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Evaluating the Series or Parallel Structure Assumption for System Reliability

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ABSTRACT The structure of a system is important for specifying a mathematical form for calculating reliability from component level data. For some systems, full-system reliability data are expensive or difficult to obtain, and using component-level data to improve the estimation of system reliability can be highly beneficial. However, if the intended structure of the system during design does not match the actual implementation after production, using the component-level data to estimate system reliability can give misleading results. Hence, we propose a test for assessing the appropriateness of assuming both independence of component failure mechanisms and a series system structure for a system when data at both the system and components level are available. Estimates are given for component and system reliabilities under the assumption of a series system. The test can also be easily adapted to test the assumption of a parallel system. Examples for both series and parallel systems are illustrated.

KEYWORDS block diagram, hypothesis testing, system reliability, system structure

INTRODUCTION

Complex systems can be formed by combining components in a number of different ways. One common structure involves systems with no redundant components, where success in a full system test requires that all of the components work. This series structure and independent functioning of the components implies that reliability for the system, $p_s = P[\text{System works}]$, is the product of all of the component reliabilities,

$$p_s = \prod_{i=1}^{N} p_i \qquad [1]$$

where N is the number of components in the series system, and p_i is the component reliability for component i. This naturally implies that a failure of the system will occur if at least one of the components fails.

A second common structure is a parallel system, where the system will work if at least one of the components works. In this case, we can summarize the reliability of the system as 1 minus the probability that all of the components fail, or

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$$p_s = 1 - \prod_{i=1}^{N}(1 - p_i) \qquad [2]$$

These two structures are special cases of coherent systems. See Rausand and Hoyland (2004) and Saunders (2007) for more details on the definition and characteristics of system structures.

In many situations doing a full-system test is destructive or prohibitively expensive, and hence testing of individual components may provide alternate sources of data to help estimate reliability more precisely at a reduced cost. See Anderson-Cook et al. (2007, 2008) and Wilson et al. (2006) for more details on approaches to data combination for complex systems. We assume that for a large population of systems, data are available from full-system tests, as well as some component level tests of all of the individual components. The number of systems or components tested for the various types of data can be different, and the observed data for each type are assumed to give a "Pass" or "Fail" response. Let X_i be the number of successes observed in n_i tests of component i, for $i = 1, \ldots, N$, and X_s be the number of successes observed in n_s full-system tests. We assume that $n_s > 0$ and $n_i > 0$ for all i.

In this article, we consider a statistical test to evaluate the validity of the series or parallel structures for a system based on independent data sets from full-system and individual component tests.

While systems may have been designed to have either a series or parallel structure, in practice there may be dependencies between components or connectivity issues that cause the assumed relationships in Eqs. [1] or [2] to be no longer be correct. For example, consider a population of complex systems, such as car engines, kitchen appliances, or rockets. The engineers who design the system intend for it to have a series structure. However, there are a number of ways that this might not be the correct structure once the system is built and data are being collected. (1) During production connectivity issues between components might mean that it is possible for two components to each work separately but not work when combined together. (2) If the performance of one component depends on receiving the correct signal or power supply from another component, we may observe apparent failures of some components even when that component is actually working. (3) The component-level tests may be incorrectly calibrated relative to the demands on that component during the full-system test. For example, it might be that a component will pass the component-level test, but the system-level test could fail because the component cannot perform adequately.

Hence, before we confidently use component-level test data to estimate system reliability, it would be beneficial to assess the compatibility of the different data sources. We first consider the series structure, and later we will adapt the new methodology to the parallel system case. For a non-aging system with constant reliabilities for components and systems, the number of passes for the system or for each component are assumed to come from binomial distributions with $X_s \sim Bin(n_s, p_s)$ and $X_i \sim Bin(n_i, p_i)$, $i = 1, \ldots, N$.

The maximum likelihood estimates for each component's reliability estimated separately is $\hat{p}_i = \frac{X_i}{n_i}$. We can estimate the reliability of the system directly from the full-system test with $\hat{p}_s = \frac{X_s}{n_s}$. If the data from the individual components are independent and the series structure is appropriate, then the reliability of the system can be estimated from component data alone, and we obtain

$$\hat{p}_{s|comp} = \prod_{i=1}^{N}\hat{p}_i = \prod_{i=1}^{N}\frac{X_i}{n_i} \qquad [3]$$

Ideally, we would hope that $\hat{p}_s$ and $\hat{p}_{s|comp}$ would be quite close. However, the estimate in Eq. [3] is only appropriate if the following assumptions are correct:

i. The system truly is a series system, where Eq. [1] correctly summarizes the relationship between system and component reliabilities.
ii. The failure mechanisms affecting each component are independent.

Engineering knowledge of the system and its components should guide our choice of what form of equation is suitable for estimating system reliability as a function of the component reliabilities. There are several ways in which the above assumptions can be incorrect. (a) The structure of the model can be incorrectly specified in (1) if there a missing component from the system. In this case, it would be possible for all of the components to work, but if the missing component not included in the model does not work, then we could see a system failure. (b) If connectivity issues between components are possible, then it would be possible to observe a system failure when all of the components are working. (c) If the tests for the components require different functionality of the component than what is required in the system test, it is also possible for the relationship in Eq. [1] to be incorrect. This could lead to either of the following outcomes: all components work, but the system test fails, or at least one of the components fails, but the system test is successful.

If the second assumption, (ii), is violated with dependent failure mechanisms between components or with correlated data, then the overall estimate of system reliability in Eq. [3] would not be appropriate. For example, consider that a single failure mechanism caused failures for several of the components simultaneously. In this case, a single system failure from this mechanism could result in lower reliabilities for multiple components. This positive correlation between component data would artificially lower the system reliability as calculated by Eq. [3].

Consider an example of a system with five components. Data are collected for each of the components and for the system, yielding the results shown in Table 1. We assume here that the component and system data are obtained from different systems, since the assumption of independence of the two sources of data is important for the assessment of compatibility of the information.

Because the full-system tests are prohibitively expensive, there is only a small sample of data available. Based on engineering knowledge of the system, our experts believe that it should be characterized as a series system with the failure mechanisms being treated independently. Figure 1 shows a reliability block diagram for how this system might commonly be characterized. We can calculate the system reliability from the full-system tests alone or as a product of the individual component reliabilities. From the small sample of direct data, our estimate of system reliability is $17/25 = 0.680$. From the individual component reliabilities assuming a series system, we estimate it as $0.925*0.960*0.950*0.980*0.988 = 0.816$. These values seem quite dissimilar, but it is difficult to know what we should consider them sufficiently different to no longer believe our assumptions are valid.

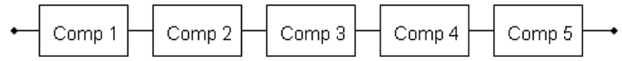


FIGURE 1 **Reliability block diagram for the five-components series system.**

TABLE 1 **Data from Five-Component System**

Data type	X_i or X_s	n_i or n_s	$\hat{p}_i$ or $\hat{p}_s$
System	17	25	0.680
Component 1	111	120	0.925
Component 2	96	100	0.960
Component 3	76	80	0.950
Component 4	98	100	0.980
Component 5	79	80	0.988

When data are available for both full-system tests and for all individual components, we can formally test the appropriateness of assumptions (i) and (ii) simultaneously. This can serve to validate our understanding of the system structure and its data. However, because of the interrelationship between the two assumptions, we will only be able to test whether our data are consistent with Eq. [1] versus the alternative that at least one of assumptions (i) or (ii) are inconsistent with our data.

In the following section, we describe how estimates for the component and system reliabilities can be obtained from a single analysis of the data if assumptions (i) and (ii) are appropriate. The next describes a test to assess the assumptions with a formal hypothesis test and this is followed by an illustration of the method for the example shown above. The final section shows results of a size and power study.

ESTIMATES FOR COMPONENT AND SYSTEM RELIABILITIES

When the assumption of a series system with independently failure mechanisms is appropriate, a single analysis using all of the data is possible to obtain estimates for all of the model parameters $p_1, \ldots, p_N$. The system reliability can then be estimated as their product. This analysis leverages information about component reliabilities from the both the individual tests, with X_i successes in n_i trials, as well as from the overall system tests with X_s successes in n_s trials. Under the assumption of Eq. [1] correctly specifying the relationship between system and components, a success in the system-level test implies that all components worked (hence each component i), while a failure implies that at least one of the components failed.

Hence, the likelihood function using the data $(X_1, X_2, \ldots, X_N, X_s)$ from $(n_1, n_2, \ldots, n_N, n_s)$ trials respectively has the form:

$$L(p_1, \ldots, p_N) = \left(\prod_{i=1}^{N} p_i\right)^{X_s} \left(1 - \prod_{i=1}^{N} p_i\right)^{(n_s - X_s)} \times \prod_{i=1}^{N} \left(p_i^{X_i}(1-p_i)^{(n_i - X_i)}\right),$$

where $1 - \prod(p_i)$ represents the probability of at least one of the components failing. The first two terms come from the full-system tests, and the remaining product comes from the component-level test. The log likelihood has the form

$$l(p_1, \ldots, p_N) = X_s \sum_{i=1}^{N} \ln p_i + (n_s - X_s) \ln\left(1 - \prod_{i=1}^{N} p_i\right) + \sum_{i=1}^{N} X_i \ln p_i + \sum_{i=1}^{N} (n_i - X_i) \ln(1 - p_i) \quad [4]$$

To obtain the maximum likelihood estimates (mle) for $p_1, \ldots, p_N$, we need to solve a nonlinear system of N equations in N unknowns obtained from taking partial derivatives of (4) with respect to each of the p_is and setting each equation equal to zero. This system of equations has the form:

$$L\frac{\partial l(p_1, \ldots, p_N)}{\partial p_i} = \frac{X_s + X_i}{p_i} - \frac{n_i - X_i}{1 - p_i} - \frac{(n_s - X_s)\left(\prod_{j \neq i} p_j\right)}{1 - p_s} \overset{set}{=} 0 \quad i = 1, \ldots, N \quad [5]$$

and p_s defined by Eq. [1]. The maximum likelihood estimates are denoted as $\tilde{p}_1, \ldots, \tilde{p}_N$. There is no known closed-form solution to this system of nonlinear equations. Alternately, we can obtain the mle's directly by maximizing the likelihood or log likelihood. One option for finding this solution is with a grid search of the component reliability space in the neighborhood of the $\hat{p}_i$s. Once the mle's have been found for the individual components, we can calculate the system reliability as

$$\tilde{p}_s = \prod_{i=1}^{N} \tilde{p}_i \quad [6]$$

This estimate will be a compromise between the two system reliability estimates, $\hat{p}_s$ and $\hat{p}_{s|comp}$. While computationally this becomes more demanding as the size of the system grows, the same procedure will continue to work for finding the maximum likelihood estimates.

HYPOTHESIS TEST FOR VALIDITY OF SERIES OF INDEPENDENT COMPONENTS

In this section we consider a formal test for assessing our assumptions of the structure of the system being a series and the independence of the failure mechanisms for our components. Three pairs of null and alternative hypotheses can be considered:

$$A: \quad H_0: p_s = \prod_{i=1}^{N} p_i \quad \text{vs} \quad H_A: p_s \neq \prod_{i=1}^{N} p_i$$

$$B: \quad H_0: p_s \leq \prod_{i=1}^{N} p_i \quad \text{vs} \quad H_A: p_s > \prod_{i=1}^{N} p_i$$

$$C: \quad H_0: p_s \geq \prod_{i=1}^{N} p_i \quad \text{vs} \quad H_A: p_s < \prod_{i=1}^{N} p_i$$

In general, the two-sided hypothesis test of A would be appropriate if there was little knowledge of how the system might deviate from the assumptions, while B and C may be sensible if there are reasons to expect particular types of deviations. For example, if it was suspected that there might be a missing failure mechanism in the system that has not been modeled, then hypothesis C might be considered.

To test the hypotheses, we compare the estimates of the system reliability from system data alone versus from the product of the component data, respectively. Hence, our test statistic is

$$t_0 = \frac{\hat{p}_s}{\prod_{i=1}^{N} \hat{p}_i} \quad [7]$$

where under the null hypothesis, we would expect values close to 1. Since this statistic does not have a known distribution, we compare it to values obtained from data generated using estimates of the system and component reliabilities under the null hypothesis. This empirical approach is straightforward to implement. We assume that the null hypothesis is true and that the mle's $\tilde{p}_1, \ldots, \tilde{p}_N$ and $\tilde{p}_s$ are appropriate estimates of the true but unknown component and system reliability estimates, $p_1, \ldots, p_N$ and p_s. We generate a large number of sets of data, $(X^*_{1k}, \ldots, X^*_{Nk}, X^*_{sk})$ for $k = 1, \ldots, N_{sim}$ with system data from a binomial distribution, $X^*_s \sim Bin(n_s, \tilde{p}_s)$, and data for each component from $X^*_i \sim Bin(n_i, \tilde{p}_i)$. The test statistic is calculated for each generated data set $(X^*_{1k}, \ldots, X^*_{Nk}, X^*_{sk})$ as $t^*_k = \hat{p}^*_{sk} / \prod_{i=1}^N \hat{p}^*_{ik}$ where $\hat{p}^*_{sk} = X^*_{sk}/n_s$ and $\hat{p}^*_{ik} = X^*_{ik}/n_i$. The population of t^*_ks provides an estimate of the distribution of what values of the test statistics we might expect to observe under the null hypothesis.

To obtain an approximate p-value for the test, we compare the proportion of t^*_k values at least as extreme as the observed t_0. For example, for hypotheses B and C, the approximate p-value would be $p_B = \#(t^*_k \geq t_0)/N_{sim}$ and $p_C = \#(t^*_k \leq t_0)/N_{sim}$, respectively. For the two-sided hypothesis A, the approximate p-value would be $p_A = 2 * \min(\#(t^*_k \geq t_0), \#(t^*_k \leq t_0))/N_{sim}$.

EXAMPLE OF A SERIES SYSTEM

For the example given in the introduction and assuming that the system does satisfy the assumptions, we estimate the system and component reliabilities using our maximum likelihood estimates. Results are shown in Table 2. The relatively low reliability estimates from the full-system data alone reduce our estimates of individual component reliabilities, while the estimate of the system reliability has been adjusted higher based on the individual component data. Note that even though the sample size for the system data is small, it still makes a substantial contribution to changing our estimates for all of the component reliabilities.

TABLE 2 **Estimated Reliabilities Using All Available Data for Five-Component System from the Maximum Likelihood Estimates**

Data type	$\tilde{p}_i$ or $\tilde{p}_s$
System	0.787
Component 1	0.916
Component 2	0.954
Component 3	0.941
Component 4	0.977
Component 5	0.980

Using the methodology of the previous section, we can test the hypothesis of whether Eq. [1] appears to be reasonable given the data that we have observed at both the system and component levels. Recall that the estimate of system reliability based on the full-system alone is 0.680, while from the product of the component reliabilities we estimate it as 0.816. Hence, the test statistic value is 0.680/0.816 = 0.833, implying that the estimate of system reliability using full-system test is more pessimistic than that obtained from the product of the component reliabilities. Using the parameter estimates shown in the second column of Table 2, Figure 2 shows a smoothed density of the t^*_k values using the "density" function in R (R Core Development Team, 2004), with a fast Fourier transform. The vertical line indicates the observed value from the original data. For the two-sided hypothesis test, the empirical p-value is 0.154, which would lead us to conclude that there is not sufficient evidence to reject the assumption of a series system with independent component failure mechanisms as incorrect. If a priori we had been interested in testing hypothesis

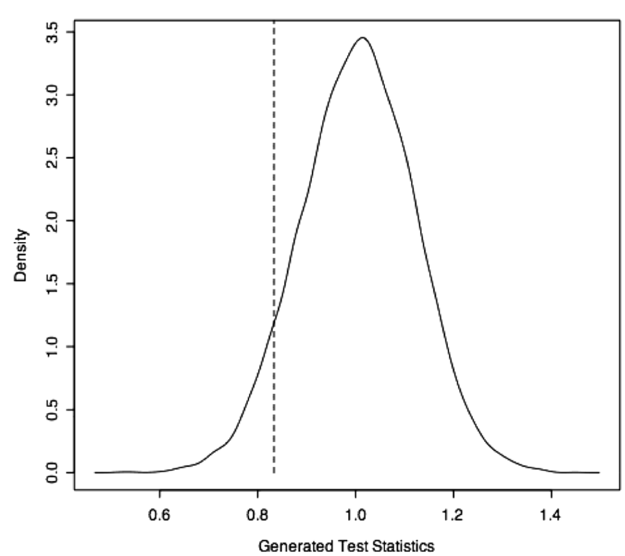


FIGURE 2 **Smoothed density plots of t^*_k values for five-components system. The observed value t_0 from the data is 0.833.**

B, the empirical p-value for this test would be 0.079, also providing insufficient evidence of a departure from our assumptions.

SIZE AND POWER OF SERIES SYSTEM TEST

To assess the size (probability of correctly accepting the null hypothesis when the assumptions are satisfied) and power (probability of correctly rejecting the null hypothesis when the assumptions are not satisfied) of this approach, a number of simulations were performed. For each case, component data $(X_1, X_2, \ldots, X_N)$ were generated that combined to give variety of system reliability values. Figure 3 shows the power curve (probability of rejecting the null hypothesis) for the two-sided hypothesis test A for a five-component system with different sample sizes (30, 50, 100) for each data type assuming that the system reliability is 0.7. A sample size of 50 means that we have 50 observations for each type of data (e.g., 50 full-system tests, 50 component 1 tests, ..., 50 component 5 tests). The chosen threshold to reject the null hypothesis was selected to be 0.05. For the case when the null hypothesis is true, each component reliability was selected as

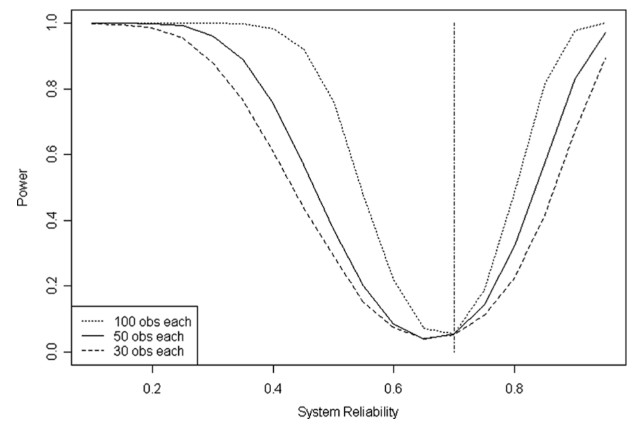
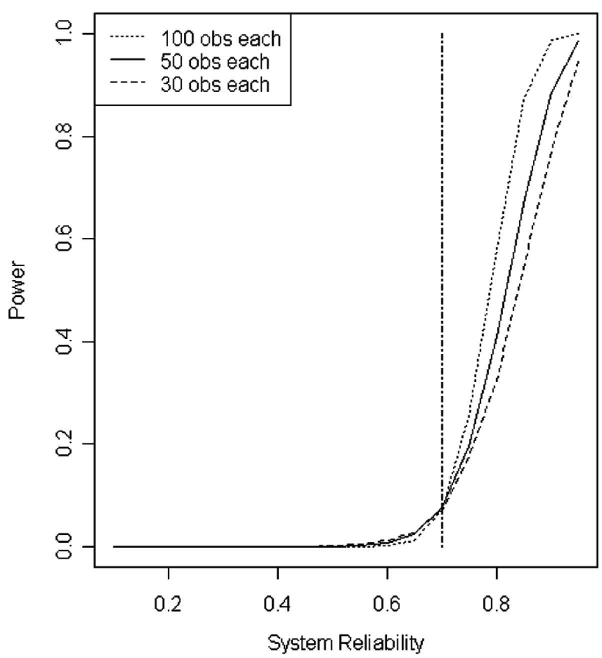


FIGURE 3 **Power curve (probability of rejecting the two-sided hypothesis) for the two-sided test of a five-component system with component reliabilities that multiply to give system reliability of 0.7. The sample size is the number of observations for each type of data.**

$0.7^{1/5} = 0.931$. For all other cases, the component reliabilities were selected as $p_s^{1/5}$.

As expected, as the sample size for each type of data increases, the power to reject the null hypothesis improves. Note that in this case (and in many others considered), for small sample sizes if the true system reliability is only slightly less than the product of the component reliabilities, the power of the test is slightly lower than 0.05. This effect



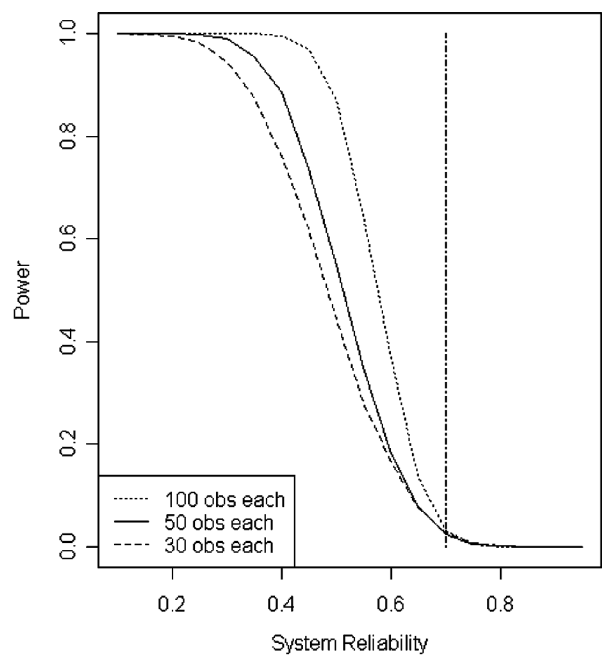


FIGURE 4 **Power curves for the one-sided test of a five-component system with component reliabilities that multiply to give system reliability of 0.7. (a) For testing alternative hypothesis $p_s > \prod_{i=1}^{N} p_i$, and (b) for testing alternative hypothesis $p_s < \prod_{i=1}^{N} p_i$.**

disappears as sample sizes or the size of the discrepancy increase.

The pattern of results was similar for series systems with different numbers of components (2, 5, and 10) and different combinations of component reliabilities $\left(\text{e.g., } (p_s^{.1}, p_s^{.15}, p_s^{.2}, p_s^{.25}, p_s^{.3})\right)$.

The power curves for the one-sided tests for the same scenario as in Figure 3 are shown for hypothesis *B* and *C* in Figures 4a and 4b, respectively. As expected, with the one-sided test, we are able to discern smaller differences between the system and the product of the component reliabilities with these test compared to the two-sided tests.

EXAMPLE OF A PARALLEL SYSTEM

The methodology presented in the previous sections provide a method for estimating the reliability of a series system from multilevel data. Suppose instead that we had a three-component system where the success of the full system depended only on a single component of the three working correctly. The appropriateness of the parallel structure for this system, shown in Figure 5, can also be evaluated with a slight adaptation of the previous methodology.

By rearranging Eq. [2] and defining "unreliability" as $u_s = 1 - p_s = P[\text{System fails}]$, $u_s = \prod_{I=1}^{n} u_i$ where $u_i = 1 - p_i = P[\text{Component } i \text{ fails}]$. Hence, the maximum likelihood estimates shown previously can now be used to estimate the reliability (from the unreliability) of the individual components and the system. Similarly, the test statistic for evaluating the assumptions of the parallel structure and independence of the failure mechanisms for the components is adapted from Eq. [7] and has the form

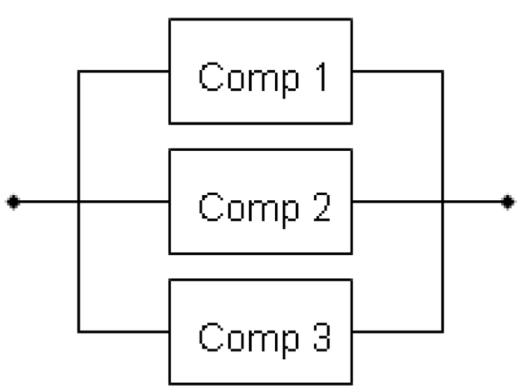


FIGURE 5 **Reliability block diagram for a three-component parallel system.**

TABLE 3 **Data From Three-Component Parallel System**

Data type	X_i or X_s	n_i or n_s	$\hat{p}_i$ or $\hat{p}_s$	$\hat{u}_i$ or $\hat{u}_s$	$\tilde{p}_i$ or $\tilde{p}_s$
System	33	40	0.825	0.175	0.923
Component 1	136	200	0.680	0.320	0.653
Component 2	54	80	0.675	0.325	0.619
Component 3	31	60	0.517	0.483	0.417

$$t_0 = \frac{\hat{u}_s}{\prod_{i=1}^{N} \hat{u}_i} = \frac{1 - \hat{p}_s}{\prod_{i=1}^{N}(1 - \hat{p}_i)} \qquad [8]$$

where under the null hypothesis, we would again expect values close to 1.

For a sample system, the second and third columns of Table 3 show the data observed for the components and system. For this system, we would estimate the reliability of the system based on the system tests alone as 0.825, while from the component data alone, we would estimate it as 1 - $(0.320*0.325*0.483) = 0.950$. The last column of Table 3 shows the maximum likelihood estimates of the various reliabilities by solving adaptations of Eq. [5]. Notice how the smaller sample sizes for components 2 and 3 lead to larger adjustments to their reliabilities for the estimates closed form estimates compared to component 1.

For the formal test of the assumptions, the test statistic from Eq. [8] would be $(0.175)/(1 - 0.950) = 3.48$. For the two-sided test where no a priori knowledge of which direction of difference between the two estimates is incorporated, the estimated empirical p-value is 0.0052. If subject matter expertise had been available to consider the appropriate one-sided test, the associated p-value would have been 0.0026. In either case, the assumption of a parallel system with independent failure mechanisms does not seem appropriate for this system given the data that we have observed.

CONCLUSIONS

Engineering understanding of a complex system can suggest that we can estimate the system reliability as a function of the component reliabilities. In the cases where all components must work for a successful system test, a series system with independent failure mechanisms is suggested. Implicit in assuming that the system reliability is the product of the component reliabilities are two assumptions that should be evaluated. If there are missing failure mechanisms, if the individual component tests are not appropriately calibrated for what is required of the component during the system test, or if one or more of the component failure mechanisms are dependent, our assumed model can be incorrect. If this is the case, the model assumptions need to be reevaluated and a new system structure developed that is consistent with the observed data. Alternately, a discrepancy measure, perhaps of the form $p_s = \delta \prod_{i=1}^{N} p_i$, can be introduced into the model and estimated. This will allow incorporation of both system and component data while more accurately reflecting the incompatibilities.

The maximum likelihood estimates for the component and system reliabilities based on the combined sets of data can be calculated numerically. A strategy for testing the null hypothesis of series system with independent failure mechanisms is provided that uses these estimates to generate data under the assumed null hypothesis. The empirical p-value for the hypothesis test can be estimated by examining the proportion of generated test statistics that are at least as extreme as our observed data.

Simulations for summarizing the power curves for the testing of the assumptions show good size and power for a variety of situations. With a simple adaptation, the methods outlined can also be used to assess the appropriateness of a parallel structure for a system. R code for estimating the maximum likelihood estimates and performing the test of hypothesis is available from the author upon request.

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