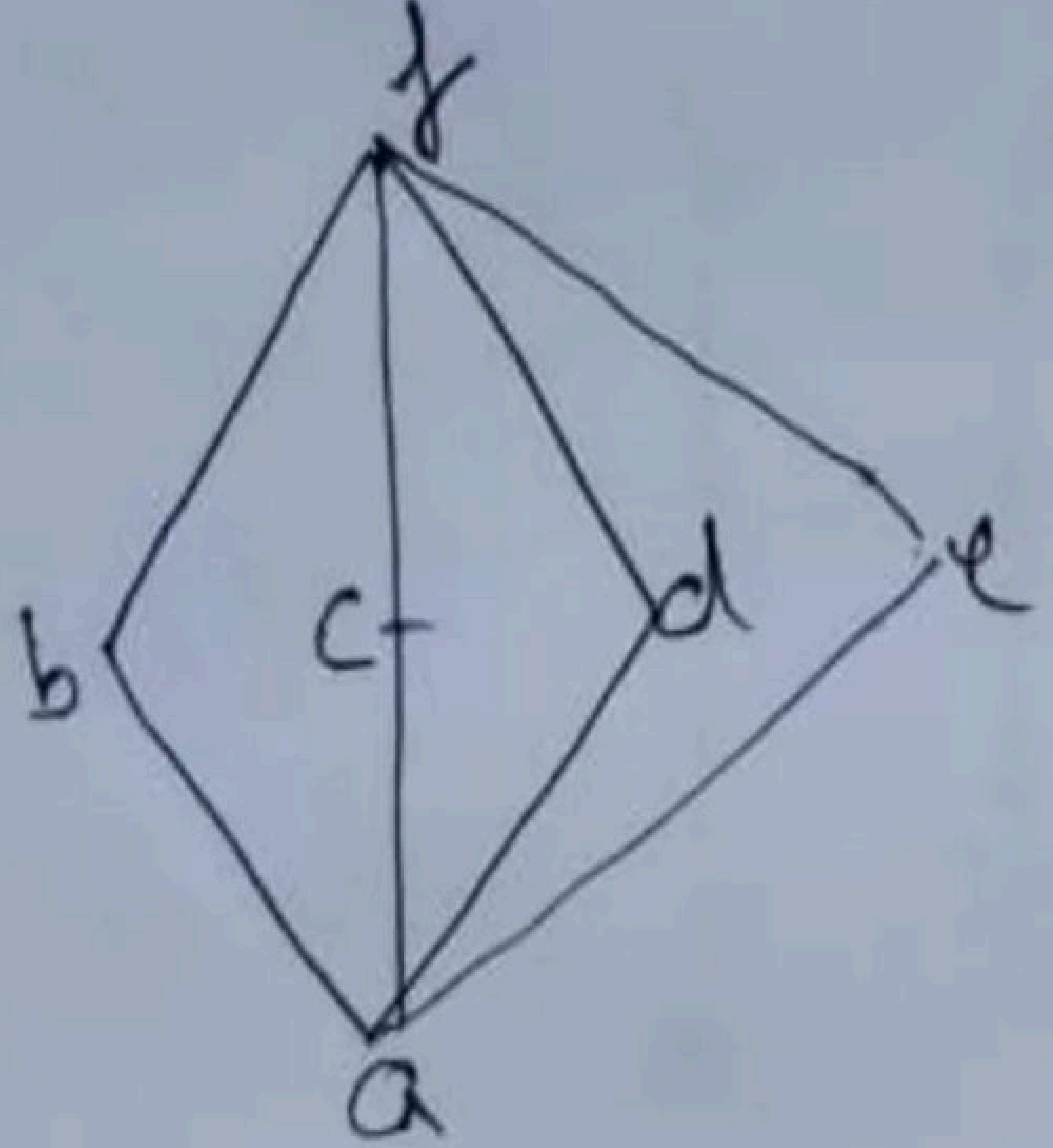
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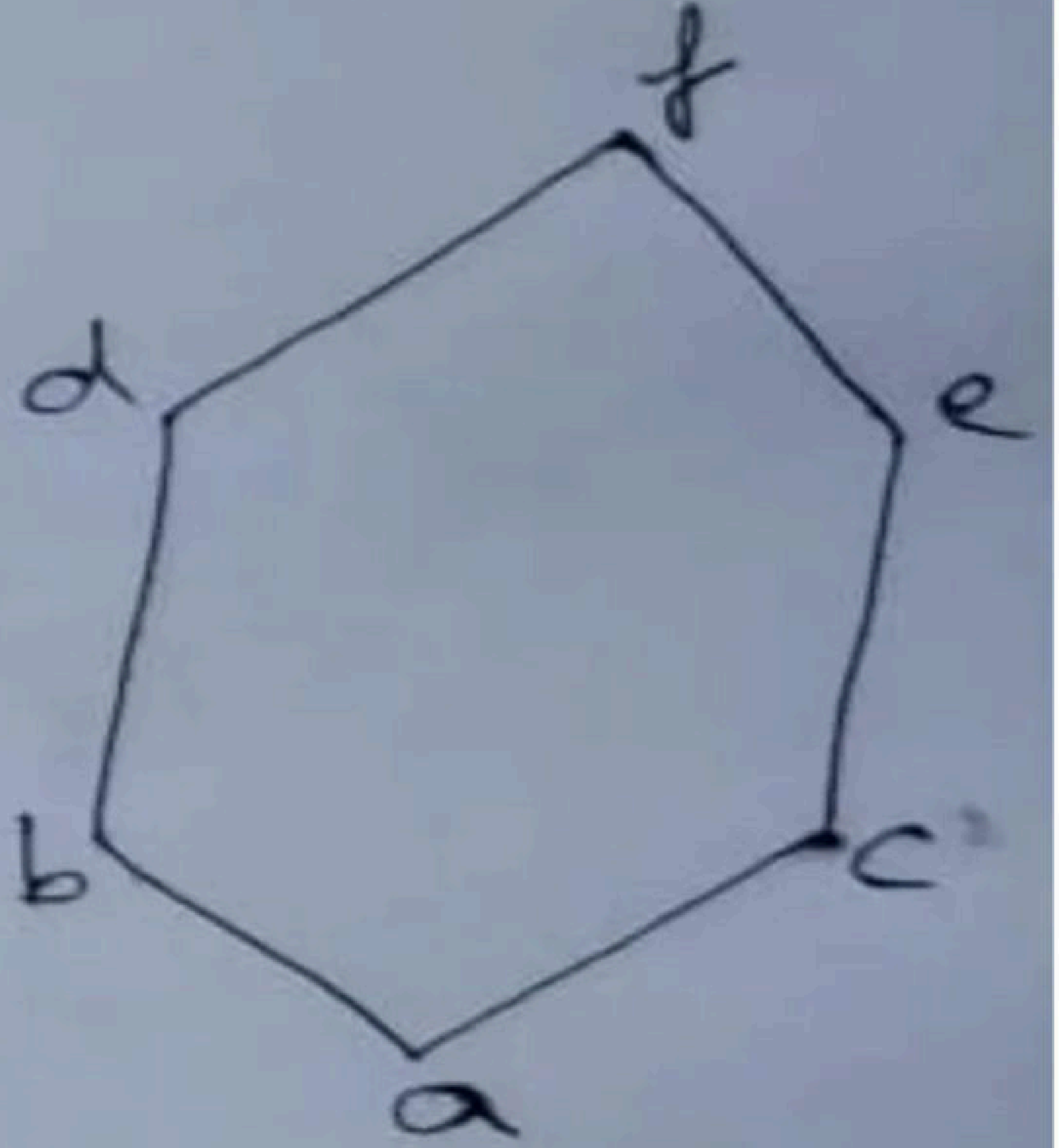
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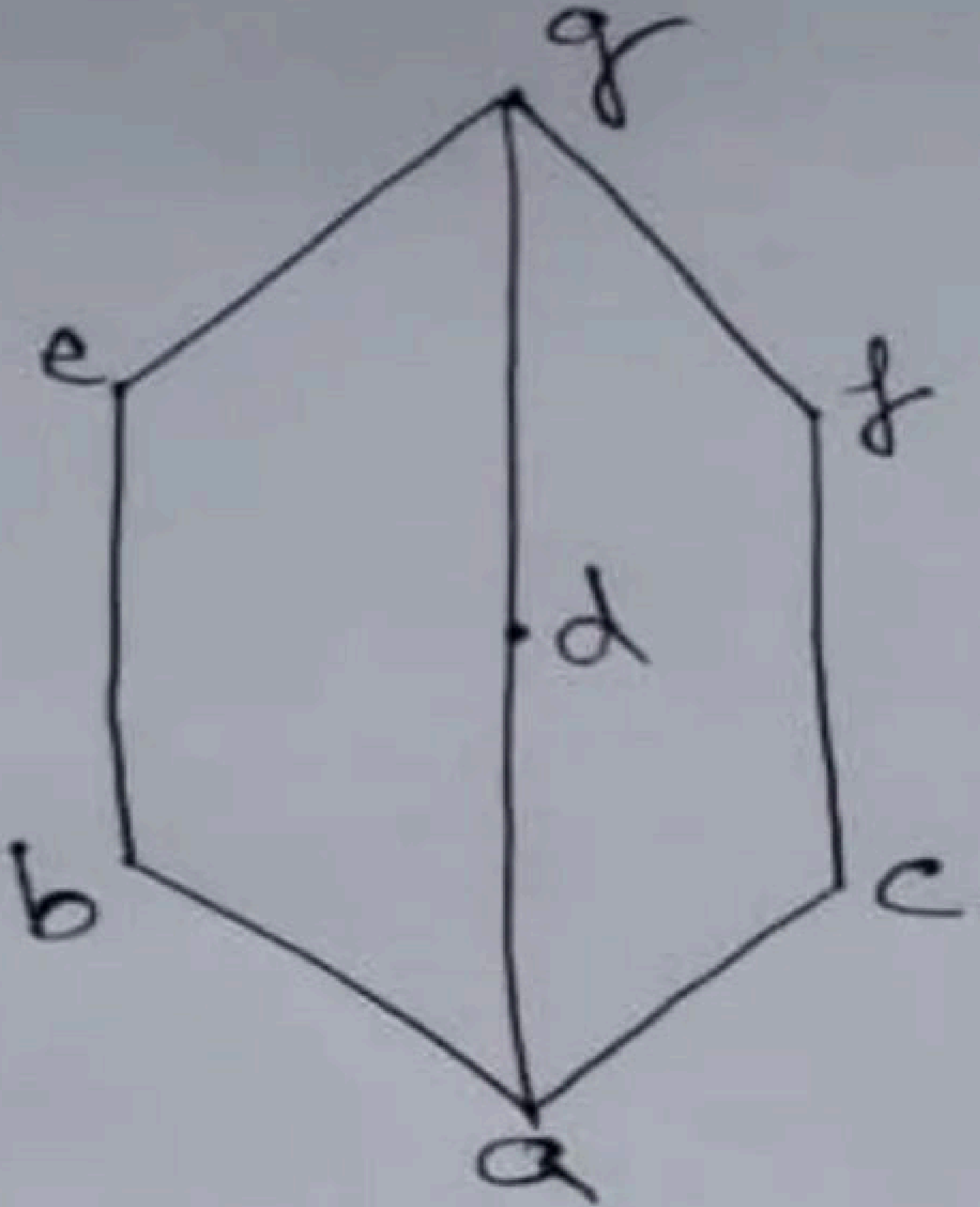
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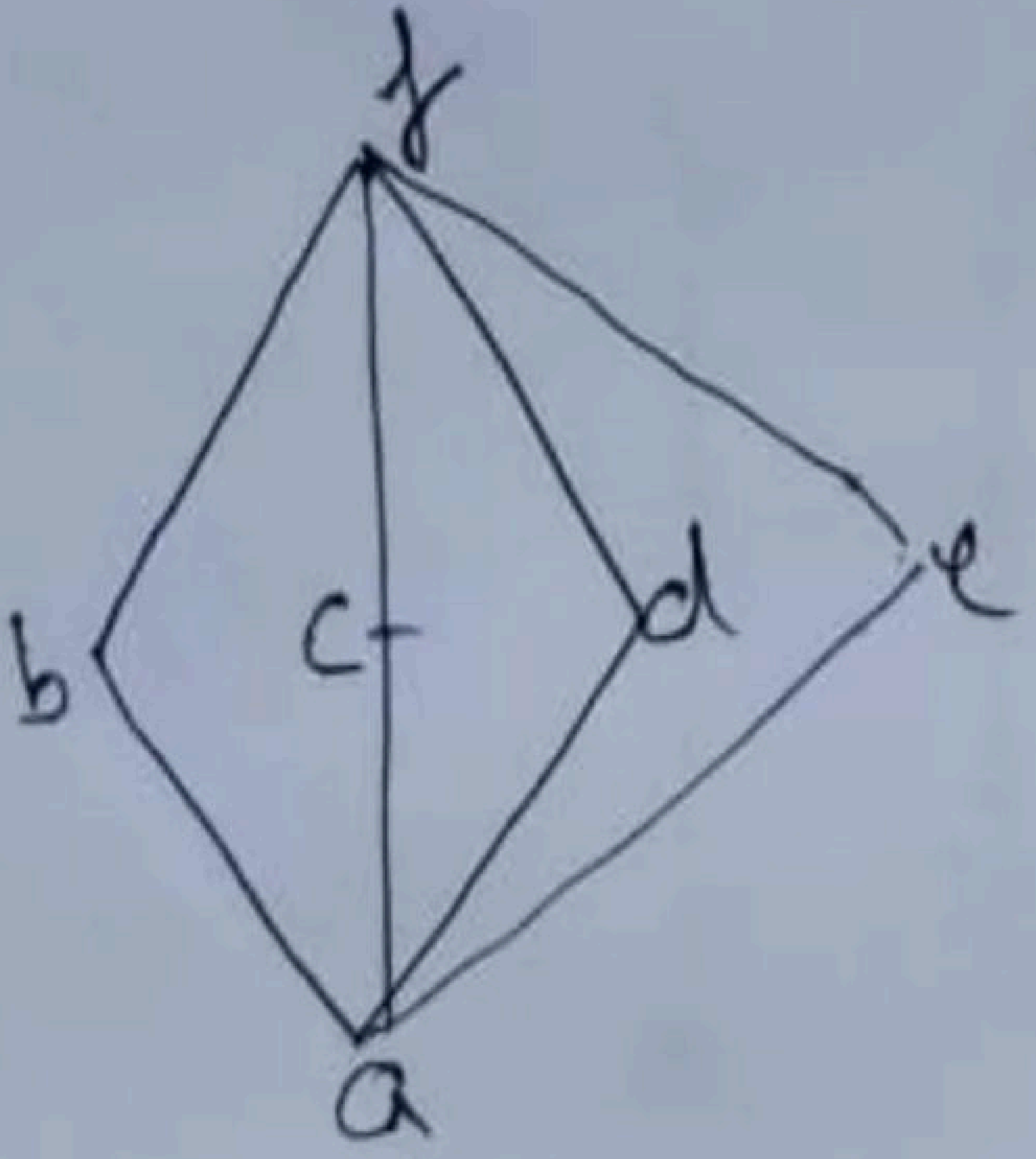
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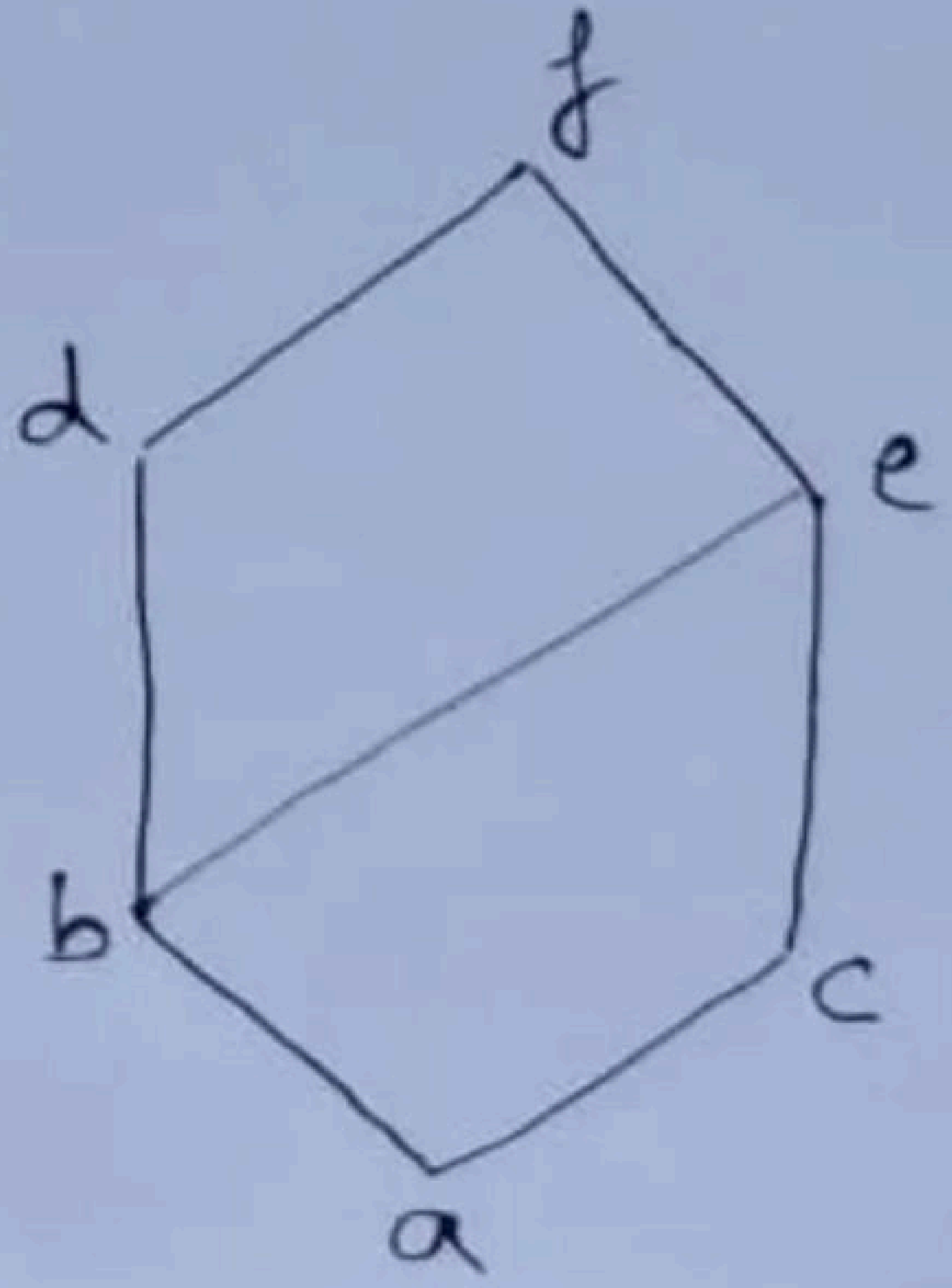

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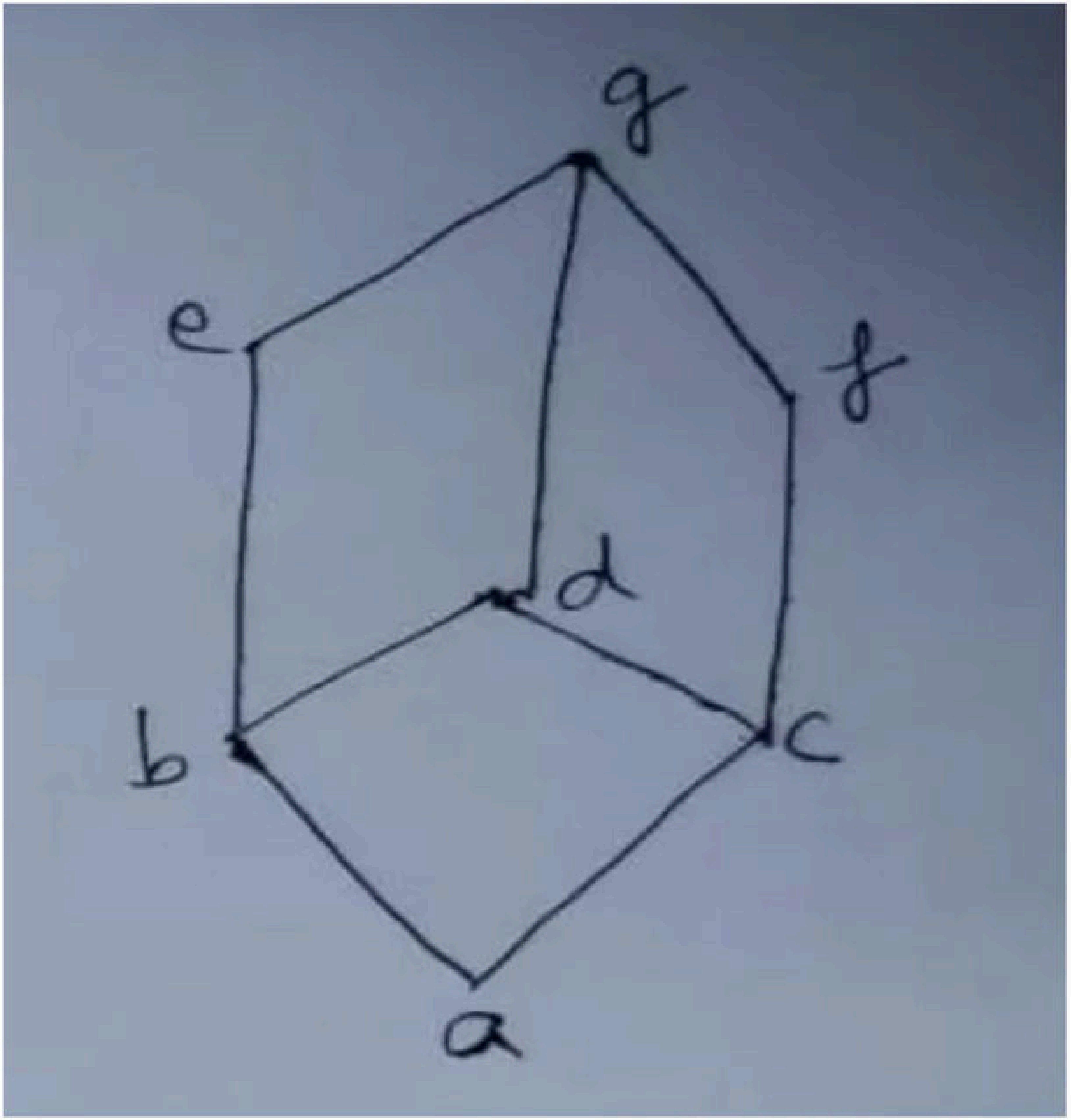
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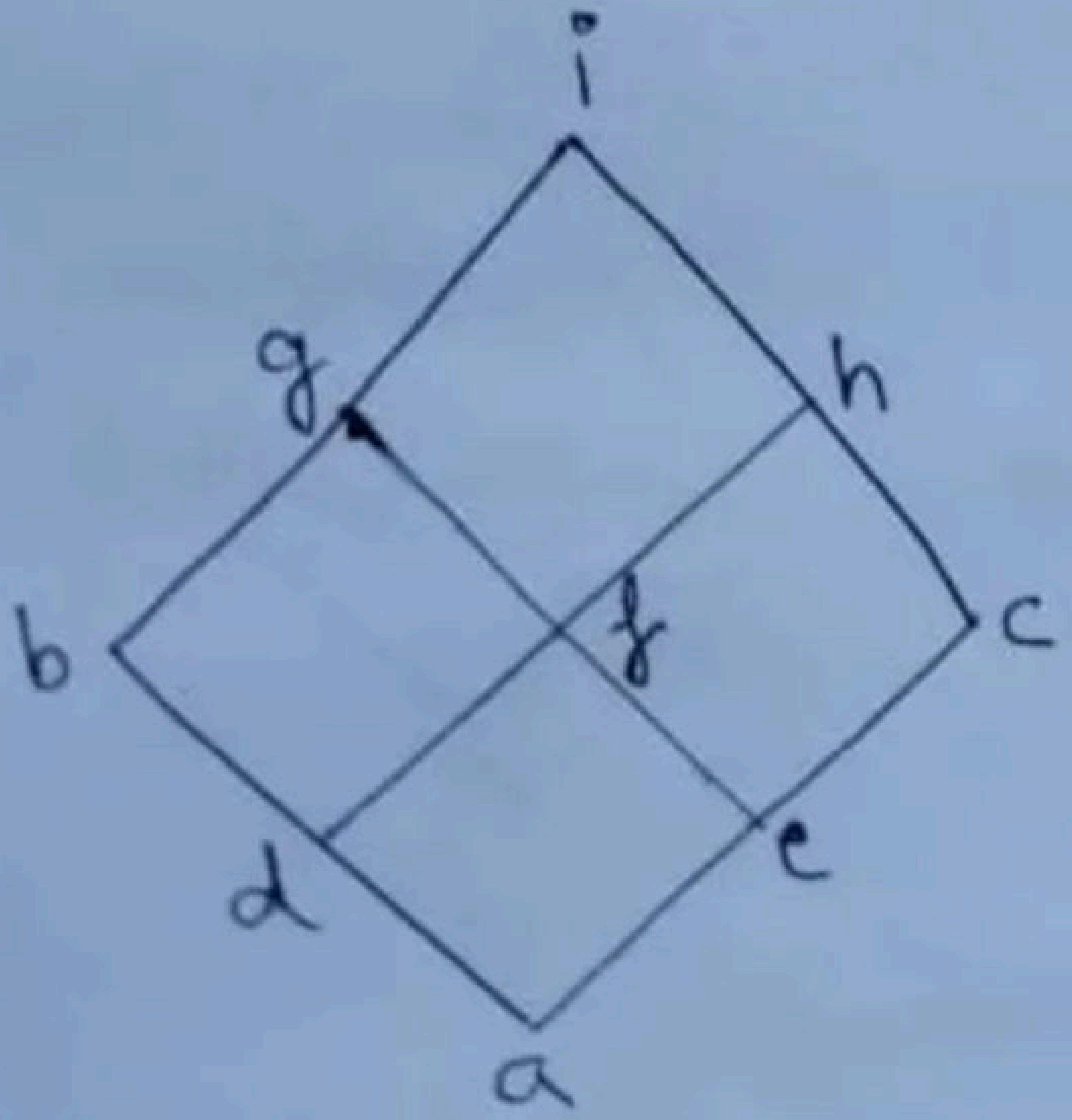
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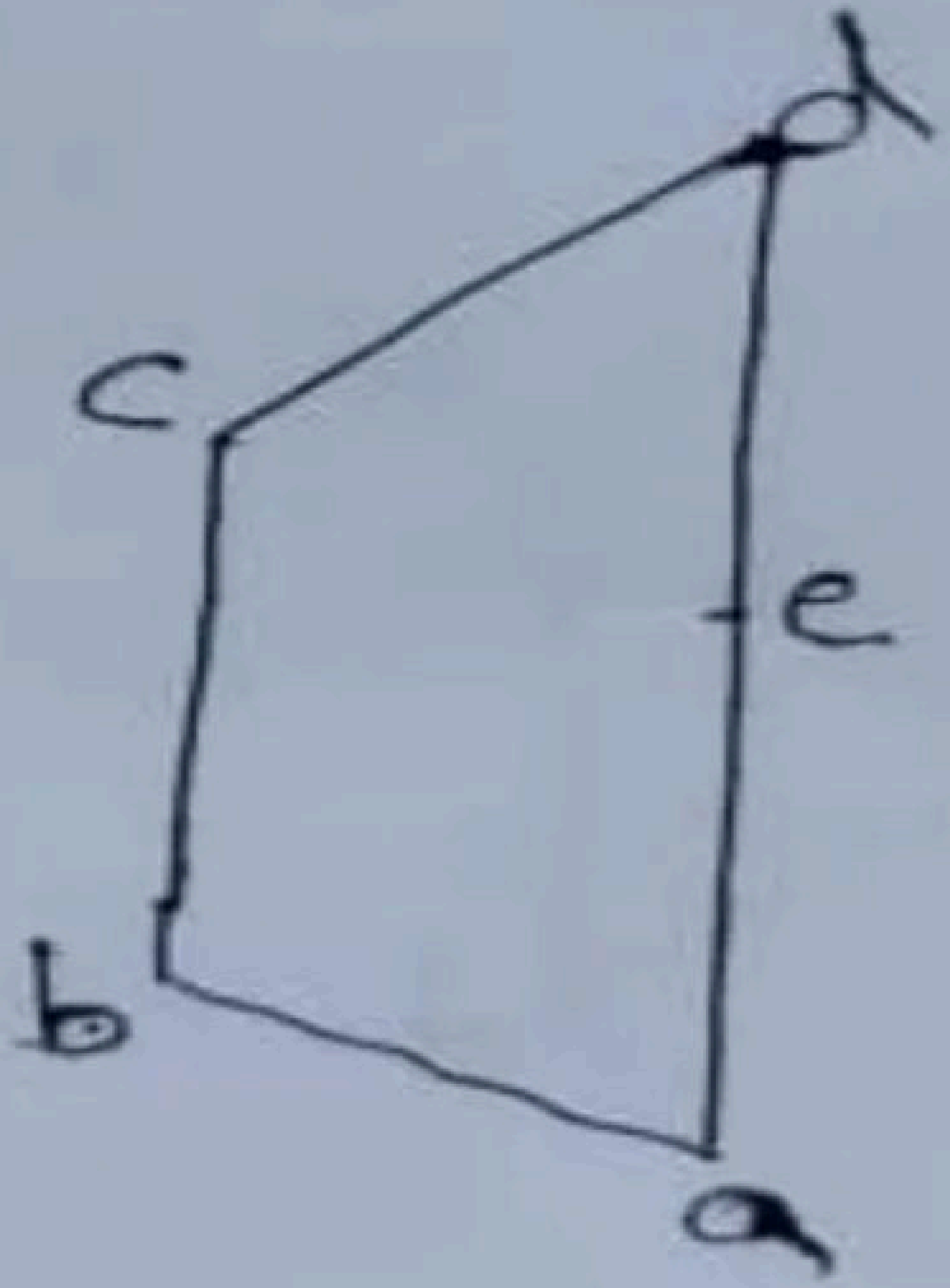
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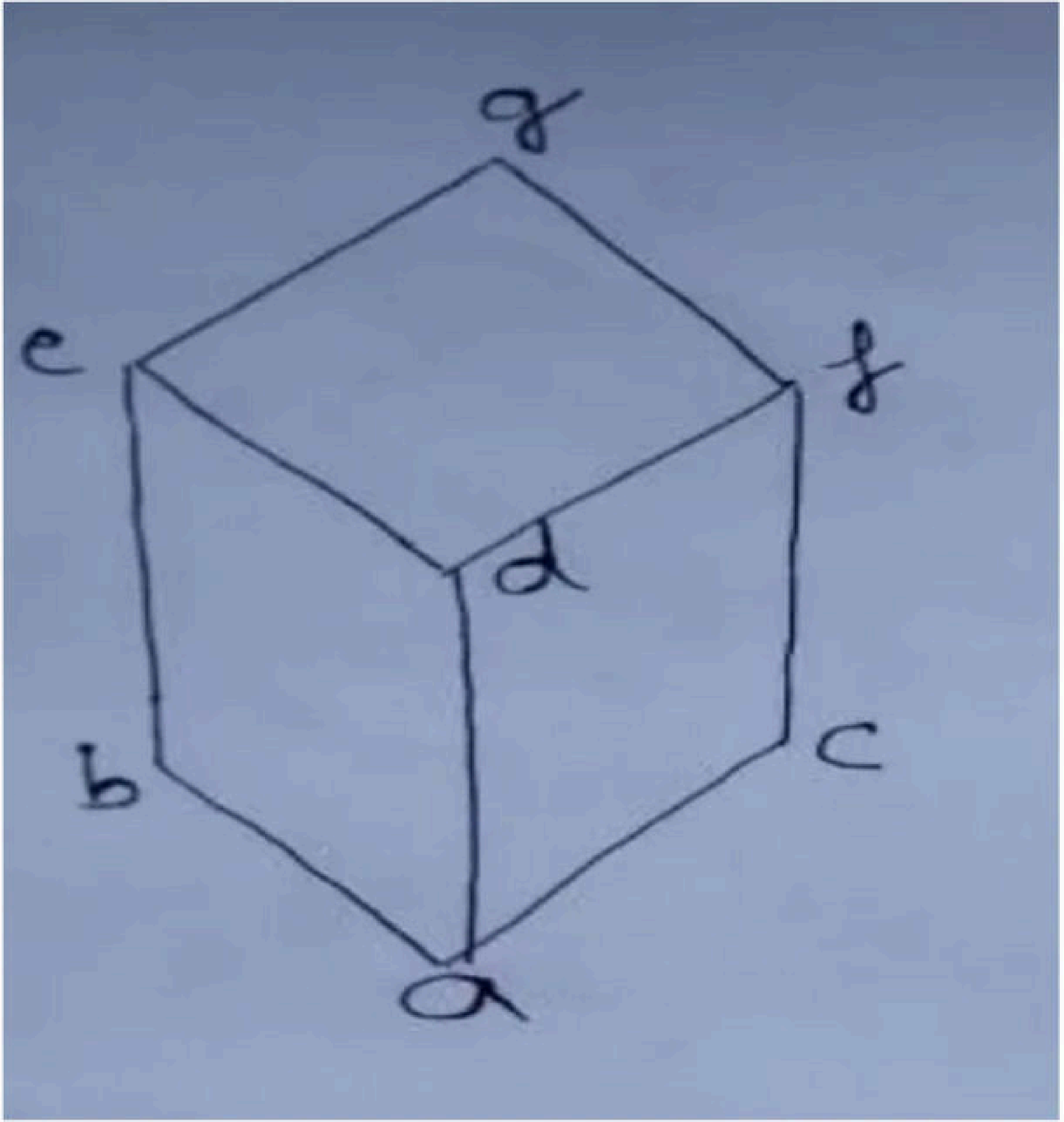
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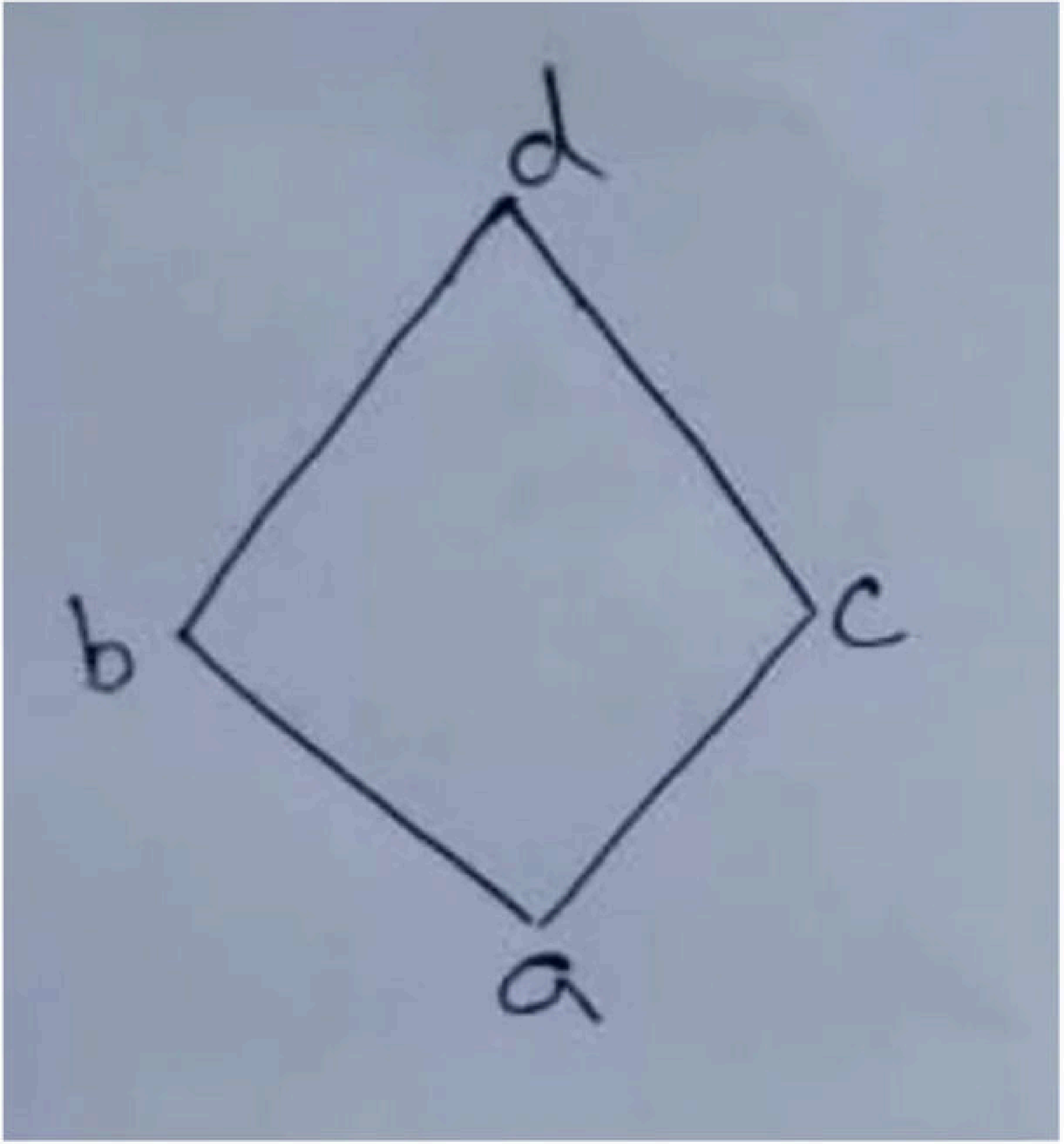
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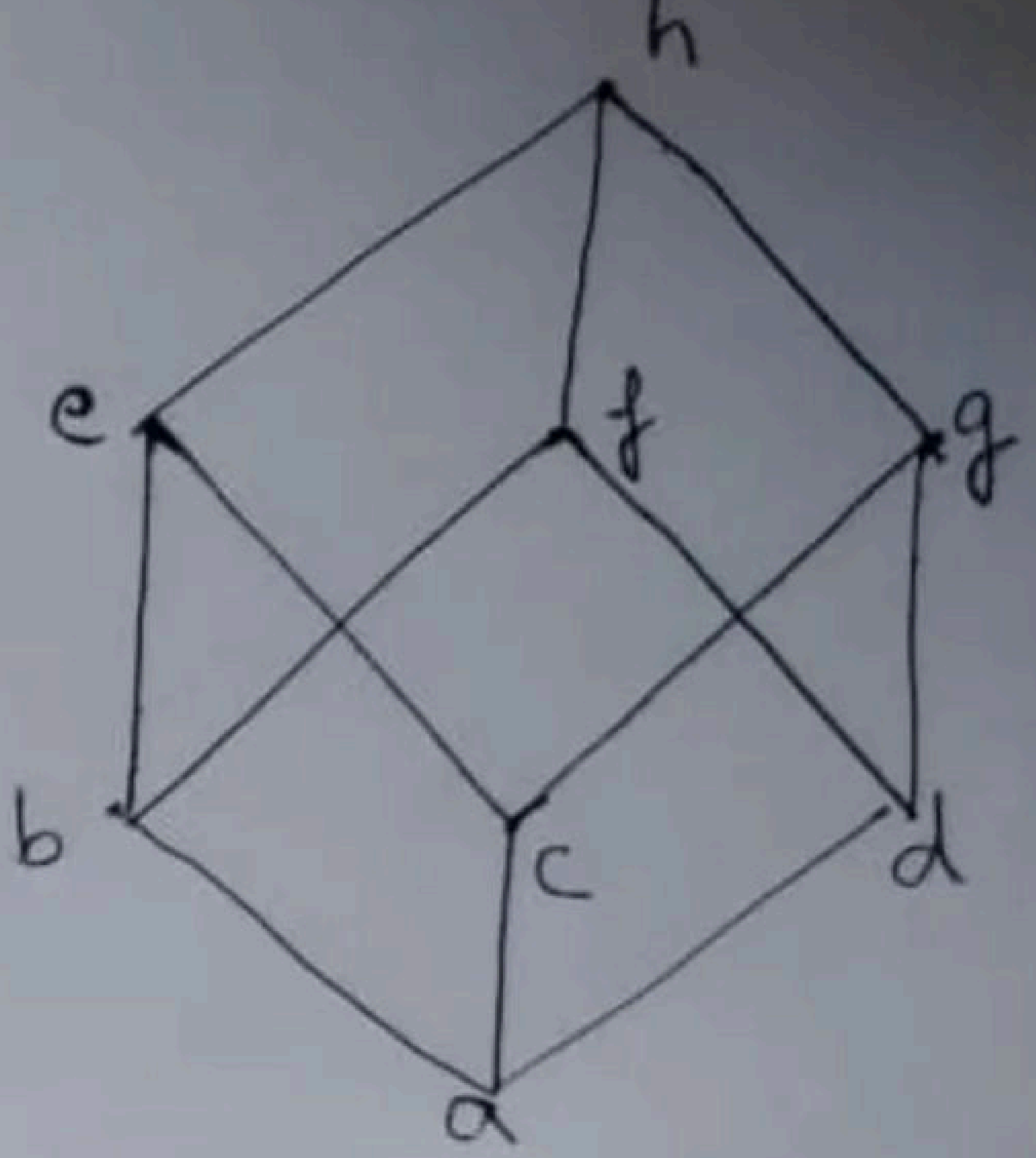
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Break

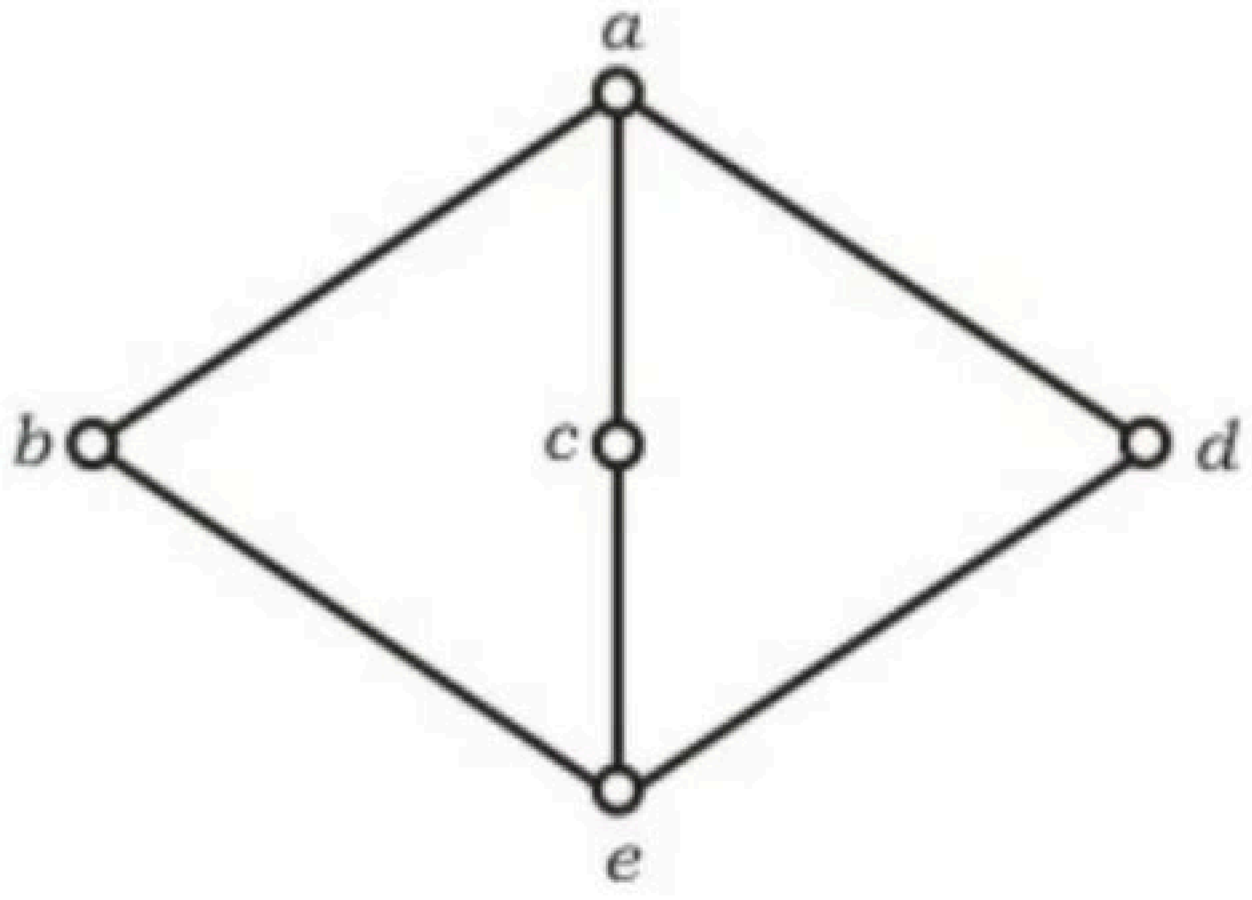
Q The following is the Hasse diagram of the Poset [{a, b, c, d, e}, ≤]
The Poset is **(GATE-2005) (1 Marks)**

(A) not a lattice

(B) a lattice but not a distributive lattice

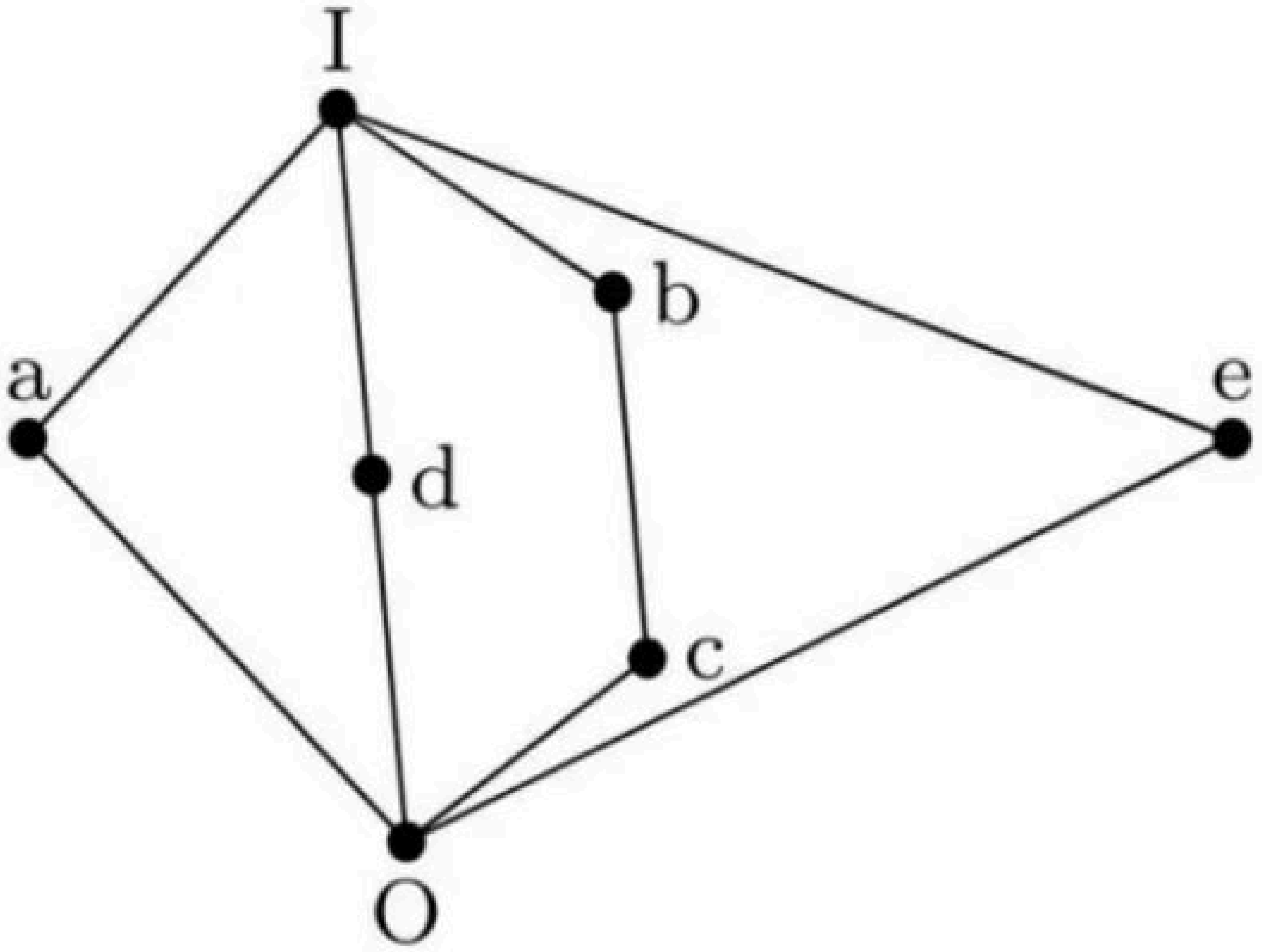
(C) a distributive lattice but not a Boolean algebra

(D) a Boolean algebra



Q The complement(s) of the element 'a' in the lattice shown in below figure is (are) ____

(GATE-1988) (2 Marks)



Q Find which of the following is a lattice and Boolean Algebra?

(1) $[D_{10}, /]$ **(2)** $[D_{12}, /]$

(3) $[D_{30}, /]$ **(4)** $[D_{45}, /]$

(5) $[D_{64}, /]$ **(6)** $[D_{81}, /]$

(7) $[D_{91}, /]$ **(8)** $[D_{110}, /]$

Q Find which of the following is a lattice and Boolean Algebra?

(1) [{1,2,3,4,6,9}, /]

(2) [{2,3,4,6,12}, /]

(3) [{1,2,3,5,30}, /]

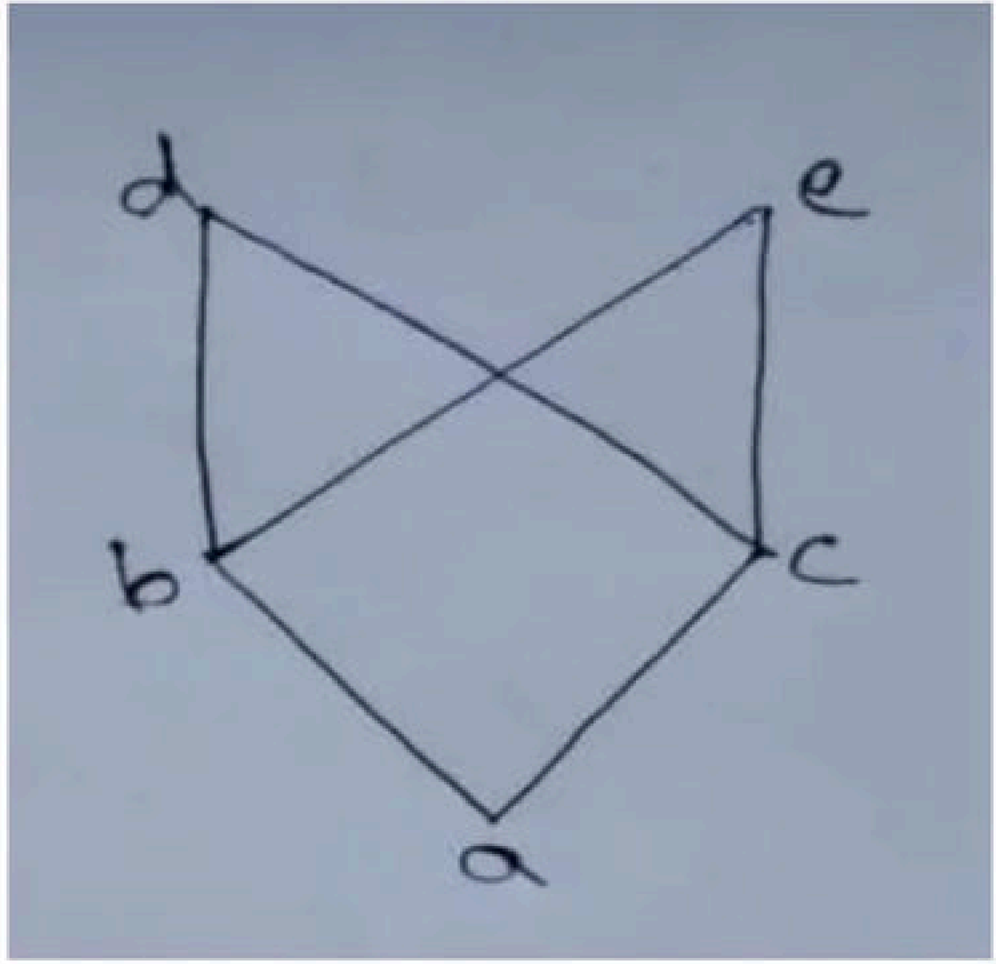
(4) [{1,2,3,6,9,18}, /]

(5) [{2,3,4,9,12,18}, /]

(6) [R, <=]

(7) [P(A), $\sqsubseteq$], A = {1,2,3}

Q Consider the following hasse diagram, find which of the following is true?



a) it is a lattice

b) subset {a, b, c , d} is a lattice

c) subset {b, c, d, e} is a lattice

d) subset {a, b, c, e} is a lattice

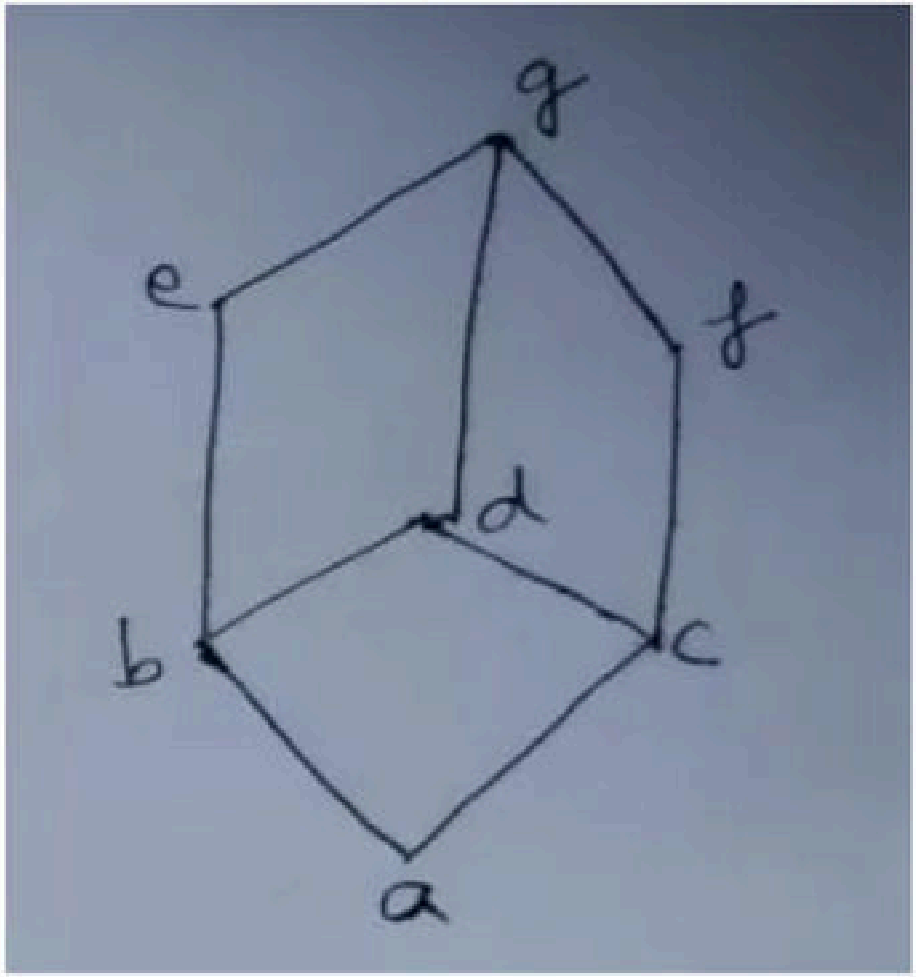
Q Consider the following hasse diagram, find which of the following is true?

a) subset {a, b, c, g} is a lattice

b) subset {a, b, f, g} is a lattice

c) subset {a, d, e, g} is a lattice

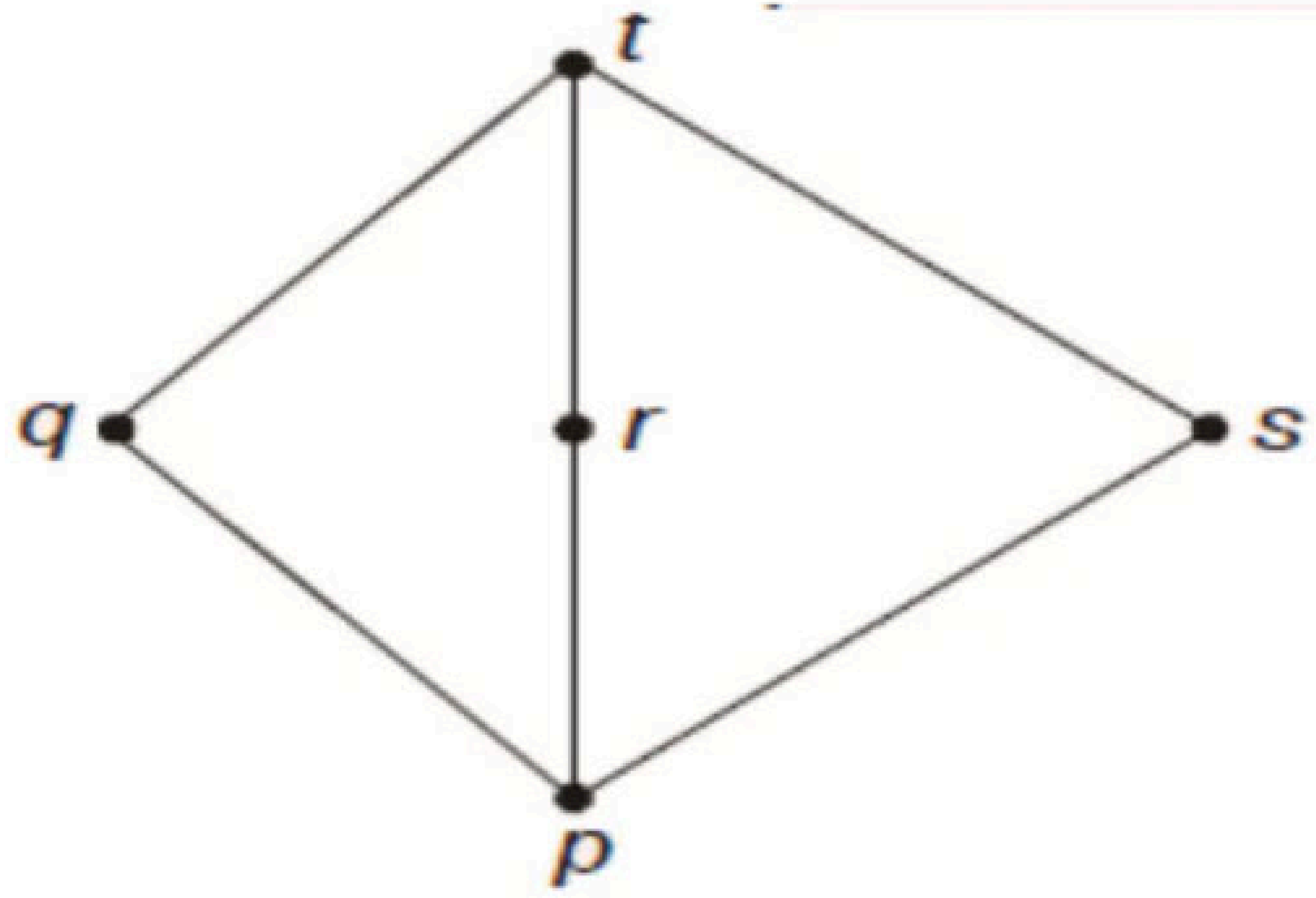
d) subset {a, c, e, g} is a lattice



Break

Q Suppose L = {p, q, r, s, t} is a lattice represented by the following Hasse diagram: For any x, y ∈ L, not necessarily distinct, $x \vee y$ and $x \wedge y$ are join and meet of x, y respectively. Let $L^3 = \{(x, y, z): x, y, z \in L\}$ be the set of all ordered triplets of the elements of L. Let P_r be the probability that an element $(x, y, z) \in L^3$ chosen equiprobably satisfies $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Then **(GATE-2015) (2 Marks)**

(A) Pr = 0 **(B)** Pr = 1 **(C)** 0 < Pr ≤ 1/5 **(D)** 1/5 < Pr < 1



Q Consider the set S = {a, b, c, d}.

Consider the following 4 partitions π_1, π_2, π_3, π_4 on

S: π_1={~~abcd~~}, π_2={~~ab~~, ~~cd~~}, π_3={~~abc~~, ~~d~~}, π_4={~~a~~, ~~b~~, ~~c~~, ~~d~~}

Let $\prec$ be the partial order on the set of partitions $S'=\{\pi_1,\pi_2,\pi_3,\pi_4\}$ defined as follows: $\pi_i \prec \pi_j$ if and only if π_i refines π_j. The Poset diagram for $(S', \prec)$ is: **(GATE-2007) (2 Marks)**