Markov Process

R.V. is a rule / function that associates outcome of an exp. to real number.

Random process is a rule / functions a time function to every outcomes.

Suppose a dice is thrown at $t=0$, then outcomes of dice $\{1,2,3,4,5,6\}$ is associated with set functions $x_1(t), x_2(t), \dots, x_6(t)$, then collection of set of functions represents a Random process.

$X(s,t) \to$ R.V.

$s \in S$ (sample space)

$t \in T$ (parameter set or index set)

$\{X(s,t)\} \to$ collection of R.V.

$\downarrow$ Random process

Set of possible values of $\{X(s,t)\}$ is called state space.

$X(s_1,t) \to x_1(t) \to$ one of member of $\{X(s,t)\}$ $\downarrow$ called sample function

Note :-

1. If 's' and 't' are fixed, then $\{X(s,t)\}$ is a number.
2. If 't' is fixed $\{X(s,t)\}$ is R.V.
3. If 's' is fixed $\{X(s,t)\}$ is a single time function
4. If 's' & 't' are variable, then $\{X(s,t)\}$ is collection of R.V. that are function of t.

Notation :-

If parameter set T is descrete then Random process note as X_n or $X(n)$

$T \to$ continuous, then $\{X(s,t)\} \to X(t)$.

Classification of Random Process :-

Depending on state space S and parameter set T

$$T \begin{cases} \rightarrow \text{Discrete} \rightarrow \text{sequence} \\ \rightarrow \text{Continuous} \rightarrow \text{process} \end{cases}$$

→ if both S & T are discrete, then discrete random sequence

→ if S is continuous & T is discrete, then continuous random sequence.

→ if S is discrete & T is continuous, then random process is called discrete random process.

→ if both S & T are continuous, then continuous random process.

$\{X(s,t)\}$ → continuous random process.

Distribution function of process $\{X(t)\}$

$$F(x,t) = P\{X(t) \le x\}$$

$$f(x,t) = \frac{\partial}{\partial x} F(x,t) \rightarrow \text{first order density of } \{X(t)\}$$

$x_1, x_2 \in S$ & $t_1, t_2 \in T$

$$F\{x_1, x_2, t_1, t_2\} = P\{X(t_1) \le x_1 ; X(t_2) \le x_2\} \rightarrow$$

joint distribution of R.Vs $X(t_1)$ & $X(t_2)$

↓

second order distribution of the process $\{X(t)\}$

$$f(x_1, x_2, t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2, t_1, t_2)$$

↓

second order density of $\{X(t)\}$

Markov Process → Markov process is a Random process in which future behaviour depends only on current value not on the past.

If $t_1 < t_2 < t_3 < \dots < t_n < t$,

$$P\{X(t) \le x / X(t_1) = x_1, X(t_2) = x_2, \dots, X(t_n) = x_n\} = P\{X(t) \le x / X(t_n) = x_n\},$$ then $\{X(t)\}$ is called a Markov Process.

A discrete parameter Markov process is called Markov Chain

Markov Chain :-

If $\forall n$, $P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_0 = a_0\} = P\{X_n = a_n / X_{n-1} = a_{n-1}\}$, then process $\{X_n\}$, $n = 0, 1, \ldots$ is called a Markov Chain

$\rightarrow$ Set of values $(a_0, a_1, a_2, \ldots, a_n, \ldots)$ are called state space/ states of Markov chain.

$\rightarrow P\{X_n = a_j / X_{n-1} = a_i\} \rightarrow$ one-step transition probability from state a_i to a_j.

$\downarrow$ $p_{ij}(n-1, n)$

does not depend on step

$\downarrow$

Markov chain called as homogeneous Markov chain / chain with stationary transition probability.

The matrix $P = \{p_{ij}\}$ is called (one-step) transition probability matrix or simply tpm.

conditions for tpm (i) $p_{ij} \geq 0$ ii) $\sum_j p_{ij} = 1$

n^{th} step transition probability is noted as $p_{ij}^{(n)}$

$\Rightarrow p_{ij}^{(1)} = p_{ij}$

Probability distribution :-

In any step, the probability of state a_i is p_i $(i = 1, 2, \ldots, k)$, then row vector $p = (p_1, p_2, \ldots, p_k)$ is called probability distribution of the process at that time.

tpm with initial probability distribution completely specifies a Markov Chain $\{X_n\}$.

Chapman - Kolmogorov Theorem :-

If P is the tpm of a homogeneous Markov chain, then the n-step tpm is equal to P^n.

i.e $\{p_{ij}^{(n)}\} = \{p_{ij}\}^n$

tpm of Markov chain → Stochastic Matrix

Regular Matrix :- A Stochastic matrix P is said to a regular matrix, if all the entries of P^m (for some positive integer) are positive.

→ A homogeneous Markov chain is said to be regular if its tpm is regular.

Statement :-

1. If $p = \{p_i\}$ is the state probability distribution at an arbitrary time, then next step probability distribution is $\underline{\underline{pP}}$, where P is tpm and after n step pP^n.

2. If a homogeneous Markov Chain is regular, then every sequence of state probability distributions approaches a unique fixed probability distribution called stationary (state) distribution/ Steady-state distribution of the Markov chain.

i.e $\lim_{n \to \infty} \{p^{(n)}\} = \pi$

$p^{(n)} = \{p_1^{(n)}, p_2^{(n)}, \dots p_k^{(n)}\}$

$\pi = \{\pi_1, \pi_2, \dots, \pi_k\}$ → stationary distribution.

3. If P is the tpm of the regular chain, then

$\underline{\underline{\pi P = \pi}}$

Prob 1. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and drove to work if and only if a 6 appeared. Find (a) the probability that he takes a train on the third day, and (b) the probability that he drives to work in the long run.

Solution :-

Current day	Next day T	Next day C
Train (T)	✗	✓
Car (C)	✓	✓

The travel pattern is a Markov chain, with state space = (train, car)

$\therefore$ The tpm of the chain is

$$P = \begin{array}{c} \\ T \\ C \end{array} \begin{array}{c} \begin{array}{cc} T & C \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1/2 & \frac{1}{2} \end{pmatrix} \end{array}$$

→ a dice is tossed to make a decision on first day

→ if he gets $\begin{cases} 6 & \text{goes by car} \\ \{1,2,3,4,5\} & \text{goes by train} \end{cases}$

$\therefore$ P(going by car) $= \frac{1}{6}$

P(going by train) $= \frac{5}{6}$

$\therefore$ The initial state probability distribution is

$$p^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$$

Next state probability distribution is

$$p^{(2)} = p^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$p^{(3)} = p^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{11}{24}, \frac{13}{24}\right)$$

$\therefore$ P(the man travels by train on the third day) $= \frac{11}{24}$

(b) Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution or stationary state distribution of the Markov chain.

$$\therefore \quad \pi P = \pi$$

$$(\pi_1, \pi_2) \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

$$\Rightarrow \left(\frac{1}{2}\pi_2 , \pi_1 + \frac{1}{2}\pi_2 \right) = (\pi_1, \pi_2)$$

$$\therefore \quad \frac{1}{2}\pi_2 = \pi_1 \quad \& \quad \pi_1 + \frac{1}{2}\pi_2 = \pi_2$$

$$\Rightarrow \pi_1 = \frac{1}{2}\pi_2$$

Above both equ^n are same

Also, we have $\pi_1 + \pi_2 = 1$ [$\because \pi$ is a probability distribution.]

Solving, $\pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$

$\therefore$ P {the man travels by car in the long run} $= \frac{2}{3}$.

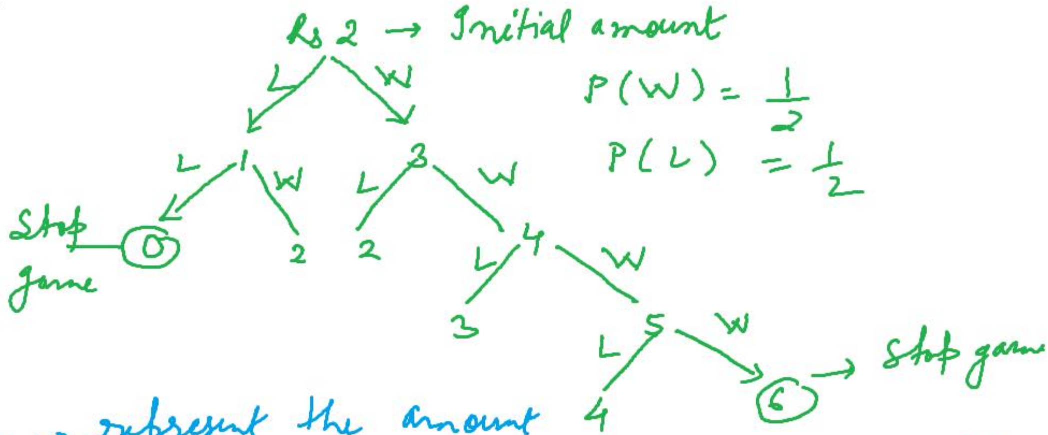
Prob 2. A gambler has Rs 2. He bets Re 1 at a time and wins Re. 1 with probability $\frac{1}{2}$. He stops playing if he loses Rs 2 or wins Rs 4.

(a) What is the tpm of the related markov chain?

(b) What is the probability that he has lost his money at the end of 5 plays?

(c) What is the probability that the game lasts more than 7 plays?

Solution :-



$P(W) = \frac{1}{2}$

$P(L) = \frac{1}{2}$

$X_n \rightarrow$ represent the amount with player at the end of n^{th} round play.

Player may have minimum Rs 0 or maximum Rs 6 at the end of game

State space of $\{X_n\}$ = (0, 1, 2, 3, 4, 5, 6)

TPM of Markov Chain given by

$$P = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 3 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 5 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Initial probability distribution of X_n is

$$p^{(0)} = (0, 0, 1, 0, 0, 0, 0)$$

$$p^{(1)} = p^{(0)} P = \left(0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0\right)$$

$$p^{(2)} = p^{(1)} P = \left(\frac{1}{4}, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, 0\right)$$

$$p^{(3)} = p^{(2)} P = \left(\frac{1}{4}, \frac{1}{4}, 0, 3/8, 0, \frac{1}{8}, 0\right)$$

$$p^{(4)} = p^{(3)} P = \left(\frac{3}{8}, 0, \frac{5}{16}, 0, \frac{1}{4}, 0, \frac{1}{16}\right)$$

$$p^{(5)} = p^{(4)} P = \left(\frac{3}{8}, \frac{5}{32}, 0, \frac{9}{32}, 0, \frac{1}{8}, \frac{1}{16}\right)$$

$\therefore$ P{player lost his money at the end of 5 plays}

$$= P\{X_5 = 0\} = \frac{3}{8}$$

Again $p^{(6)} = p^{(5)} P = \left(\frac{29}{64}, 0, \frac{7}{32}, 0, \frac{13}{64}, 0, \frac{1}{8}\right)$

$$p^{(7)} = p^{(6)} P = \left(\frac{29}{64}, \frac{7}{64}, 0, \frac{27}{128}, 0, \frac{13}{128}, \frac{1}{8}\right)$$

P{the game lasts more than 7 rounds}

= P{the process is neither in 0 & 6 state after 7th round}

$$= P\{X_7 = 1, 2, 3, 4 \text{ or } 5\}$$

$$= P\{X_7 = 1\} + P\{X_7 = 2\} + P\{X_7 = 3\} + P\{X_7 = 4\} + P\{X_7 = 5\}$$

$$= \frac{7}{64} + 0 + \frac{27}{128} + 0 + \frac{13}{128}$$

$$= \frac{27}{64}$$

Prob 3. The transition probability matrix of a Markov chain $\{X_n\}$, $n = 1, 2, 3, \ldots$ having 3 states 1, 2, 3 is

$$P(X_3 = 1 | X_2 = 2) \qquad P(X_2 = 1 | X_1 = 1)$$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \end{array}$$

$X_{n-1} \to X_n$ $\quad P(A/B) = P(A)$

and the initial distribution $p^{(0)} = (0.7, 0.2, 0.1)$.

Find (a) $P\{X_2 = 3\}$ and (b) $P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$

Solution:

$$P^{(1)} = P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$P^{(2)} = P^2 = P \cdot P$$

$$P\{X_n = 3 / X_{n-2} = 1\} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix} \end{array}$$

(a) $$P\{X_2 = 3\} = \sum_{i=1}^{3} P\{X_2 = 3 / X_0 = i\} \times P\{X_0 = i\}$$

$$= P\{X_2 = 3 / X_0 = 1\} \cdot P\{X_0 = 1\} + P\{X_2 = 3 / X_0 = 2\} \cdot P\{X_0 = 2\} + P\{X_2 = 3 / X_0 = 3\} \cdot P(X_0 = 3)$$

$$= p_{13}^{(2)} P(X_0 = 1) + p_{23}^{(2)} P(X_0 = 2) + p_{23}^{(2)} P(X_0 = 3)$$

$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1$$

$$= 0.279$$

$P(A \cap B) = P(A) P(B)$

(b) $$P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$= P\{X_3 = 2 / X_2 = 3\} \cdot P\{X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$= p_{32}^{(1)} \cdot P\{X_2 = 3 / X_1 = 3\} \cdot P\{X_1 = 3, X_0 = 2\}$$

$$= p_{32}^{(1)} \cdot p_{33}^{(1)} \cdot P\{X_1 = 3 / X_0 = 2\} \cdot P\{X_0 = 2\}$$

$$= p_{32}^{(1)} \, p_{33}^{(1)} \cdot p_{23}^{(1)} \times P(X_0 = 2) = 0.4 \times 0.3 \times 0.2 \times 0.2 = 0.0048$$

Prob 4. A fair dice is tossed repeatedly. If X_n denotes the maximum of the numbers occurring in the first n tosses, find the transition probability matrix P of the Markov chain $\{X_n\}$.
Find also P^2 and $P\{X_2 = 6\}$.

Solution :- TPM

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rough

Upto 4 tosses (1,1,1,1) Max 1 — 5th toss Max (1,2,3,...,6) $\to \frac{1}{6}$

Max 2, (1,2) $\frac{2}{6}$ — Max (3,4,5,6) $\to 1/6$

Max 3, (1,2,3) . (4,5,6) — $\frac{3}{6}$, $\frac{1}{6}$

$$P^2 = \frac{1}{36}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

Initial state probability distribution is

$$p^{(0)} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

$$P\{X_2 = 6\} = \sum_{i=1}^{6} P\{X_2 = 6 / X_0 = i\} \times \underbrace{P\{X_0 = i\}}_{\to \frac{1}{6}}$$

$$= \frac{1}{6} \sum_{i=1}^{6} P_{i6}^{(2)}$$

$$= \frac{1}{6} \times \frac{1}{36} (11 + 11 + 11 + 11 + 11 + 36)$$

$$= 91/216$$

Classification of States of a Markov Chain :-

→ If $p_{ij}^{(n)} > 0$, for some n and for all i and j, then every state can be reached from every other state.

→ If above condition statisfied, the Markov chain is said to irreducible.

→ The tpm of an irreducible chain is irreducible matrix otherwise reducible / non irreducible

→ State 'i' of a Markov chain is called a return state, if $p_{ii}^{(n)} > 0$ for some $n > 1$.

→ The period d_i of a return state i is defined as the greatest common divisor of all m such that $p_{ii}^{(m)} > 0$, i.e. $d_i = GCD\{m : p_{ii}^{(m)} > 0\}$

→ State 'i' is said to be periodic with period d_i if $d_i > 1$ otherwise aperiodic if $d_i = 1$.

→ The probability that the chain returns to state i, having started from state i, for the first time at the nth step (or after n transitions) is denoted by $f_{ii}^{(n)}$ and called the first return time probability or the recurrence time probability.

→ $\{n, f_{ii}^{(n)}\}$, $n = 1, 2, 3, \ldots$, is the distribution of recurrence times of the state i.

→ The return to state 'i' is certain if $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$

→ The mean recurrence time of the state 'i' is

$$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$$

→ A state 'i' is said to be persistent or recurrent if the return to state 'i' is certain, i.e if $F_{ii} = 1$.

→ The state 'i' is said to be transient if the return to state 'i' is uncertain, i.e. if $F_{ii} < 1$.

→ If the mean recurrence time μ_{ii} is finite, then state 'i' is said to be nonnull persistent.

→ If $\mu_{ii} = \infty$, then state 'i' is said to be null persistent.

→ A non-null persistent and aperiodic state is called ergodic

Theorems for classification of states:-

1. If a Markov chain is irreducible, all its states are of the same type. They are all transient, all null persistent or all non-null persistent. All its states are either aperiodic or periodic with the same period.
2. If a Markov chain finite irreducible, all its states are non-null persistent.

Prob 1. Find the nature of the states of the Markov chain with the tpm

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Solution :- We have

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^3 = P, \quad \therefore P^4 = P^2 \text{ and so on}$$

In general $P^{2n} = P^2$, $P^{2n+1} = P$

Now, we can observe that

$$p_{00}^{(2)} > 0, \; p_{01}^{(1)} > 0, \; p_{02}^{(2)} > 0$$

$$p_{10}^{(1)} > 0, \; p_{11}^{(2)} > 0, \; p_{12}^{(1)} > 0$$

$$p_{20}^{(2)} > 0, \; p_{21}^{(1)} > 0, \; p_{22}^{(2)} > 0$$

$\therefore$ Markov chain is irreducible.

Rough

step 1/2 (non-zero)

$$\begin{matrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{matrix}$$

$\forall$ some n

All enteries are non-zero

Also, $p_{ii}^{(2)} = p_{ii}^{(4)} = p_{ii}^{(6)} \dots > 0$, for all i,

$$p_{ii}^{(n)} > 0, \text{ for some } n > 1$$

Each state(i) of Markov chain is return state.

$$d_i = GCD\{n : p_{ii}^{(n)} > 0\}$$

$$= GCD\{2, 4, 6, \dots\} = 2$$

$\therefore$ State i is periodic with period 2

Since, the chain is finite and irreducible.

Therefore, all its states are non-null persistent

Again, states are non-null persistent but periodic with period 2.

Hence, all states are not ergodic.

Prob 2. Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states.

Solution :-

$P(X_n = B \mid X_{n-1} = A) = 1, \quad P(X_n = C \mid X_{n-1} = B) = 1$

$P(A \mid C) = \frac{1}{2}, \quad P(B \mid C) = \frac{1}{2}$

The transition probability matrix of the process $\{X_n\}$ is given below :-

$$P = \begin{array}{cc} & \text{state } X_n \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{state } X_{n-1} \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

State of X_n depends only on states of X_{n-1} not on earlier states of X_{n-2}, X_{n-1} and so on.

Therefore, $\{X_n\}$ is a Markov chain.

Now, $P^2 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $P^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

Now, $p_{11}^{(3)} > 0$, $p_{12}^{(1)} > 0$, $p_{13}^{(2)} > 0$

$p_{21}^{(2)} > 0$, $p_{22}^{(2)} > 0$, $p_{23}^{(1)} > 0$

$p_{31}^{(1)} > 0$, $p_{32}^{(1)} > 0$, $p_{33}^{(2)} > 0$

$\therefore$ Markov chain is irreducible

$P^4 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$, $P^5 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{pmatrix}$, $P^6 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{8} & \frac{1}{2} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \end{pmatrix}$

Now, $p_{ii}^{(2)}, p_{ii}^{(3)}, p_{ii}^{(4)}, \ldots > 0 \quad \forall\, i = 2, 3$

$\therefore d_i = GCD\{m : p_{ii}^{(m)} > 0\}$

$= GCD\{2, 3, 4, 5, \ldots\} = 1$

$\therefore$ States 2 & 3 (i.e B & C) are periodic with period 1.

i.e aperiodic.

Now, $p_{ii}^{(3)}, p_{ii}^{(5)}, p_{ii}^{(6)}, \ldots > 0 \quad \forall\, i = 1$

$d_1 = GCD\{m : p_{11}^{(m)} > 0\}$

$= GCD\{3, 5, 6, \ldots\} = 1$

$\therefore$ State 1 (i.e. A) also aperiodic

Hence, all states are aperiodic.

Now, chain is finite and irreducible so all its states are non-null persistent.

Also, all states are non-null persistent and aperiodic.

Hence, all the states are ergodic.