The discrete 1D Fourier transform

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The discrete Fourier transform

If we have a real signal $x(n), n = 1, \ldots N$, where N is a power of 2, the discrete Fourier transform (the matlab function fft) produces an array $X(1 : N)$ of N complex numbers. These numbers express the signal as a sum of sine and cosine functions as explained below.

Complex numbers

Complex numbers $a + ib$ have a real part a and an imaginary part b. See appendix and figure 2 for an example of addition and multiplication of two complex numbers. The complex conjugate of the complex number $z = a + ib$ is denoted $\overline{z} = \overline{a + ib}$, and is the complex number $a - ib$. We let ω denote the N'th complex root of unity, see figure 1.

$$
\begin{aligned}
\omega &= e^{i\frac{2\pi}{N}} & (1) \\
&= \cos(\frac{2\pi}{N}) + i\sin(\frac{2\pi}{N}) \quad (\text{Euler}'\text{s formula}) & (2)
\end{aligned}
$$

For later use we note that

$$
\begin{aligned}
\omega^N &= 1 \\
\omega^{N-k} &= \omega^{-k}
\end{aligned}
$$

Complex numbers are often convenient for expressing relations in the real domain. An example are the addition formulae in trigonometry, which are a consequence of Eulers formulae

$$
\begin{aligned}
e^{i\phi}e^{i\theta} &= e^{i(\phi+\theta)} \\
(\cos\phi + i\sin\phi)(\cos\theta + i\sin\theta) &= \cos(\phi+\theta) + i\sin(\phi+\theta)
\end{aligned}
$$

Multiplicative expansion of the left hand side and equating the real and imaginary parts gives us the addition formulae of trigonometry.

$$
(\cos\phi\cos\theta - \sin\phi\sin\theta) + i(\sin\phi\cos\theta + \cos\phi\sin\theta) = \cos(\phi+\theta) + i\sin(\phi+\theta)
$$

$$
\begin{aligned}
\cos(\phi+\theta) &= \cos\phi\cos\theta - \sin\phi\sin\theta \\
\sin(\phi+\theta) &= \sin\phi\cos\theta + \cos\phi\sin\theta
\end{aligned}
$$

The Fourier transform

Fourier descriptors are best understood by means of the inverse Fourier transform

$$
\begin{aligned}
f(n) &= \sum_{k=1}^{N} F(k)\omega^{(k-1)(n-1)} \qquad (3) \\
&= F(1) + F(2)\omega^{(n-1)} + F(3)\omega^{2(n-1)} + \ldots + F(N)\omega^{(N-1)(n-1)} \quad (4)
\end{aligned}
$$

$F(k)$ are complex numbers which are defined by the (forward) Fourier transform (11)

It can be shown that when the numbers in x are real - which there are in our case - and not complex, there is a relation between the coefficients: $F(1) \in R$, $F(2) = \overline{F(N-1)}$, $F(3) = \overline{F(N-2)}$, etc. Thus, we have

$$
\begin{aligned}
f(n) &= F(1) + F(2)\omega^{(n-1)} + F(3)\omega^{2(n-1)} + \ldots + \overline{F(2)}\omega^{-(n-1)} \qquad (5) \\
&= F(1) + F(2)\omega^{(n-1)} + F(3)\omega^{2(n-1)} + \ldots + \overline{F(2)\omega^{(n-1)}} \qquad (6)
\end{aligned}
$$

The formulas (5), $n = 1, 2, \ldots, N$ express the sequence $f(n)$, $n = 1, 2, \ldots, N$ as the sum of N sequences, each of length N, when we interpret each term $\omega^{(k-1)(n-1)}$, $n = 1, 2, \ldots, N$ as a sequence. Below an explicit listing of the sequences for $N = 5$. The sequences are represented as column vectors.
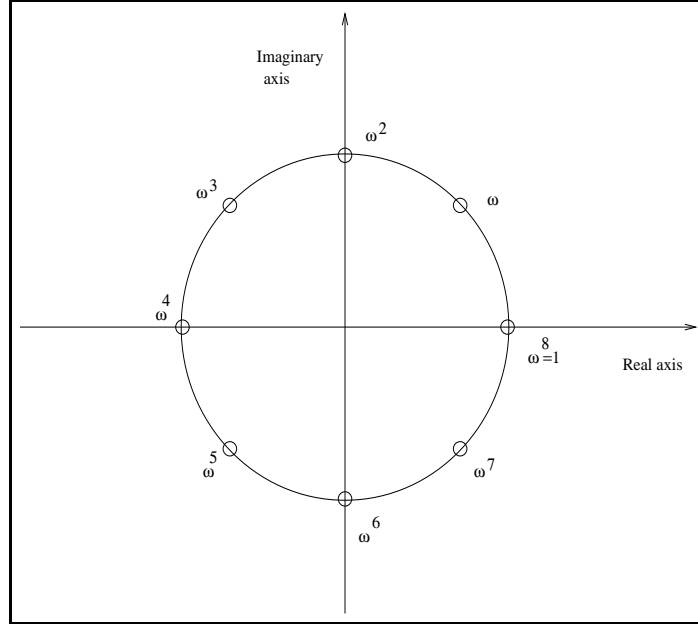
Figure 1: The 8th unit root ω and its powers.

$$\begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \end{pmatrix} = F(1)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + F(2)\begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + F(3)\begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ \omega^8 \end{pmatrix} + \qquad (7)$$

$$F(4)\begin{pmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega^9 \\ \omega^{12} \end{pmatrix} + F(5)\begin{pmatrix} 1 \\ \omega^4 \\ \omega^8 \\ \omega^{12} \\ \omega^{16} \end{pmatrix} \qquad (8)$$

In the real case we have

$$\begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \end{pmatrix} = F(1)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + F(2)\begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} + F(3)\begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ \omega^8 \end{pmatrix} + \qquad (9)$$

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$$F(4)\begin{pmatrix}\dfrac{1}{\omega^2}\\[4pt]\omega^4\\[2pt]\omega^6\\[2pt]\omega^8\end{pmatrix}+\overline{F(2)}\begin{pmatrix}\dfrac{1}{\omega^1}\\[4pt]\omega^2\\[2pt]\omega^3\\[2pt]\omega^4\end{pmatrix}\tag{10}$$

Let us consider these sequences in turn.

The sequence $F(1)\{1,1,\ldots,1\}$ is special, in that it does not depend on n. It is a sequence of length n with all elements equal to the real constant $F(1)$.

Let us consider the geometry of $F(2)\{1,\omega,\omega^2\ldots,\omega^{N-1}\}$ when plotted in the complex plane. Assume first $F(2)=1$. The sequence consists of N equidistant points on the unit circle in the complex plane. When $F(2)\neq 1$, all sequence elements are multiplied by the same complex number. Multiplication by a complex number consists of scaling by its absolute value followed by addition of its argument. The net effect is that equidistant points on the unit circle are still equidistant, but lie on a circle with a different radius. Their location on the circle have all been rotated by the same angle.

We now take advantage of the fact that our sequence are real numbers, because when we look at the last sequence $\overline{F(2)}\{1,\overline{\omega^1},\overline{\omega^2},\overline{\omega^3},\overline{\omega^4}\}$ we observe that the sequence consists of the complex conjugate points on the circle, and multiplied by a complx conjugate number. the effect is that the points are equidistant on a circle with the same radius $|F(2)|$ and rotated the opposite angle. the net effect when adding the points on the two cirles are that the points all fall on the real axis. Perhaps it is easier to understand when we simply add the elements of the two sequences, since the sum of two complex conjugate numbers is a real number.

We then have to interpret the sum of the two sequences. Let $F(2)=re^{i\theta}$ and let $w=e^{i\frac{2\pi}{5}}=e^{i\phi}$. Then $\overline{F(2)}=re^{-i\theta}$ and $\overline{w}=e^{-i\frac{2\pi}{5}}=e^{-i\phi}$. Their sum is

$$\begin{aligned}&re^{i\theta}\{1,e^{i\phi},e^{2i\phi},e^{3i\phi},e^{4i\phi}\}+re^{-i\theta}\{1,e^{-i\phi},e^{-i2\phi},e^{-3i\phi},e^{-4i\phi}\}\\[4pt]=\ &r\{e^{i\theta},e^{i(\phi+\theta)},e^{i(2\phi+\theta)},e^{i(3\phi+\theta)},e^{i(4\phi+\theta)}\}+\\&r\{e^{-i\theta},e^{-i(\phi+\theta)},e^{-i(2\phi+\theta)},e^{-i(3\phi+\theta)},e^{-i(4\phi+\theta)}\}\\[4pt]=\ &r\{e^{i\theta}+e^{-i\theta},e^{i(\phi+\theta)}+e^{-i(\phi+\theta)},e^{i(2\phi+\theta)}+e^{-i(2\phi+\theta)},\\&\quad e^{i(3\phi+\theta)}+e^{-i(3\phi+\theta)},e^{i(4\phi+\theta)}+e^{-i(4\phi+\theta)}\}\end{aligned}$$

Using Eulers formula we have

$$\begin{aligned}e^{ib}+e^{-ib}&=\{\cos(b)+i\sin(b)\}+\{\cos(-b)+i\sin(-b)\}\\&=2\cos b\end{aligned}$$

As a consequence the sum of the two sequences becomes

$$2r\{\cos\theta,\cos(\phi+\theta),\cos(2\phi+\theta),\cos(3\phi+\theta),\cos(4\phi+\theta)\}$$

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where θ and r are computed from $F(2)$ as $\theta = \arg F(2)$ and $r = |F(2)|$. We insert $\phi = \frac{2\pi}{5}$ to get the sequence

$$2r\{\cos\theta, \cos(\frac{2\pi}{5} + \theta), \cos(2\frac{2\pi}{5} + \theta), \cos(3\frac{2\pi}{5} + \theta), \cos(4\frac{2\pi}{5} + \theta)\}$$

What has happened is the we have found a function $2r\cos(n\frac{2\pi}{5} + \theta)$ that - sampled at the values $n = 0, 1, 2, 3, 4$ - gives the sum of the two sequences. This function is a cosine with amplitude $2r$, with frequency $\frac{2\pi}{5}$, and with phase angle θ. Adding the third and fourth sequence would give a function with double frequency $\frac{2\pi}{5}$ with its own amplitude and phase angle.

Result

Any sequence of length 5 can be written as samples at $n = 0, 1, 2, 3, 4$ of the function

$$f(n) = a_0 + a_1 \cos(n\frac{2\pi}{5} + \theta_1) + a_2 \cos(2n\frac{2\pi}{5} + \theta_2)$$

In general: A sequence of N real numbers can be written as samples at $n = 0, 1, \ldots, N$ of the function

$$f(n) = a_0 + \sum_{k=1}^{(N-1)/2} a_k \cos(kn\frac{2\pi}{N} + \theta_k)$$

If N is an even number the upper limit is $(N+1)/2$

If a sequence is measurements of a phenomenon which is a mixture of periodic signals, computation of a_k and θ_k tells the size and the delay of the periodic contributions.

Appendix: The forward Fourier transform

$$F(n) = \frac{1}{N}\sum_{k=1}^{N} f(k)\omega^{-(k-1)(n-1)} \tag{11}$$

$$= \frac{1}{N}\{f(1) + f(2)\omega^{-(n-1)} + F(3)\omega^{-2(n-1)} + \ldots + f(n)\omega^{-(N-1)(n-1)}\}$$

Appendix: Complex addition and multiplication

Two complex numbers can be perceived as vectors in the complex plane. Addition is vector addition. Multiplication also has a geometric interpretation. The absolute value of the product is the product of the absolute values. The angle from the real axis to the product is the sum of the angles of the factors.
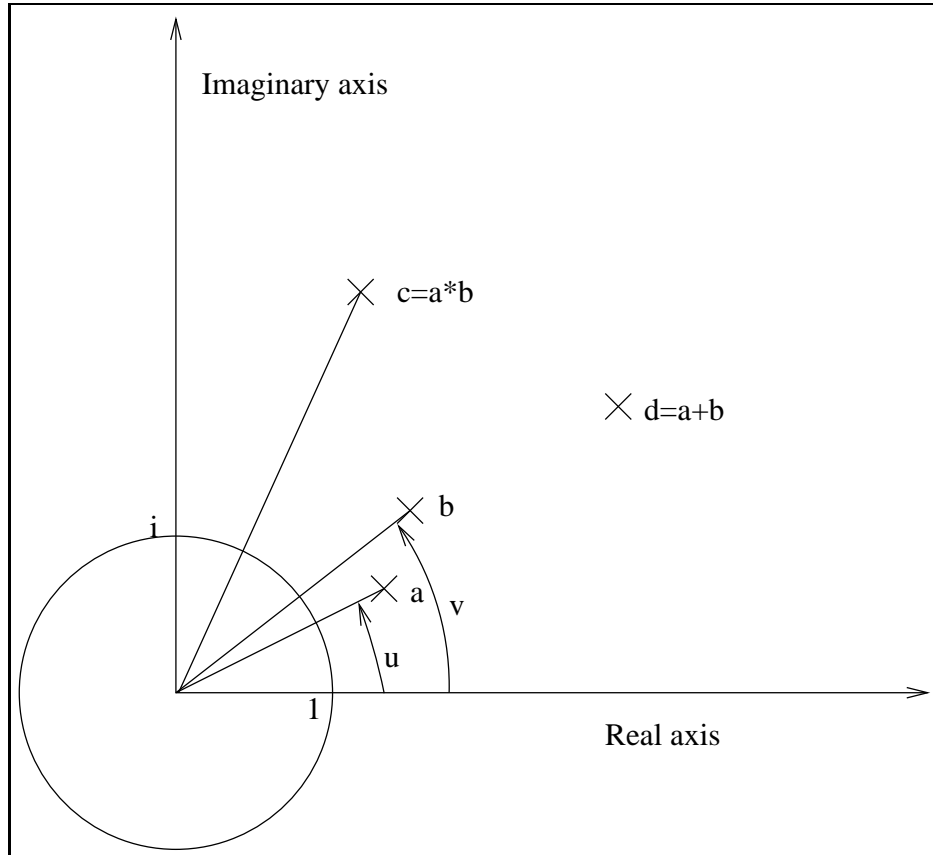
Figure 2: The complex plane with the unit circle. The numbers $a = (4 + 2i)/3$ and $b = (9 + 7i)/6$ have absolute value 1.49 and 1.90 and angle $u = 26.5^o$ and $v = 37.9^o$. Their product is c and their sum is d. The product of the absolute values is 2.83 and the sum of the angles is 64.4^o. The vector sum is $d = (17 + 11 * i)/6$