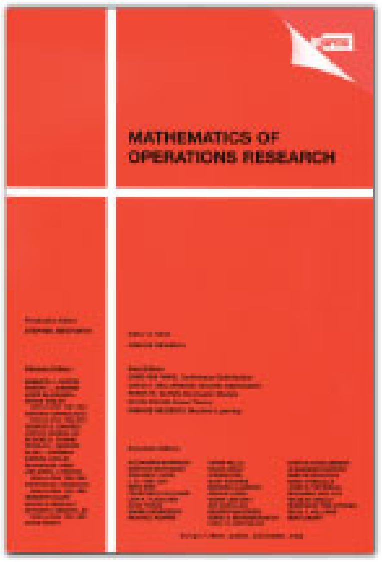

Mathematics of Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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MATHEMATICS OF OPERATIONS RESEARCH
Vol. 21, No. 2, May 1996
Printed in U.S.A.

THE DEGREE OF AN EXACT ORDER MATRIX

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The classes of exact order k matrices for any positive integer k, were defined and studied by Mohan, Parthasarathy and Sridhar (1992). Here, we prove results on the linear complementarity problem LCP(q, M), for M belonging to the class of exact order $k, k \geq 3$, using the concepts of degree theory. Our main result in this paper consists in proving that a matrix $M \in R^{n \times n}$ of exact order k, for any positive integer $n \geq k + 3$, belongs to the class Q if and only if the degree of M is either $+1$ or -1. Also, a complete characterization of exact order 2 matrices is presented, in terms of their inverse structure.

1. Introduction. Given an n-vector q and a matrix $M \in R^{n \times n}$, the linear complementarity problem, denoted by LCP(q, M), is that of finding vectors $w \in R^n$, $z \in R^n$ such that

$$(1) \qquad w - Mz = q,$$

$$w^t z = 0, \qquad w \geq 0, z \geq 0.$$

A pair of vectors (w, z), that satisfies (1) is said to be a solution for the LCP(q, M).

The linear complementarity problem arises in many diversified fields. For more details on linear complementarity, we refer to Cottle, Pang and Stone (1992) and Murty (1988). If for all $q \in R^n$ the LCP(q, M) has a solution, then the matrix M is said to be a Q-matrix. An efficient condition which characterizes the class of Q-matrices is not yet known. However, several sufficient conditions are identified under which a matrix M belongs to the class Q. See Cottle, Pang and Stone (1992) and Murty (1988).

The problem (1) had been considered for some special classes of matrices that are defined based on the signs of the principal minors of M. Some of the well-known classes in the literature are P-matrices, N-matrices, almost P and almost N-matrices. For some equivalent characterizations on these classes, we refer to the books by Murty (1972) and Cottle, Pang and Stone (1992). Olech, Parthasarathy and Ravindran (1989, 1991) studied the classes of almost N and almost P-matrices. Among the class of nondegenerate matrices, i.e., matrices whose principal minors are nonzero, only the following subclasses are known to be Q-matrices: P, N with a positive entry, almost N and almost P-matrices with positive minimax values. Recently, Mohan, Parthasarathy and Sridhar (1992) introduced classes of matrices known as exact order matrices, which in a way extend these classes. They established results on the Q-nature of the class of exact order 2, and posed the problem of proving similar results for the class of exact order $k, k \geq 3$. This paper attempts to extend such results for the class of exact order $k, k \geq 3$ using some recent concepts.

One of the approaches of studying the linear complementarity problem is using degree theory, by a reformulation of the LCP(q, M) as a piecewise linear map. There

Received October 15, 1992; revised October 5, 1994.
AMS 1991 subject classification. Primary: 90C33.
OR/MS Index 1978 subject classification. Primary: 622 Programming/Complementarity.
Key words. Linear complementarity, exact order matrix, degree theory.

0364-765X/96/2102/0427/$01.25

are many interesting articles that describe this method, to quote a few, by Ha (1987), Gowda (1993), Gowda and Pang (1991), Howe and Stone (1983), Robinson (1979) and Morris (1990). In this paper, the degree of a matrix M is defined using the Stewart's extension formula (Gowda 1991 and Stewart 1993).

It is well known in the literature of linear complementarity, that if the degree of a matrix is nonzero, then M is a Q-matrix. But there are Q-matrices whose degree is zero. See Chapter 6 of Cottle, Pang and Stone (1992) for examples. Our motivation behind this paper is to answer the following question: Can we identify the classes of matrices for which the degree being nonzero is both necessary and sufficient for them to belong to the class Q? It is proved here that the classes of exact order matrices fall in this category. Besides this, our main result in this paper consists in proving that a matrix $M \in R^{n \times n}$ of exact order k, for any positive integer k, $n \geq k + 3$, belongs to the class Q if and only if the modulus of the degree of M is 1.

The next section defines some terminology and presents the basic results that are required. A complete characterization of exact order 2 matrices is given in §3, in terms of their inverse structure. In §4, we analyse the sign of the minimax value of an exact order k matrix. It is one of the necessary tools in knowing whether a matrix M is a Q-matrix. Section 5 describes the intricacy of the LCP(q, M) for an M in exact order $k, k \geq 3$, using degree theory. We conclude this paper with some open problems.

2. Preliminaries. Here, we introduce the terminology and the essential results needed in this paper.

Let $M \in R^{n \times n}$ be nonsingular. For subsets $J, K \subseteq \{1, \dots, n\}$, we denote by M_{JK} and M^{JK}, the submatrices of M and M^{-1} respectively, with rows and columns corresponding to the index sets J and K. For $J = \{1, \dots, n\}$, M_{JK} is written for simplicity as $M_{\cdot K}$. The matrix M_{JJ} for $J \subseteq \{1, \dots, n\}$ denotes a principal submatrix of M. When $|J| = k$, M_{JJ} is called the principal submatrix of order k. Then, the determinant of M_{JJ} denoted by $\det M_{JJ}$, is called a principal minor of order k. The (i, j)th entry of M and M^{-1} are denoted by m_{ij} and m^{ij} respectively. For any $J \subseteq \{1, \dots, n\}$, $\bar{J}$ denotes the set $\{1, 2, \dots, n\} \setminus J$.

By $v(M)$, we denote the minimax value of the two-person zero-sum game, with M as the pay-off matrix. When $M \in Q$, it is clear that $v(M)$ is positive. In linear complementarity, matrices for which the minimax value is positive (nonnegative) are known as $S(S_0)$-matrices. However, we will not make use of this notation here, because we require more results from game theory, based on the minimax value of the payoff matrix M, which we state below.

A mixed strategy x of a player is said to be *completely mixed*, if no coordinate of x is zero. If the optimal strategies in a game with a payoff M are completely mixed, then we call such a game a *completely mixed game*. We make use of the following results on completely mixed games, the first of which is due to Kaplansky (1945).

THEOREM 1. *Let M denote the payoff matrix of order m by n, of a two person zero-sum game.*

(1) *If player 1 has a completely mixed optimal strategy $p = (p_1, p_2, \dots, p_m)$, then any optimal strategy $q = (q_1, q_2, \dots, q_n)$ for player 2 satisfies $\Sigma_j m_{ij} q_j = v$, $\forall i = 1, \dots, m$.*

(2) *If $m = n$, and the game is not completely mixed, then both the players have optimal strategies that are not completely mixed.*

(3) *A game with value zero is completely mixed if and only if*

(*a*) *its matrix is square, i.e., $m = n$ and has rank $(n - 1)$ and*

(*b*) *all the cofactors M_{ij} are different from zero and have the same sign (cofactor M_{ij} denotes $(-1)^{i+j}$ times the determinant of the principal submatrix of M got by deleting the ith row and the jth column).*

(4) *The value v of a completely mixed game is given by*

$$v = \frac{|M|}{\Sigma\Sigma M_{ij}}$$

where $|M|$ is the determinant of M.

(5) *Let $V = ((V_{ij}))$ denote the matrix of order $m \times n$ where V_{ij} is the value of a game whose pay-off matrix is obtained from M by omitting its ith row and jth column. Then the game with pay-off matrix M is not completely mixed if and only if the game with pay-off matrix V has a pure saddle point, that is, if and only if there exists a pair (i_0, j_0) such that*

$$v_{i_0 j} \le v_{i_0 j_0} \le v_{i j_0}, \qquad \forall i, j,$$

and $v_{i_0 j_0} = v(M)$.

To state the next theorem on completely mixed games, we need the following notation. Let $M \in R^{n \times n}$. By B_i, we denote the principal submatrix of M got by deleting the ith row and the ith column. $v(B_i)$ stands for the value of the game with B_i as the payoff matrix.

THEOREM 2 (MOHAN, PARTHASARATHY AND SRIDHAR 1992). *Let M of order n by n be the payoff matrix of a zero-sum two-person game. Let $v(B_i) < 0$, for all $i = 1, \dots, n$. Then the following statements are equivalent*:

(i) $v(M) > 0$.

(ii) $M \in Q$.

(iii) *M is nonsingular and $M^{-1} > 0$.*

For any matrix M satisfying the hypotheses of the above theorem, if any of the conditions from (i) to (iii) hold, the game with the payoff M will be completely mixed. This is easy to verify using Statement (5) of Theorem 1.

Now, we introduce some terminologies relating to linear complementarity. For any set $J \subseteq \{1, \dots, n\}$, C_J denotes a *complementary matrix* of $[I : -M]$ which is defined as (if necessary after a principal rearrangement),

$$C_J = \begin{bmatrix} -M_{JJ} & 0 \\ -M_{\bar{J}J} & I_{\bar{J}\bar{J}} \end{bmatrix}. \tag{2}$$

If $J = \phi$, then $C_J = I$. The nonnegative cone generated by a complementary matrix C_J of $[I : -M]$ denoted by $\text{pos}(C_J)$, is defined as

$$\text{pos}(C_J) = \{y : y = C_J x,\ x \ge 0\}.$$

It is known as a *complementary cone* of $[I: -M]$. For any $q \in R^n$, LCP(q, M) has a solution if and only if there exists an index set $J \subseteq \{1, \dots, n\}$, such that $q \in \text{pos}(C_J)$. We refer to Cottle, Pang and Stone (1992) and Murty (1988) for more details on complementary cones. For a principal submatrix M_{JJ} of M, we say that M_{JJ} *gives rise to a solution* for the LCP(q, M), whenever $q \in \text{pos}\, C_J$.

Corresponding to a solution (w, z) of (1), let us define the following index sets:

$$J = \{i\colon z_i > 0\}; \qquad K = \{i\colon w_i > 0\}; \qquad L = \{i\colon z_i = w_i = 0\}. \tag{3}$$

Let us denote the Schur complement of M_{JJ} in

$$\begin{bmatrix} M_{JJ} & M_{JL} \\ M_{LJ} & M_{LL} \end{bmatrix}$$

by M^S_{JJ}, which is defined as $M^S_{JJ} = M_{LL} - M_{LJ}M^{-1}_{JJ}M_{JL}$. When J is empty, $M^S_{JJ} = M_{LL}$ and when L is empty, M^S_{JJ} is taken as the identity matrix of appropriate order.

A solution (w, z) for the LCP(q, M) is said to be *nondegenerate*, if it has exactly n coordinates positive. The solution (w, z) is called *semi-nondegenerate*, if the corresponding principal submatrix M_{JJ} of M where J is defined as in (3), is nonsingular. A vector $q \in R^n$ is said to be *semi-nondegenerate* (*nondegenerate*) with respect to M, if every solution of LCP(q, M) is semi-nondegenerate (nondegenerate). From Murty (1972), it follows that for a matrix whose principal minors are nonzero (called a *nondegenerate matrix*), every $q \in R^n$ is semi-nondegenerate with respect to M and that the set of all solutions of LCP(q, M) is finite.

We define the degree of M for nondegenerate matrices. Let $M \in R^{n\times n}$ be a nondegenerate matrix. For a vector $q \in R^n$ nondegenerate with respect to M, define the number

$$\deg M = \sum \operatorname{sgn} \det M_{JJ}$$

where the summation is taken over all the index sets $J \subseteq \{1, \dots, n\}$ such that $q \in \operatorname{pos}(C_J)$ and 'sgn' stands for the sign of any scalar. This number is the same for all $q \in R^n$ nondegenerate with respect to M and is called the *degree* of M. Stewart (1993) gave an extension of this formula for vectors semi-nondegenerate with respect to M. For any $q \in R^n$, q semi-nondegenerate with respect to M, it is stated as

$$\deg M = \sum (\operatorname{sgn} \det M_{JJ}) \deg M^S_{JJ} \tag{4}$$

where the summation is as defined before. Indeed, since M is a nondegenerate matrix, the above definition of degree is valid for any $q \in R^n$. We remark at this juncture that the definition of degree in cited papers varies; we have adapted the one as mentioned in Gowda (1991). Also, it is stated here for nondegenerate matrices only, as the classes we deal with in this paper, are nondegenerate.

It is well known that if $\deg M$ is nonzero, then M is a Q-matrix. Also, from Theorem 6.6.23 of Cottle, Pang and Stone (1992), if $\deg M$ is r, then $\deg M^{-1}$ is either r or $-r$ depending on whether the determinant of M is positive or negative.

A matrix $M \in R^{n\times n}$ is called a P-matrix (N-matrix), if all its principal minors are positive (negative). For several properties of these classes, see Murty (1972), Saigal (1972b), Mohan and Sridhar (1992) and Parthasarathy and Ravindran (1990). The classes of exact order matrices, which are defined as generalizations of these classes, are as follows:

DEFINITION 1. A matrix $M \in R^{n\times n}$ is called an *N-matrix* (*P-matrix*) *of exact order* k, $k < n$, if every principal submatrix of order $(n-k)$ is an N-matrix (P-matrix), and if every principal minor of order r, $n - k < r \leq n$, is positive (negative). M is called a *matrix of exact order* k, if it is either a P-matrix of exact order k or an N-matrix of exact order k.

We denote the classes of P and N of exact order k, by E^+_k and E^-_k respectively and the class of exact order k by simply E_k. Any proper principal submatrix of an E_k-matrix is an E_r-matrix, for some $0 \leq r < k$. In particular, when $M \in E_k$, for convenience of notation, we call an $(n-k+r) \times (n-k+r)$ principal submatrix of M as an E_r principal submatrix of M for $0 \leq r < k$.

The following is an example of a matrix of exact order k, for any positive integer k.

EXAMPLE 1. Let M be an n by n matrix of the form,

$$M = \begin{bmatrix} 1 & x & x & \cdots \\ x & 1 & x & \cdots \\ x & x & 1 & \cdots \\ \vdots & & & \end{bmatrix}.$$

For $x \neq 1$, $\det(M) = (1-x)^{n-1}(1+(n-1)x)$. When $-1/(n-k-1) < x < -1/(n-k)$, $M \in E_k^+$.

Constructing examples of E_k^--matrices for $k \geq 3$ is a more tedious task. We give below an example of an E_3^--matrix:

EXAMPLE 2. The matrix

$$M = \begin{bmatrix} -1 & 1 & 1.15 & 1.1 & 2 & -1 \\ 2 & -1 & -3 & -3 & -1.5 & 1.5 \\ 1.7 & -3 & -1 & -3 & -1.5 & 1.5 \\ 2 & -3 & -3 & -1 & -1.5 & 2 \\ 2 & -2 & -1.5 & -1.5 & -1.1 & 10 \\ -15 & 1.6 & 1.2 & 1.1 & 1.1 & -1 \end{bmatrix}$$

is an E_3^- matrix. By taking the vector $x' = (.0057, 31.5588, 31.5597, 16.7749, 141.4083, 259.457)$, we see that $x > 0$ and $Mx > 0$. Hence, the minimax value $v(M)$ is positive.

The following lemmas present the signs of the principal minors of the inverse of an E_k-matrix.

LEMMA 1. *Let $M \in R^{n\times n} \cap E_k^+$, $1 < k < n$. Then $M^{-1} \in E_{n-k+1}^+$.*

PROOF. From the Schur complement formula, we have for any $J \subseteq \{1, \ldots, n\}$,

$$\det M^{JJ} = \frac{\det M_{\bar{J}\bar{J}}}{\det M}. \tag{5}$$

Since $\det M < 0$, $\operatorname{sgn} \det M^{JJ} = -\operatorname{sgn} \det M_{\bar{J}\bar{J}}$. Hence M^{-1} has principal minors up to order $(k-1)$ positive and from order k onwards up to $\det M^{-1}$ negative. This implies that $M^{-1} \in E_{n-k+1}^+$. □

LEMMA 2. *Let $M \in R^{n\times n} \cap E_k$, $1 < k < n$. Let D be an $r \times r$ principal submatrix of M^{-1}, for $k < r < n$. Then $D^{-1} \in E_k^+$.*

A proof of this lemma is given in Mohan, Parthasarathy and Sridhar (1992).

We recall below the three different categories of E_2 as defined in Mohan, Parthasarathy and Sridhar (1992). Throughout this article, we denote by B_i, the principal submatrix of M gotten by deleting its ith row and the ith column.

DEFINITION 2. Let $M \in R^{n\times n}$ be an E_2-matrix. We say that M is of *the first category*, if $\forall i$, $1 \leq i \leq n$, $v(B_i) > 0$, except possibly for one index k, $1 \leq k \leq n$, for which $B_k < 0$. We say that M is of the *second category*, if $\forall i$, $v(B_i) < 0$. M is said to be of *the third category*, if there are indices, $i, j \in \{1, \ldots, n\}$, such that $v(B_i) > 0$ and $B_j \not< 0$, $v(B_j) < 0$.

As mentioned in Mohan, Parthasarathy and Sridhar (1992), we may assume, for $M \in E_2^-$ that the order of the matrix M is at least 5. For $M \in E_2^+$, it is enough to have the order of M to be greater than or equal to 3.

Earlier Mohan, Parthasarathy and Sridhar (1992) proved the following result on the Q-nature of E_2:

THEOREM 3. *Let $M \in R^{n \times n}$ be an E_2^- (E_2^+)-matrix, for $n \geq 5$ ($n \geq 3$) with $v(M) > 0$. Then $M \in Q$ if and only if the cardinality of the set*

$$L = \{i: B_i \nless 0, B_i^{-1} < 0, 1 \leq i \leq n\} \tag{6}$$

is either 0 *or* 2.

From Olech et al. (1991), an exact order one matrix M (which is called as an almost N-matrix in Olech et al. (1991)) is said to be of the second category, if either $M < 0$ or $M^{-1} < 0$. Therefore, the index set L defined above and the index set given in Theorem 4.8 of Mohan, Parthasarathy and Sridhar (1992) are one and the same. We refer to Theorem 4.9 of Mohan, Parthasarathy and Sridhar (1992), for a proof. From the proof of this theorem, the following lemma can be easily seen.

LEMMA 3. *Let $M \in R^{n \times n}$ be an E_2-matrix with a positive minimax value, then the cardinality of the set L in* (6) *is either* 0, 1 *or* 2.

The following results on exact order two matrices, which we quote from Mohan, Parthasarathy and Sridhar (1992), are needed for our discussion:

LEMMA 4. *Let $M \in R^{n \times n}$ be an E_2^+ (E_2^-)-matrix of exact order* 2, *with $n \geq 3$ ($n \geq 5$) and $v(M) > 0$. Suppose $B_1 \nless 0$ is a matrix of exact order* 1 *of the second category. Then $M^{\cdot 1} > 0$, and $M^1 > 0$.*

LEMMA 5. *Let $M \in R^{n \times n}$ be an E_2^- (E_2^+)-matrix, with $n \geq 5$ ($n \geq 3$) and $v(M) > 0$. Let M^{-1} have every row a negative entry. Then, M is of the first category.*

3. A characterization of E_2. Our aim in this section is to characterize the class of Q-matrices among E_2, just by observing the inverse sign patterns. In turn, the inverse positivity of the class E_2 is completely characterized. Such results have been studied for some subclasses of P-matrices. See Johnson (1982) for a survey on this. A result on the inverse sign patterns could easily be observed for the class of E_1-matrices (from Olech, Parthasarathy and Ravindran (1991)) as follows: An E_1^+-matrix belongs to the class Q if and only if its inverse has a positive entry. Similarly an E_1^--matrix belongs to Q if and only if both the matrix and its inverse have positive entries.

Before proceeding further, we make a crucial observation on the degree of an E_2-matrix, from Theorem 3, that it lies between -1 and $+1$. This we present below as a remark for future reference.

REMARK 1. We will explain here that the degree lies between -1 and $+1$ for an E_2^+-matrix and for E_2^-, it can be established in a similar way. Let $M \in E_2^+$. If M is not a Q-matrix, its degree is zero. Now, let $M \in Q$ and $|L| = 0$, i.e., M is of the first category. It follows from the proof of Theorem 4.5 of Mohan, Parthasarathy and Sridhar (1992), that for a $q > 0$, LCP(q, M) has a unique solution. Hence the degree of M is 1. Suppose that $M \in Q$ and $|L| = 2$, and without loss of generality, assume that $L = \{1, n\}$. As before, consider the LCP(q, M) for a $q > 0$. We claim that the principal submatrix B_i, for $i = 1, n$, gives rise to a solution for the LCP(q, M). We will demonstrate this for B_n. Let the complementary matrix C_J for $J = \{1, \ldots, n-1\}$ be written as

$$C_J = \begin{bmatrix} -B_n & 0 \\ -c^t & 1 \end{bmatrix}.$$

Then $C_J^{-1}q > 0$, since $-B_n^{-1} > 0$ and $-c^t B_n^{-1} > 0$ from Lemma 4. Hence $q \in \text{pos}(C_J)$ and B_n gives rise to a solution for the LCP(q, M). We can prove in a similar way that B_1 gives rise to a solution for the LCP(q, M) and that it has no other solution, except the trivial one, *viz.*, $w = q, z = 0$. This observation along with $\det B_i < 0$ for any $i = 1, \dots, n$ implies that $\deg M = 1 - 1 - 1 = -1$. Hence, when M is an E_2-matrix, we see that the degree of M is either 0 or ± 1. Especially, for $M \in E_2^+$, when $|L| = 0$, i.e., M is of the first category, $\deg M = 1$; when $|L| = 2$, $\deg M = -1$. Similarly, one can note for $M \in E_2^-$, that $\deg M = -1$ when $|L| = 0$ and when $|L| = 2$, the degree of $M = 1$.

When $M \in E_2$, M^{-1} has no zero entry, as from Lemma 1, the diagonal entries of M^{-1} are positive, and all its order 2 principal minors are negative. Under the assumption of value positive of an E_2-matrix, we can identify the cardinality of the set L defined in (6) in terms of the sign structure of its inverse. This is the main result in this section and is stated below.

THEOREM 4. *Let $M \in R^{n \times n}$ be an E_2^- (E_2^+)-matrix, with $n \geq 5$ ($n \geq 3$) and $v(M) > 0$. Let L be the index set, as defined in* (6). *Then we have the following*:

(i) $|L| = 2$ *if and only if* $M^{-1} > 0$.

(ii) $|L| = 1$ *if and only if* $M^{-1} \ngtr 0$ *has at least one row* (*and one column*) *of positive entries*.

(iii) $|L| = 0$ *if and only if* M^{-1} *has every row* (*and every column*) *a negative entry*.

Now, it will follow from the above theorem and Theorem 3, that an E_2-matrix with a positive minimax value belongs to the class Q only when its inverse is either a positive matrix, or has every row a negative entry.

We are going to prove Theorem 4 through several lemmas. Some parts of this theorem have already been observed in Mohan, Parthasarathy and Sridhar (1992). This is discussed below.

Let $M \in E_2$. When $M^{-1} > 0$ as stated in (i), we have $\deg M^{-1} = 1$. Then degree of M is either 1 or -1 depending on whether M is an E_2^--matrix or an E_2^+. Accordingly, one can notice that the cardinality of the set L is 2. Hence, the 'if' part of (i) follows. The 'if' statement of (iii) in Theorem 4 follows from Lemma 5.

The 'only if' of statement (i) is proved in the next lemma.

LEMMA 6. *Let $M \in R^{n \times n}$ be an E_2^- (E_2^+)-matrix, with $n \geq 5$ ($n \geq 3$) and $v(M) > 0$. If $|L| = 2$, where L is as defined in* (6), *then $M^{-1} > 0$.*

PROOF. Without loss of generality, let $L = \{1, 2\}$. Then M^{-1} has the first two rows and two columns positive using Lemma 4. Since $|L| = 2$, we note from Remark 1, that $\deg M = -1$ if $M \in E_2^+$ and $\deg M = 1$ if $M \in E_2^-$. In either case, the degree of M^{-1} equals 1.

Let A be the principal submatrix of M^{-1} leaving the first two rows and the first two columns. Then from 4.9 of Murty (1972), we notice that A is a Q-matrix. If A has any more rows positive, then we repeat this argument, until we get a principal submatrix D of M^{-1}, such that every row of D has a negative entry. We note that D is a Q-matrix. Let r denote the order of the matrix D. Assume without loss of generality, that D is obtained from M^{-1}, by deleting its first $(n - r)$ rows and $(n - r)$ columns. Let a vector $q \in R^n$ be defined as having its first $(n - r)$ coordinates as 1's and the rest of the elements zero. It can be seen that LCP(q, M^{-1}) has a unique solution, *viz.*, $w = q, z = 0$. Now, using this vector q in the calculation of degree of M^{-1}, we see that $\deg D$ is the same as $\deg M^{-1}$. Hence $\deg D = 1$.

As for the order r of the matrix D is concerned, we first note that $r = 2$ is impossible. This is due to the fact that a 2 by 2 matrix with every row having a negative entry will have degree 1, only when it is a P-matrix. But on the contrary,

M^{-1} has order 2 principal minors negative. So, $r > 2$ and D^{-1} is a P-matrix of exact order 2 from Lemma 2; from Theorem 6.6.23 of Cottle, Pang and Stone (1992), we get deg $D^{-1} = -1$. But from Lemma 5, since D has every row a negative entry and $v(D) > 0$, D^{-1} is an E_2^+-matrix of the first category. Hence, deg $D^{-1} = 1$ which leads to a contradiction. □

The next lemma will provide us with a proof of parts (ii) and (iii) of Theorem 4.

LEMMA 7. *Let $M \in R^{n\times n} \cap E_2$. Let $A = M^{-1}$. If A has only the first row (and the first column) positive, then B_1, the principal submatrix of M leaving the first row and the first column, is of the second category. Also, the degree of M equals zero.*

PROOF. Let $M \in E_2^+$ and $A = M^{-1}$ satisfy the hypotheses of the lemma. We will prove the lemma by induction on the order of A. For $n = 3$ the sign pattern of A is given by,

$$A = \begin{bmatrix} + & + & + \\ + & + & - \\ + & - & + \end{bmatrix}.$$

Clearly, deg $A = 0$, as all the 2×2 principal minors of A are negative. Thus deg $M = 0$, and we can see that B_1 of M is of the second category.

Let $n > 3$ be fixed. Let the lemma be true for all $(n-1) \times (n-1)$ order E_2^+-matrices. Suppose $M \in R^{n\times n}$ is an E_2^+-matrix and A is its inverse, with the first row of A having all its entries positive.

As every row of A except the first row has a negative entry, assume without loss of generality, that $a_{n2} < 0$. Let $D \in R^{(n-1)\times(n-1)}$ be the principal submatrix of A obtained by deleting its last row and the last column. Since $D^{-1} \in E_2^+$, D^{-1} has exactly one second category, *viz.*, the principal submatrix gotten by deleting the first row and the first column of D^{-1}. Hence there exists a nonnegative vector $y \in R^{(n-1)}$, with $y_1 = 0$, such that $D(x_1, x_2, 0, \dots, 0)^t = y$, for some $x_1 > 0$ and $x_2 < 0$. Also, we have $y_i > 0$, $\forall 2 \le i \le (n-1)$. Now by extending the vector $x^t = (x_1, x_2, 0, \dots, 0)$ to R^n, by taking the nth coordinate of x as 0, we have

$$Ax = \begin{bmatrix} 0 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

where $y_i > 0$, $\forall 2 \le i \le (n-1)$. Also $y_n > 0$, since $a_{n2} < 0$. This implies $A^{-1}y = My = x$ and we have $B_1(y_2, \dots, y_n)^t \le 0$. Thus, the minimax value of B_1^t is negative, and B_1 is of the second category. That it has no other second category E_1 principal submatrix follows from Lemma 4.

When $M \in E_2^-$, $n \ge 5$, with $A = M^{-1}$, every $(n-1) \times (n-1)$ principal submatrix of A has its inverse, an E_2^+-matrix, from Lemma 2. Now, by considering D to be the principal submatrix of A, deleting its last row and the last column, as done in the earlier paragraph, we can see that there exists a vector $y \in R^{(n-1)}$, $y > 0$ such that $B_{1y} < 0$. Hence the minimax value of B_1^t is negative, and B_1 is a second category E_1 principal submatrix of M. Let L be the index set as defined in (6), denoting the number of E_1 principal submatrices in M. We observe using Lemma 6, that if $|L| = 2$, then M^{-1} is entrywise positive which contradicts our hypothesis. $|L|$ cannot be greater than 2 also follows from Lemma 3. Hence $|L| = 1$ which implies that M has exactly one second category E_1 principal submatrix. Now, from Remark 1, it follows that the deg M is zero. This concludes the proof. □

If $M \in E_2$ with $v(M) > 0$ and $|L| = 1$, where L is defined as in (6), then there exists a row (and a column) of M^{-1} which is positive, from Lemma 4.

Conversely, let M^{-1} have r rows positive, where $1 \le r < n$. For simplicity of our discussion, assume that $r = 2$ and that the first two rows of M^{-1} are positive. Let D be the principal submatrix of M^{-1} leaving the first row and the first column of M^{-1}. Then from the above lemma, we can see that deg $D = 0$. We notice that LCP($I_{.1}$, M^{-1}) has a unique solution, and when the degree of M^{-1} is calculated with respect to the vector $I_{.1}$, we get

$$\deg M^{-1} = \deg D = 0.$$

Thus deg $M = 0$ and $|L| = 1$. If M^{-1} has in general r rows positive, $1 \le r < n$, we can repeat this argument and conclude the same. Hence, when $M^{-1} \ngtr 0$ and has at least one row positive, we get $|L| = 1$, and part (ii) follows.

Now, having proved the parts (i) and (ii) of Theorem 2 completely, the 'only if' of statement (iii) follows directly. And with this, we conclude the proof of Theorem 4.

4. The minimax value of E_k. Here, we introduce the three different categories of E_k and study the signs of their minimax values.

In a similar manner as done in §3, one can classify the exact order k matrices $k \ge 3$, into three different categories, based on the minimax values of the exact order one principal submatrices present in them:

DEFINITION 3. Let $M \in R^{n \times n}$. Suppose $M \in E_k^+$ ($M \in E_k^-$), $n \ge k + 1$ ($n \ge k + 3$). M is of *the first category* if $M \nless 0$ and every E_1 principal submatrix of M, which has a positive entry, has the minimax value positive; we say that M is of *the second category* if all the E_1 principal submatrices of M have their minimax values negative. M is said to be of *the third category* if there are at least two E_1 principal submatrices of M, such that one of them has value positive and the other with a positive entry, has value negative.

REMARK 2. Due to Lemma 3.1 of Mohan and Sridhar (1992), which states that M can be written in a partitioned form if M has up to order 3 principal minors negative, we assume for $M \in R^{n \times n} \cap E_k^-$, that n is greater than or equal to $k + 3$. For P-matrices of exact order k, it is enough to have $n \ge k + 1$. Hence, whether mentioned or not, all our results below bear this assumption. Without this, some of our theorems may not hold.

For the first category E_k-matrices, the minimax value is always positive. This is proved in the next theorem.

THEOREM 5. *Let $M \in R^{n \times n}$ be an E_k-matrix of the first category. Then $v(M) > 0$.*

PROOF. At first, we observe, since M is of the first category, that $v(M) \ne 0$. If, for $k = 3$, $v(M) = 0$, then the game is completely mixed from Theorem 1 and there exists a probability vector $x > 0$ such that $Mx = 0$. But this is impossible since M is nonsingular. Similarly one can prove that for any order k of the first category, the value is nonzero. We will in fact prove, inductively over k, that $v(M) > 0$. We know for $k = 2$ from Mohan, Parthasarathy and Sridhar (1992), that $v(M) > 0$. If, for $k = 3$, $v(M)$ is negative, then the game is completely mixed and using Theorem 2, $M^{-1} < 0$. But using the determinantal expression given in (5), we see that the inverse of an exact order k matrix ($k \ge 2$) has all the diagonal entries positive and hence, $M^{-1} < 0$ is not possible. By repeated use of this argument, we can see that $v(M) > 0$. □

A subclass of second category exact order matrices will not belong to the class Q. This is observed in the next theorem.

THEOREM 6. *Let $M \in R^{n \times n}$ be an E_k-matrix of the second category with each E_1 principal submatrix of M having at least one positive entry. Then $v(M) < 0$.*

PROOF. We give a proof for $k = 3$ and for $k > 3$, the theorem can be proved in a similar way. Let M be written in a partitioned form, as

$$M = \begin{bmatrix} A & b & c \\ d & m_{(n-1)(n-1)} & m_{(n-1)n} \\ f & m_{n(n-1)} & m_{nn} \end{bmatrix},$$

where A is a matrix of exact order one. Doing a principal pivot transform of M with respect to A, the resulting matrix $\overline{M}$ is given by

$$\overline{M} = \begin{bmatrix} A^{-1} & -A^{-1}b & -A^{-1}c \\ A^{-1}d & & \\ A^{-1}f & & (M/A) \end{bmatrix}.$$

See Cottle (1968), for details on principal pivot transforms. From Theorem 4.6 of Mohan, Parthasarathy and Sridhar (1992), we have $-A^{-1}b < 0$ and $-A^{-1}c < 0$. Since $A^{-1} < 0$, $v(\overline{M}) < 0$ and hence the theorem follows. □

From the above theorem, we have $v(M) < 0$ when $M \in E_k^+$, $1 < k < n$ is of the second category. For the rest of the cases in E_k, i.e., when $M \in E_k$, $k \geq 3$, is of the third category, or of the second category with at least one E_1 principal submatrix of M being entrywise negative, we are not able to assert this though we believe that the value of M is nonzero.

5. The degree of E_k. This section deals with the analysis of the Q-nature of E_k. At first, we present results based on the categories of E_k, and later a complete characterization is given using the degree of M.

As mentioned in the earlier section, we remark that for $M \in E_k^+$, we need to assume that the order of the matrix M is at least $k + 1$ and for $M \in E_k^-$, it is necessary that the order of the matrix M is at least $k + 3$. This is assumed in the outset of each result proved in this section.

Let $M \in R^{n \times n}$ be an E_k-matrix of the first category. If M is an E_k^+-matrix, from Theorem 5, it is clear that $v(M_{JJ}) > 0$ for any principal submatrix of M. Hence M is an L_1-matrix (Cottle, Pang and Stone 1992) and LCP(q, M) has a unique solution for any $q > 0$. Since E_k-matrices are nondegenerate by definition, from Murty (1972) and Cottle, Pang and Stone (1992), it follows that M is a Q-matrix. Indeed, we notice that M is a completely Q-matrix as defined by Cottle (1980). Hence, there are efficient algorithms that can process the LCP(q, M). See also Chapter 3 of Murty (1988). If $M \in E_k^-$, we observe from Theorem 5, that the value of M_{LL} is positive for all $M_{LL} \nleq 0$, $L \subseteq \{1, \ldots, n\}$. That M belongs to the class Q follows from the next theorem, which is proved for a matrix with a specified partitioned form.

THEOREM 7. *Let $M \in R^{n \times n}$ have a partitioned form (if necessary, after a principal rearrangement of its rows and columns) which is given by*

$$M = \begin{bmatrix} M_{JJ} & M_{J\bar{J}} \\ M_{\bar{J}J} & M_{\bar{J}\bar{J}} \end{bmatrix} \tag{7}$$

for $\phi \neq J \subset \{1, \ldots, n\}$, where $M_{JJ} < 0$, $M_{\bar{J}\bar{J}} < 0$ and $M_{J\bar{J}}$ and $M_{\bar{J}J}$ have all entries positive. Let for all $K \subseteq \{1, \ldots, n\}$, with $M_{KK} \not< 0$, $v(M_{KK}^t) > 0$. Then M is a Q-matrix.

PROOF. Under the assumptions stated, we first claim that the problem LCP$(0, M)$ has a unique solution. Suppose on the contrary there exists a solution (w, z) for the LCP$(0, M)$ with $z \neq 0$. Let $L = \{i: z_i > 0\}$. Then the solution (w, z) can be written as

$$\begin{bmatrix} -M_{LL} & 0 \\ -M_{\bar{L}L} & I_{\bar{L}\bar{L}} \end{bmatrix} \begin{bmatrix} z_L \\ w_{\bar{L}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies that the system $M_{LL} z_L = 0, z_L > 0$ has a solution. It is clear that neither $L \subseteq J$ nor $L \subseteq \bar{J}$. Hence, the index set L is such that $L \cap J \neq \phi$ and $L \cap \bar{J} \neq \phi$ which implies, $M_{LL} \not< 0$. But the system $z_L > 0, M_{LL} \leq 0$ has a solution implies that $v(M_{LL}^t) \leq 0$, which is contrary to the hypothesis. Hence LCP$(0, M)$ has a unique solution.

Let us consider the LCP(q, M) for a $q > 0$, where q_J and $q_{\bar{J}}$ are so chosen that q_J is nondegenerate with respect to M_{JJ} and $q_{\bar{J}}$ is nondegenerate with respect to $M_{\bar{J}\bar{J}}$. By Theorem 4.3 of Saigal (1972a), it can be seen that LCP(q_J, M_{JJ}) has an even number of nondegenerate solutions. As $M_{\bar{J}J} > 0$, each of these solutions gives rise to a nondegenerate solution for the LCP(q, M). In a similar manner, it can be seen that every solution of the subproblem LCP$(q_{\bar{J}}, M_{\bar{J}\bar{J}})$ gives rise to a nondegenerate solution for the LCP(q, M). We can see that LCP(q, M) has no other solution, as given by the argument in the earlier paragraph. Hence, it follows that LCP(q, M) has in all an odd number of distinct nondegenerate solutions (for the solution $(w = q, z = 0)$ gets repeated once). Now by Corollary 3.2 of Saigal (1972a), it follows that M is a Q-matrix. □

As mentioned in §4, for the second category and the third category E_k-matrices, we are not sure about whether the minimax values of the principal submatrices of M are nonzero or not. Thus the arguments through the minimax values of principal submatrices do not lead us to characterize the subclass $E_k \cap Q$, except in the case of the first category E_k. However, we will prove that the deg M being nonzero is both necessary and sufficient for $M \in E_k \cap Q$.

Our next theorem ensures that the modulus of deg M for $M \in E_k$ is at most 1.

THEOREM 8. *Let $M \in R^{n \times n}$ be an E_k^+- (E_k^-)-matrix for $n \geq k + 1$ ($n \geq k + 3$). Then* deg M *is either* $-1, 0$ *or* 1.

PROOF. We treat the classes E_k^+ and E_k^- separately.

Let $M \in E_k^+$. Consider a $q \in R^n$ such that $q > 0$ and q is nondegenerate with respect to M. If LCP(q, M) has a nontrivial solution, i.e., if $q \in \text{pos}(C_J)$, for any $\phi \neq J \subseteq \{1, \ldots, n\}$, then clearly det $M_{JJ} < 0$. Otherwise, M_{JJ} will be a P-matrix, which will contradict the fact that the subproblem LCP(q_J, M_{JJ}) has a nontrivial solution. So, when the degree of M is computed with respect to this vector $q > 0$, only the cone corresponding to the positive orthant gives rise to an index $+1$ and the rest of the indices will be -1; hence, it can be seen that $\deg M \leq 1$. From Lemma 1, $M^{-1} \in E_{n-k+1}^{+}$. By our earlier argument, $\deg M^{-1} \leq 1$. But $\deg M = -\deg M^{-1} \geq -1$. Hence for M a P-matrix of exact order, the degree of M lies between -1 and $+1$.

Now, let $M \in E_k^-$. As $n \geq k+3$, there exists a partitioned form of M as given in (7) for $\phi \neq L \subseteq \{1, \ldots, n\}$. If $L = \phi$ or $L = \{1, \ldots, n\}$, then degree of M is zero, as $M < 0$. So let us consider the case when L is a nonempty proper subset of $\{1, \ldots, n\}$. Choose a $q > 0, q \in R^n$ nondegenerate with respect to M. As in the proof of Theorem 7, we can notice that LCP(q, M) has solutions from the subproblems LCP(q_L, M_{LL}) and LCP$(q_{\bar{L}}, M_{\bar{L}\bar{L}})$, extended in a natural way to the original problem. Thus

$$\sum_{q \in \operatorname{pos}(C_J)} (\operatorname{sgn} \det M_{JJ}) = -1$$

where the index J is such that either $J \subseteq L$ or $J \subseteq \bar{L}$.

If LCP(q, M) has any other solutions, i.e., if $q \in \operatorname{pos}(C_J)$, for $\phi \neq J \subseteq \{1, \ldots, n\}$, for which $M_{JJ} \nless 0$, then clearly, $\det M_{JJ} > 0$. This follows from Theorem 3.1 of Mohan and Sridhar (1992) on N-matrices of the first category: Hence

$$\deg M = \sum (\operatorname{sgn} \det M_{JJ}) \deg M_{JJ}^S \geq -1. \tag{8}$$

Consider the LCP(q, M^{-1}) for a vector $q > 0$. M^{-1} has principal minors up to order $(k-1)$ positive and from order k onwards up to $(n-1)$ negative. Hence, if $q \in \operatorname{pos}(C_J)$ where $\operatorname{pos}(C_J)$ is a complementary cone of $[I : -M^{-1}]$ for $J \subseteq \{1, \ldots, n\}$, then $|J| \geq k$. Also $\det M^{JJ} < 0$. Thus we see that $\deg M^{-1}$ and in turn $\deg M$ is less than or equal to 2.

Suppose $\deg M$ is exactly equal to 2. Then $\forall q > 0$, q nondegenerate with respect to M^{-1}, LCP(q, M^{-1}) has a solution in the complementary cone $\operatorname{pos}(-M^{-1})$. Since $\operatorname{pos}(-M^{-1})$ is closed, this implies that $\operatorname{pos}(I) \subseteq \operatorname{pos}(-M^{-1})$. We then have for every $j \in \{1, \ldots, n\}$, $M_{\cdot j} \leq 0$, which implies that M is not a Q-matrix. This leads to a contradiction as $\deg M = 2$ would mean that $M \in Q$. This completes the proof. □

Our next task lies in proving that if $M \in E_k \cap Q$, where $M \in R^{n \times n}$ with $n \geq k+3$, then the degree of M is nonzero. We note that this need not in general be true for a nondegenerate Q-matrix. We refer to an example of Kelly and Watson (1979) and also examples in Chapter 6 of Cottle, Pang and Stone (1992).

We prove below, a result that characterizes the E_k-matrices of the first category.

THEOREM 9. *Let $M \in R^{n \times n}$ be an E_k-matrix with $n \geq k+1$, for some positive integer k.*

(a) *If $M \in E_k^+$, then M is of the first category if and only if* $\deg M = 1$.

(b) *If $M \in E_k^-$ with $n \geq k+3$, then M is of the first category if and only if* $\deg M = -1$.

PROOF. (a) When $M \in E_k^+$ of the first category, we can see that LCP(q, M) has a unique solution for any vector $q > 0$. Hence $\deg M = 1$.

Conversely suppose $\deg M = 1$. We claim that LCP$(I_{\cdot j}, M)$ has a unique solution $\forall j$, $j \in \{1, \ldots, n\}$. Whenever $q \in \operatorname{pos}(C_J)$, where $q = I_{\cdot j}$, for some $\phi \neq J \subseteq \{1, \ldots, n\}$. Then clearly $|J| > (n-k)$ and $\det M_{JJ} < 0$. The degree of M calculated with respect to the vector $I_{\cdot j}$ is

$$\deg M = \deg B_j + \sum (\operatorname{sgn} \det M_{JJ}) = 1.$$

The degree of M_{JJ}^S can be seen to be always 1, when $|J| > (n-k)$. Hence, it is not mentioned in the above equation. If the expression in the summation is nonzero, i.e., if LCP($I_{.j}$, M) has a nontrivial solution, then deg $B_j \geq 2$, which contradicts Theorem 8. Therefore when deg M is 1, LCP($I_{.j}$, M) has a unique solution for all $j = 1, \ldots, n$. This in turn implies that deg $B_j = 1$, $\forall j = 1, \ldots, n$.

Now each B_j is an E_{k-1}^+-matrix with degree 1. So, we can apply the above argument for all B_j. By repeatedly using this, we arrive at the conclusion that the degree of every E_1^+ principal submatrix of M has degree 1. In other words, M belongs to E_k^+ of the first category.

(b) Let $M \in E_k^-$. If M is of the first category, it follows from Theorem 3 that deg $M = -1$.

If M has degree -1, we can proceed as discussed in the earlier paragraphs except for the fact that LCP($I_{.j}$, M) has a unique solution only for those $j \in \{1, \ldots, n\}$ for which there exists an $l \in \{1, \ldots, n)$, $l \neq j$, with $m_{jl} < 0$. Ultimately, it can be seen that every principal submatrix M_{LL} to M, $M_{LL} \not< 0$, has degree -1. Hence all the E_1^- principal submatrices of M which have a positive entry, are of the first category and the conclusion follows. □

Next, we prove our desired theorem.

THEOREM 10. *Let $M \in R^{n \times n}$ be an E_k-matrix for any nonnegative integer k such that $n \geq k+3$. Then $M \in Q$ if and only if* deg $M \neq 0$.

PROOF. Let $M \in R^{n \times n}$ be an E_k-matrix for some nonnegative integer k such that $n \geq k+3$. For $k \leq 2$, this theorem is known. See Gowda (1993) and Cottle, Pang and Stone (1992). We require to prove that when $M \in Q$, M has degree nonzero for $k \geq 3$.

We deal separately below with the cases when M is a P-matrix of exact order k and when M is an N-matrix of exact order k.

Let $M \in E_k^+$. Let us assume on the contrary, that M is a Q-matrix with deg $M = 0$. We then claim that the LCP($I_{.j}$, M) has at most two solutions, for each $j \in \{1, \ldots, n\}$. This can be asserted as in the proof of Theorem 9.

Suppose there exists a $j \in \{1, \ldots, n\}$ for which LCP($I_{.j}$, M) has a unique solution. Then from 4.9 of Murty (1972), we have $B_j \in Q$. Also, B_j is an E_{k-1}-matrix with deg B_j = deg $M = 0$.

We can repeat this argument until we arrive at an $r \times r$ principal submatrix A of M which has the following properties:

(i) $A \in Q$.
(ii) deg $A = 0$.
(iii) The problem LCP($I_{.j}$, A) has exactly two solutions for each $j \in \{1, \ldots, r\}$.

We have assumed here, without loss of generality, that A is the leading $r \times r$ principal submatrix of M. Since A is a principal submatrix of M, A is an exact order matrix. From (i) and (ii), we have $A \in E_s$ for $s \geq 3$ and $r \geq s+1$.

Let $j \in \{1, \ldots, r\}$ be fixed; as LCP($I_{.j}$, A) has exactly two solutions, if (w, z) is a nontrivial solution to the LCP($I_{.j}$, A), then det $A_{JJ} < 0$, where the index set J is as defined in (3). Then the degree of A when calculated using the semi-nondegenerate vector $I_{.j}$, could be written as

$$\deg A = \deg A_j + (-1) = 0,$$

where A_j stands for the principal submatrix of A leaving the jth row and the jth column. This implies deg $A_j = 1$, $\forall j \in \{1, \ldots, r\}$. Using Theorem 9, we notice that every exact order one principal submatrix of A_j is of the first category, $\forall j \in \{1, \ldots, r\}$. But this is impossible, for then deg $A = 1$, contradicting (ii). Hence deg $M \neq 0$ if $M \in E_k^+ \cap Q$.

Let $M \in E_k^-$, $n \geq k+3$. We can proceed as before, except that for the principal submatrix A of M, LCP(I_j, A) has exactly two solutions only for those $j \in \{1, \ldots, r\}$ for which there exists an $l \in \{1, \ldots, r\}$, $l \neq j$, such that $a_{lj} < 0$. Hence we get for every A_j of A, for which $A_j \nless 0$, deg $A_j = -1$. This can be repeated for the principal submatrices of A and until we arrive at the fact that every E_1^- principal submatrix of A which is not entrywise negative has degree -1. But this again implies from Theorem 9, that A is a matrix of the first category and deg $A = -1$ which leads to a contradiction. Hence the proof is complete. □

Finally, the characterization of the class $E_k \cap Q$ in terms of the degree can be stated as follows:

THEOREM 11. *Let $M \in R^{n \times n}$ be an E_k-matrix for nonnegative integer k with $n \geq k+3$. Then $M \in Q$ if and only if the* deg M *is either* -1 *or* $+1$.

6. Conclusion and open problems. Though our aim in this paper was to characterize the Q-nature of exact order matrices, we also noticed in §4 that the minimax value of E_k is nonzero in some special cases. It is not yet known whether there exists a matrix $M \in R^{n \times n}$, such that $M \in E_k$, $n \geq k+3$, for which $v(M)$ equals zero. Also, it is of interest to know whether the minimax values of M and M^t keep the same sign when M ($\in E_k$) is of the second/third category. These have been observed for $k = 0, 1, 2$ (Mohan, Parthasarathy and Sridhar 1992). We believe that this is true in general, but we are unaware of a proof of this.

It was proved in Mohan, Parthasarathy and Sridhar (1992), that when $M \in E_2$ of the second category with every E_1 principal submatrix of M having a positive entry, the off-diagonal entries of M^{-1} are less than or equal to zero. Such matrices are called Z-matrices and are well studied in connection with linear complementarity (Cottle, Pang and Stone 1992). This result does not carry over as we go up the hierarchy of exact order. The following is an example of a matrix in E_3^- of the second category, whose inverse is not a Z-matrix.

Let

$$M = \begin{bmatrix} -1 & 1 & 1.15 & 1.1 & -2 & -1 \\ 2 & -1 & -3 & -3 & 1.15 & 1.15 \\ 1.7 & -3 & -1 & -3 & 1.15 & 1.5 \\ 2 & 3 & -3 & -3 & 1.5 & 2 \\ -2 & 2 & 1.5 & 1.5 & -1.1 & -10 \\ -15 & 1.6 & 1.2 & 1.1 & -1.1 & -1 \end{bmatrix}.$$

Then one can verify that M is an E_3^--matrix of the second category with every E_1 principal submatrix of M having a positive entry. But $M^{-1} \notin Z$, as $m^{56} = .005$ which is positive. But $-M$ is a Q-matrix. Hence, we can pose the following: Let $M \in R^{n \times n}$. When $M \in E_k$ ($n \geq k+3$) of the second category with $v(M) < 0$, is $-M$ a Q-matrix?

The behaviour of the class E_k^+ for any fixed k, is almost similar to that of E_k^-. The difference between these two classes with regard to the number of solutions for the LCP(q, M) occurs only in the positive orthant. For instance, when M is a P-matrix, the LCP(q, M) has an unique solution for every $q \in R^n$. If $M \nless 0$ is an N-matrix, from Mohan and Sridhar (1992), LCP(q, M) has a unique solution for all $q \ngtr 0$ and has exactly three solutions for any $q > 0$. We can compare similarly the almost P-matrices (Olech, Parthasarathy and Ravindran 1989) and almost N-matrices with a positive entry (Olech, Parthasarathy and Ravindran 1991) and notice such a disparity in the number of solutions to the LCP(q, M) for any fixed $q \in R^n$, in the positive orthant only. This makes one raise the following question for matrices in E_k, $k \geq 3$:

Consider two $n \times n$ matrices M^1 and M^2 such that $M^1 \in E_k^+$ and $M^2 \in E_k^-$, $n \geq k + 3$. Let M^1 and M^2 be of the same category with every E_1 principal submatrix of M^2 having a positive entry. Then for a $q \in R^n$, $q \ngeq 0$, is it true that LCP(q, M^1) and LCP(q, M^2) have the same number of solutions?

One of the essential problems that has not been attempted here is to characterize the class $E_k \cap Q$ in terms of the number of second category E_1 principal submatrices present. This for the class E_2 has been proved in Mohan, Parthasarathy and Sridhar (1992) which is stated in Theorem 3.

Acknowledgements. I would like to thank Professors S. R. Mohan and T. Parthasarathy of the Indian Statistical Institute for their constant encouragement and useful discussions. I am very much indebted to two anonymous referees, whose suggestions improved the presentation of this paper to a great extent.

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