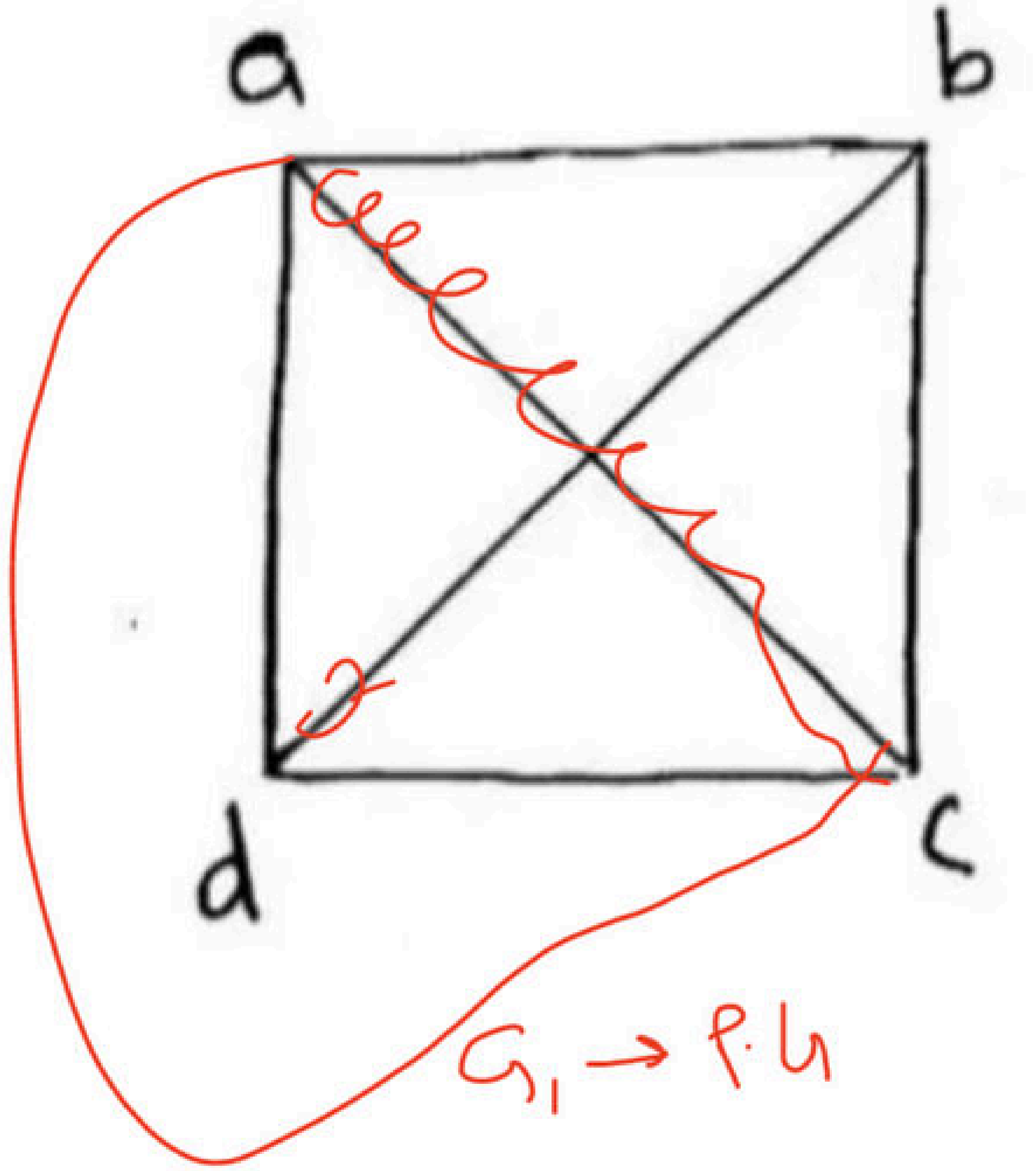
# Doubt Clearing Session

Course on Discrete Mathematics for GATE 2023

Sanchit Jain • Lesson 32 • Oct 10, 2022

# Planer Graph

**Planer Graph: -** A graph is called a planer graph if it can be drawn on a plan in such a way that no edges cross each other, otherwise it is called non-planer. Application: civil engineering, circuit designing

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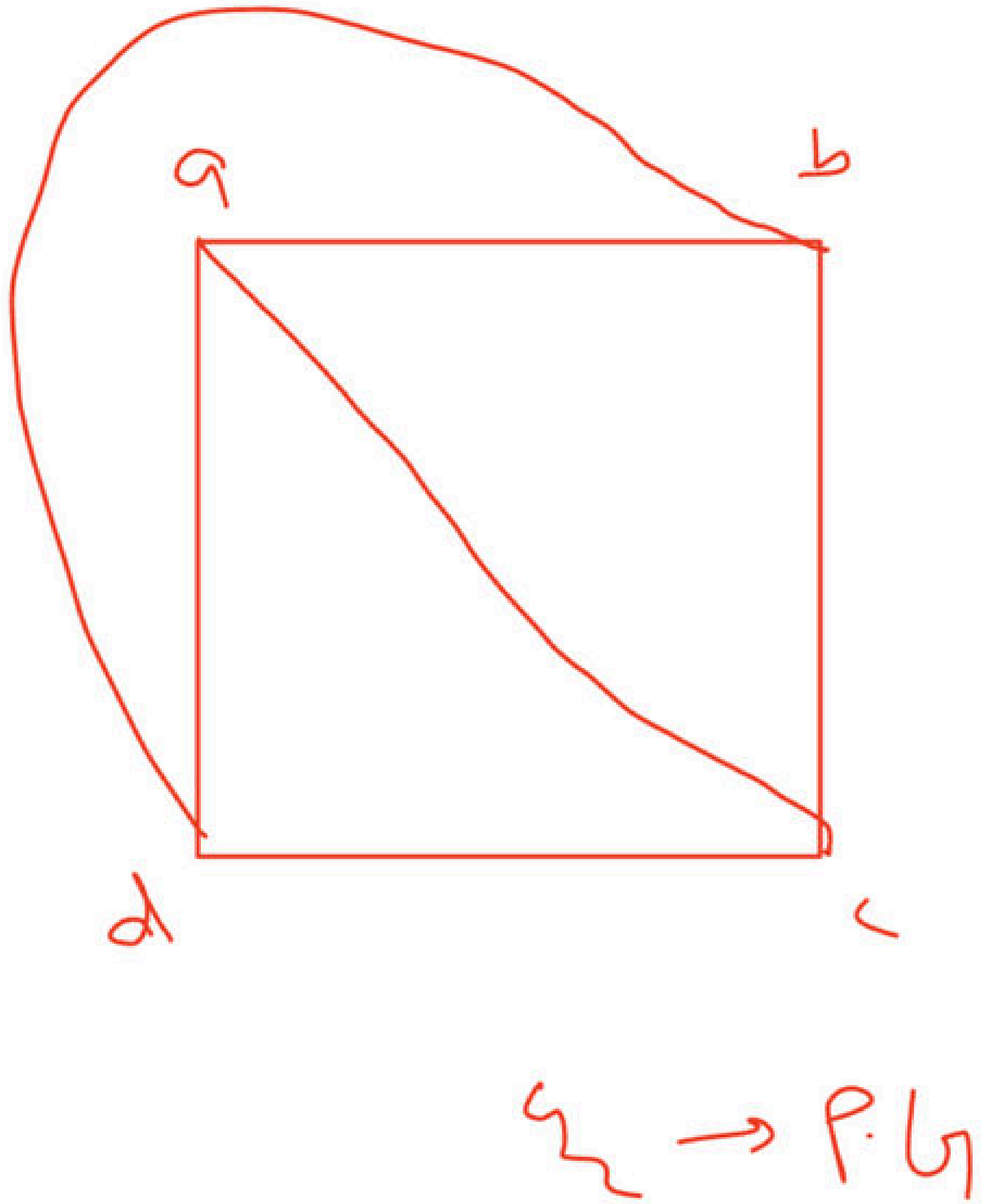
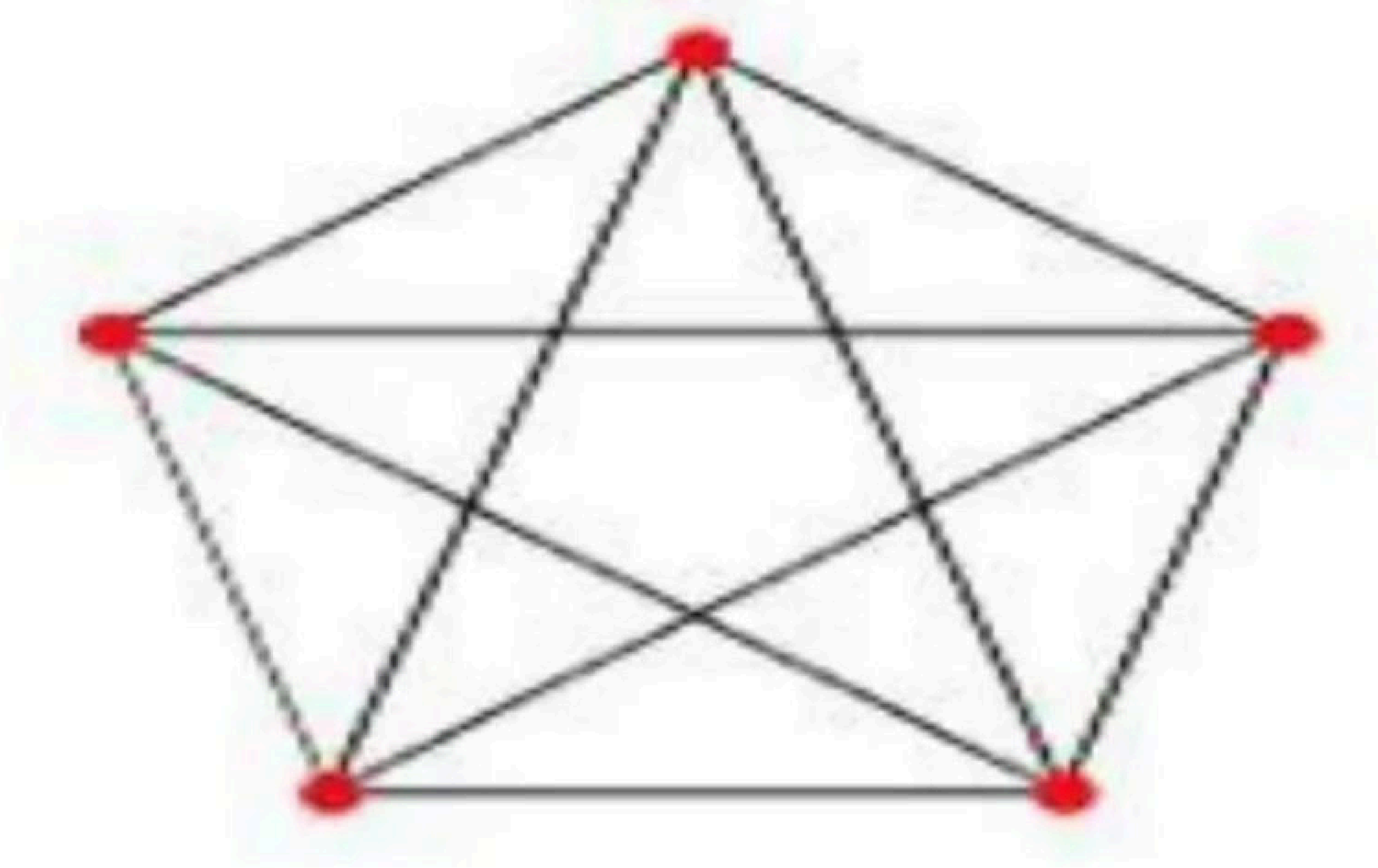
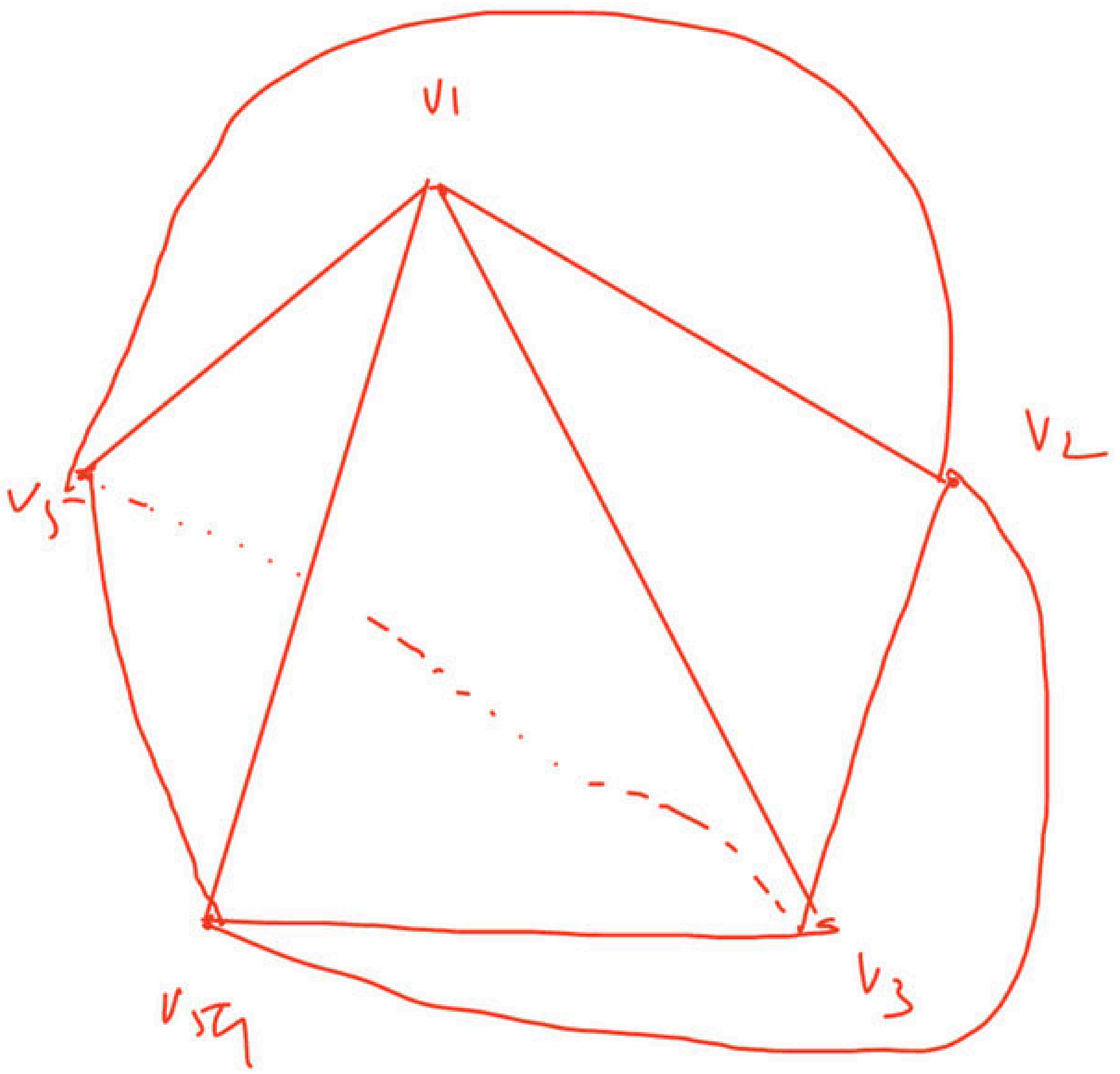


$K_4$



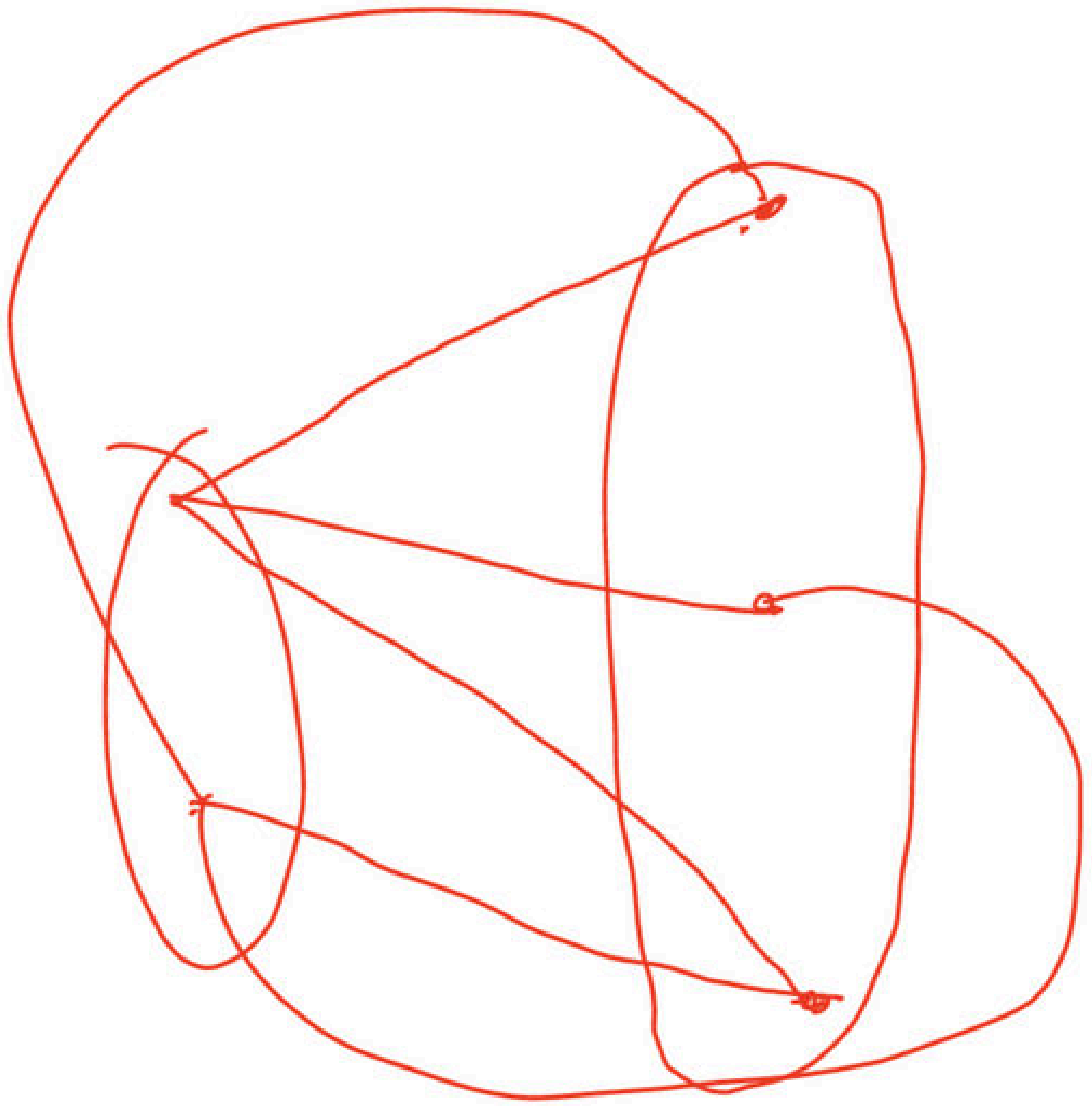
$K_5$

$\boxed{|V|=5}$

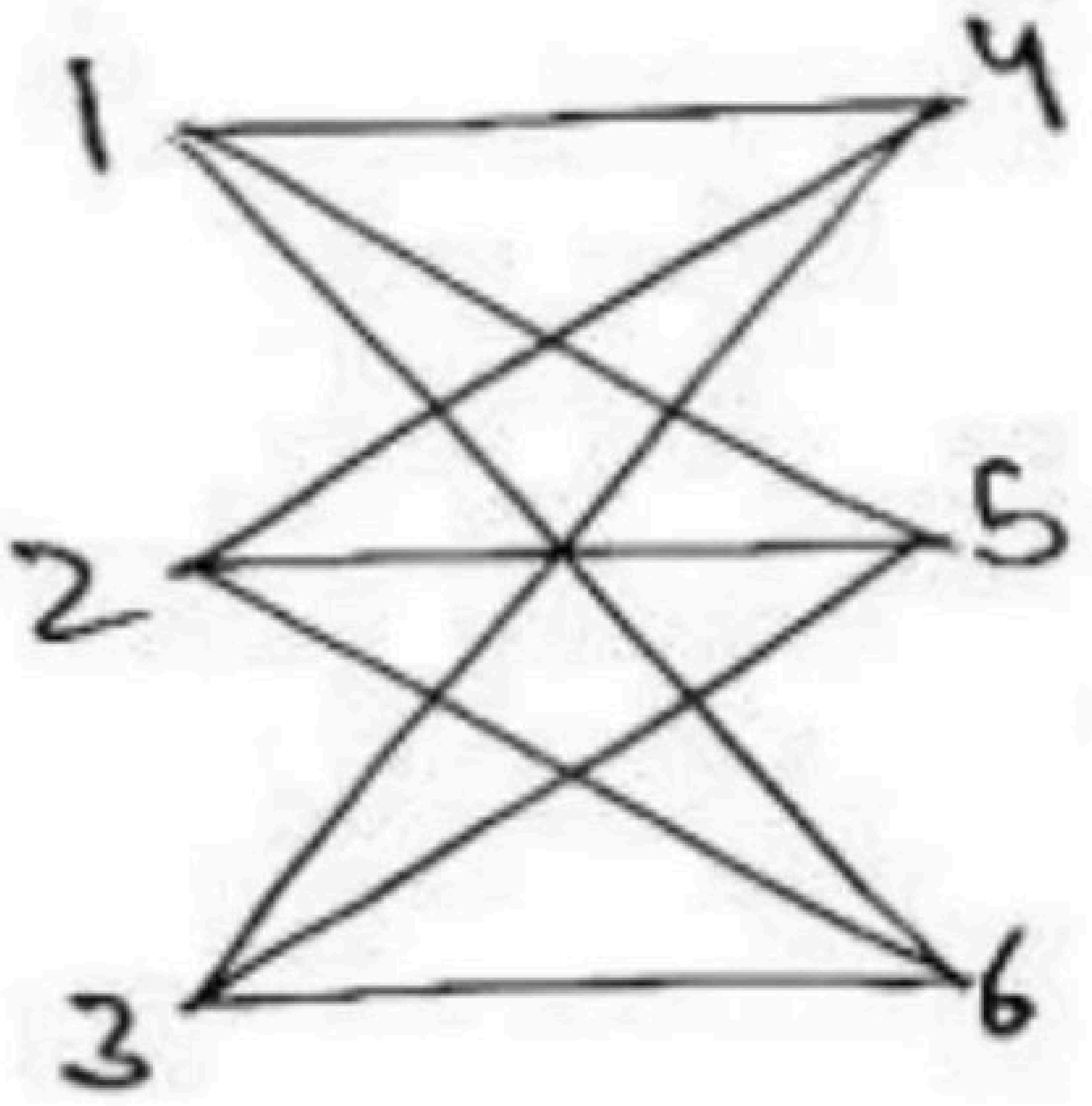


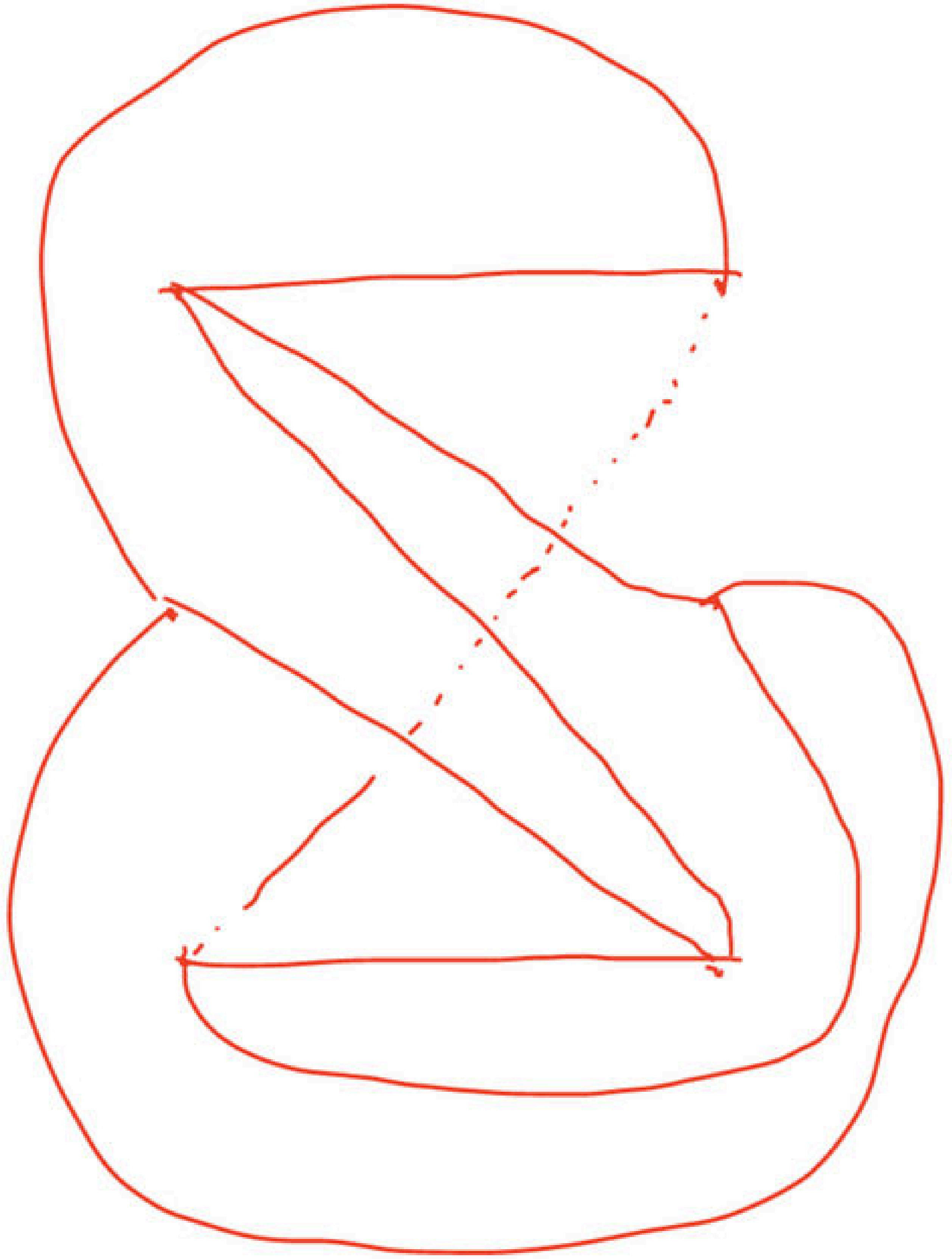

$T = 31$

$F = 64$



$K_{2,3}$



$K_{100}$

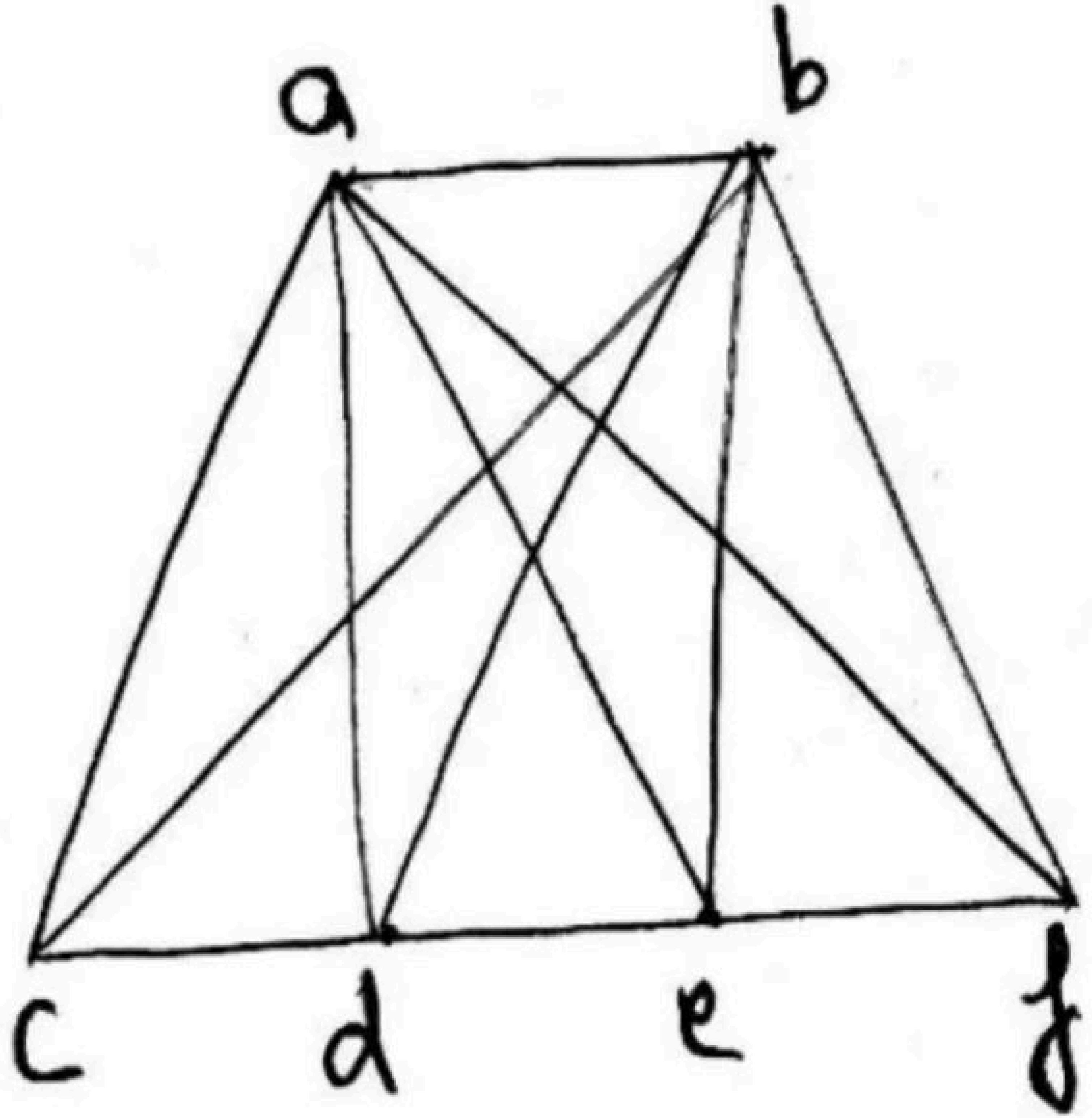
$K_{3,3}$

# Simplest Non-Planer Graphs

1. **Kuratowski's case I:** - $K_5$ → $|V| = 5$ $|E| = 10$ → min → V
2. **Kuratowski's case II:** - $K_{3,3}$ → $|V| = 6$ $|E| = 9$ → min → $|E|$
3. Both are simplest non-planer graph
4. Both are regular graph
5. If we delete either an edge or a vertex from any of the graph, they will become planer

- **Kazimierz Kuratowski** ( 2 February 1896 – 18 June 1980) was a Polish mathematician and logician.

# Break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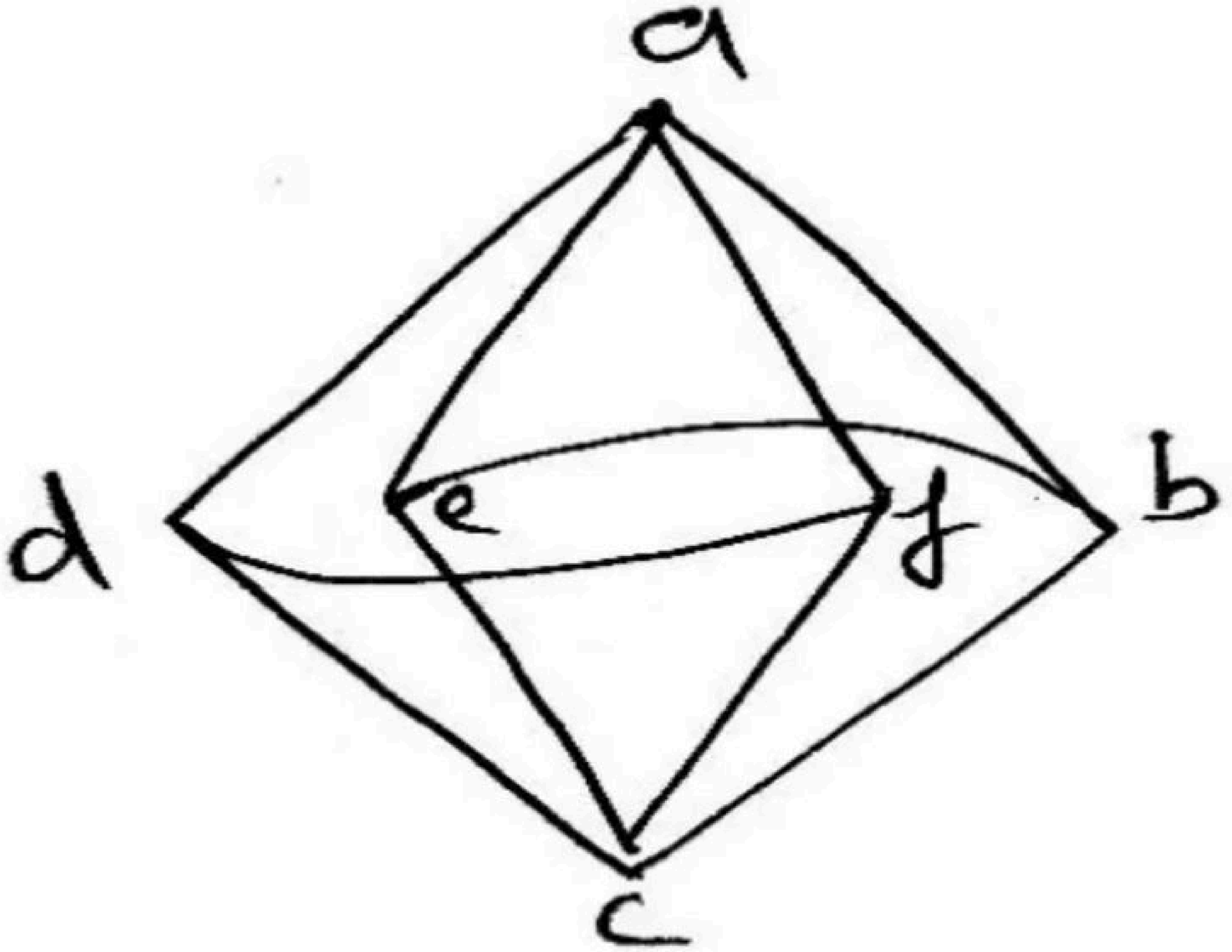

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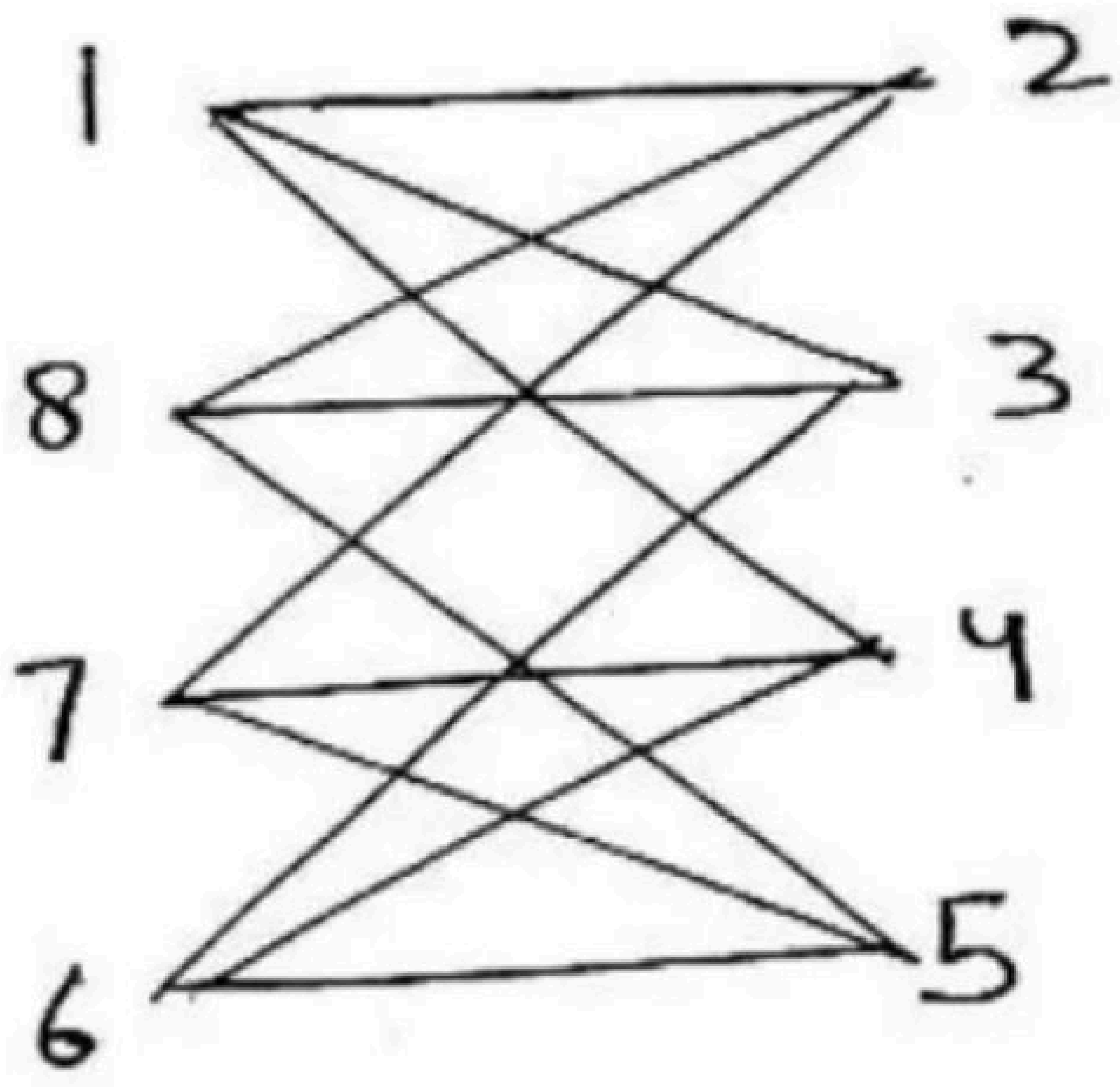
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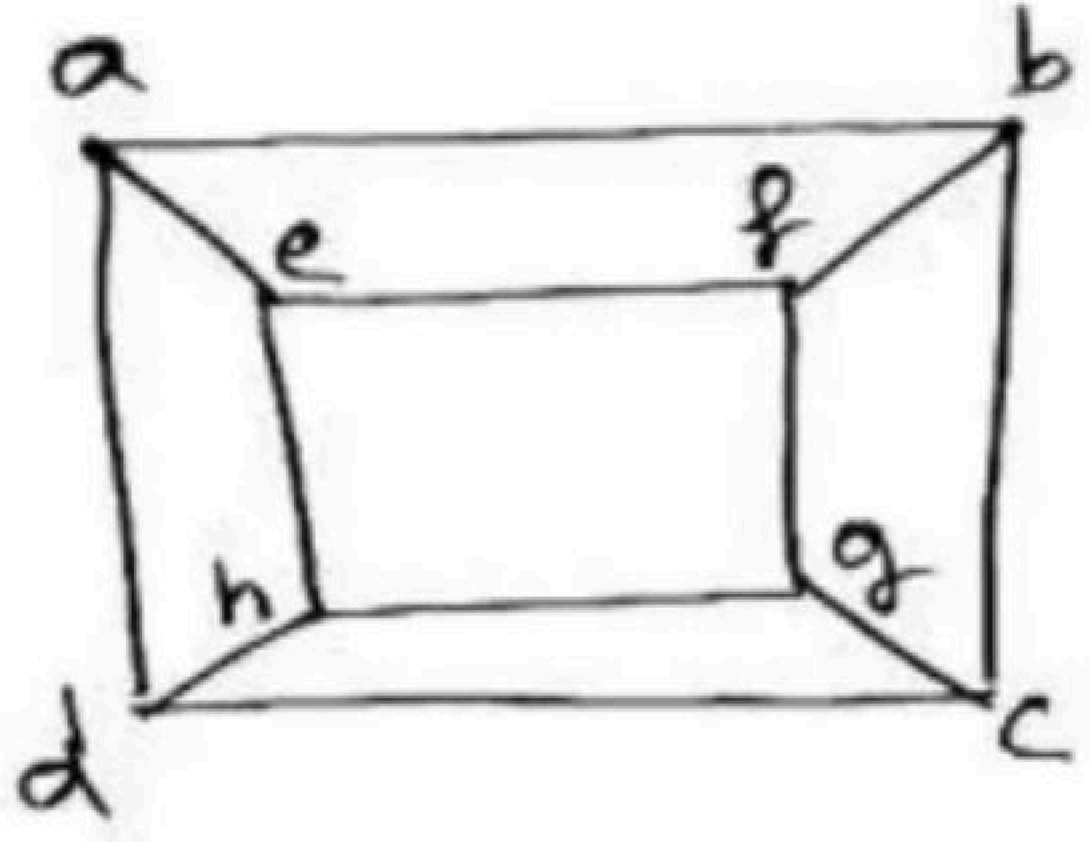
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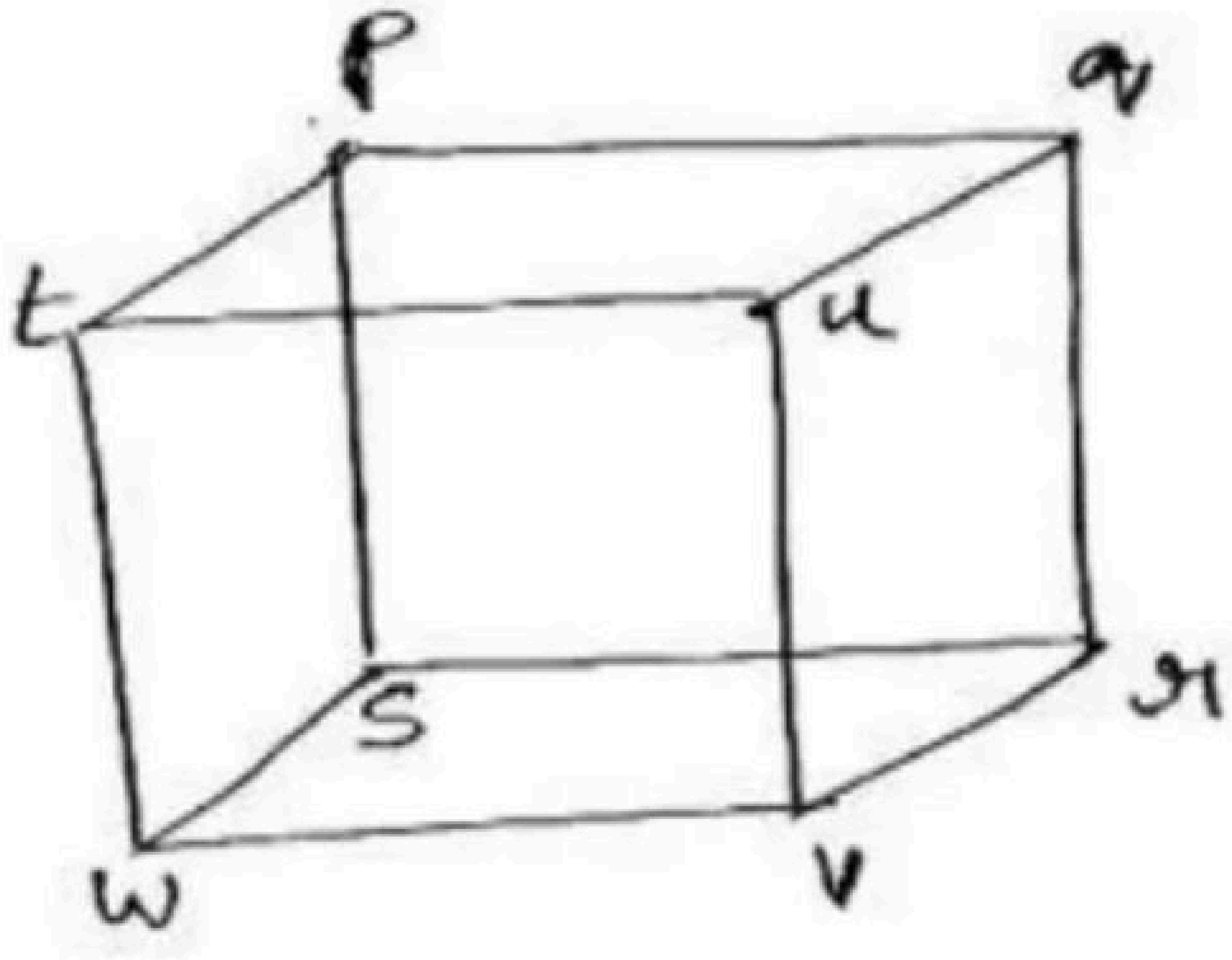
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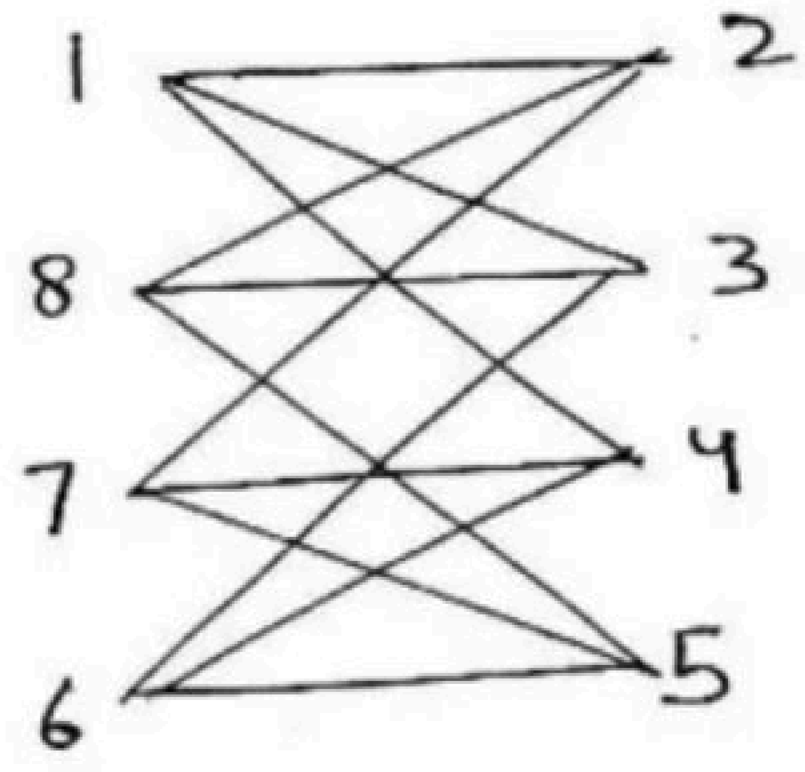
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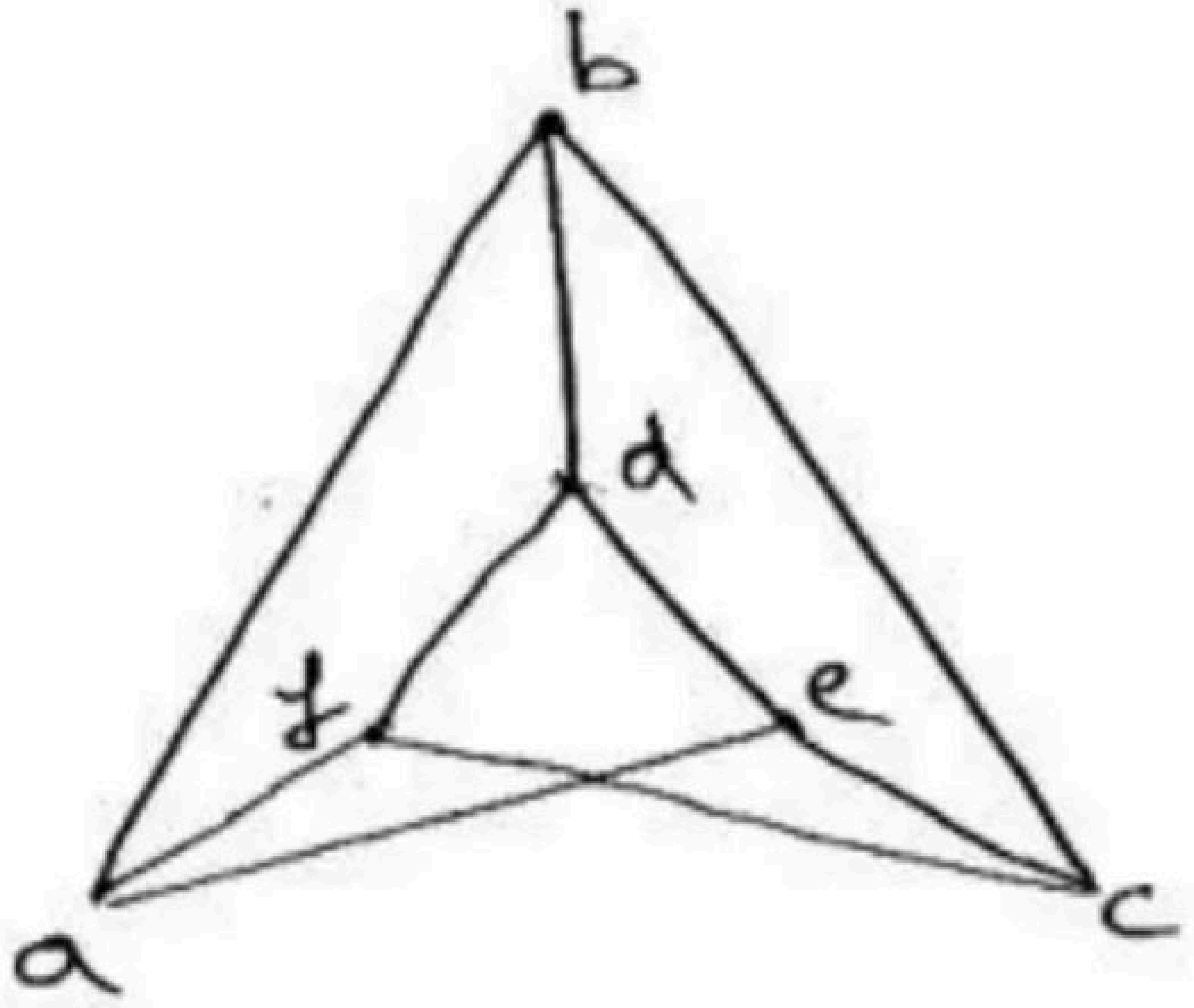
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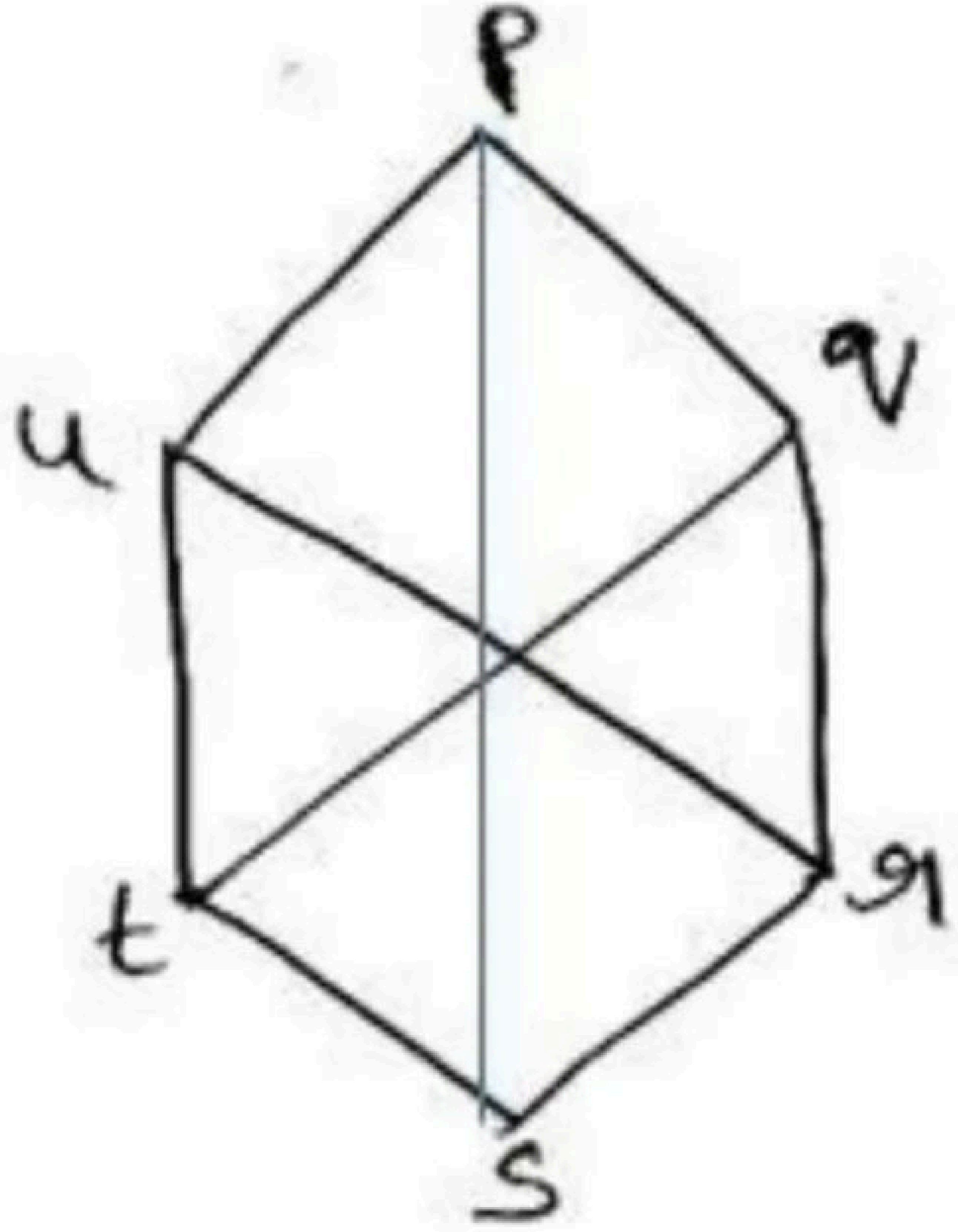
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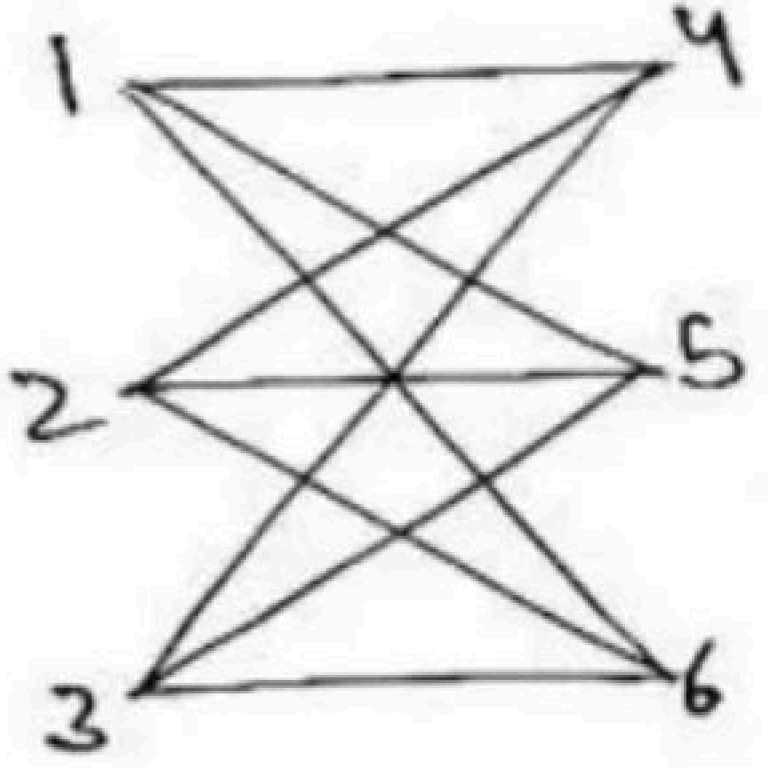
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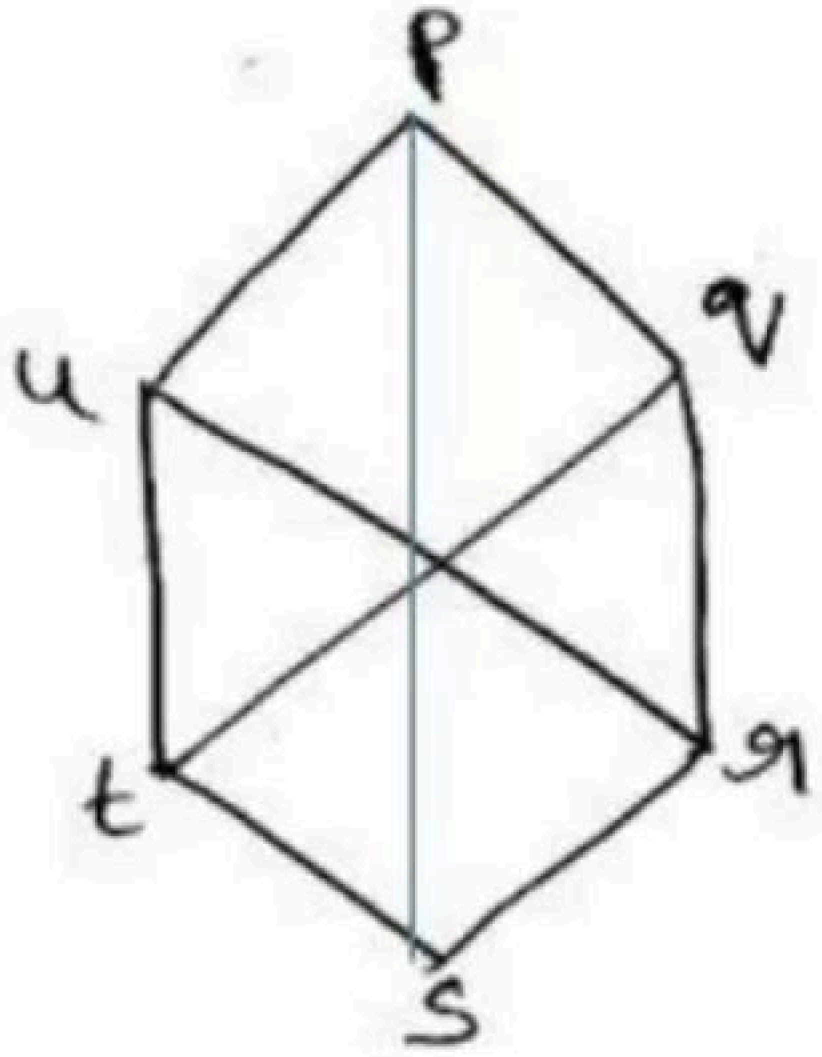
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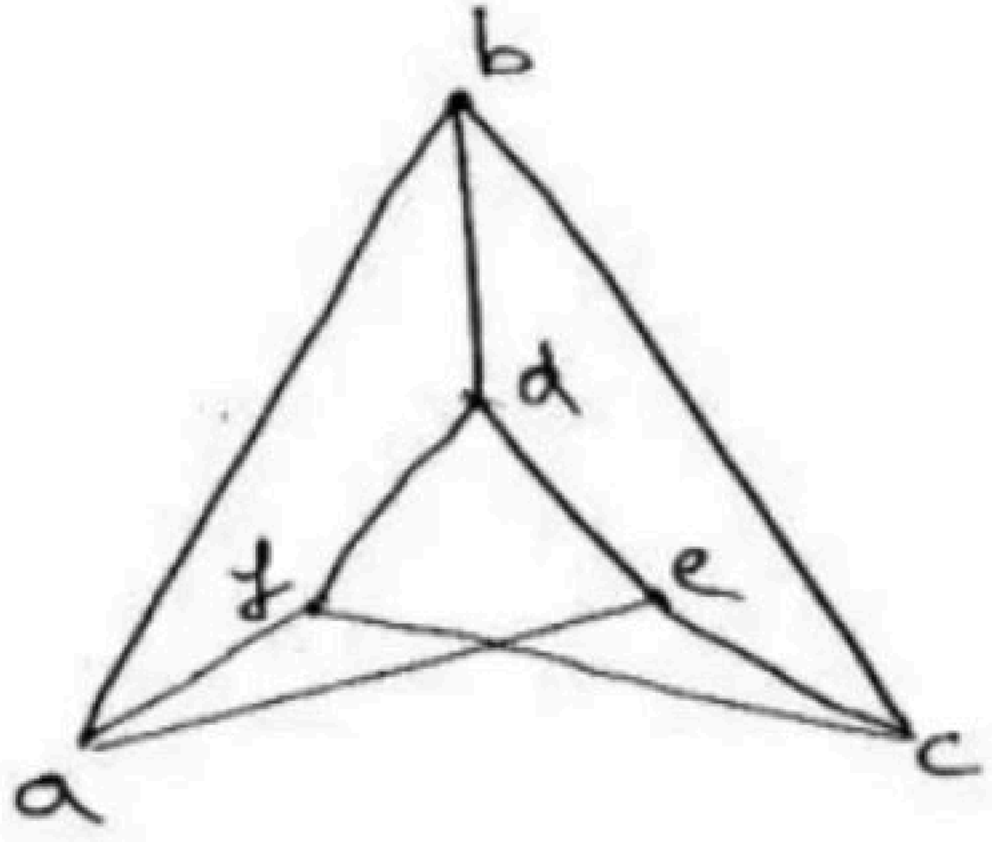
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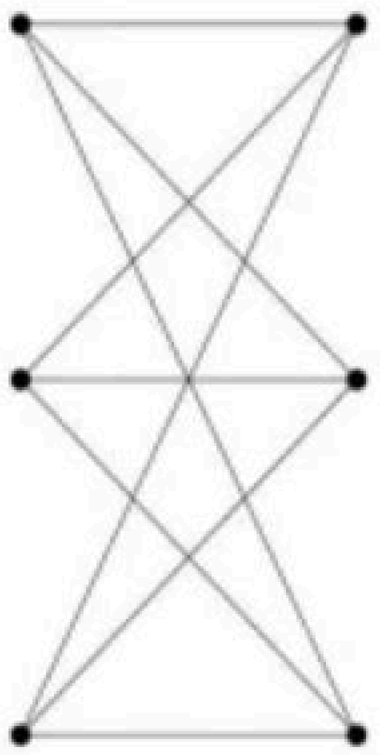
- How to find weather a graph is planer or non-planer
  - A finite graph is planar if and only if it does not contain a subgraph that is a subdivision(homorphism) of the complete graph $K_5$ or the complete bipartite graph. In practice, it is difficult to use Kuratowski's criterion to quickly decide whether a given graph is planar.

- This is a well-studied problem in computer science for which many practical algorithms have emerged, many taking advantage of novel data structures. Most of these methods operate in $\underline{O}(n)$ time (linear time), where n is the number of edges (or vertices) in the graph, which is asymptotically optimal.

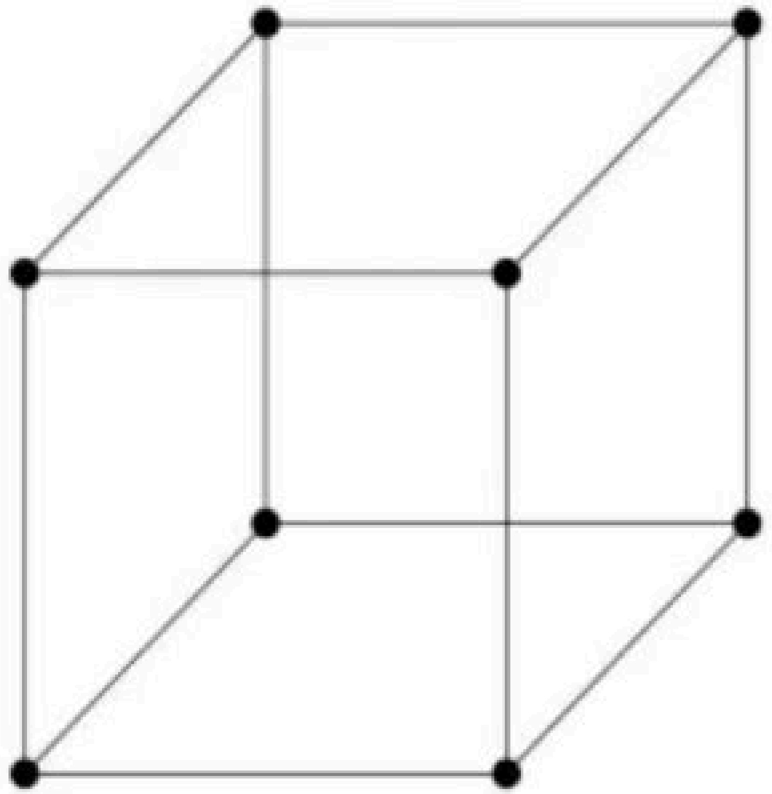
# Break

**Q** Which of the following graphs is/are planner? **(GATE-19) (2 Marks)**

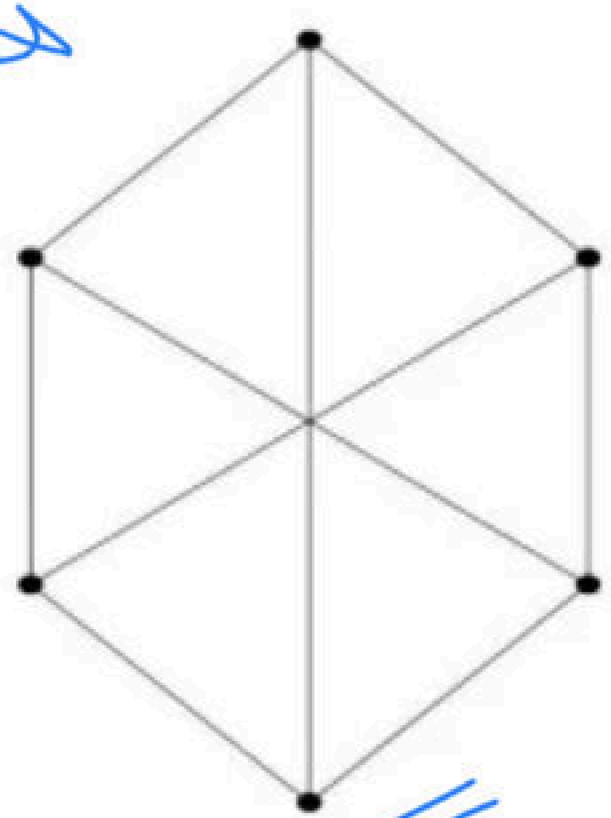
K3,3



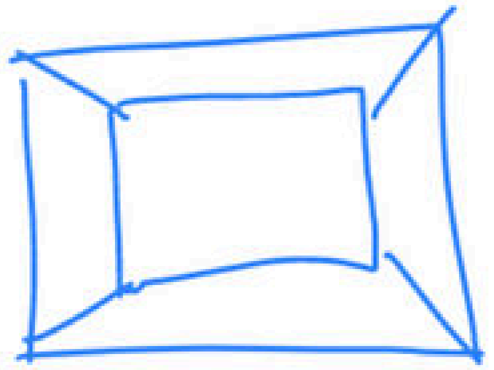
$G_1$

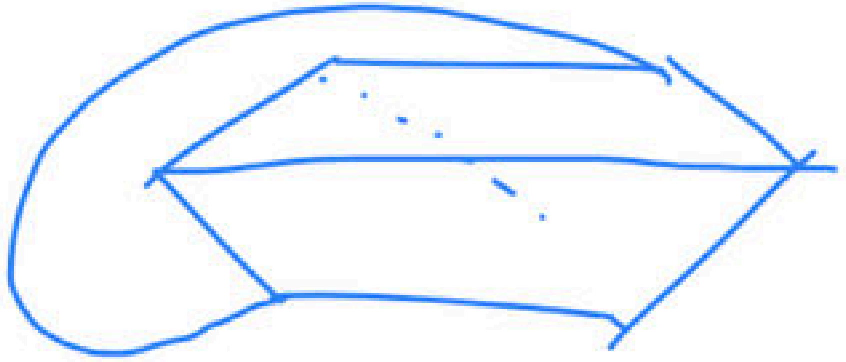


$G_2$



$G_3$





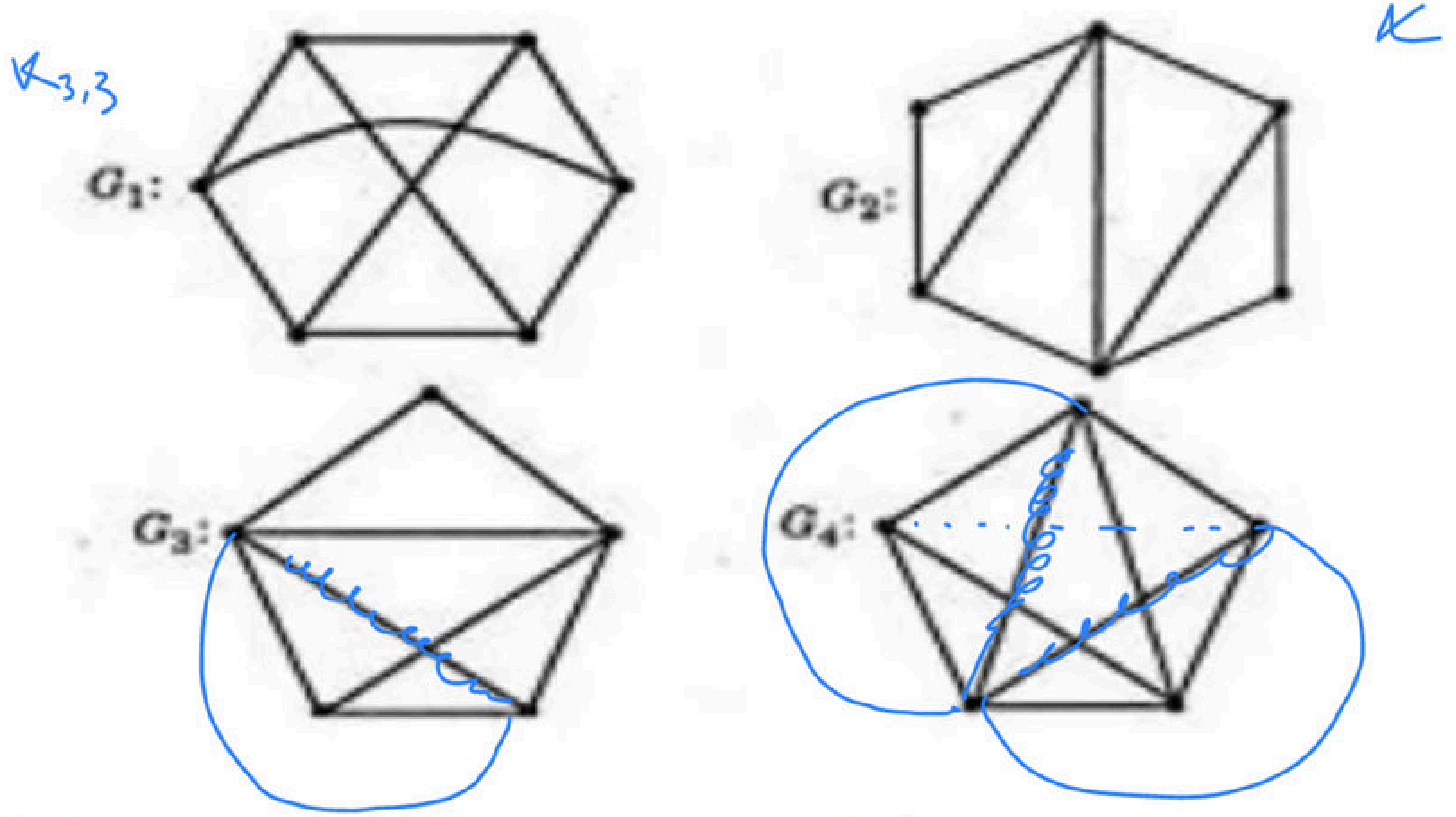
a) G1 only — 6

**b)** G1 and G2 — 3

**c)** G2 only — 61

**d)** ~~G2 and G3~~ — 30

**Q** Which one of the following graphs is NOT planar? **(GATE-2005) (2 Marks)**

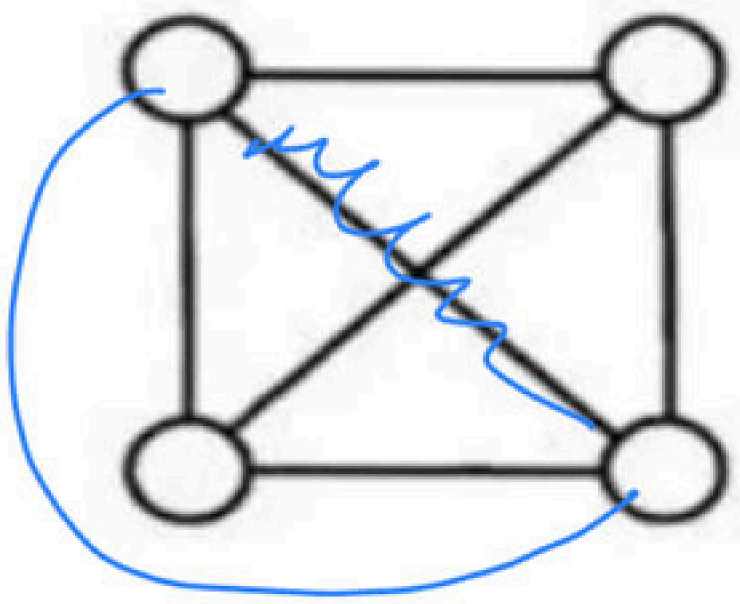



$K_5$ $K_{3,3}$

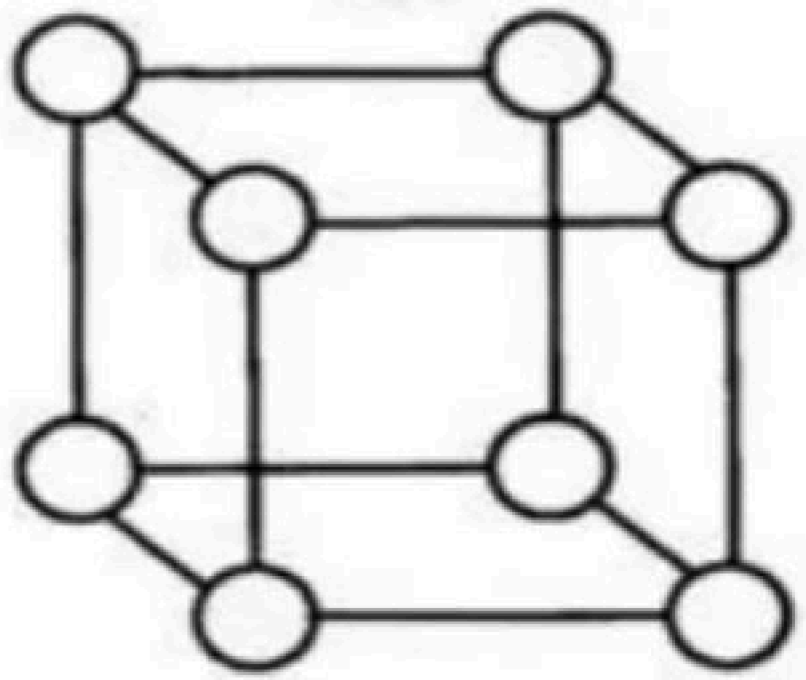
**(A)** ~~G1~~ 59 **(B)** G2 6 **(C)** G3 4 **(D)** G4 31

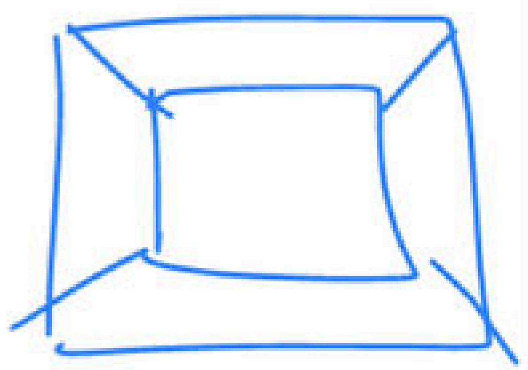
# Q (GATE-2010) (2 Marks)

K4



Q3





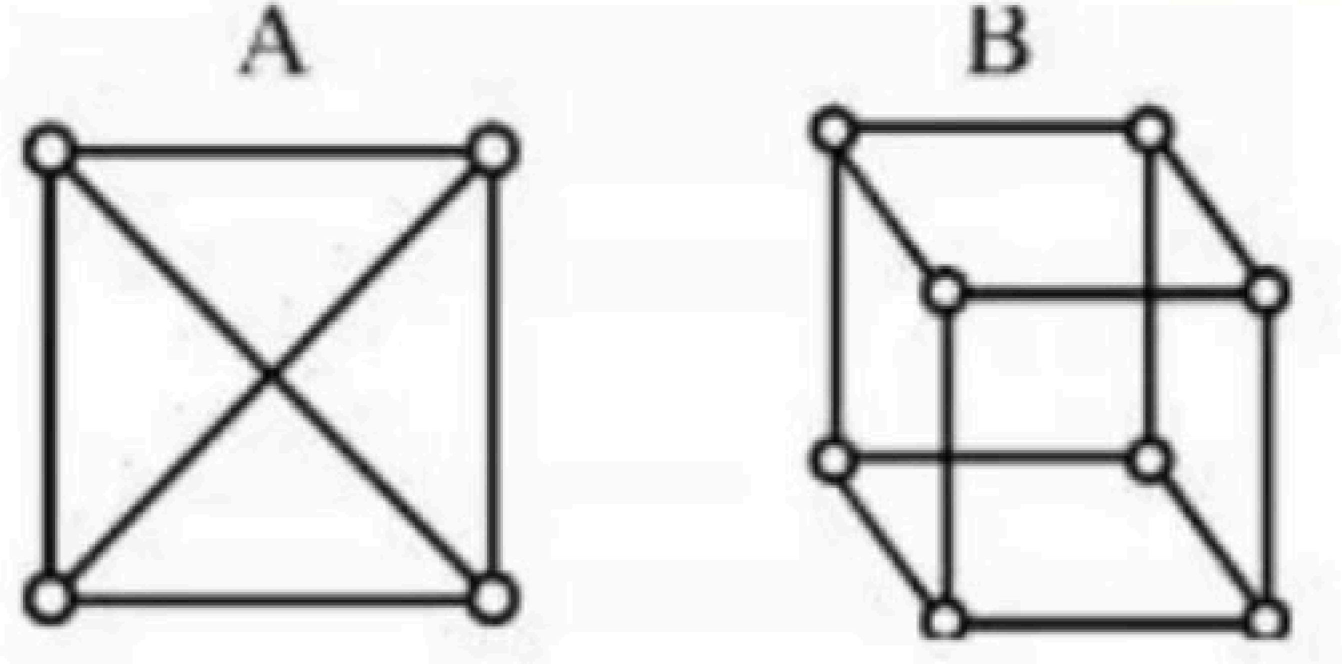
**(A)** K4 is planar while Q3 is not ← 16

~~**(B)** Both K4 and Q3 are planar~~ ← 84

**(C)** Q3 is planar while K4 is not – 75

**(D)** Neither K4 nor Q3 are planar – 0

**Q** Two graphs A and B are shown below: Which one of the following statements is true? **(NET-DEC-2015)**




**(a)** Both A and B are planar – 91

**b)** Neither A nor B is planar – 3

**(c)** A is planar and B is not – 0

**d)** B is planar and A is not – 1

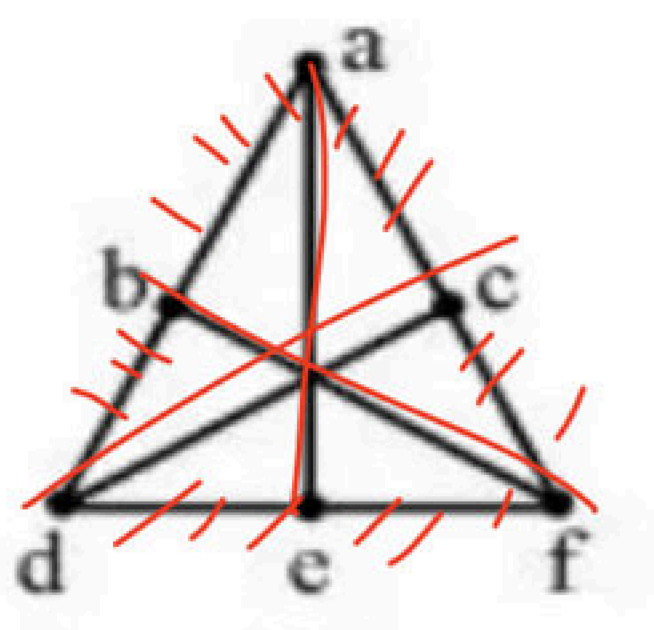
**Q** G1 and G2 are two graphs as shown: **(NET-JUNE-2012)**

**a)** Both G1 and G2 are planar graphs — 10

**b)** Both G1 and G2 are not planar graphs — 54

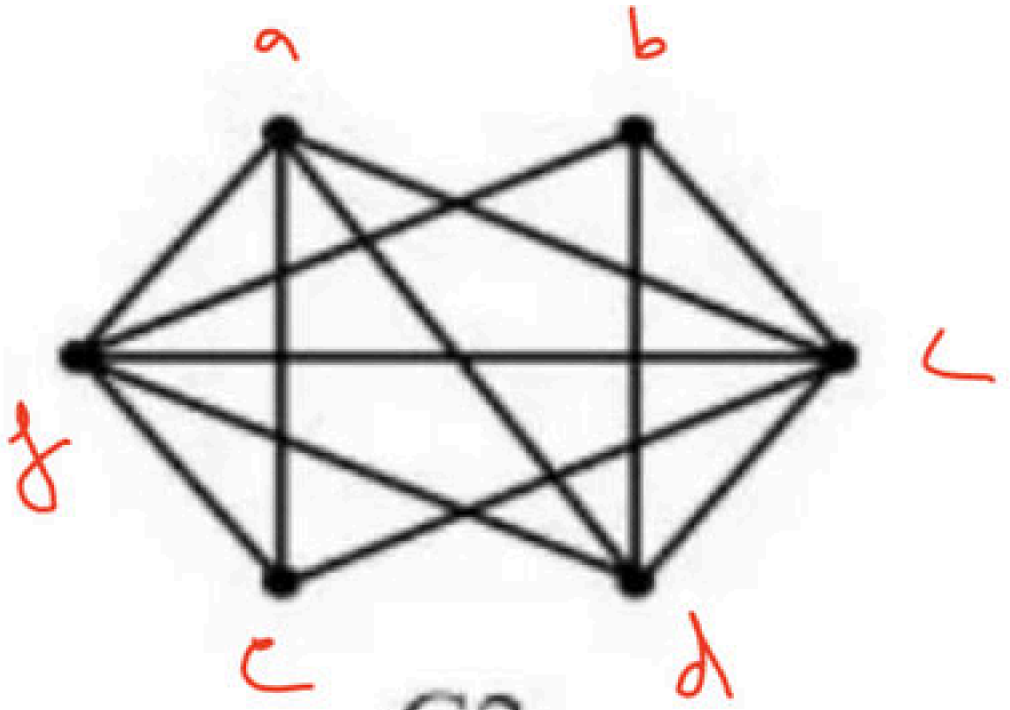
**c)** G1 is planar and G2 is not planar — 10

**d)** G1 is not planar and G2 is planar — 26

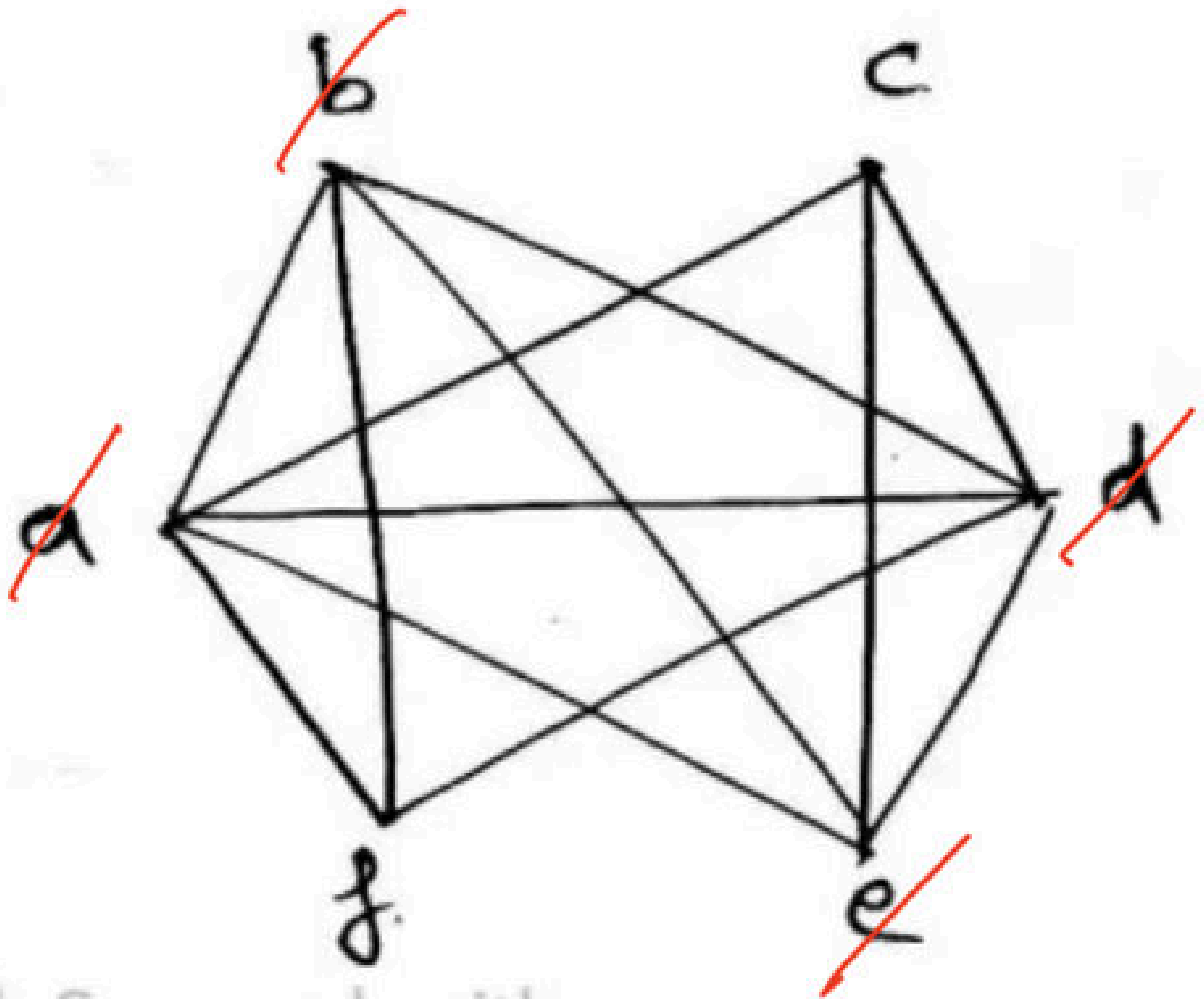


G1

$K_{3,3}$



G2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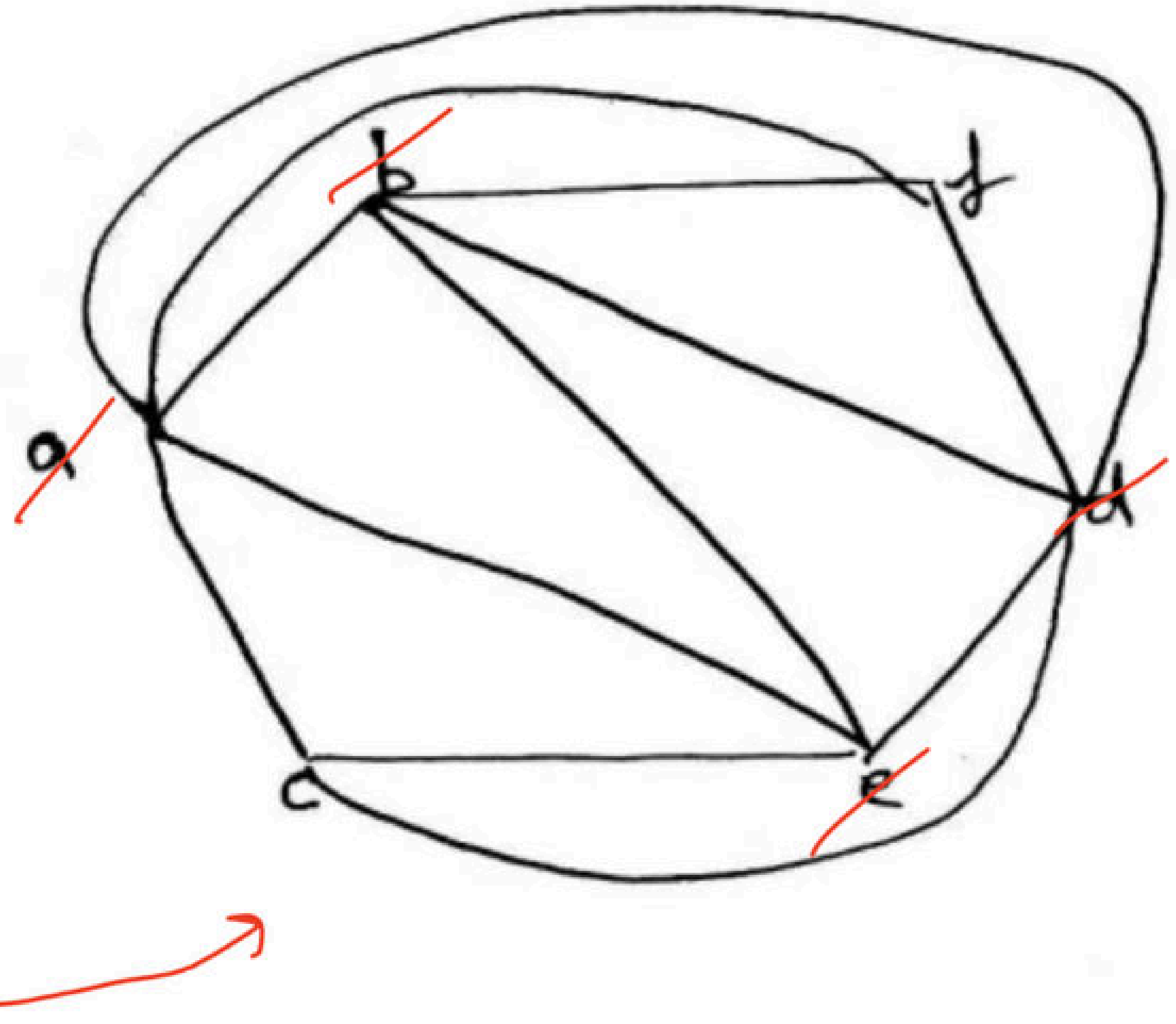
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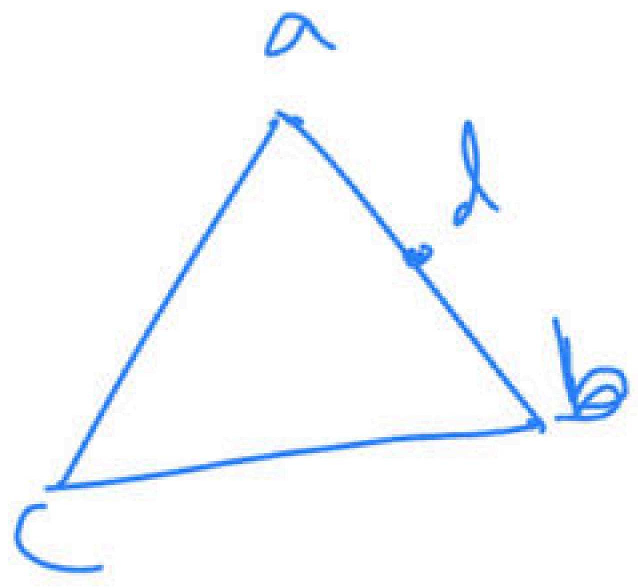
**Q** Let G be the non-planar graph with the minimum possible number of edges. Then G has **(GATE-1992) (1 Marks) (GATE-2007) (1 Marks)**

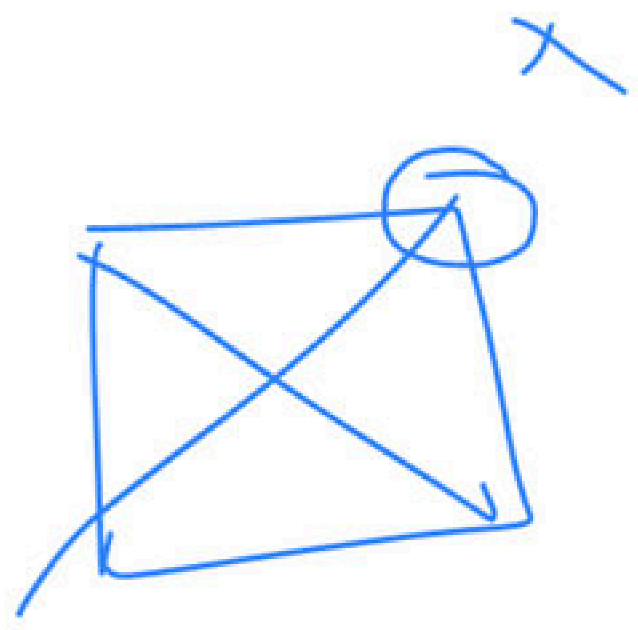
**(A)** ~~9 edges and 5 vertices~~

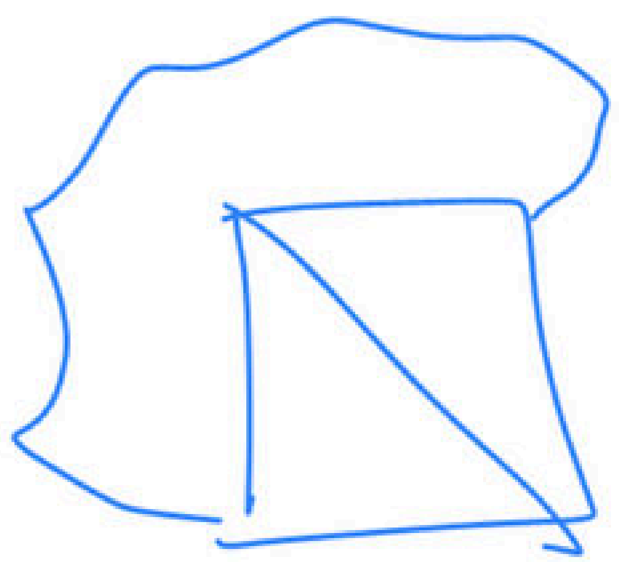
**(B)** 9 edges and 6 vertices

**(C)** ~~10 edges and 5 vertices~~

**(D)** ~~10 edges and 6 vertices~~





$C_3 \longleftrightarrow C_4 \longleftrightarrow C_5 \longleftrightarrow 6_5$ $\cancel{K_5}$ $\cancel{K_{3,3}}$

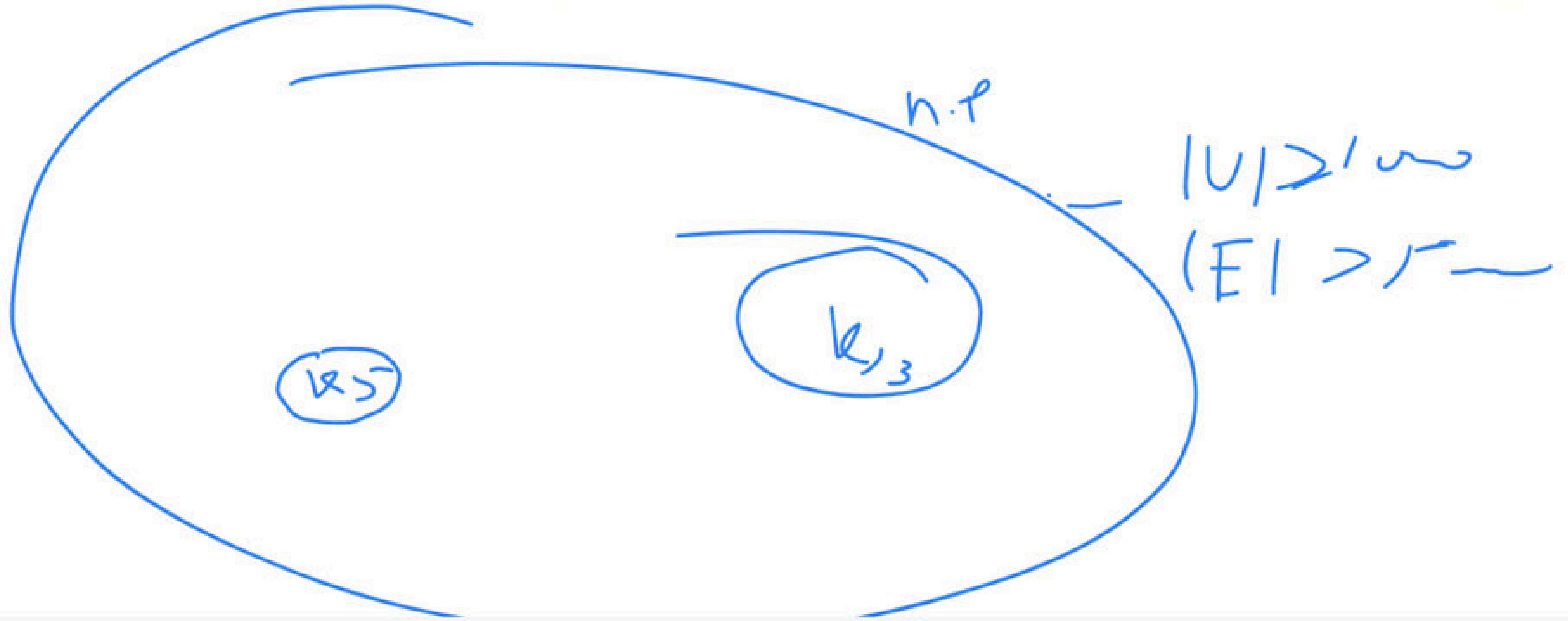
**Q** A graph is planar if and only if, **(GATE-1990) (2 Marks)**

**a)** It does not contain subgraphs homeomorphic to k5 ~~and~~ k3,3. — 41

**b)** It does not contain subgraphs isomorphic to k5 or k3,3. — 7

**c)** It does not contain a subgraph isomorphic to k5 or k3,3 — 11

**d)** It does not contain a subgraph homeomorphic to k5 or k3,3. — 41



**Q** A graph is non-planar if and only if it contains a subgraph homomorphic to **(NET-DEC-2013)**

**(A)** $K_{3,\,2}$ or $K_5$ **(B)** $K_{3,\,3}$ and $K_6$ **(C)** $K_{3,\,3}$ or $K_5$ **(D)** $K_{2,\,3}$ and $K_5$

# Break

# Euler's formula

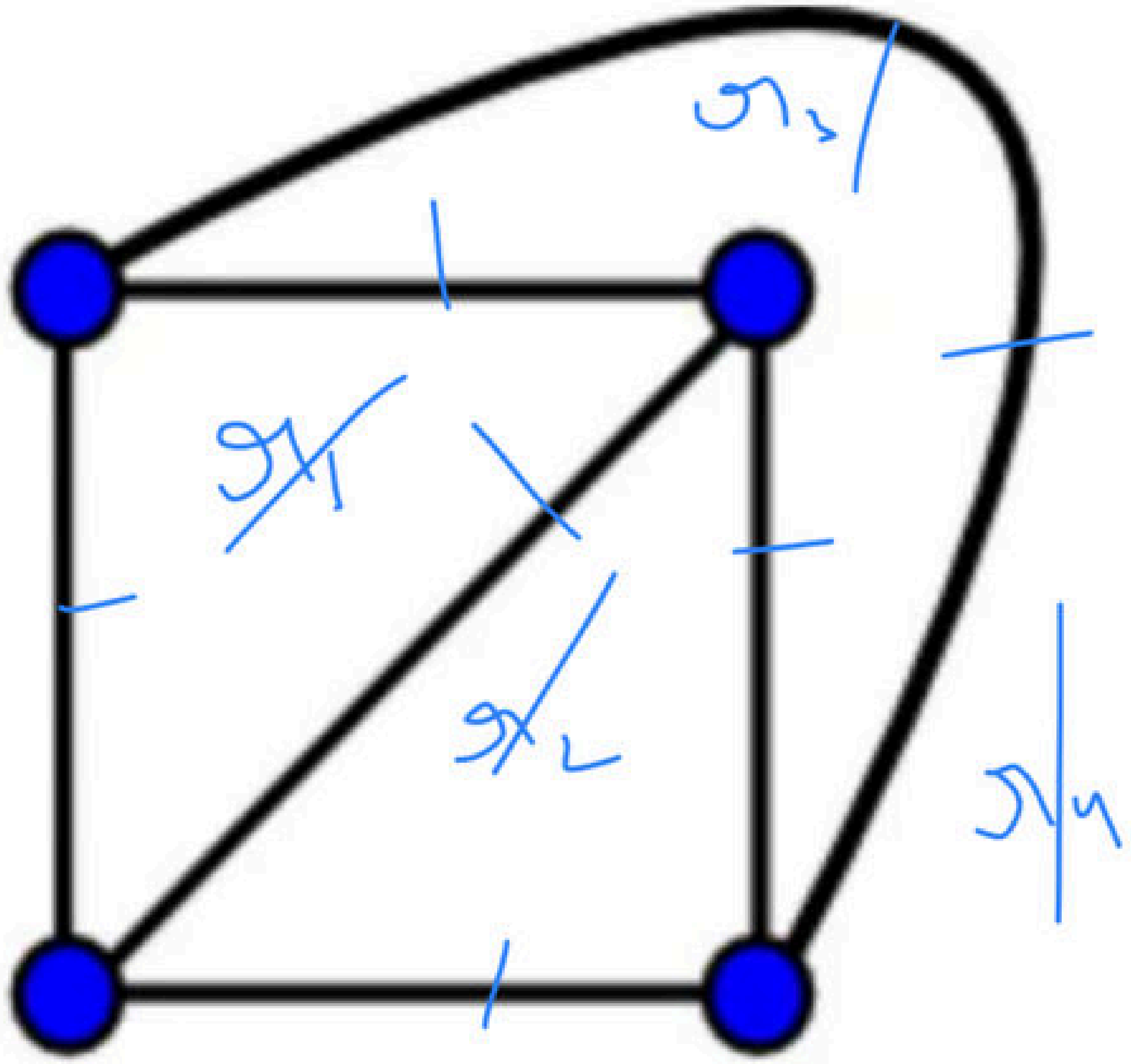
- A planer graph divides the plane into number or regions (faces, planer embedding), which are further divided into bounded(internal) and unbounded region(external).
- **Euler's formula** states that if a finite, connected, planar graph with *v* is the number of vertices, *e* is the number of edges and *r* is the number of faces (regions bounded by edges, including the outer, infinitely large region), then
- $r = e - v + 2$
- Euler's formula can be proved by mathematical induction

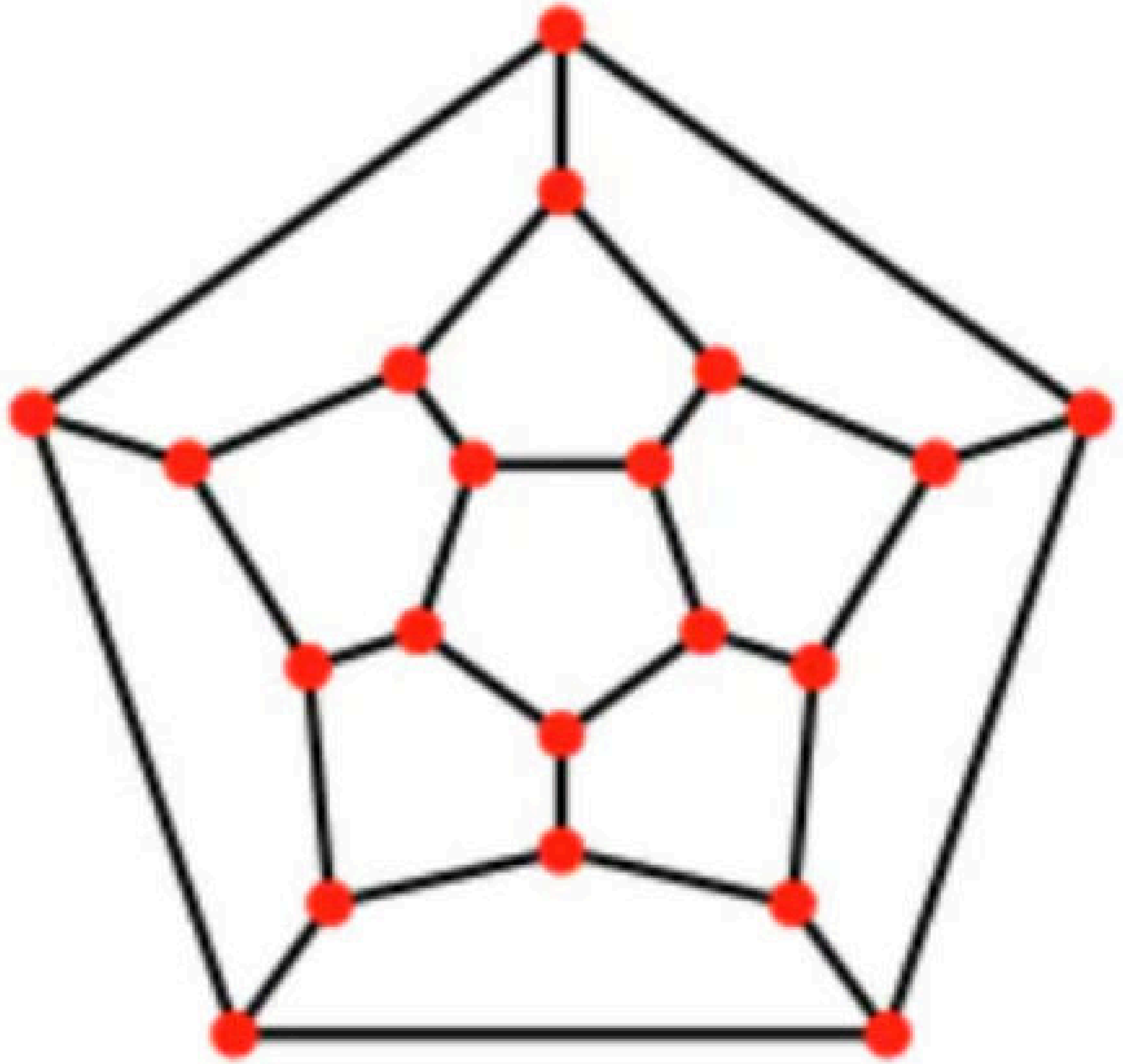
$r = e - v + 2$

$\cancel{6 - 4} + 2$

$2 + 2$

$= 4$

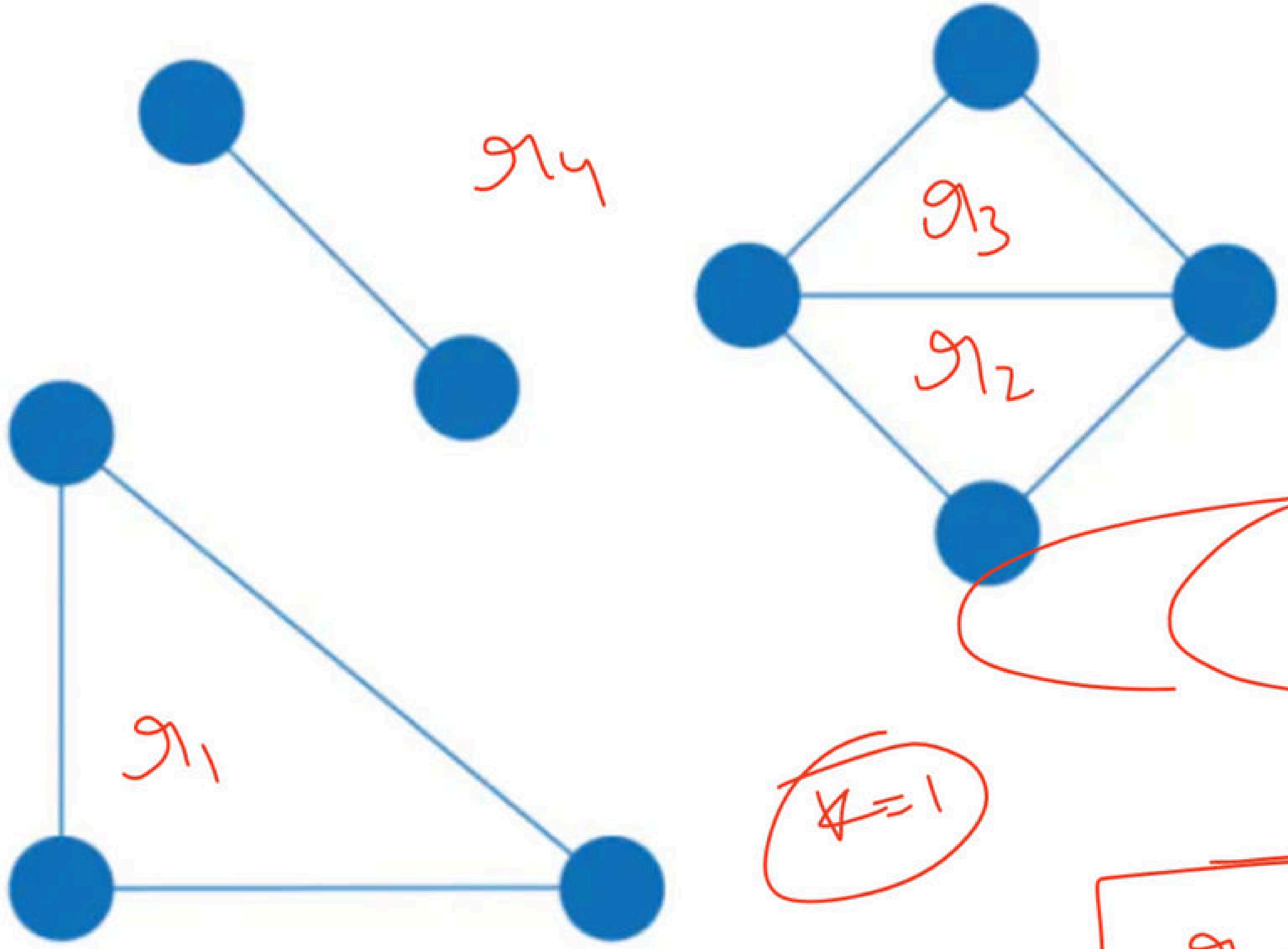


- Euler's formula (Disconnected graph): $\boxed{V - e + r - k = 1}$

$\cancel{9} - \cancel{9} + r - 3 = 1$

$r = 4$




$V - e + r - k = 1$

$k = 1$

$V - e + r - 1 = 1$

$\boxed{r = e - v + 2}$

In an undirected connected planar graph $G$, there are eight vertices and five faces. The number of edges in $G$ is ___11___.

**(Gate-2021) (1 Marks)**

$$r = e - v + 2$$

$$5 = e - 8 + 2$$

$$-6$$

$$e = 11$$

**Q** suppose that a connected planar graph has six vertices, each of degree four, into how many regions is the plane divided by a planner representation of this graph?

**(NET-JULY-2019)**

**a)** 6 — 5

**b)** 8 — 46

**c)** 12 — 14

**d)** 20 — 35

$$r = e - v + 2$$

$$= \cancel{12} - \cancel{6} + 2$$

$$6 + 2$$

$$r = 8$$

$$\sum_{i=1}^{n} dg(v_i) = 2 \cdot E$$

$$\cancel{6 \times 4}$$

$$\cancel{24/2} \quad 12$$

**Q** Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is **(GATE-2005) (2 Marks)**

**(A)** 6 **(B)** 8 **(C)** 9 **(D)** 13

6 93

$r = e - v + 2$

$= \cancel{19} - \cancel{13} + 2$

$6 + 2$

$= 8$

**Q** Let G be a simple undirected planar graph on 10 vertices with 15 edges. If G is a connected graph, then the number of bounded faces in any embedding of G on the plane is equal to **(GATE-2012) (1 Marks)**

**(A)** 3      (B) 4      (C) 5      ~~(D) 6~~

4      2      11      83

$$r = e - v + 2$$

$$= \cancel{15} - \cancel{10} + 2$$

$$5 + 2$$

$$= 7$$

$r$

$r-1$      $1$

# Break

# Other formula derived from Euler's formula

- Connected planar graphs with more than one edge obey the inequality $2e>=3r$, because each face has at least three face-edge incidences and each edge contribute exactly two incidences.

- Degree of the region is number of edges covering the region. Sum of degree of regions = $2|E|$

Using r = e – v + 2 and 3r<=2e
eliminating r we get, $e \leq 3v - 6$

Using r = e – v + 2 and 3r<=2e
Eliminating e we get, $r \leq 2v - 4$

**Q** maximum number of edges in a planar graph with n vertices **(GATE-1992) (1 Marks)**

**Q** A graph $G = (V, E)$ satisfies $|E| \leq 3|V| - 6$. The min-degree of G is defined as

$$\min_{v \in V} \{\text{degree}(v)\}$$

Therefore, min-degree of G cannot be **(GATE-2003) (2 Marks)**

**(A)** 3 **(B)** 4 **(C)** 5 **(D)** 6

**Q** Let $\delta$ denote the minimum degree of a vertex in a graph. For all planar graphs on n vertices with $\delta \geq 3$, which one of the following is TRUE? **(GATE-2014) (2 Marks)**

**(A)** In any planar embedding, the number of faces is at least $n/2 + 2$

**(B)** In any planar embedding, the number of faces is less than $n/2 + 2$

**(C)** There is a planar embedding in which the number of faces is less than $n/2 + 2$

**(D)** There is a planar embedding in which the number of faces is at most $n/(\delta + 1)$

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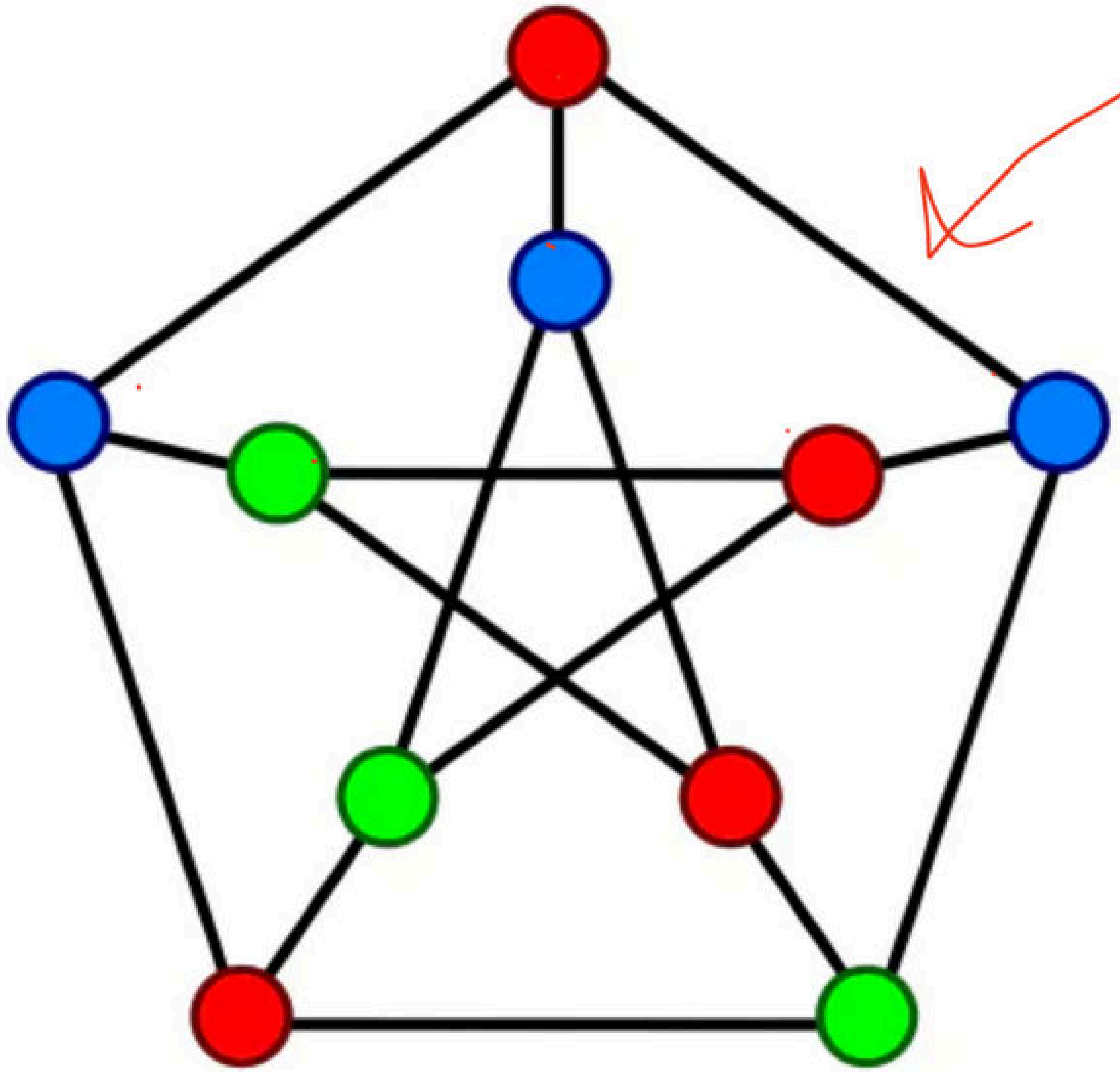
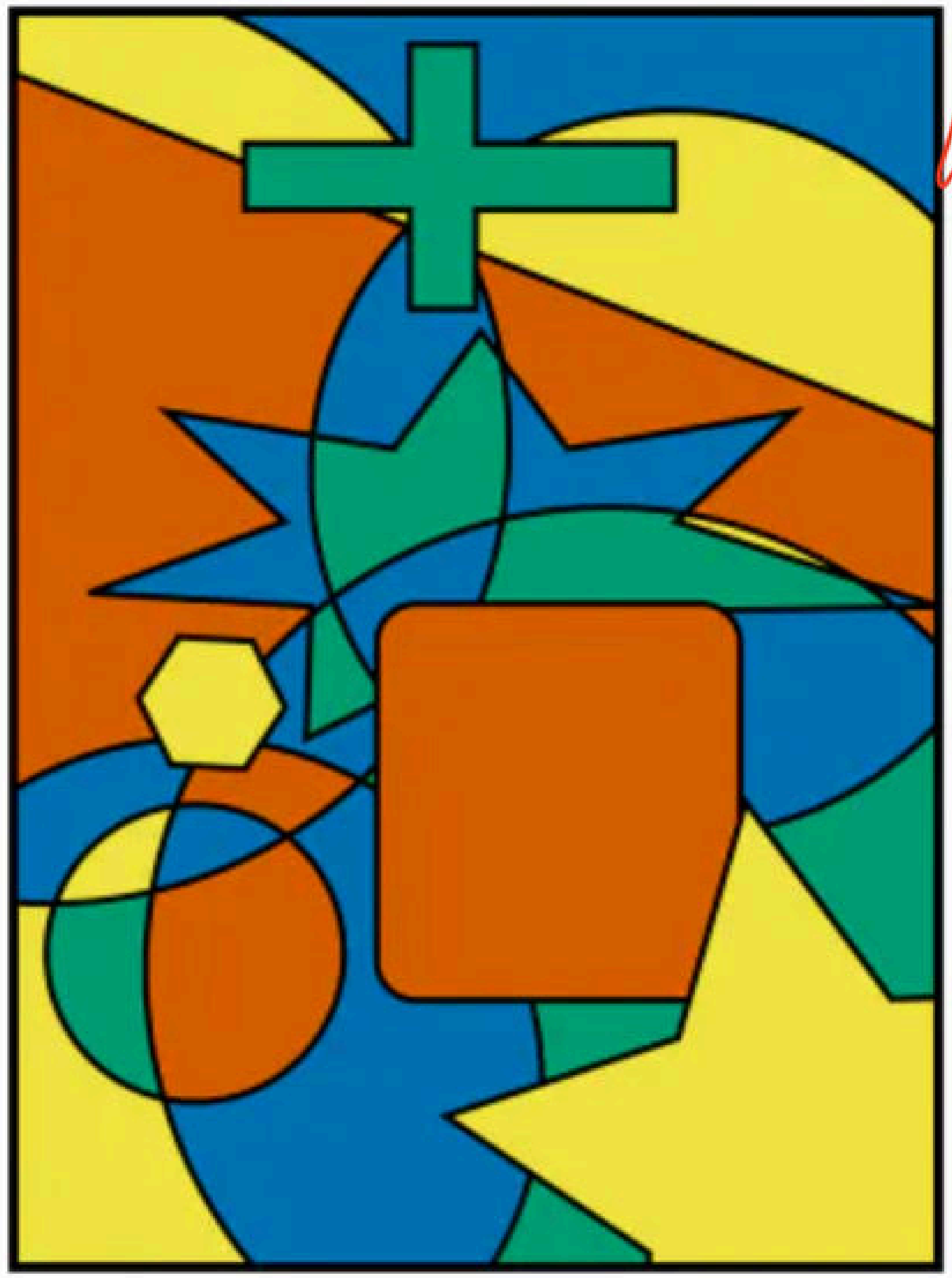
# Graph Coloring

- Graph coloring can be of two types vertex coloring and region coloring.

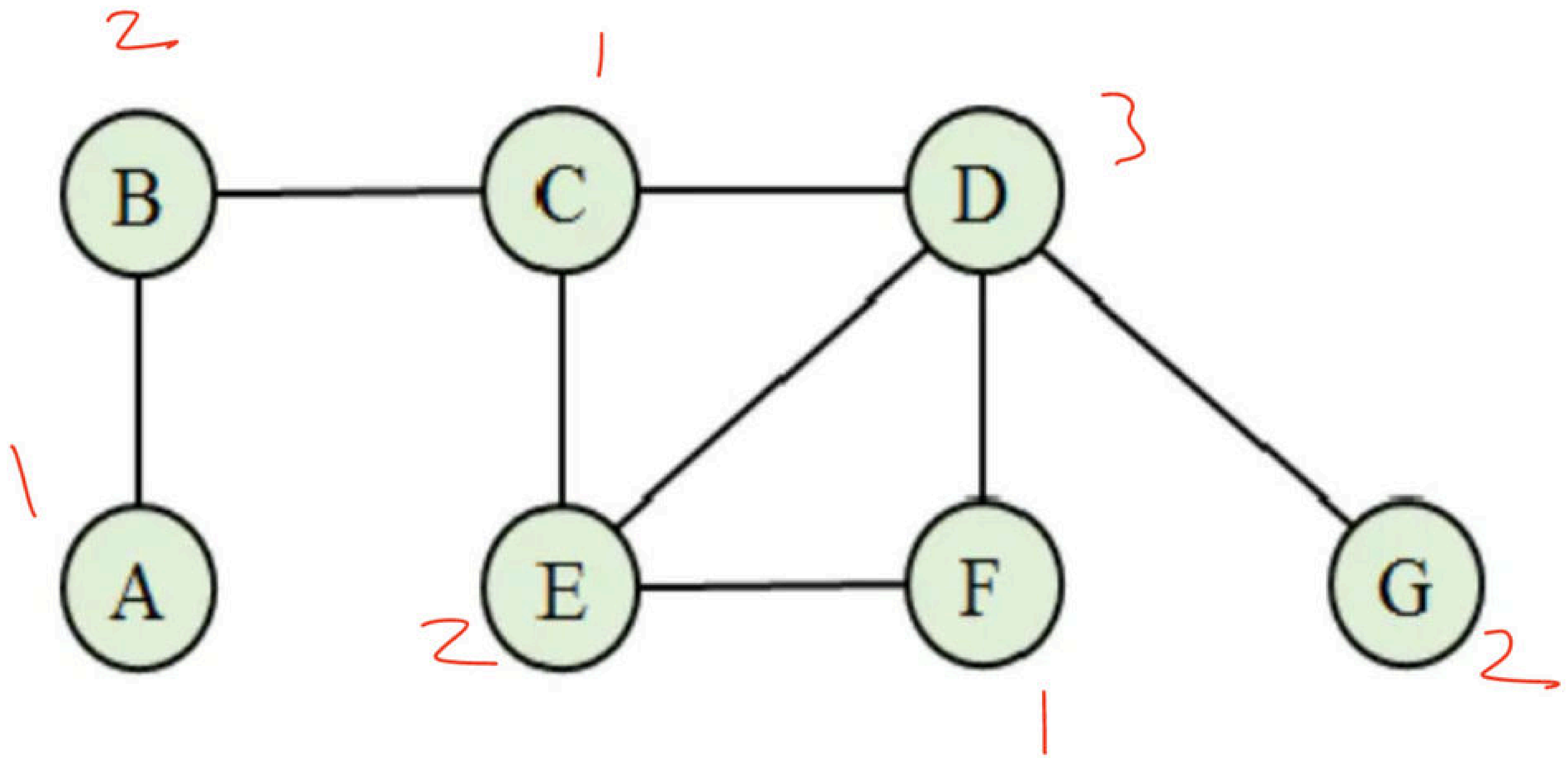
P/NP





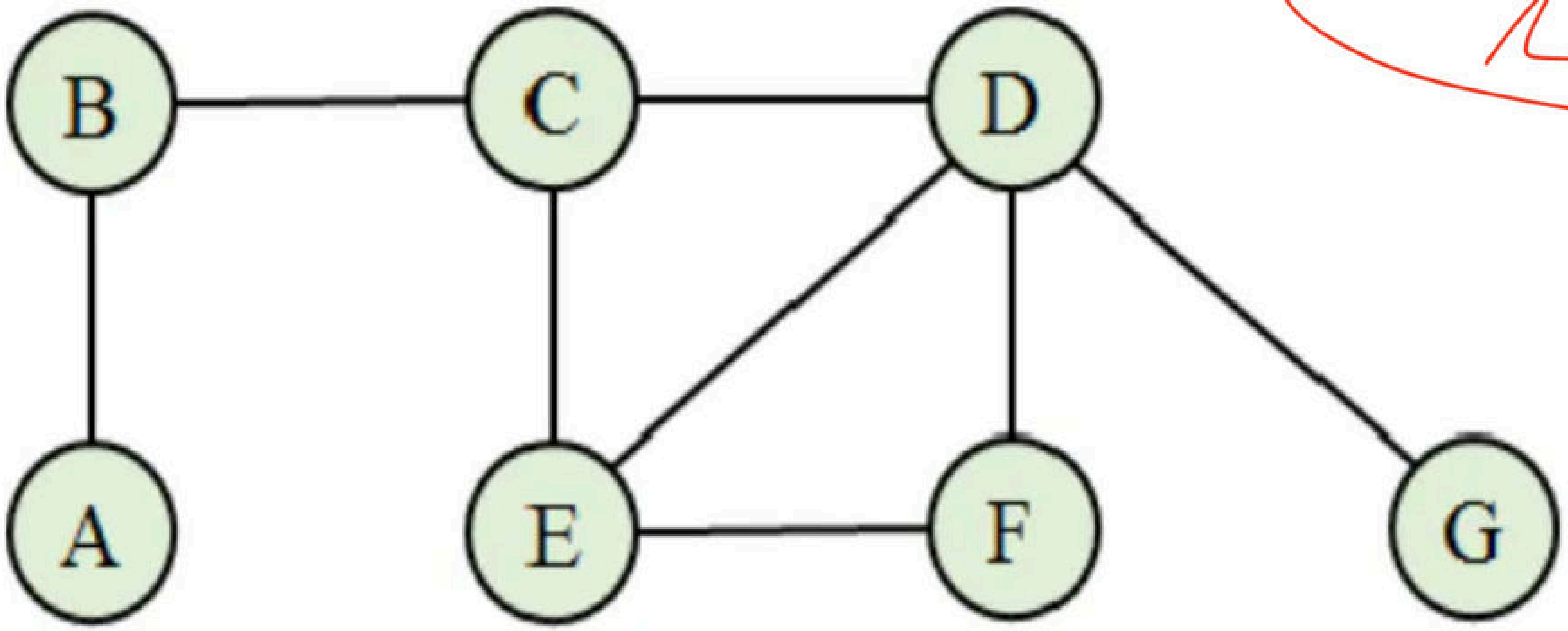
- Associating a color with each vertex of the graph is called vertex coloring.

- **Proper Vertex coloring:** - Associating all the vertex of a graph with colors such that no two adjacent vertices have the same color is called proper vertex coloring.

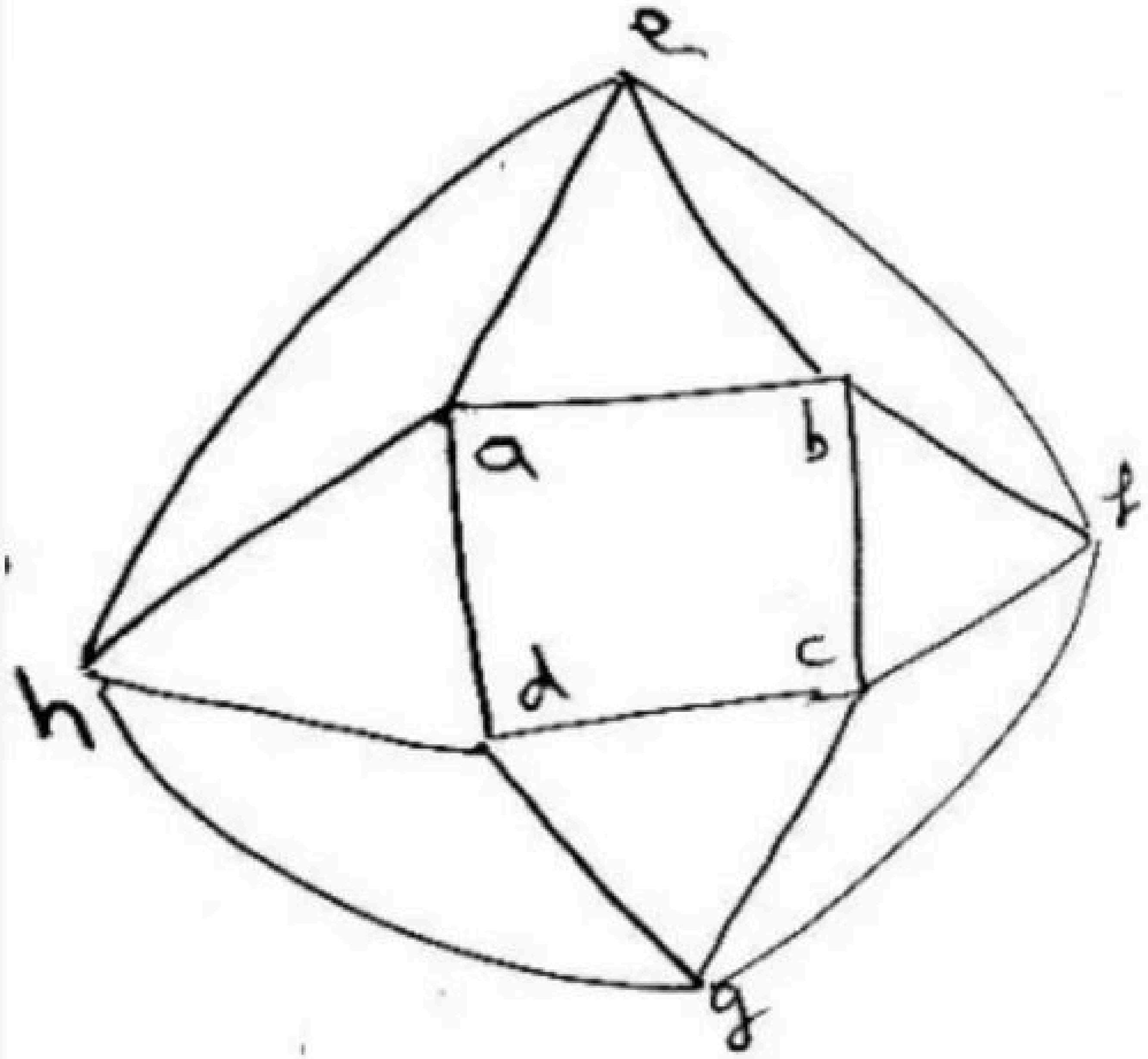


- **Chromatic number of the graph:** - Minimum number of colors required to do a proper vertex coloring is called the chromatic number of the graph, denoted by $\chi(G)$. the graph is called K-chromatic or K-colorable.

$\chi(G) = 3$



- Cost of finding chromatic number is an NPC problem and there exists no polynomial algorithm to do that. There exists some greedy approach which try to solve it in P time, but they do not guarantee optimal solution.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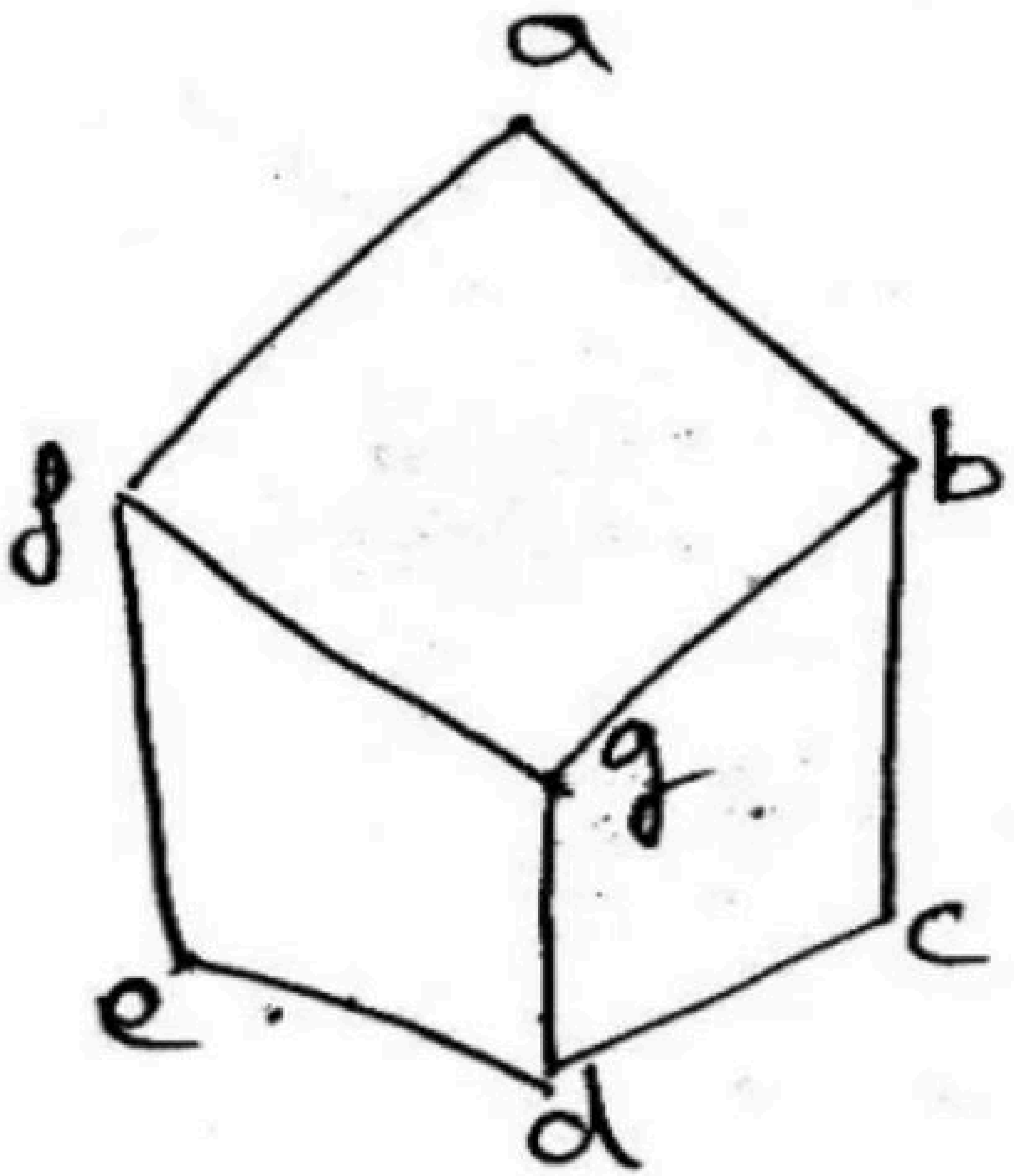
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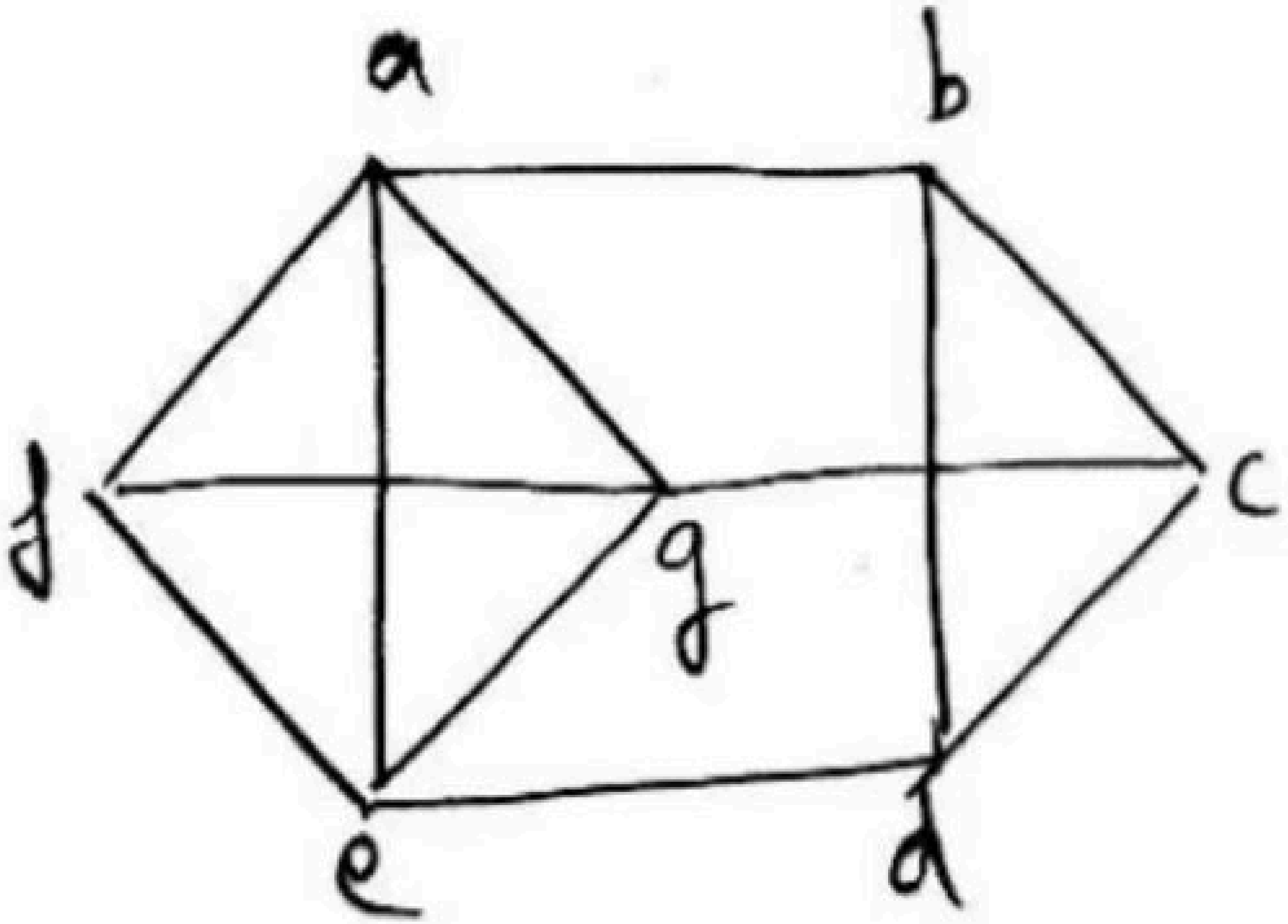
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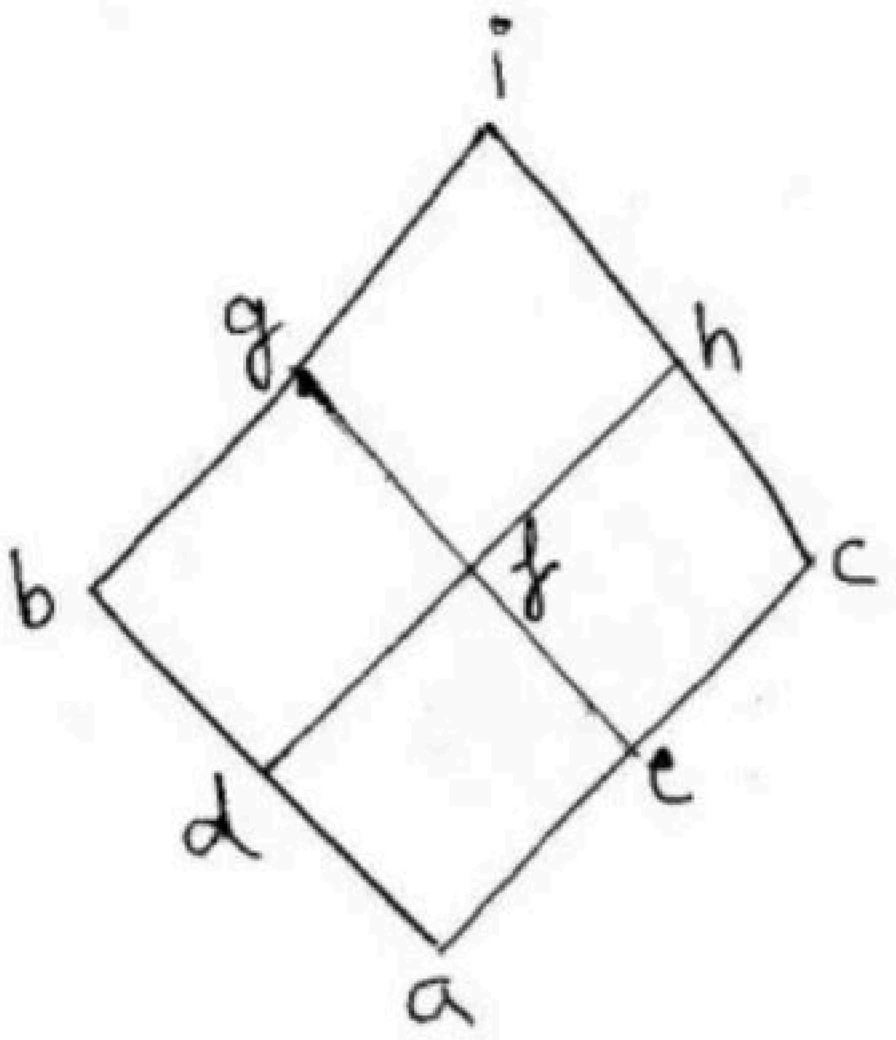
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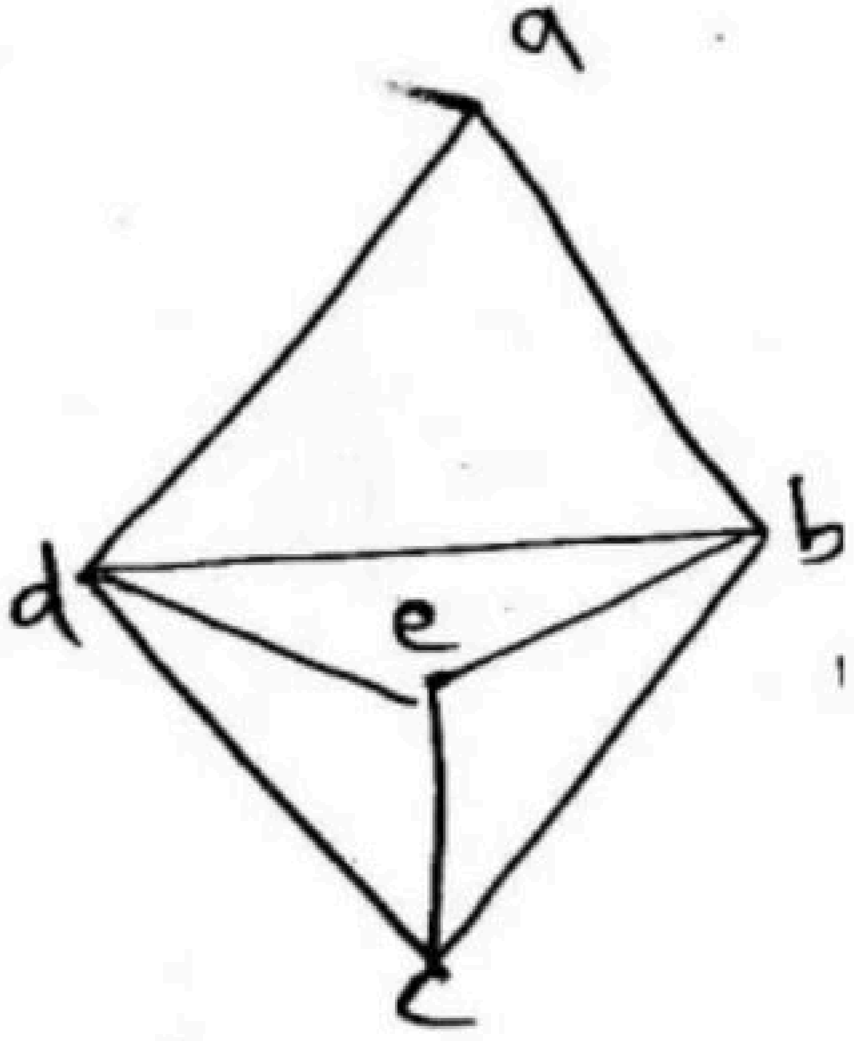
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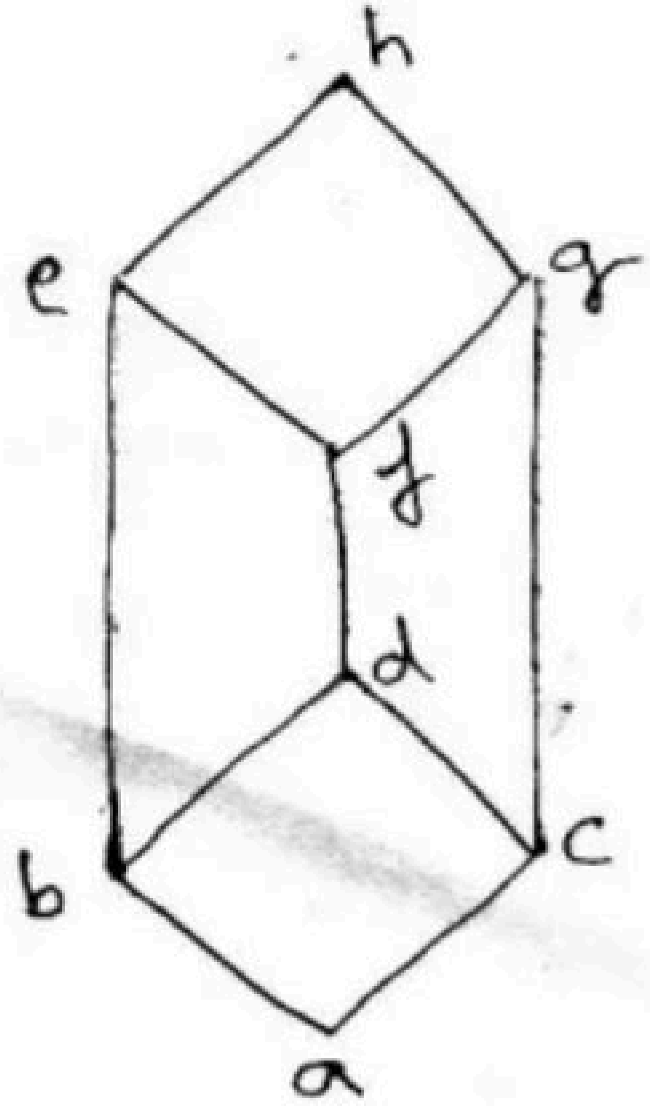
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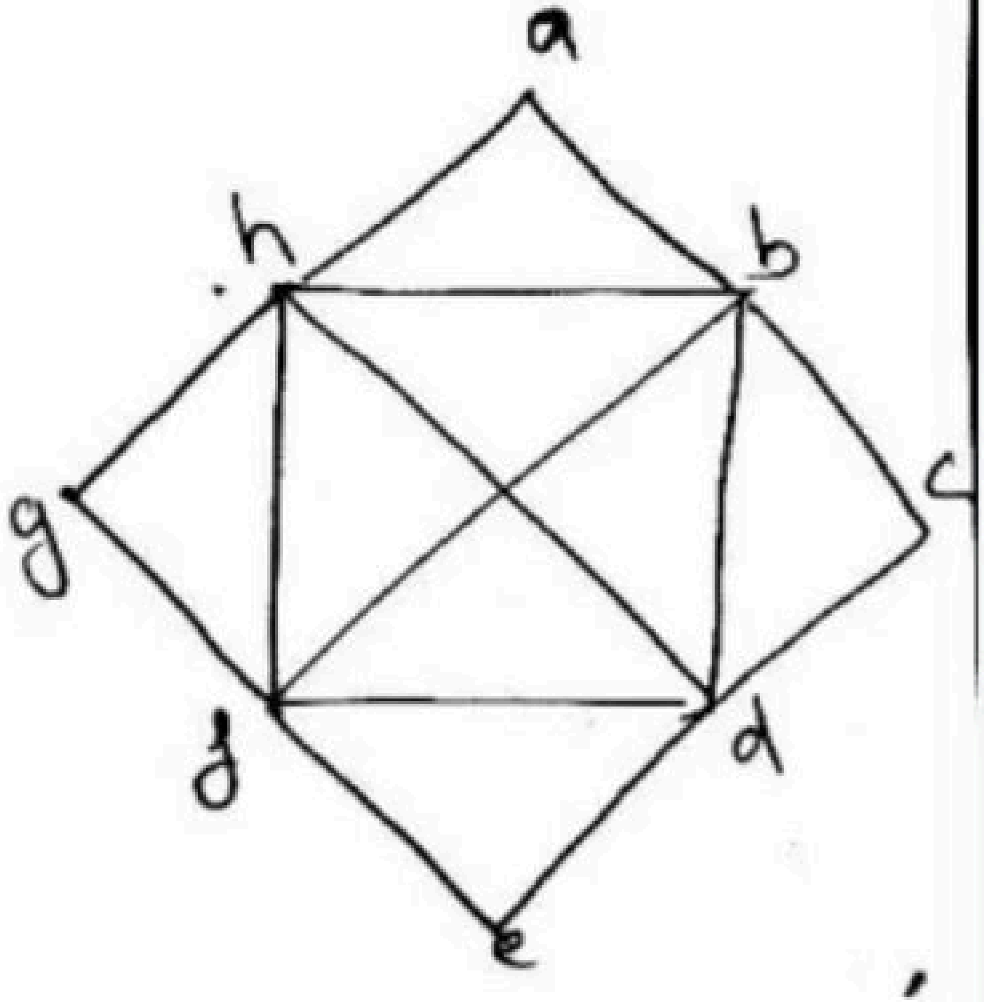
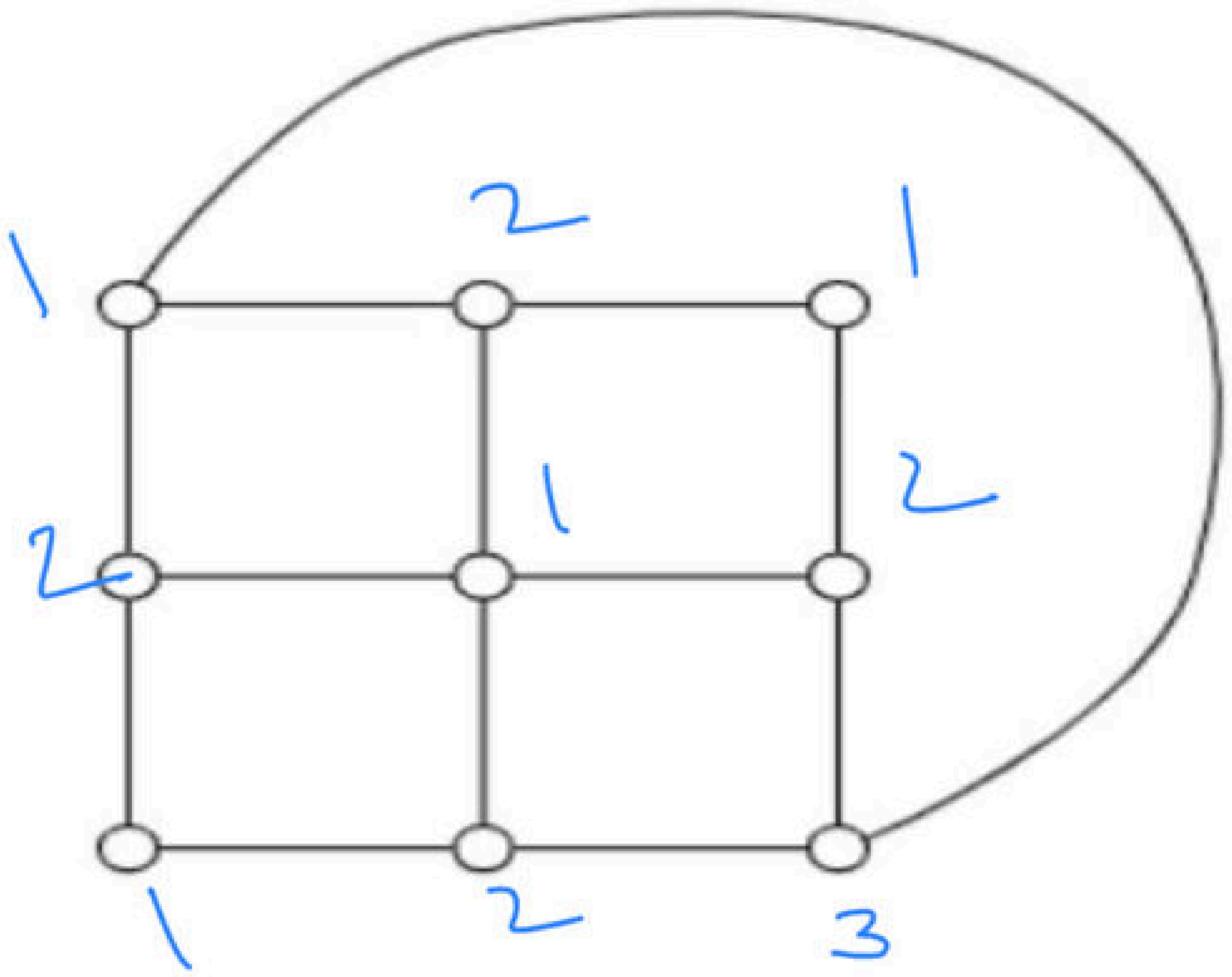

a
h
b
g
c
f
d
e

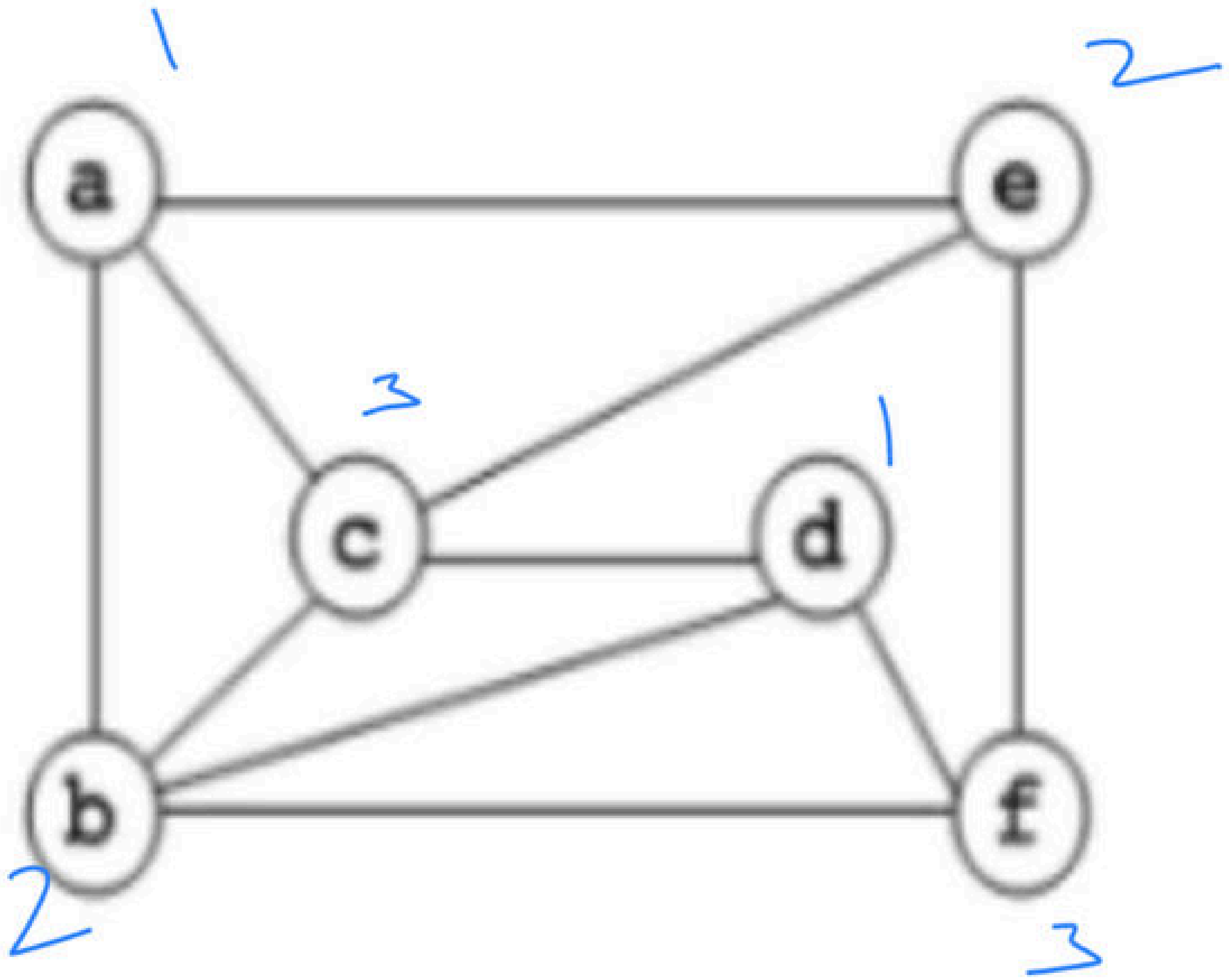
**Q** What is the chromatic number of the following graph? **(GATE-2008) (1 Marks)**




~~**(A)** 2~~ **(B)** ~~3~~ **(C)** 4 **(D)** 5

11 76 11 7

**Q** The chromatic number of the following graph is ________ **(GATE-2018) (2 Marks)**



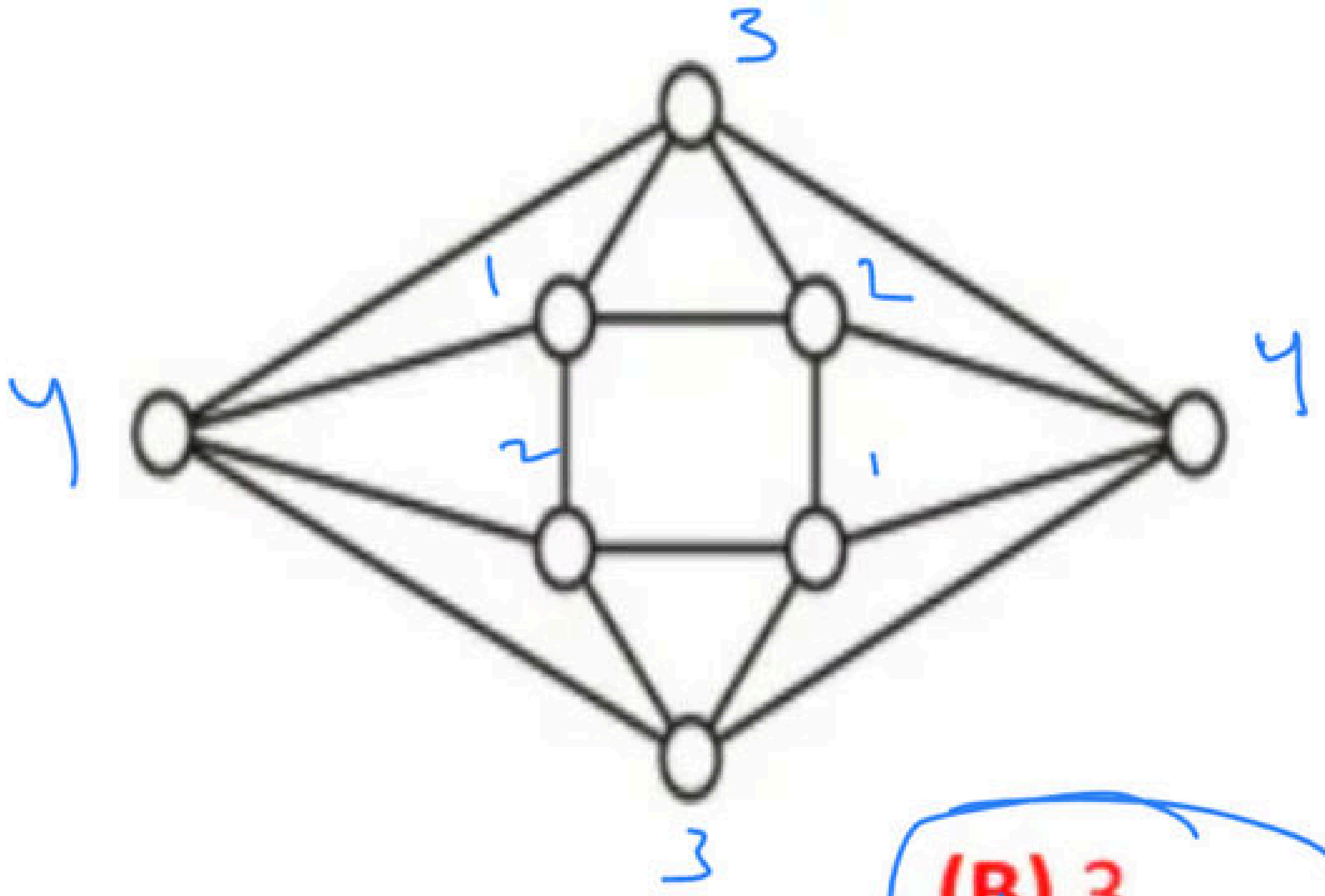

a) 3 ⟸ 70

b) 4 — 29

c) 5 — 2

d) 6 — 0

**Q** The minimum number of colors required to color the following graph, such that no two adjacent vertices are assigned the same color, is **(GATE-2004) (2 Marks)**




~~(A) 2~~

(B) 3

(C) 4

~~(D) 5~~

**Q** The minimum number of colors that is sufficient to vertex color any planar graph is _______________ **(GATE-2016) (1 Marks)**

**Q** What is the chromatic number of an n-vertex simple connected graph which does not contain any odd length cycle? Assume n >= 2. **(GATE-2009) (1 Marks)**

**(A)** 2 **(B)** 3 **(C)** n-1 **(D)** n

**Q** The minimum number of colors required to color the vertices of a cycle with η nodes in such a way that no two adjacent nodes have the same color is **(GATE-2002) (1 Marks)**

**(A)** 2 **(B)** 3 **(C)** 4 **(D)** $n - 2\lfloor n/2 \rfloor + 2$

**Q** The number of colors required to properly color the vertices of every planer graph is **(NET-JUNE-2012)**

| | | | |
|---|---|---|---|
| **a)** 2 | **b)** 3 | **c)** 4 | **d)** 5 |

**Q** In k-coloring of an undirected graph G = (V, E) is a function

c: V → {0, 1, ..., K-1} such that c(u) != c (v) for every edge (u,v) ∈ E.

Which of the following is not correct? **(NET-DEC-2018)**

**a)** G is bipartite

**b)** G is 2-colorable

**c)** G has cycles of odd length

**d)** G has no cycles of odd length

# Break

- Trivial graph is 1-chromatic
- A graph with 1 or more edge is at least 2-chromatic
- A complete graph $K_n$ is n-chromatic

- Tree is always 2-chromatic
- Bi-partite graph is 2-chromatic
- $C_n$ is 2-chromatic if n is even, $C_n$ is 3-chromatic if n is odd

- 5-color theorem-any planer graph is at most 5-chromatic
- 4-colour theorem/hypothesis- any planer graph is 4-chromatic
- If $\Delta(G)$ is the maximum degree of any vertex in a graph then, $\chi(G) <= 1 + \Delta(G)$