Revised: 2/6/2021

Probability II

Michael C. Hackett

Assistant Professor, Computer Science

Community College of Philadelphia

Marginal Probability

- The contingency table below is for a hypothetical test of a machine learning algorithm's ability to predict if an image is a picture of a car.
- We could use the contingency table to find probabilities.

		Image		
		Car	Not a car	Total
ML Algorithm	Car	178	17	195
	Not a car	132	524	656
	Total	310	541	851

Marginal Probability

- Probability the ML algorithm classified a picture as being an image of a car:

$$P(\text{Prediction is Car}) = \frac{195}{851} \sim 0.23$$

		Image		
		Car	Not a car	Total
ML Algorithm Prediction	Car	178	17	$\boxed{195}$
	Not a car	132	524	656
	Total	310	541	$\boxed{851}$

Marginal Probability

- Probability of a picture being an image of a car:

$$P(\text{Image is Car}) = \frac{310}{851} \sim 0.36$$

		Image		
		Car	Not a car	Total
ML Algorithm Prediction	Car	178	17	195
	Not a car	132	524	656
	Total	$\boxed{310}$	541	$\boxed{851}$

Marginal Probability

- Both are examples of a **marginal probability**.
  - Marginal probabilities are based only on one variable

P(Prediction is Car)

- Probability is based only on the Algorithm's choice

P(Image is Car)

- Probability is based only on the content of the Image

Joint Probability

- Probability the ML algorithm classified a picture of a car and the Image was of a car:

$$P(\text{Prediction is Car and Image is Car}) = \frac{178}{851} \sim 0.21$$

		Image		
		Car	Not a car	Total
ML Algorithm Prediction	Car	$\boxed{178}$	17	195
	Not a car	132	524	656
	Total	310	541	$\boxed{851}$

- This is an example of a **joint probability**.
    - Probabilities based on two or more variables

Joint Probability

- Probability the ML algorithm classified a picture as not of a car and the Image was not of a car:

$$P(\text{Prediction is Not a car, Image is Not a car}) = \frac{524}{851} \sim 0.62$$

		Image		
		Car	Not a car	Total
ML Algorithm Prediction	Car	178	17	195
	Not a car	132	$\boxed{524}$	656
	Total	310	541	$\boxed{851}$

Joint Probability

- The contingency table converted to proportions

		Image		
		Car	Not a car	Total
ML Algorithm Prediction	Car	0.21	0.02	0.23
	Not a car	0.15	0.62	0.77
	Total	0.36	0.64	1.00

Joint Probability

- Joint probability distribution

Joint outcome	Probability
Prediction is Car, Image is Car	0.21
Prediction is Car, Image is Not a car	0.02
Prediction is Not a car, Image is Car	0.15
Prediction is Not a car, Image is Not a car	0.62
Total	1.00

- Does not indicate the accuracy of the algorithm

Conditional Probability

- Of the images that were actually an image of a car, what was the probability that ML algorithm classified it as a car?:

$$P(\text{Prediction is Car given Image is Car}) = \frac{178}{310} \sim 0.57$$

		Image		
		Car	Not a car	Total
ML Algorithm Prediction	Car	$\boxed{178}$	17	195
	Not a car	132	524	656
	Total	$\boxed{310}$	541	851

- This is an example of a **conditional probability**.
  - One probability based on another probability.
  - “Given” is often written “|”

Conditional Probability

- The conditional probability of A given B is computed using:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

  - *P*(Prediction is Car | Image is Car)
  - *P*(A and B) = 0.2092 (from Slide 6)
  - *P*(B) = Image is Car = 0.3643 (from Slide 4)

$$P(\text{Prediction is Car} \mid \text{Image is Car}) = \frac{0.2092}{0.3643} \sim 0.57$$

  - Same result from the previous slide

Conditional Probability

- Of the images the ML algorithm classified as a car, how many were actually an image of a car?:

$$P(\text{Image is Car} \mid \text{Prediction is Car}) = \frac{178}{195} \sim 0.91$$

		Image		
		Car	Not a car	Total
ML Algorithm Prediction	Car	$\boxed{178}$	17	$\boxed{195}$
	Not a car	132	524	656
	Total	310	541	851

Conditional Probability

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

- *P*(Image is Car | Prediction is Car)
- *P*(A and B) = 0.2092 (from Slide 8)
- *P*(B) = Prediction is Car = 0.23 (from Slide 3)

$$P(\text{Image is Car} \mid \text{Prediction is Car}) = \frac{0.21}{0.23} \sim 0.91$$

- Same result from the previous slide

Conditional Probability

$$P(\text{Prediction is Car} \mid \text{Image is Car}) = \frac{0.2092}{0.3643} \sim 0.57$$

The probability the ML algorithm will predict “Car” given that the image is a car: ~57%

- In other words: If an image is of a Car, there is a ~57% probability the algorithm will correctly predict Car

$$P(\text{Image is Car} \mid \text{Prediction is Car}) = \frac{0.21}{0.23} \sim 0.91$$

The probability that an image is of a car given that the ML algorithm predicted “Car”: ~91%

- In other words: If the algorithm predicted Car, there is a ~91% probability it was correct (the image is of a Car)

General Multiplication Rule

- Recall from the previous lecture that the multiplication rule is used for calculating probabilities for multiple independent events occurring:
  - Where A and B are two different and independent events

$$P(A \text{ and } B) = P(A) \times P(B)$$

- The **General Multiplication Rule** allows calculating probabilities for events that might not be independent:
  - Where A and B are two different events

$$P(A \text{ and } B) = P(A \mid B) \times P(B)$$

General Multiplication Rule

- We previously determined that:

$$P(\text{Prediction is Car and Image is Car}) = \frac{178}{851} \sim 0.21$$

		Image		
		Car	Not a car	Total
ML Algorithm Prediction	Car	$\boxed{178}$	17	195
	Not a car	132	524	656
	Total	310	541	$\boxed{851}$

- We can calculate/verify this using only the General Multiplication Rule

General Multiplication Rule

$$P(A \text{ and } B) = P(A \mid B) \times P(B)$$

- *P*(Prediction is Car | Image is Car)
- *P*(A | B) = 0.57 (from Slide 11)
- *P*(B) = Image is Car = 0.3643 (from Slide 4)

$$P(\text{Prediction is Car and Image is Car}) = 0.57 \ \times 0.364 \sim 0.21$$

- Same result from the previous slide

General Multiplication Rule

- The formula previously shown for calculating conditional probabilities is simply a re-arrangement of the general multiplication rule.
  - General Multiplication Rule

$$P(A \text{ and } B) = P(A \mid B) \times P(B)$$

  - Conditional Probability Formula

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

The Gambler’s Fallacy

- In the previous lecture, we saw that the events of a Roulette wheel are independent.
  - Knowing the previous number has no effect on the next number.

- Let’s say 8 red numbers came up in a row.
  - The probability of 8 red numbers showing up in a row is

$$\left(\frac{18}{37}\right)^8 \sim 0.003 \sim 0.3\%$$

- The probability of 9 red numbers showing up in a row is

$$\left(\frac{18}{37}\right)^9 \sim 0.001 \sim 0.1\%$$

- With only a 0.1% probability of 9 red numbers appearing in a row, one might think there is an almost definite chance the next number will be black

The Gambler's Fallacy

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(\mathit{Black}\,|\,\mathit{8\ Reds\ in\ a\ row}) = \frac{P(\mathit{Black} \text{ and } \mathit{8\ Reds\ in\ a\ row})}{P(\mathit{8\ Reds\ in\ a\ row})}$$

$P(\mathit{8\ Reds\ in\ a\ row})$ = $(\frac{18}{37})^8 \sim 0.003137$

$P(\mathit{Black})$ = $\frac{18}{37} \sim 0.486486$

$P(\mathit{Black} \text{ and } \mathit{8\ Reds\ in\ a\ row}) = P(\mathit{Black}) \times P(\mathit{8\ Reds\ in\ a\ row}) \sim 0.001526$

$\boldsymbol{P(\mathit{Black}\,|\,\mathit{8\ Reds\ in\ a\ row}) = \frac{0.001526}{0.003127} \sim .486}$

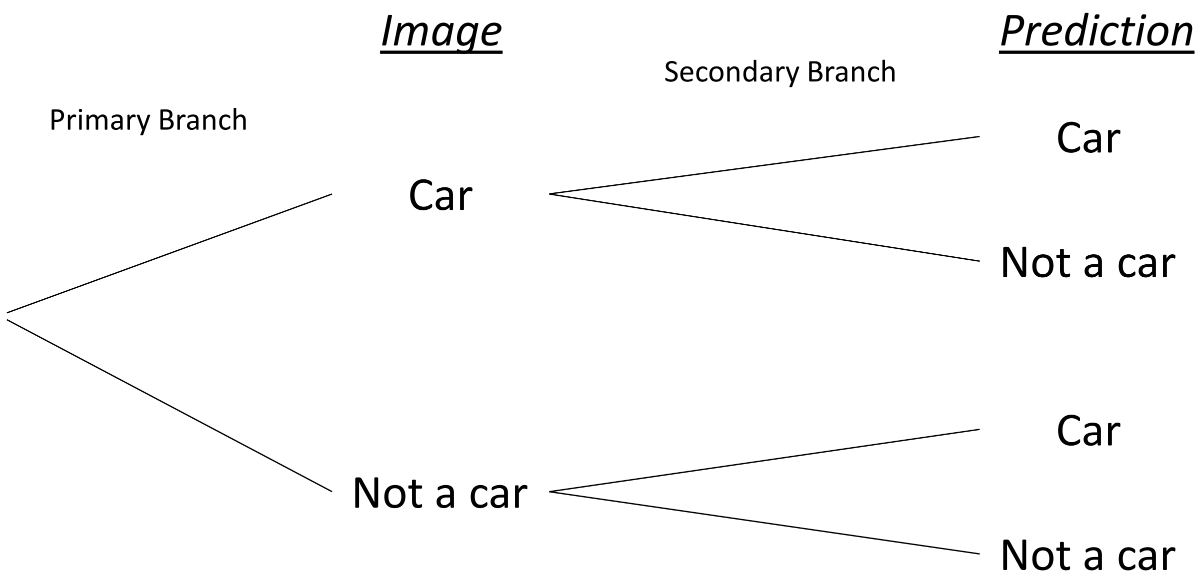
The Gambler’s Fallacy

- The probability of a black number given 8 reds appearing in a row is still .486
  - The same probability we determined for a black number appearing on any spin

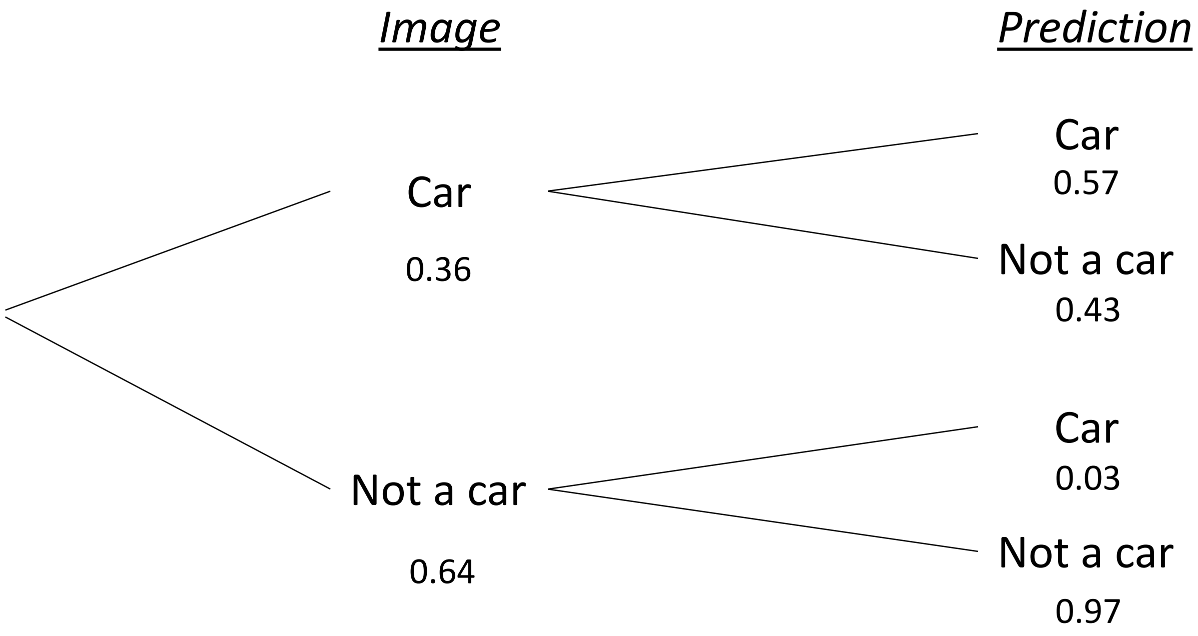
- Thus, whether no preceding spins, 8 red numbers preceding a spin, or 100 red numbers preceding a spin, the probability of a black number appearing on the next spin is still .486
  - Mathematically shows that each event resulting from a spin is independent.

Tree Diagrams

- Tree diagrams visually organize outcomes and probabilities.



Tree Diagrams



$0.36 \times 0.57 = 0.21$

$0.36 \times 0.43 = 0.15$

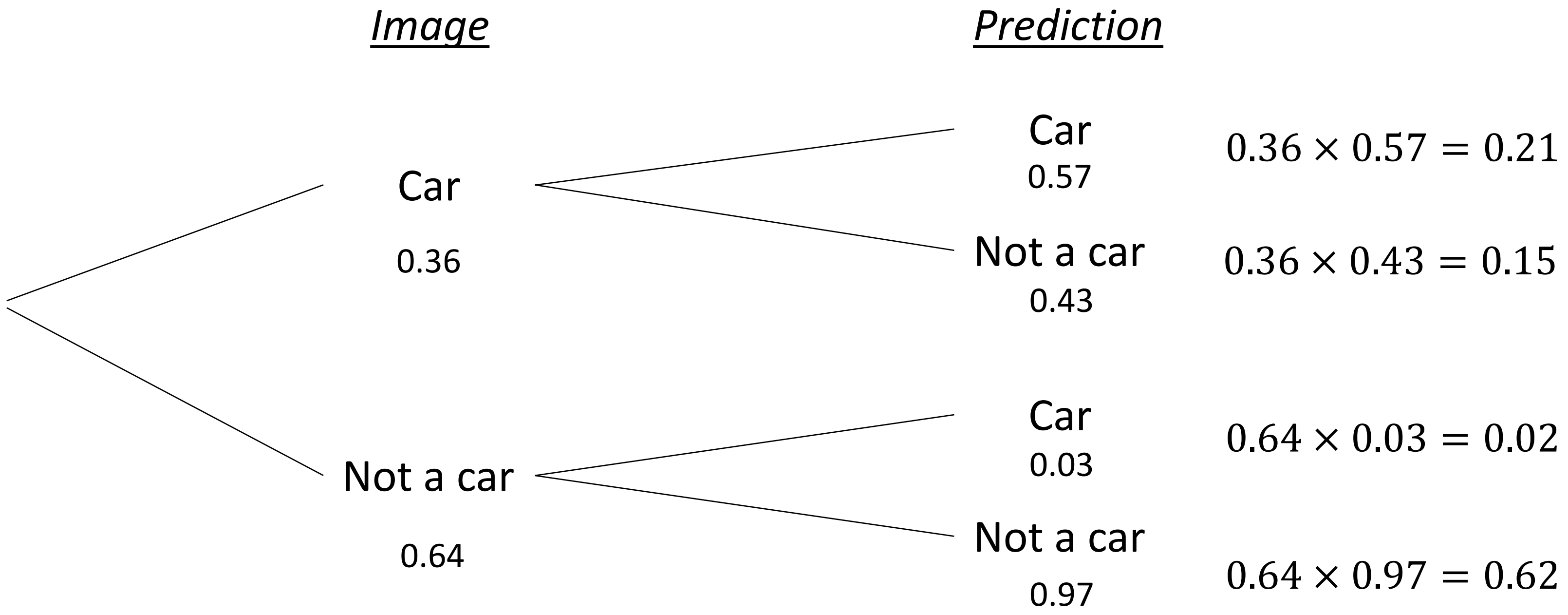
$0.64 \times 0.03 = 0.02$

$0.64 \times 0.97 = 0.62$

Tree Diagrams

$P(\text{Prediction is Car} \mid \text{Image is Car}) = 0.57$

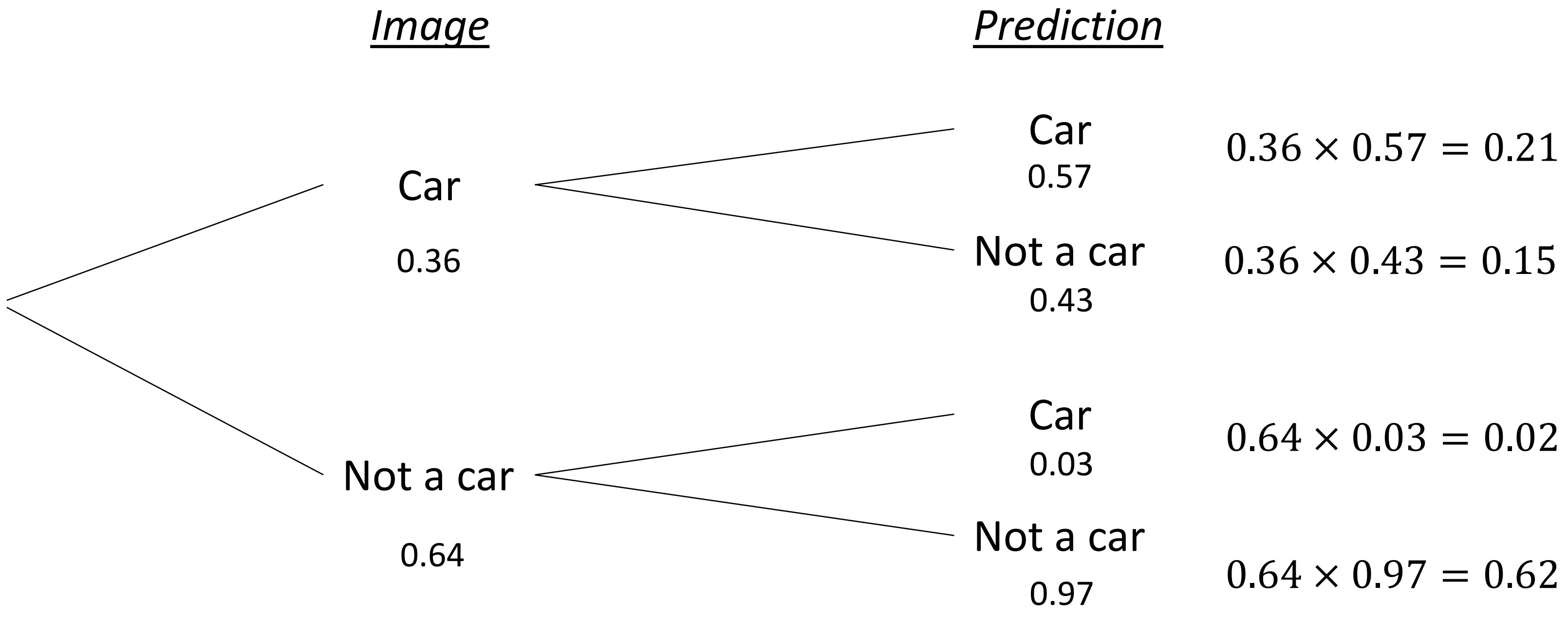
$P(\text{Image is Car and Prediction is Car}) = 0.21$



Tree Diagrams

$P(\text{Prediction is Car} \mid \text{Image is Not a car}) = 0.03$

$P(\text{Image is Not a car and Prediction is Car}) = 0.62$



Tree Diagrams

- The last slides gave the conditional probability

$$P(\text{Prediction is Car} \mid \text{Image is a car})$$
$$P(\text{Prediction is Car} \mid \text{Image is Not a car})$$

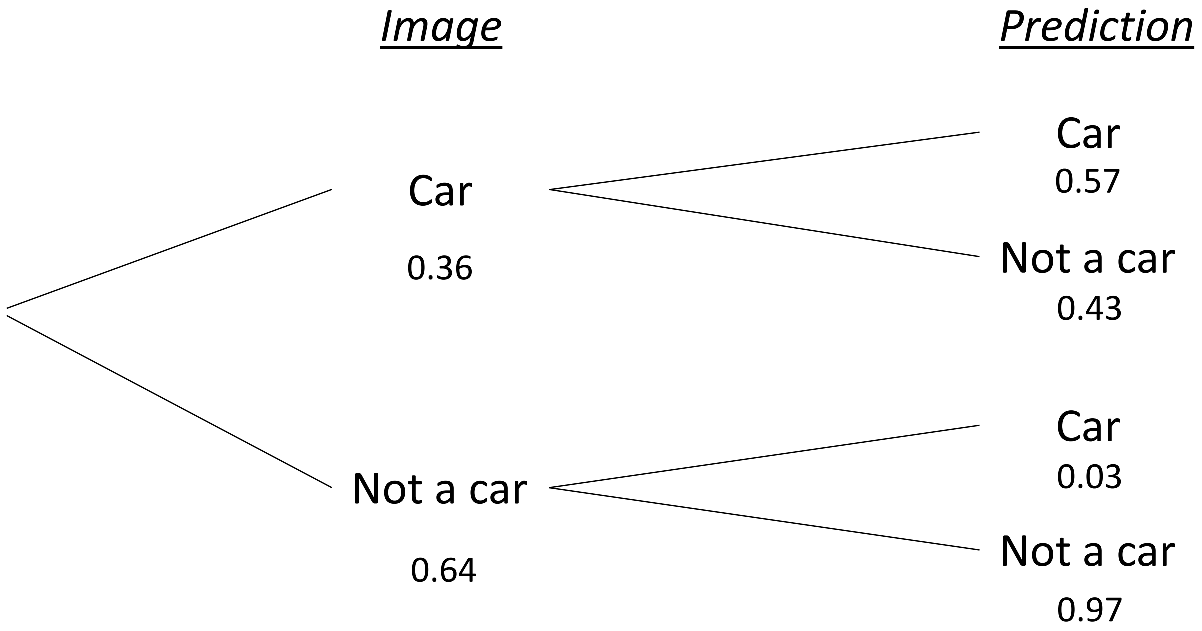
- What if we wanted to find the inverse?

$$P(\text{Image is a car} \mid \text{Prediction is Car})$$
$$P(\text{Image is Not a car} \mid \text{Prediction is Car})$$

Tree Diagrams

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(\text{Image is a car} \mid \text{Prediction is a car}) = \frac{0.21}{0.23} = 0.91$$



$$0.36 \times 0.57 = 0.21$$

$$0.36 \times 0.43 = 0.15$$

$$0.64 \times 0.03 = 0.02$$

$$0.64 \times 0.97 = 0.62$$

Bayes' Theorem

- Bayes' Theorem is a generalization of inverting a conditional probability

$$P(A_1 \mid B) = P(\text{Image is a car} \mid \text{Prediction is a car})$$

$$P(B \mid A_1) = P(\text{Prediction is a car} \mid \text{Image is a car})$$

$$P(B \mid A_1) = \frac{P(B \mid A_1) \times P(A_1)}{P(B \mid A_1) \times P(A_1) + P(B \mid A_2) \times P(A_2) + \cdots + P(B \mid A_n) \times P(A_n)}$$

  - Where A_1, A_2, …, A_n are the different outcomes of the first variable

$P(A_1 | B) = P(\text{Image is a car} \mid \text{Prediction is a car})$
$P(B \mid A_1) = P(\text{Prediction is a car} \mid \text{Image is a car})$

Bayes' Theorem

- To apply Bayes' Theorem:
  - First, identify the marginal probabilities of each possible outcome of the first variable: $P(A_1), P(A_2), \ldots, P(A_n)$

    $P(A_1) = P(\text{Image is a car}) = 0.36$

    $P(A_2) = P(\text{Image is Not a car}) = 0.64$

$P(A_1 | B) = P(\text{Image is a car} \mid \text{Prediction is a car})$
$P(B \mid A_1) = P(\text{Prediction is a car} \mid \text{Image is a car})$

Bayes' Theorem

- To apply Bayes' Theorem:
  - Second, identify the probability of the outcome B, conditioned on each possible scenario for the first variable: $P(B \mid A_1)$, $P(B \mid A_2)$, …, $P(B \mid A_n)$

$$P(B \mid A_1) = P(\text{Prediction is a car} \mid \text{Image is a car}) = \frac{P(\text{Prediction is a car and Image is a car})}{P(\text{Image is a car})} = 0.57$$

$$P(B \mid A_2) = P(\text{Prediction is a car} \mid \text{Image is Not a car}) = \frac{P(\text{Prediction is a car and Image is Not a car})}{P(\text{Image is a car})} = 0.03$$

$$P(B \mid A_1) = \frac{P(B \mid A_1) \times P(A_1)}{P(B \mid A_1) \times P(A_1) + P(B \mid A_2) \times P(A_2) + \cdots + P(B \mid A_n) \times P(A_n)}$$

Bayes' Theorem

$P(A_1 \mid B) = P(\text{Image is a car} \mid \text{Prediction is a car})$

$P(A_2 \mid B) = P(\text{Image is Not a car} \mid \text{Prediction is a car})$

$P(B \mid A_1) = P(\text{Prediction is a car} \mid \text{Image is a car})$

$P(B \mid A_2) = P(\text{Prediction is a car} \mid \text{Image is Not a car})$

$$P(B \mid A_1) = \frac{\mathbf{0.57 \times 0.36}}{0.57 \times 0.36 + 0.03 \times 0.64} = \frac{0.21}{0.21 + 0.02} = \frac{0.21}{0.23} = 0.91$$

$$P(B \mid A_2) = \frac{\mathbf{0.03 \times 0.64}}{0.57 \times 0.36 + 0.03 \times 0.64} = \frac{0.02}{0.21 + 0.02} = \frac{0.02}{0.23} = 0.09$$

- Notice the numerator is the only term that changes
  - This is a useful for when it is too cumbersome to create tree diagrams

Sampling Small Populations

- When sampling large populations, its unlikely we sample the same cases more than once.
  - For example, picking five numbers between 1 and 1 million
- With small populations, it becomes more likely that we might sample the same cases more than once.
  - For example, picking five numbers between 1 and 10

Sampling Small Populations

- If we sample *with replacement* it means a case may be sampled more than once.

- Imagine a bag of M&M's with an equal number (5) of red, brown, blue, green, orange, and yellow candies (30 candies total)

- The probability of drawing a red M&M is $\frac{5}{30}$
  - If we put the red M&M back in the bag, the chance of drawing a red M&M (or any other color) remains $\frac{5}{30} \sim .17$
  - The events of drawing M&Ms are independent.

Sampling Small Populations

- If we sample *without replacement* it means a case will only be sampled once.

- Imagine the same bag of M&M's
- The probability of drawing a red M&M is $\frac{5}{30}$
  - We remove/eat the M&M
  - The chance of drawing a red M&M is now $\frac{4}{29} \sim .14$
  - The chance of drawing any other color is now $\frac{5}{29} \sim .17$
  - The events of drawing M&Ms are no longer independent

Sampling Small Populations

- The probability of drawing red then blue then green then orange
  - With replacement

$$\frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \sim .0007$$

  - Without replacement

$$\frac{5}{30} \times \frac{5}{29} \times \frac{5}{28} \times \frac{5}{27} \sim .0009$$

- Drawing without replacement has a greater probability of drawing red then blue then green then orange

Sampling Small Populations

- The probability of drawing five red M&M's in a row
  - With replacement

$$\frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \sim .0001286$$

  - Without replacement

$$\frac{5}{30} \times \frac{4}{29} \times \frac{3}{28} \times \frac{2}{27} \times \frac{1}{26} \sim .0000070$$

- Drawing with replacement has a greater probability of drawing five red M&M's in a row