Revised: 2/8/2021

Probability I

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Probability

- **Probability** is the proportion of times an outcome occurs in a random process if we observed the process an infinite number of times.

- Probability is used to describe and better understand apparent randomness.

- A random process is used to generate outcomes.
    - Such as flipping a coin.

Probability

- When you flip a (fair and unbiased) coin, it has and equal chance of landing on heads or tails (or “face up”/“face down”).
  - We’ll use **H** for heads and **T** for tails

- Probability is a proportion; The probability of an event happening is between 0 and 1
  - Or, between 0% and 100% chance of happening.

Probability

- There are only two possible outcomes when flipping a coin
  - It lands on heads or it lands on tails

- Only <u>one</u> of the <u>two</u> possible outcomes result in the coin landing on heads
  - Probability of landing on heads = $\frac{1}{2}$ = 0.50 = 50%
  - $P(\mathrm{H}) = 0.5$

- Likewise, only <u>one</u> of the <u>two</u> possible outcomes result in the coin landing on tails
  - Probability of landing on tails = $\frac{1}{2}$ = 0.50 = 50%
  - $P(\mathrm{T}) = 0.5$

- The probabilities of all possible outcomes must add up to 1 (100%)
  - $P(\mathrm{H}) + P(\mathrm{T}) = 0.5 + 0.5 = 1.0$

Probability

- Rolling a (fair and unbiased) die demonstrates a comparable example.
  - Only six possible outcomes when rolling a die; numbers 1 through 6 all have an equal chance of being rolled

- Each number has the same probability ($\frac{1}{6} \sim 0.167 \sim 16.7\%$) of being rolled
  - $P(1)$ = ~0.167
  - $P(2)$ = ~0.167
  - $P(3)$ = ~0.167
  - $P(4)$ = ~0.167
  - $P(5)$ = ~0.167
  - $P(6)$ = ~0.167

Probability

- A Roulette wheel is another example used to demonstrate probability.
- Equally sized slots on the wheel are numbered 0 through 36
  - Excluding 0, half of the numbers are even, and the other half are odd.
  - 0 is colored green, half of the other numbers are colored red, and the other half are colored black.

Probability

- Each number has the same probability ($\frac{1}{37} \sim 0.027 \sim 2.7\%$) of being chosen
  - $P(0) = \frac{1}{37}$ = ~0.027
  - $P(7)$ = ~0.027
  - $P(36)$ = ~0.027

- There are 18 red numbers. The probability of a *red* number being chosen is:
  - $P(\text{Red}) = \frac{18}{37}$ = ~0.468 = ~48.6%

Probability

- We’ve established that if we flip a fair and unbiased coin once, there is a 50%-50% chance of either outcome (heads or tails) being observed.
  - If we flip it again, there is still a 50%-50% chance of either outcome being observed.
  - If we flip it seven more times, there is still a 50%-50% chance of either outcome being observed on each flip.

- If we flipped the coin ten times, and each outcome has a 50%-50% chance of being observed, then we would expect half of the flips result in heads and the other half of the flips result in tails.

Probability

- Despite this, it’s reasonable to believe that if you flipped a coin ten times, you might not get an equal number of heads and tails outcomes.
    - You might get 6 heads and 4 tails, or maybe 8 heads and 2 tails

- Similarly, it’s reasonable to believe that if you rolled a die six times, you might not roll one of each number.
    - You might get three 4’s, two 1’s, and a 5.

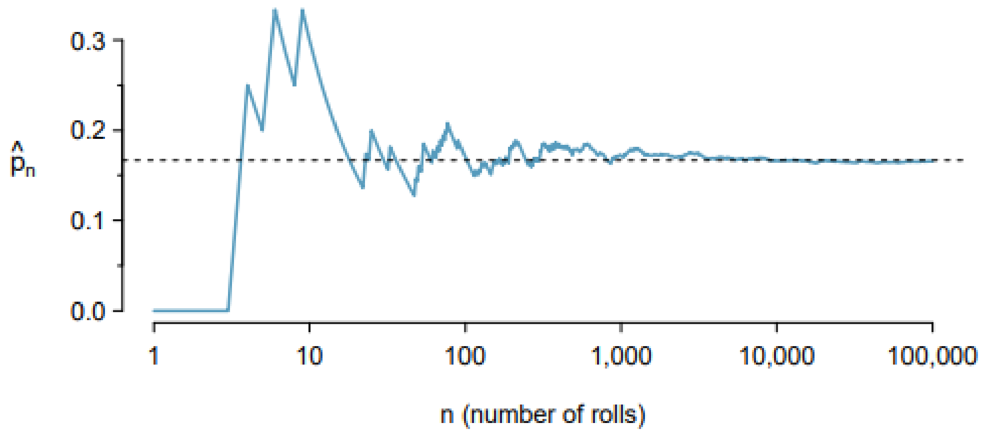
Probability

- Recall the definition of probability:
  - *The proportion of times an outcome occurs in a random process if we observed the process* ***an infinite number of times***.
  - Flipping a coin ten times or rolling a die six times is nowhere near an *infinite* number of times

- **Law of Large Numbers**
  - As more observations are collected, the proportion $\hat{p}_n$ of occurrences with a particular outcome converges to the probability of that outcome.
  - Where $\hat{p}_n$ is the proportion of a particular outcome after n observations

Probability

- Going back to the example of rolling a die:
  - If we let $\hat{p}_n$ be the probability of rolling a 1 after the first n rolls, then as n (the number of rolls) increases, $\hat{p}_n$ will converge on the probability of rolling a 1 ($\sim .167 \sim$ 16.7%)



- As an infinite number of rolls is approached, $\hat{p}_n$ will stabilize closer and closer to the probability of that outcome

Disjoint Outcomes

- Outcomes are **disjoint** (or *mutually exclusive*) if they cannot happen simultaneously.
- Flipping a coin results in heads or tails; They are <u>disjoint</u> outcomes because they cannot both occur at the same time.
- If you roll a die, the outcomes of “rolling a 2” and “rolling an even number” are <u>not disjoint</u> outcomes; Both outcomes can occur simultaneously when a 2 is rolled.

Disjoint Outcomes

- **Addition Rule**
  - The probability that one of multiple disjoint outcomes occurs is the sum of their separate probabilities.

  - Where A_1 and A_2 are two disjoint outcomes

$$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$

Disjoint Outcomes

- Probability of rolling a 1 or a 5 on a single roll
  - $P(1) = \frac{1}{6} = {\sim}.167$
  - $P(5) = \frac{1}{6} = {\sim}.167$

  - $P(1 \text{ or } 5) = P(1) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = {\sim}.33 = {\sim}33\%$

Disjoint Outcomes

- Probability of rolling an odd number on a single roll
  - $P(1) = \frac{1}{6} = \sim.167$
  - $P(3) = \frac{1}{6} = \sim.167$
  - $P(5) = \frac{1}{6} = \sim.167$

  - $P(1 \text{ or } 3 \text{ or } 5) = P(1) + P(3) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} = 50\%$

Disjoint Outcomes

- Probability of a 6 or an odd number on a single roll
  - $P(\text{odd}) = P(1 \text{ or } 3 \text{ or } 5) = \frac{1}{2} = .50$
  - $P(6) = \frac{1}{6} = \sim .167$

  - $P\big((1 \text{ or } 3 \text{ or } 5) \text{ or } 6\big) = P(1 \text{ or } 3 \text{ or } 5) + P(6) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = \sim .667 = \sim 66\%$

Disjoint Outcomes

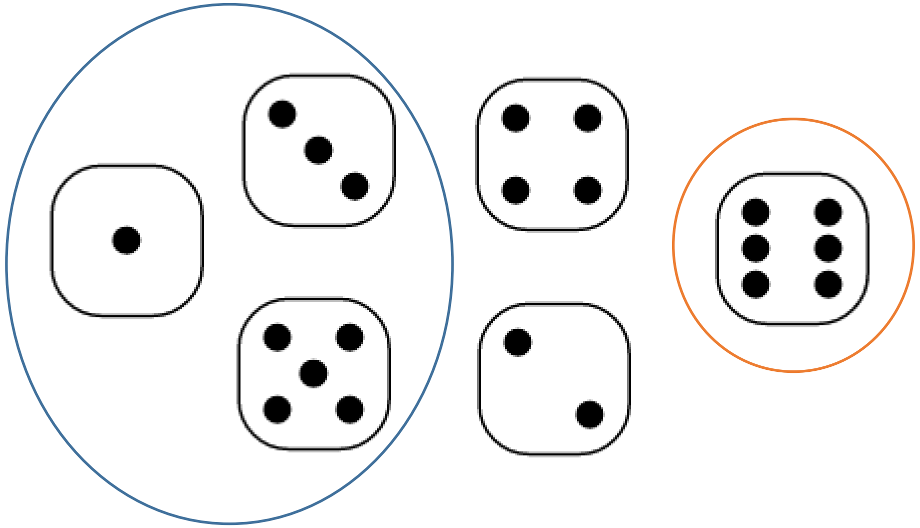
- Probability of landing on a red number or zero on a single spin
  - $P(\text{Red}) = \frac{18}{37} = \sim.486 = \sim48.6\%$
  - $P(0) = \frac{1}{37} = \sim.027 = \sim2.7\%$
  - $P(\text{Red or } 0) = \frac{18}{37} + \frac{1}{37} = \sim.514 = \sim51.4\%$

Events

- An **event** is simply a set of possible outcomes.
  - The set of all possible outcomes is called the *sample space*.
    - Sample space of rolling a die is {1, 2, 3, 4, 5, 6}
- The previous example of rolling an odd number can be described as an event.
  - This event would be written as {1, 3, 5}
- Like outcomes, events can be disjoint
  - Rolling a 6 or an odd number are disjoint events: {6} and {1, 3, 5}
  - Rolling a 4 or an even number are not disjoint events: {**4**} and {2, **4**, 6}

Events

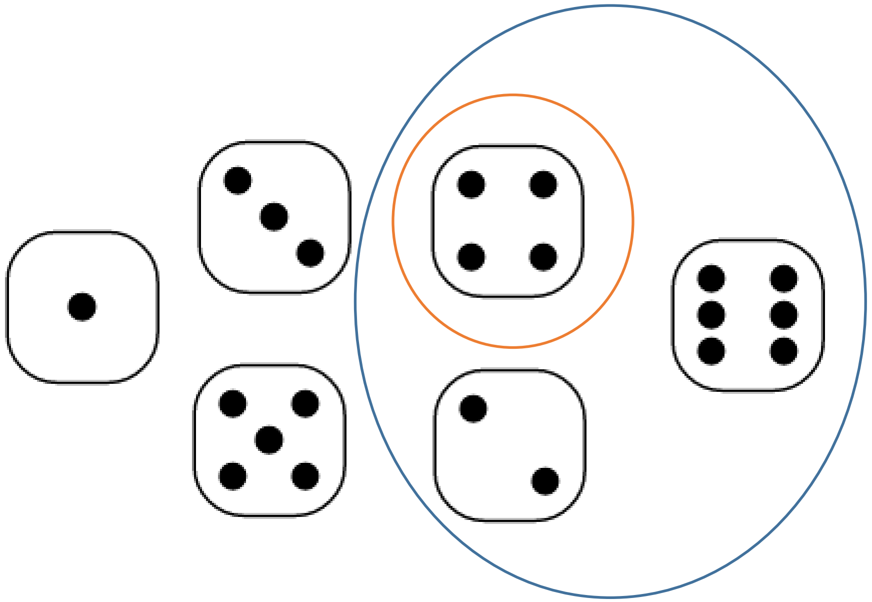
- Venn Diagrams are useful for visualizing if events are disjoint.
- Rolling a 6 or an odd number:



- The rings do not overlap, meaning the events are disjoint.

Events

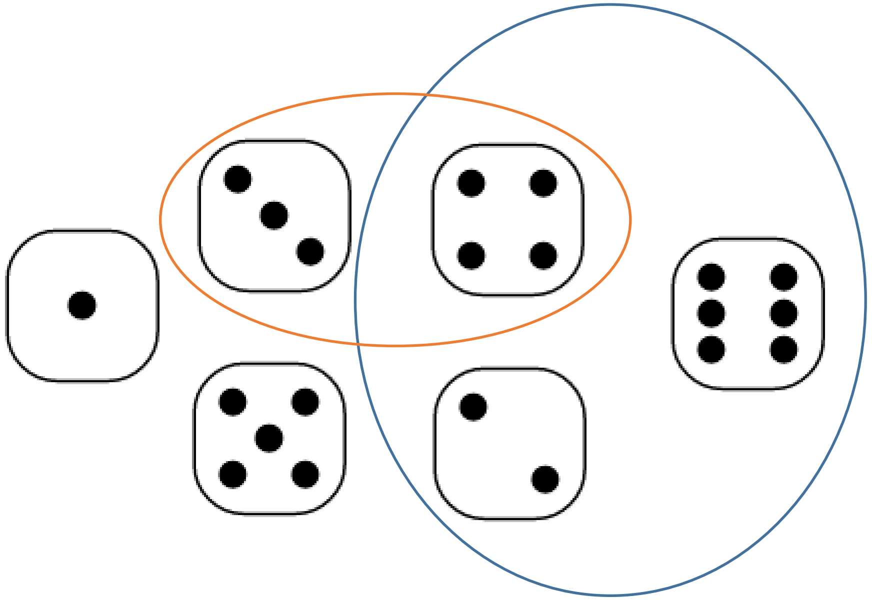
- Rolling a 4 or an even number:



- The rings do overlap, meaning the events are not disjoint.

Events

- Rolling a 3, a 4, or an even number:



- The rings do overlap, meaning the events are not disjoint.

Events

- We previously saw the Addition Rule for calculating probabilities for disjoint outcomes and events.

- The **General Addition Rule** calculates probabilities for both disjoint *and not disjoint* events:
  - Where A and B are any two events

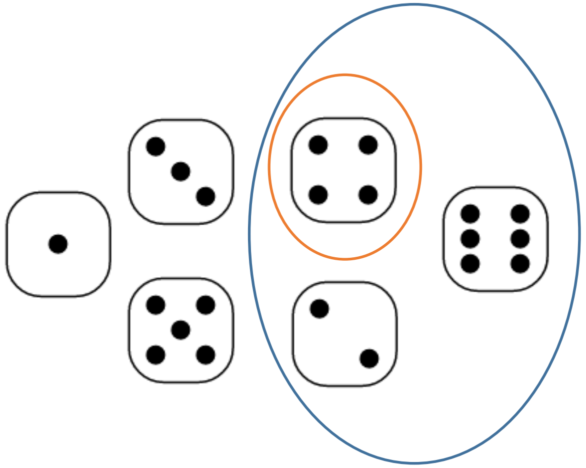
$$P(A \text{ or } B) = P(A) + P(B) \ - P(A \text{ and } B)$$

  - and where $P(A \text{ and } B)$ is the probability that **both** events occur

Events

- Rolling a 4 (A) or an even number (B):
  - (Not disjoint events)
  - $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
  - $P(A) = \frac{1}{6} = \sim .167$
  - $P(B) = \frac{3}{6} = \frac{1}{2} = .50$
  - $P(A \text{ and } B) = \frac{1}{6} = \sim .167$

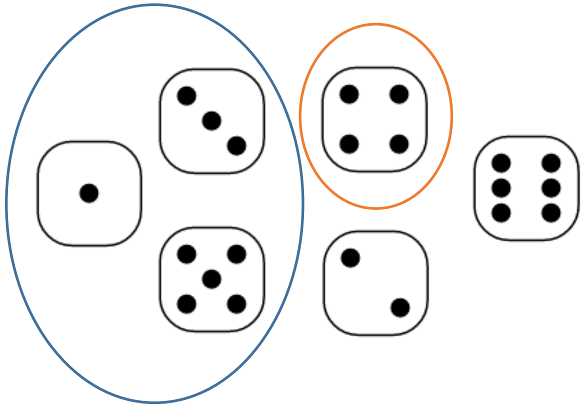
  - $P(A \text{ or } B) = \frac{1}{6} + \frac{3}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2} = .50 = 50\%$



Events

- Rolling a 4 (A) or an odd number (B):
  - (Disjoint events)
  - $P(A \text{ or } B) = P(A) + P(B) \; - P(A \text{ and } B)$
  - $P(A) = \frac{1}{6} = \sim.167$
  - $P(B) = \frac{3}{6} = \frac{1}{2} = .50$
  - $P(A \text{ and } B) = \frac{0}{6} = 0$ *(always 0 for disjoint events)*

  - $P(A \text{ or } B) = \frac{1}{6} + \frac{3}{6} - \frac{0}{6} = \frac{4}{6} = \frac{2}{3} = \sim.667 = \sim 66.7\%$



Events

- Probability of landing on a red number(A) or even number(B)
  - (Not disjoint events)
  - $P(A \text{ or } B) = P(A) + P(B) \; - P(A \text{ and } B)$
  - $P(A) = \frac{18}{37} = \sim.486 = \sim48.6\%$
  - $P(B) = \frac{18}{37} = \sim.486 = \sim48.6\%$
  - $P(A \text{ and } B) = \frac{8}{18} = \sim.444 = \sim44\%$

  - $P(A \text{ or } B) = \frac{18}{37} + \frac{18}{37} - \frac{8}{18} = \sim.529 = \sim52.9\%$

Events

- The **complement** of an event is the set of outcomes that are not in the event.
  - Sample space: {1, 2, 3, 4, 5, 6}
  - Event: Rolling a 6 or an odd number = {6, 1, 3, 5}
  - Event complement: {2, 4}

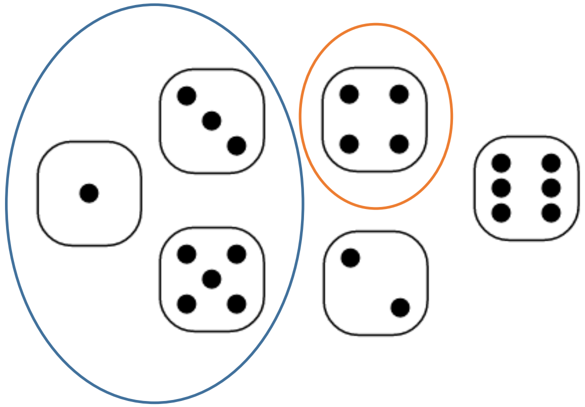
- Where A is the event and A^C is the complement:

$$P(A) + P(A^C) = 1$$

Events

- Rolling a 4 or an odd number:
  - A = {1, 3, 4, 5}
  - A^C = {2, 6}

  - $P(A) = \frac{4}{6} = \frac{2}{3} = \sim.667 = \sim66.7\%$
  - $P(A^C) = \frac{2}{6} = \frac{1}{3} = \sim.334 = \sim33.4\%$
    - *(The probability of **not** rolling a 4 or an odd number)*



Probability Distributions

- A probability distribution is a table of all possible outcomes and their corresponding probabilities.

Coin Side	Heads	Tails
Probability	$\frac{1}{2}$	$\frac{1}{2}$

Die Face	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- A probability distribution has the following characteristics:
  - Only lists outcomes that are disjoint
  - Each outcome’s probability is between 0 and 1
  - The sum of all probabilities must be 1

Probability Distributions

- Probability of landing on the different colors.

Color	Green	Red	Black
Probability	$\frac{1}{37}$	$\frac{18}{37}$	$\frac{18}{37}$

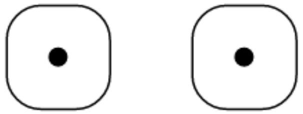
- Probability of landing on an even or odd number

Evenness	Neither (0)	Even	Odd
Probability	$\frac{1}{37}$	$\frac{18}{37}$	$\frac{18}{37}$

Probability Distributions

- If we roll two dice and sum the numbers, the smallest sum we can have is 2 (1+1) and the largest sum we can have is 12 (6+6)
  - In fact, the only way we can have a sum of two is by rolling two 1's and a sum of twelve by rolling two 6's.

  2 : 1+1



  3 :

  4 :

  5 :

  6 :

  7 :

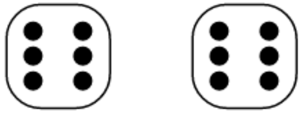
  8 :

  9 :

  10 :

  11 :

  12 : 6+6

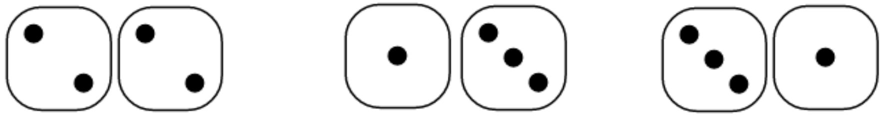


Probability Distributions

- The ways in which the sum is three
  - 2 : 1+1
  - 3 : **2+1, 1+2**
  - 4 :
  - 5 :
  - 6 :
  - 7 :
  - 8 :
  - 9 :
  - 10 :
  - 11 :
  - 12 : 6+6

Probability Distributions

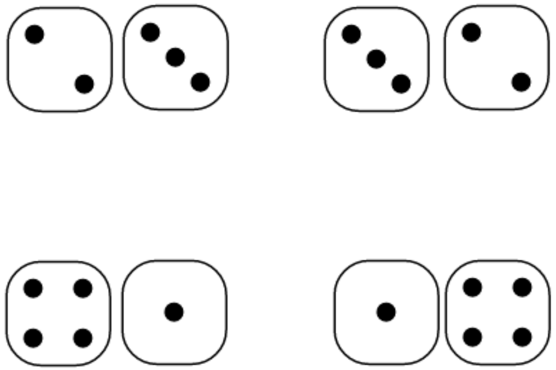
- The ways in which the sum is four
  - 2 : 1+1
  - 3 : 2+1, 1+2
  - 4 : **2+2, 1+3, 3+1**
  - 5 :
  - 6 :
  - 7 :
  - 8 :
  - 9 :
  - 10 :
  - 11 :
  - 12 : 6+6



Probability Distributions

- The ways in which the sum is five

  2 : 1+1
  3 : 2+1, 1+2
  4 : 2+2, 1+3, 3+1
  5 : **1+4, 4+1, 3+2, 2+3**
  6 :
  7 :
  8 :
  9 :
  10 :
  11 :
  12 : 6+6



Probability Distributions

- The ways in which the sum is six

  2 : 1+1

  3 : 2+1, 1+2

  4 : 2+2, 1+3, 3+1

  5 : 1+4, 4+1, 3+2, 2+3

  6 : **5+1, 1+5, 4+2, 2+4, 3+3**

  7 :

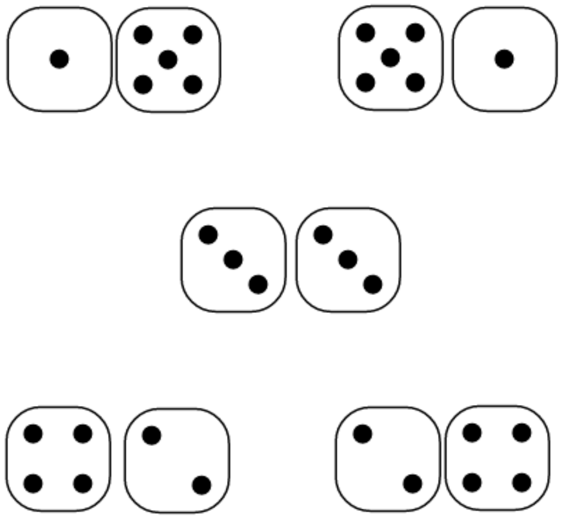
  8 :

  9 :

  10 :

  11 :

  12 : 6+6



Probability Distributions

- The ways in which the sum is seven

  2 : 1+1

  3 : 2+1, 1+2

  4 : 2+2, 1+3, 3+1

  5 : 1+4, 4+1, 3+2, 2+3

  6 : 5+1, 1+5, 4+2, 2+4, 3+3

  7 : **6+1, 1+6, 5+2, 2+5, 4+3, 3+4**

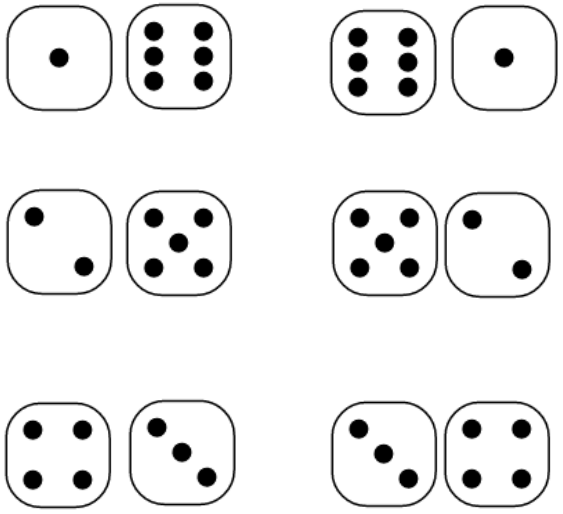
  8 :

  9 :

  10 :

  11 :

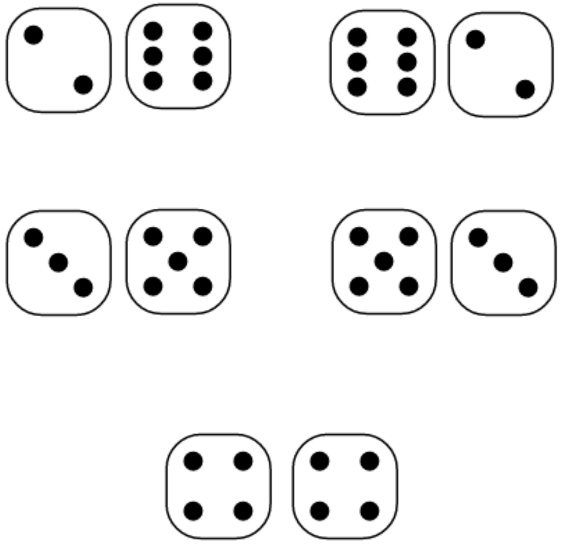
  12 : 6+6



Probability Distributions

- The ways in which the sum is eight

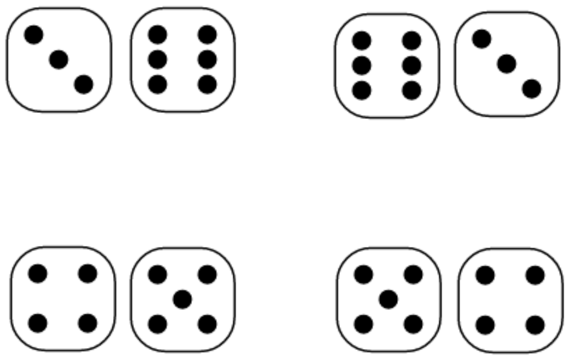
  2 : 1+1
  3 : 2+1, 1+2
  4 : 2+2, 1+3, 3+1
  5 : 1+4, 4+1, 3+2, 2+3
  6 : 5+1, 1+5, 4+2, 2+4, 3+3
  7 : 6+1, 1+6, 5+2, 2+5, 4+3, 3+4
  8 : **6+2, 2+6, 5+3, 3+5, 4+4**
  9 :
  10 :
  11 :
  12 : 6+6



Probability Distributions

- The ways in which the sum is nine

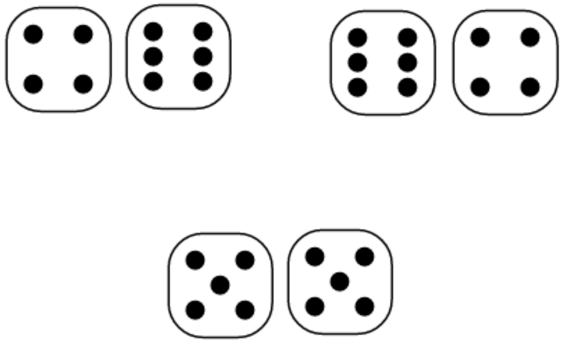
  2 : 1+1
  3 : 2+1, 1+2
  4 : 2+2, 1+3, 3+1
  5 : 1+4, 4+1, 3+2, 2+3
  6 : 5+1, 1+5, 4+2, 2+4, 3+3
  7 : 6+1, 1+6, 5+2, 2+5, 4+3, 3+4
  8 : 6+2, 2+6, 5+3, 3+5, 4+4
  9 : **6+3, 3+6, 5+4, 4+5**
  10 :
  11 :
  12 : 6+6



Probability Distributions

- The ways in which the sum is ten

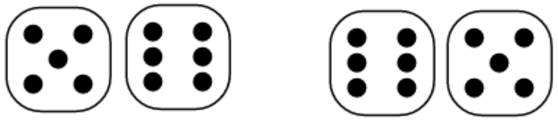
  2 : 1+1
  3 : 2+1, 1+2
  4 : 2+2, 1+3, 3+1
  5 : 1+4, 4+1, 3+2, 2+3
  6 : 5+1, 1+5, 4+2, 2+4, 3+3
  7 : 6+1, 1+6, 5+2, 2+5, 4+3, 3+4
  8 : 6+2, 2+6, 5+3, 3+5, 4+4
  9 : 6+3, 3+6, 5+4, 4+5
  10 : **6+4, 4+6, 5+5**
  11 :
  12 : 6+6



Probability Distributions

- The ways in which the sum is eleven

  2 : 1+1
  3 : 2+1, 1+2
  4 : 2+2, 1+3, 3+1
  5 : 1+4, 4+1, 3+2, 2+3
  6 : 5+1, 1+5, 4+2, 2+4, 3+3
  7 : 6+1, 1+6, 5+2, 2+5, 4+3, 3+4
  8 : 6+2, 2+6, 5+3, 3+5, 4+4
  9 : 6+3, 3+6, 5+4, 4+5
  10 : 6+4, 4+6, 5+5
  11 : **6+5, 5+6**
  12 : 6+6



Probability Distributions

- Combinations

2 : 1+1	1 possible combination
3 : 2+1, 1+2	2 possible combinations
4 : 2+2, 1+3, 3+1	3 possible combinations
5 : 1+4, 4+1, 3+2, 2+3	4 possible combinations
6 : 5+1, 1+5, 4+2, 2+4, 3+3	5 possible combinations
7 : 6+1, 1+6, 5+2, 2+5, 4+3, 3+4	6 possible combinations
8 : 6+2, 2+6, 5+3, 3+5, 4+4	5 possible combinations
9 : 6+3, 3+6, 5+4, 4+5	4 possible combinations
10 : 6+4, 4+6, 5+5	3 possible combinations
11 : 6+5, 5+6	2 possible combinations
12 : 6+6	1 possible combination
	36 possible combinations

Probability Distributions

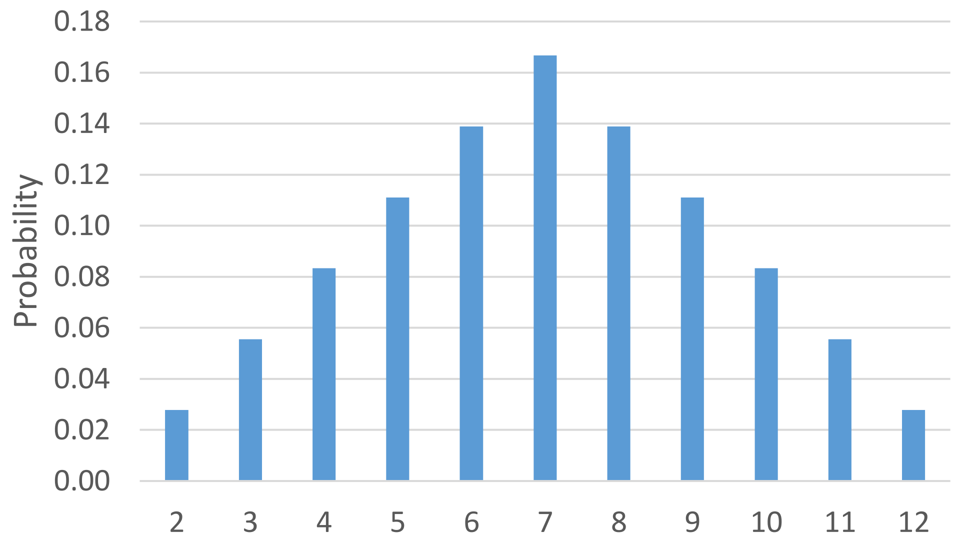
- The probability distribution for the sum of two dice.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

- The probability that the sum is a prime number:
  - $P(\text{Prime}) = P(2) + P(3) + P(5) + P(7) + P(11)$
  - $P(\text{Prime}) = \frac{1}{36} + \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{2}{36} = \frac{15}{36} = \sim.417 = \sim 41.7\%$

Probability Distributions

- The probability distribution for the sum of two dice (bar plot).



Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Independence

- Two processes are **independent** if knowing the outcome of one has no effect on the outcome of the other.
- Coin flips are independent
  - The previous flip has no effect on the next flip
    - There is still a 50% chance of heads and a 50% chance of tails on the next flip
- Dice rolls are also independent
  - Rolling a 6 has no effect on which number is rolled next

Independence

- The **Multiplication Rule** calculates probabilities for multiple independent events occurring:
  - Where A and B are two different and independent events

$$P(A \text{ and } B) = P(A) \times P(B)$$

Independence

- Probability of tails happening twice in a row in two consecutive coin flips:
  - $P(T) = \frac{1}{2} = 0.50$
  - $P(T \text{ and } T) = P(T) \times P(T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = 0.25 = 25\%$

Independence

- Probability of tails-heads-tails happening in three consecutive coin flips:
  - $P(T) = \frac{1}{2} = 0.50$
  - $P(H) = \frac{1}{2} = 0.50$

  - $P(T \text{ and } H \text{ and } T) = P(T) \times P(H) \times P(T) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = 0.125 = 12.5\%$

Independence

- Probability of black-black-black happening in three consecutive spins of a Roulette wheel:
  - $P(Black) = \frac{18}{37} = \sim 0.486$
  - $P(Black \text{ and } Black \text{ and } Black) =$
  - $P(Black) \times P(Black) \times P(Black) = \frac{18}{37} \times \frac{18}{37} \times \frac{18}{37} = \sim 0.115 = \sim 11.5\%$

Independence

- Probability of red-green-red happening in three consecutive spins of a Roulette wheel:
  - $P(Red) = \frac{18}{37} = \sim 0.486$
  - $P(Green) = \frac{1}{37} = \sim 0.027$

  - $P(Red \text{ and } Green \text{ and } Red) =$
  - $P(Red) \times P(Green) \times P(Red) = \frac{18}{37} \times \frac{1}{37} \times \frac{18}{37} = \sim 0.006 = \sim 0.6\%$

Sampling Small Populations

- If we sample *with replacement* it means a case may be sampled more than once.
  - With large populations, its unlikely we would sample the same cases more than once. (Picking five numbers between 1 and 1 million)
  - With small populations, it becomes more likely that we would sample the same cases more than once. (Picking five numbers between 1 and 10)

Sampling Small Populations

- Imagine a bag of M&M's with an equal number (5) of red, brown, blue, green, orange, and yellow candies (30 candies total)

- The probability of drawing a red M&M is $\frac{5}{30}$
  - If we put the red M&M back in the bag, the chance of drawing a red M&M (or any other color) remains $\frac{5}{30} \sim .17$
  - We are sampling with replacement
  - The events of drawing M&Ms are independent.

Sampling Small Populations

- If we sample *without replacement* it means a case will only be sampled once.

- Imagine the same bag of M&M's
- The probability of drawing a red M&M is $\frac{5}{30}$
  - We remove/eat the M&M
  - The probability of drawing a red M&M is now $\frac{4}{29} \sim .14$
  - The probability of drawing any other color is now $\frac{5}{29} \sim .17$
  - The events of drawing M&Ms are no longer independent

Sampling Small Populations

- The probability of drawing red then blue then green then orange
  - With replacement

    $$\frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \sim .0007$$

  - Without replacement

    $$\frac{5}{30} \times \frac{5}{29} \times \frac{5}{28} \times \frac{5}{27} \sim .0009$$

- Drawing without replacement has a greater probability of drawing red then blue then green then orange

Sampling Small Populations

- The probability of drawing five red M&M's in a row
  - With replacement

    $$\frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \times \frac{5}{30} \sim .0001286$$

  - Without replacement

    $$\frac{5}{30} \times \frac{4}{29} \times \frac{3}{28} \times \frac{2}{27} \times \frac{1}{26} \sim .0000070$$

- Drawing with replacement has a greater probability of drawing five red M&M's in a row