DIGITAL SIGNAL PROCESSING

A. Nagoor Kani

Second Edition

- 1500+ Solved Examples and Exercise Problems
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DIGITAL SIGNAL PROCESSING

Second Edition

A. Nagoor Kani
Founder, RBA Educational Group
Chennai

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Preface

The main objective of this book is to explore the basic concepts of digital signal processing in a simple and easy-to-understand manner.

This text on digital signal processing has been suitably crafted and designed to meet student's requirements. Considering the highly mathematical nature of this subject, more emphasis has been given on the problem- solving methodology. Considerable effort has been made to elucidate mathematical derivations in a step-by-step manner. Exercise problems with varied difficulty levels are given in the text to help students get an intuitive grasp on the subject.

This book with its lucid writing style and germane pedagogical features will prove to be a master text for engineering students and practitioners.

Salient Features

The salient features of this book on Digital Signal Processing are,

- proof of properties of transforms are clearly highlighted by shaded boxes
- wherever required, problems are solved by multiple methods
- additional explanations for solutions and proofs are provided in separate boxes
- different types of fonts are used for text, proof and solved problems for better clarity
- keywords are highlighted by bold, italic fonts

Organization

In this book, the concepts of discrete time signals and their transforms are organized in four chapters and two chapters are devoted to digital filter design. One chapter is devoted to each topic in digital signal processing like finite word length effects, mutirate DSP, spectrum analysis, digital signal processors and applications of DSP. Each chapter provides the foundations and practical implications with a large number of solved numerical examples for better understanding.

The important concepts are summarized at the end of each chapter which can help in quick reference. Another significant aspect of this book is MATLAB based computer exercises with complete explanations given in each chapter. This will be of great assistance to both instructors and students.

Chapter 1 deals with a general introduction about various aspects of digital signal processing and its importance in real life. Basic definitions of discrete time signals and systems, mathematical representation of discrete time systems and significance of time and frequency domain analysis are presented in brief. Introduction to various topics of digital signal processing like FIR filters, IIR filters, finite word length effects, multirate DSP, power spectrum, digital signal processors, applications of digital signal processing and usage of MATLAB in this course are also presented in a brief manner.

Chapter 2 is devoted to concepts of discrete time signals and systems and is more concerned with generation, representation, classification, mathematical operations of discrete time signals and systems, block diagram and signal flow graph notations.

The chapter also presents the methods of obtaining responses of LTI discrete time systems and various convolution methods. The deconvolution, correlation techniques and the inverse systems are clearly explained with solved numericals. In addition, the concept of sampling and its importance are dealt with briefly.

Chapter 3 explains $\mathcal{Z}$-transform and its application to discrete time signals and systems. All the important properties of $\mathcal{Z}$-transform are presented explicitly. Inverse $\mathcal{Z}$-transforms and solutions of difference equations describing the discrete time systems are demonstrated with numerical examples. Also, the structures for realization of IIR and FIR systems are provided.

Chapter 4 is dedicated to discrete time Fourier series and Fourier transform which form the basics for frequency domain analysis of discrete time signals and systems. In the first half of this chapter, the discrete time Fourier series and the frequency spectrum using discrete time Fourier series are discussed with relevant examples.

The second half of the chapter details the development of discrete time Fourier transform from discrete time Fourier series, frequency spectrum, various properties of Fourier transform, and Fourier transform of some standard discrete time signals. In addition, the computation of frequency responses of LTI discrete time systems using Fourier transform are also explained with examples. The relation between Fourier transform and $\mathcal{Z}$-transform of discrete time signals is also discussed in the chapter.

Chapter 5 extends the understanding of the concepts of Discrete time Fourier transform(DTFT) to DFT (Discrete Fourier transform) and FFT (Fast Fourier Transform). Development of DFT from DTFT, properties of DFT, relation between DFT and $\mathcal{Z}$-transform, analysis of the LTI systems using DFT and FFT are extensively discussed.

Chapter 6 focuses on frequency response of FIR filters and characteristics of various windows used for FIR filter design. Also, design of linear phase FIR filters by windowing and frequency sampling techniques are presented with suitable examples.

Chapter 7 explains the techniques for transforming analog filter to digital filter and the characteristics of analog Butterworth and Chebyshev filters. Also, design of Butterworth and Chebyshev digital IIR filters are presented with examples.

Chapter 8 discusses the quantization and representation of digital/binary number systems. The effects due to finite precision of filter coefficients and products, and various types of overflow in recursive computations are also discussed with appropriate examples.

Chapter 9 focuses on sampling rate conversion by decimation and interpolation and their effects on frequency spectrum. Implementation of sampling rate conversion in filters and application of multirate digital signal processing are also discussed in the chapter.

Chapter 10 is concerned with the estimation of energy spectrum of discrete time signals and power spectrum of random processes. The various nonparametric methods, power spectrum estimation and their performance characteristics are presented.

Chapter 11 focuses on architecture and programming of special purpose processors for digital signal processing with particular concentration to Texas Instruments digital signal processors, TMS320C5x and TMS320C54x processors.

Chapter 12 provides a brief discussion on some applications of digital signal processing in speech, musical sound, audio/video, communication and biomedical signals.

The author has taken care to present the concepts of Digital Signal Processing in a simple manner and hopes that the teaching and student community will welcome the book. The readers can feel free to convey their criticism and suggestions to **kani@vsnl.com** for further improvement of the book.

A.Nagoor Kani

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A. Nagoor Kani

List of Symbols and Abbreviations

Symbols

A	-	Number of integer digit
A_s	-	Gain at stopband edge frequency
A_p	-	Gain at passband edge frequency
B	-	Bandwidth in Hz
b	-	Size of binary excluding sign bit
c_k	-	Fourier coefficients of exponential form of Fourier series of x(t)
D	-	Sampling rate reduction factor
E	-	Energy of a signal
E_r	-	Relative error due to rounding
E_t	-	Relative error due to truncation
e_r	-	Rounding error
f	-	Frequency of discrete time signal (or digital frequency) in cycles/sample
F	-	Frequency of continuous time signal (or analog frequency) in Hz
f_o	-	Fundamental frequency of discrete time signal in cycles/sample
F_o	-	Fundamental frequency of continuous time signal in Hz
F_m	-	Maximum frequency of continuous time signal in Hz
F_s	-	Sampling frequency of continuous time signal in Hz
I	-	Sampling rate mulitplication factor
j	-	complex operator, $\sqrt{-1}$
L	-	Number of segments
$\mathcal{M}$	-	Figure of merit
M	-	Mantissa

N	-	Fundamental period
N	-	Order of the filter
N_f	-	Floating point binary number
N_{tf}	-	Truncated floating point number
P	-	Power of a signal
p	-	Pole
$P_{xx}(f)$	-	Power spectrum
$P_{xx}^{B}(f)$	-	Bartlett power spectrum estimate
$P_{xx}^{BT}(f)$	-	Blackman-Tukey power spectrum estimate
$P_{xx}^{per}(f)$	-	Periodogram power spectrum estimate
$P_{xx}^{W}(f)$	-	Welch power spectrum estimate
q	-	Quantization step size
Q	-	Quality factor
R	-	Range of decimal number
r	-	Radix or base
S	-	Sign bit
$S_{xx}(f)$	-	Energy spectrum
t	-	Time in seconds
T	-	Time period in seconds
ν	-	Variabiltiy
W	-	Phase factor or Twiddle factor
x(n)	-	Discrete time signal or Ergodic random process
X(n)	-	Random process
z	-	Complex variable (z = u + jv)
z	-	Unit advance operator or Zero
z^{-1}	-	Unit delay operator
$\in$	-	Attenuation costant

Ω	-	Angular frequency of continuous time signal in rad/sec
Ω_o	-	Center frequency
Ω_s	-	Stop band edge analog frequency in rad/sec
Ω_p	-	Pass band edge analog frequency in rad/sec
ω	-	Angular frequency of discrete time signal in rad/sample
ω_k	-	Sampling frequency point
ω_p	-	Pass band edge digital frequency in rad/sample
ω_s	-	Stop band edge digital frequency in rad/sample
σ^2	-	Variance
σ^2_{eoi}	-	Steady state output noise power due to input quantization error
α_p	-	Attenuation at a pass band frequency
α_s	-	Attenuation at a stop band frequency
$*$	-	Convolution operator
$*$	-	Circular convolution operator
$\int$	-	Integration operator
$\frac{d}{dt}$	-	Differentiation operator

Standard/Input/Output Signals

$\|A(\omega)\|$	-	Magnitude function
$h(n)$	-	Impulse response of discrete time system
$h'(n)$	-	Impulse response of inverse system
$h_d(n)$	-	Desired impulse response
$r_{xy}(m)$	-	Crosscorrelation sequence of x(n) and y(n)
$r_{xx}(m)$	-	Autocorrelation sequence of discrete time signal
$r_{xx}(m)$	-	Autocorrelation sequence of random process with finite data
$\gamma_{xx}(m)$	-	Autocorrelation sequence of random process with infinite data

$\bar{r}_{xx}(m)$	-	Circular autocorrelation sequence of x(n)
$\bar{r}_{xy}(m)$	-	Circular crosscorrelation sequence of x(n) and y(n)
$u(n)$	-	Discrete time unit step signal
$w_R(n)$	-	Rectangular window sequence
$w_r(n)$	-	Bartlett or triangular window sequence
$w_C(n)$	-	Hanning window sequence
$w_H(n)$	-	Hamming window sequence
$w_B(n)$	-	Blackman window sequence
$w_K(n)$	-	Kaiser window sequence
$x(n)$	-	Discrete time signal
$x(n)$	-	Input of discrete time system
$x_o(n)$	-	Odd part of discrete time signal x(n)
$x_e(n)$	-	Even part of discrete time signal x(n)
$x(n-m)$	-	Delayed or linearly shifted x(n) by m units
$x((n-m))_N$	-	Circularly shifted x(n) by m units, where N is period
$x(Dn)$	-	Down sampled version of x(n)
$x(n/I)$	-	Upsampled version of x(n)
$x_P(n)$	-	Periodic extension of x(n)
$y(n)$	-	Output / Response of discrete time system
$y(n - m)$	-	Delayed output / Response of discrete time system
$y_p(n)$	-	Particular soultion of discrete time system
$y_n(n)$	-	Homogenous solution of discrete time system
$y_{zs}(n)$	-	Zero state response of discrete time system
$y_{zi}(n)$	-	Zero input response of discrete time time system
$\delta(n)$	-	Discrete time impulse signal
$\delta(n - m)$	-	Delayed impulse signal

τ_p	-	Phase delay
τ_g	-	Group delay
$\theta(\omega)$	-	Phase function

Transform Operators and Functions

$\mathcal{DFT}$	-	Discrete Fourier transform (DFT)
$\mathcal{DFT}^{-1}$	-	Inverse DFT
$E\{X\}$	-	Expected value of random variable
$\mathcal{F}$	-	Fourier transform
$\mathcal{F}^{-1}$	-	Inverse Fourier transform
$\mathcal{H}$	-	System operator
$\mathcal{H}^{-1}$	-	Inverse system operator
$H(z)$	-	Transfer function
$H(e^{j\omega})$	-	Frequency response of the digital filter
$H_N(z)$	-	Normalized transfer function
$H_d(e^{j\omega})$	-	Desired or ideal frequency response
$Q[\ ]$	-	Quantization operations
$X(e^{j\omega})$	-	Discrete time Fourier transform of x(n)
$X_r(e^{j\omega})$	-	Real part of $X(e^{j\omega})$
$X_i(e^{j\omega})$	-	Imaginary part of $X(e^{j\omega})$
$X(j\Omega)$	-	Fourier transform of x(t)
$X(k)$	-	Discrete Fourier transform of x(n)
$X_r(k)$	-	Real part of X(k)
$X_i(k)$	-	Imaginary part of X(k)
$X(z)$	-	$\mathcal{Z}$-transform of x(n)
$\mathcal{Z}$	-	$\mathcal{Z}$-transform
$\mathcal{Z}^{-1}$	-	Inverse $\mathcal{Z}$-transform

Chapter 1

Introduction to Digital Signal Processing

1.1 Introduction

Digital **S**ignal **P**rocessing (DSP) refers to processing of signals by digital systems like **P**ersonal **C**omputers (PC) and systems designed using digital **I**ntegrated **C**ircuit**s** (ICs), microprocessors and microcontrollers. DSP gained popularity in the 1960s. Earlier, DSP systems were limited to general purpose non-real-time scientific and business applications. The rapid advancement in computers and IC fabrication technology leads to complete domination of DSP systems in both real-time and non-real-time applications in all fields of engineering and technology.

The basic components of a DSP system are shown in fig 1.1. The ***DSP system*** involves conversion of analog signal to digital signal, then processing of the digital signal by a digital system and then conversion of the processed digital signal back to analog signal.

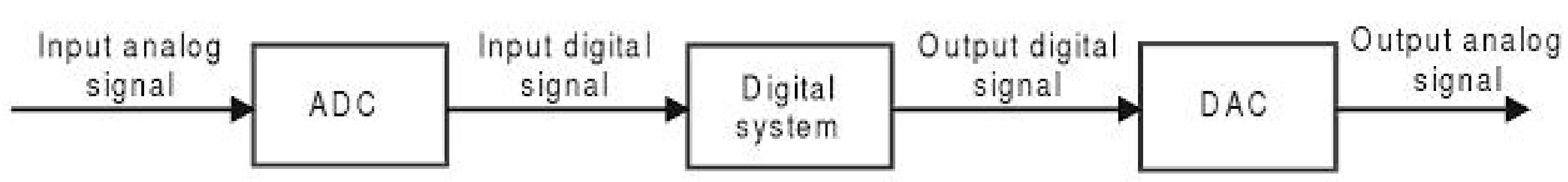


***Fig 1.1** : Basic components of a DSP system.*

The real-world signals are analog, and only for processing by digital systems, the signals are converted to digital. For conversion of signals from analog to digital, an ADC (**A**nalog to **D**igital **C**onverter) is employed. The various steps in analog to digital conversion process are sampling and quantization of analog signals, and then converting the quantized samples to suitable binary codes. The digital signals in the form of binary codes are fed to digital system for processing, and after processing, it generates an output digital signal in the form of binary codes. The output analog signal is constructed from the output binary codes using a DAC (**D**igital to **A**nalog **C**onverter).

The processing of signals are basically spectrum analysis to determine the various frequency components of a signal and filtering the signal to extract the required frequency component of the signal.

The ***digital system*** can be a specially designed programmable hardware for DSP or an algorithm/ software running on a general purpose digital system like **P**ersonal **C**omputer (PC).

Advantages of Digital Signal Processing

Some of the advantages of digital processing of signals are,

1. The digital hardware are compact, reliable, less expensive, and programmable.
2. Since the DSP systems are programmable, the performance of the system can be easily upgraded/ modified.
3. By employing high speed, sophisticated digital hardware higher precision can be achieved in processing of signals.
4. The digital signals can be permanently stored in magnetic media so that they are transportable and can be processed in non-real-time or off-line.

1.2 Signal

Any physical phenomenon that conveys or carries some information can be called a ***signal***. The music, speech, motion pictures, still photos, heart beat, etc., are examples of signals that we normally encounter in day-to-day life.

When a signal is defined continuously for any value of an independent variable, it is called an ***analog*** or ***continuous signal***. Most of the signals encountered in science and engineering are analog in nature. When the dependent variable of an analog signal is time, it is called a ***continuous time signal*** and it is denoted as "x(t)".

When a signal is defined for discrete intervals of an independent variable, it is called a ***discrete signal***. When the dependent variable of a discrete signal is time, it is called ***discrete time signal*** and it is denoted by "x(n)". Most of the discrete signals are either sampled versions of analog signals for processing by digital systems or output of digital systems.

The quantized and coded version of the discrete time signals are called ***digital signals***. In digital signals the value of the signal for every discrete time "n" is represented in binary codes. The process of conversion of a discrete time signal to digital signal involves quantization and coding.

Normally, for binary representation, a standard size of binary is chosen. In m-bit binary representation, we can have 2^m binary codes. The possible range of values of the discrete time signals are usually divided into 2^m steps called ***quantization levels***, and a binary code is attached to each quantization level. The values of the discrete time signals are approximated by rounding or truncation in order to match the nearest quantization level.

1.3 Discrete Time System

Any process that exhibits cause and effect relation can be called a ***system***. A system will have an input signal and an output signal. The output signal will be a processed version of the input signal. A system is either interconnection of hardware devices or software / algorithm.

A system which can process a discrete time signal is called a ***discrete time system***, and so the input and output signals of a discrete time system are discrete time signals.

A discrete time system is denoted by the letter $\mathcal{H}$. The input of discrete time system is denoted as "x(n)" and the output of discrete time system is denoted as "y(n)". The diagrammatic representation of a discrete time system is shown in fig 1.2.

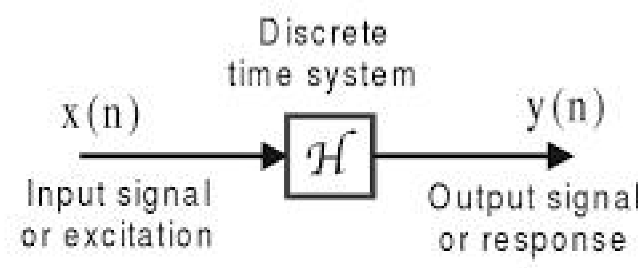


Fig 1.2 : Representation of discrete time system.

The operation performed by a discrete time system on input to produce output or response can be expressed as,

Response, $y(n) = \mathcal{H}\{x(n)\}$

where, $\mathcal{H}$ denotes the system operation (also called system operator).

When a discrete time system satisfies the properties of linearity and time invariance then it is called ***LTI (Linear Time Invariant) discrete time system*** .

The input-output relation of an LTI discrete time system is represented by constant coefficient difference equation shown below.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

where, N = Order of the system, and $M \leq N$.

The solution of the above difference equation is the response y(n) of the discrete time system, for the input x(n).

1.4 Analysis of Discrete Time System

Mostly, the discrete time systems are designed for analysis of discrete time signals. Physically, the discrete time systems are realized in time domain. In time domain, the discrete time systems are governed by difference equations. The analysis of discrete time signals and systems in time domain involves solution of difference equations. The solution of difference equations are difficult due to assumption of a solution and then solving the constants using initial conditions.

In order to simplify the task of analysis, the discrete time signals can be transformed to some other domain, where the analysis may be easier. One such transform exists for discrete time signals is $\mathcal{Z}$-transform.The $\mathcal{Z}$-transform, will transform a function of discrete time "n" into a function of complex variable "z", where $z = re^{jw}$ .Therefore, ***Z-transform*** of a discrete time signal will transform the time domain signal into z-domain signal.

On taking $\mathcal{Z}$ -transform of the difference equation governing the discrete time system, it becomes algebraic equation in "z" and the solution of algebraic equation will give the response of the system as a function of "z" and it is called z-domain response. The inverse $\mathcal{Z}$ -transform of the z-domain response, will give the time domain response of the discrete time system. Also, the stability analysis of the discrete systems are much easier in z-domain.

The ratio of Z -transform of output and input is called ***transfer function*** of the discrete time system. The inverse Z -transform of the system gives the ***impulse response*** of the system, which is used to study the characteristics of a system.

Another important characteristic of any signal is frequency, and for most of the applications the frequency content of the signal is an important criteria. The frequency range of some of the signals are listed in table 1.1 and 1.2.

Table 1.1 : Frequency Range of Some Electromagnetic Signals

Type of signal	**Wavelength (m)**	**Frequency range (Hz)**
Radio broadcast	10^{4} to 10^{2}	3 ´ 10^{4} to 3 ´ 10^{6}
Shortwave radio signals	10^{2} to 10^{-2}	3 ´ 10^{6} to 3 ´ 10^{10}
Radar / Space communications	1 to 10^{-2}	3 ´ 10^{8} to 3 ´ 10^{10}
Common-carrier microwave	1 to 10^{-2}	3 ´ 10^{8} to 3 ´ 10^{10}
Infrared	10^{-3} to 10^{-6}	3 ´ 10^{11} to 3 ´ 10^{14}
Visible light	3.9´10^{-7} to 8.1´10^{-7}	3.7 ´ 10^{14} to 7.7 ´ 10^{14}
Ultraviolet	10^{-7} to 10^{-8}	3 ´ 10^{15} to 3 ´ 10^{16}
Gamma rays and x-rays	10^{-9} to 10^{-10}	3 ´ 10^{17} to 3 ´ 10^{18}

Table 1.2 : Frequency Range of Some Biological and Seismic Signals

Type of Signal	**Frequency Range (Hz)**
Electroretinogram	0 to 20
Electronystagmogram	0 to 20
Pneumogram	0 to 40
Electrocardiogram (ECG)	0 to 100
Electroencephalogram (EEG)	0 to 100
Electromyogram	10 to 200
Sphygmomanogram	0 to 200
Speech	100 to 4000
Wind noise	100 to 1000
Seismic exploration signals	10 to 100
Earthquake and nuclear explosion signals	0.01 to 10
Seismic noise	0.1 to 1

The frequency contents of a discrete time signal can be studied by taking Fourier transform of the discrete time signal. The Fourier transform of discrete time signal is a particular class of Z-transform in which $z = e^{jw}$,where "w" is the frequency of the discrete time signals.

The ***Fourier transform***, will transform a function of discrete time "n" into a function of frequency "w". Therefore, Fourier transform of a discrete time signal will transform the discrete time signal into frequency domain signal. The Fourier transform of the discrete time signal, is also called ***frequency spectrum*** of the discrete time signal. The Fourier transform of the impulse response of a system is called ***frequency response*** of the system. The frequency spectrum is a complex function of "w" and so can be expressed as magnitude spectrum and phase spectrum. The magnitude spectrum is used to study the various frequency components of the discrete time signal.

The frequency spectrum obtained via Fourier transform will be a continuous spectrum and so cannot be computed by digital systems, Therefore, the samples of Fourier transform can be computed at sufficient number of points by digital systems. The samples of Fourier transform can also be directly computed using DFT (**D**iscrete **F**ourier **T**ransform). The computation of DFT involves a large number of calculations. In order to reduce the computational task of DFT, a number of methods/algorithms are developed which are collectively called ***FFT*** (**F**ast **F**ourier **T**ransform). The DFT of discrete time signal will give the ***discrete frequency spectrum*** of the signal.

1.5 Filters

The filters are frequency selective devices. The two major types of digital filters are FIR (**F**inite **I**mpulse **R**esponse) and IIR (**I**nfinite **I**mpulse **R**esponse) filters.

Generally, the filter specification will be a desired frequency response. The inverse Fourier transform of the frequency response will be the impulse response of the filter, and it will be an infinite duration signal. The digital filters designed by choosing finite samples of impulse rseponse are called ***FIR filters***, and the filters designed by considering all the infinite samples are called ***IIR filters***.

Since, an FIR filter is designed from the finite samples of impluse response, the direct design of FIR filter is possible in which the transfer function of the filter is obtained by taking $\mathcal{Z}$ -transform of impulse response.

Note : Mathematically, the filter design is design of transfer function of the filter.

Since, an IIR filter is designed by considering / preserving the infinite samples of impulse response, the direct design of IIR filter is not possible. Therefore, the IIR filter is designed via analog filter. For designing IIR filter, first the specifications of IIR filter is transformed to specifications of analog filter using bilinear or impulse-invariant transformation, then an analog filter transfer function is designed using Butterworth or Chebychev approximation. Finally the analog filter transfer function is transfered to digital filter transfer function using the transformation chosen for transforming the specifications.

1.6 Finite Word Length Effects

In digital representation the signals are represented as an array of binary numbers, and the digital system employ a fixed size of binary called "word size or word length" for number representation. This finite word size for number representation leads to errors in input signals, intermediate signals in computations and in the final output signals. In general, the various effects due to finite precision representation of numbers in digital systems are called ***finite word length effects***.

Some of the finite word length effects in digital systems are given below.

- Errors due to quantization of input data.
- Errors due to quantization of filter coefficients.

- Errors due to rounding the product in multiplication.
- Errors due to overflow in addition.
- Limit cycles in recursive computations.

1.7 Multirate DSP

In many communication systems, the sampling rate conversion is a vital requirement. Some of the systems that employ sampling rate conversion are video receivers that receive both NTSC and PAL signals, audio systems that can play CDs recorded in different standards, etc.

The processing of discrete time signals at different sampling rates in different parts of a system is called ***multirate DSP***. In digital systems, the sampling rate conversion is achieved by either decimation or interpolation. In decimation, the sampling rate is reduced, whereas in interpolation the sampling rate is increased. The multirate DSP systems leads to reduction in computations, memory requirement and errors due to finite word length effects.

1.8 Energy and Power Spectrum

There are many situations where the signals are corrupted by noise like sonar signals corrupted by ambient ocean noise, speech signal from cockpit of an airplane corrupted by engine noise, etc. When the signals are corrupted by noise, then the energy or power spectrum will be useful to identify the signal from noise.

The ***energy spectrum*** can be computed for deterministic signals, and it is given by square of magnitude of Fourier transform of the signal. Alternatively, the energy spectrum is given by Fourier transform of the autocorrelation sequence of the signal.

The power spectrum can be estimated for nondeterministic signals or random process/signals. The power spectrum estimation methods can be broadly classified into two groups, namely, nonparametric methods and parametric methods.

In ***nonparametric methods***, first an estimate of autocorrelation of the random process is determined which represents the average behaviour of the signal, then the Fourier transform of estimated autocorrelation is determined, which is the power spectrum estimate of the random process.

In ***parametric methods***, first an appropriate model is selected for the given random process, then the parameters of the model are computed using the available data of the random process. Finally, the power spectrum is estimated from the constructed model.

1.9 Digital Signal Processors

The ***digital signal processors*** are specially designed microprocessors/microcontrollers for DSP applications.

The importance of special purpose processors for signal processing applications were realised in 1980s, and many companies started releasing special processors for DSP applications. The pioneers among them are Texas Instruments and Analog Devices. The Texas Instruments has released a large variety of processors in the family name TMS320Cxx and Analog Devices has released processors in the family name ADSPxx.

Some of the special features of digital signal processors are given below.

- Modified Harvard architecture with two or more internal buses for simultaneous access of code and one or two data.
- Specialized addressing modes like circular addressing and bit reversed addressing suitable for computations like convolution, correlation and FFT.
- MAC unit for performing multiply-accumulate computations involved in convolution, correlation and FFT in single clock cycle.
- Larger size ALU and accumulators with guard bits to accommodate the overflow in computation.
- Pipelining of instructions to execute different phases of four or six instructions in parallel.
- VLIW architecture to fetch and execute multiple instructions in parallel.
- Multiprocessor architecture by integrating multiple processors on a single piece of silicon for parallel processing.

1.10 Importance of Digital Signal Processing

The technology advancement in programmable digital signal processors, helps to implement more and more real time applications in digital systems.

The digital processing of signal plays a vital role in almost every field of Science and Engineering. Some of the applications of digital processing of signals in various field of Science and Engineering are listed here.

1. Biomedical

- ECG is used to predict heart diseases.
- EEG is used to study normal and abnormal behaviour of the brain.
- EMG is used to study the condition of muscles.
- X-ray images are used to predict the bone fractures and tuberculosis.
- Ultrasonic scan images of kidney and gall bladder is used to predict stones.
- Ultrasonic scan images of foetus is used to predict abnormalities in a baby.
- MRI scan is used to study minute inner details of any part of the human body.

2. Speech Processing

- Speech compression and decompression to reduce memory requirement of storage systems.
- Speech compression and decompression for effective use of transmission channels.
- Speech recognization for voice operated systems and voice based security systems.
- Speech recognization for conversion of voice to text.
- Speech synthesis for various voice based warnings or annoucements.

3. Audio and Video Equipments

- The analysis of audio signals will be useful to design systems for special effects in audio systems like stereo, woofer, karoke, equalizer, attenuator, etc.
- Music synthesis and composing using music keyboards.
- Audio and video compression for storage in DVDs.

4. Communication

- The spectrum analysis of modulated signals helps to identify the information bearing frequency component that can be used for transmission.
- The analysis of signals received from radars are used to detect flying objects and thier velocity.
- Generation and detection of DTMF signals in telephones.
- Echo and noise cancellation in transmission channels.

5. Power electronics

- The spectrum analysis of the output of coverters and inverters will reveal the harmonics present in the output, which in turn helps to design suitable filter to eliminate the harmonics.
- The analysis of switching currents and voltages in power devices will help to reduce losses.

6. Image processing

- Image compression and decompression to reduce memory requirement of storage systems.
- Image compression and decompression for effective use of transmission channels.
- Image recognition for security systems.
- Filtering operations on images to extract the features or hidden information.

7. Geology

- The seismic signals are used to determine the magnitude of earthquakes and volcanic eruptions.
- The seismic signals are also used to predict nuclear explosions.
- The seismic noises are also used to predict the movement of earth layers (tectonic plates).

8. Astronomy

- The analysis of light received from a star is used to determine the condition of the star.
- The analysis of images of various celestial bodies gives vital information about them.

1.11 Use of MATLAB in Digital Signal Processing

MATLAB (**MAT**rix **LAB**oratory) is a software developed by The MathWork Inc, USA, which can run on any windows platform in a PC (**P**ersonal **C**omputer). This software has a number of tools for the study of various engineering subjects. It includes various tools for digital signal processing also. Using these tools, a wide variety of studies can be made on discrete time signals and systems. Some of the analysis that is relevant to this particular textbook are given below.

- Sketch or plot of discrete time signals as a function of independent variable.
- Spectrum analysis of discrete time signals.
- Solution of LTI discrete time systems.
- Perform convolution and deconvolution operations on discrete time signals.
- Perform various transforms on discrete time signals like Fourier transform, $\mathcal{Z}$-transform, **F**ast **F**ourier **T**ransform (FFT), etc.
- Design and frequency response analysis of FIR and IIR filters.
- Decimation and interpolation of discrete time signals.
- Estimation of energy and power spectrum of discrete time signals.

Chapter 2

Discrete Time Signals and Systems

2.1 Introduction

In today's world, digital systems are employed for almost every application. The digital systems can process only discrete signals. This chapter deals with time domain analysis of discrete time signals and systems. In the first part of this chapter, the generation, representation, classification and mathematical operations on discrete time signals are discussed in detail. In the second part of this chapter, the representation, classification and response of discrete time systems are discussed in detail. The concept of LTI systems are highlighted wherever necessary.

Discrete Signal and Discrete Time Signal

The ***discrete signal*** is a function of a discrete independent variable. The independent variable is divided into uniform intervals and each interval is represented by an integer. The letter "n" is used to denote the independent variable. The discrete or digital signal is denoted by x(n).

The discrete signal is defined for every integer value of the independent variable "n". The magnitude (or value) of discrete signal can take any discrete value in the specified range. Here both the value of the signal and the independent variable are discrete. The discrete signal can be represented by a one-dimensional array as shown in the following example.

Example :

$x(n) = \{ 2, 4, -1, 3, 3, 4 \}$

Here the discrete signal x(n) is defined for, $n = 0, 1, 2, 3, 4, 5$

\ $x(0) = 2$; $x(1) = 4$; $x(2) = -1$; $x(3) = 3$; $x(4) = 3$; $x(5) = 4$.

When the independent variable is time t, the discrete signal is called ***discrete time signal***. In discrete time signal, the time is divided uniformly using the relation $t = nT$, where T is the sampling time period. (The sampling time period is the inverse of sampling frequency). The discrete time signal is denoted by x(n) or x(nT).

Since the discrete signals have a sequence of numbers (or values) defined for integer values of the independent variable, the discrete signals are also known as ***discrete sequence***. In this book, the term sequence and signal are used synonymously. Also in this book, the discrete signal is referred as discrete time signal.

Digital Signal

The ***digital signal*** is same as discrete signal except that the magnitude of the signal is quantized. The magnitude of the signal can take one of the values in a set of quantized values. Here quantization is necessary to represent the signal in binary codes.

The generation of a discrete time signal by sampling a continuous time signal and then quantizing the samples in order to convert the signal to digital signal is shown in the following example.

Let, $x(t)$ = Continuous time signal

T = Sampling time

A typical continuous time signal and the sampling of this continuous time signal at uniform interval are shown in fig 2.1a and fig 2.1b respectively. The samples of the continuous time signal as a function of sampling time instants are shown in fig 2.1c. (In fig 2.1c, 1T, 2T, 3T, ...etc., represents sampling time instants and the value of the samples are functions of this sampling time instants).

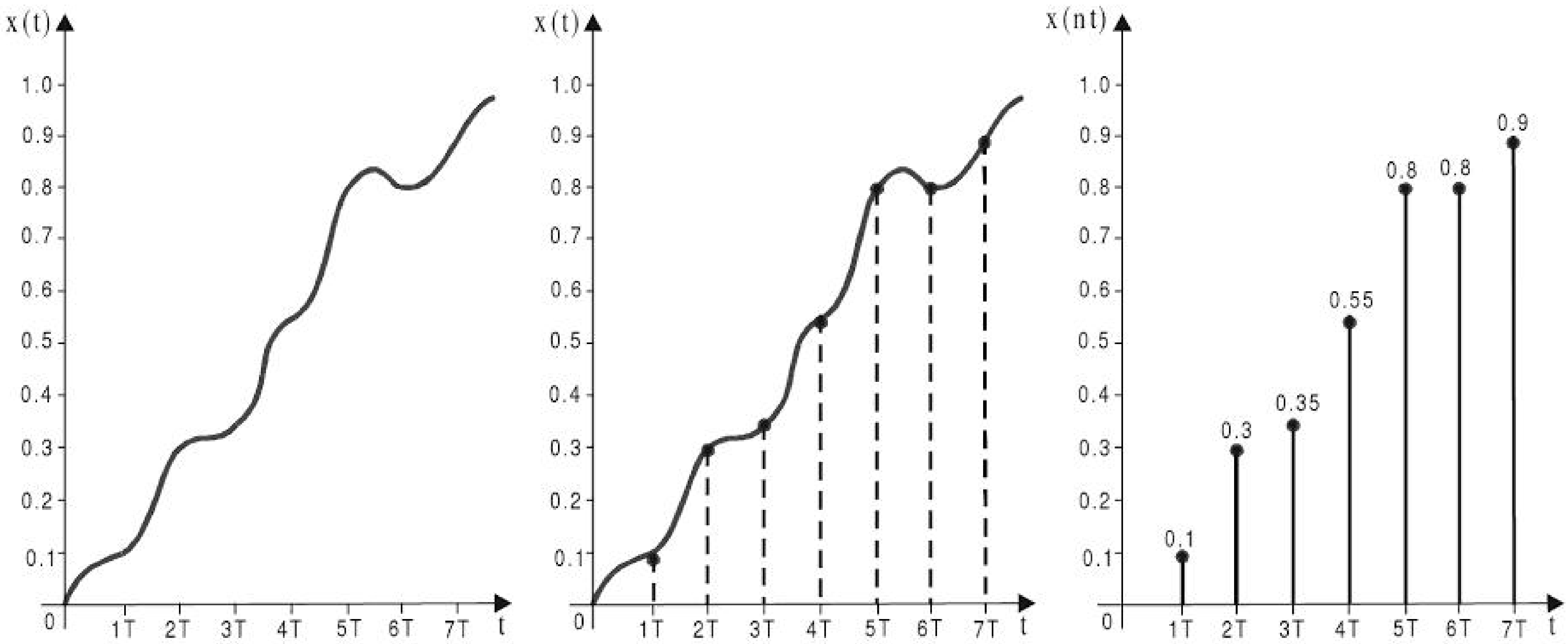


Fig 2.1a. *Fig 2.1b.* *Fig 2.1c.*

Fig 2.1 : Sampling a continuous time signal to generate discrete time signal.

When $t = 0$; $x(t) = 0$	When $t = 4T$; $x(t) = 0.55$
When $t = 1T$; $x(t) = 0.1$	When $t = 5T$; $x(t) = 0.8$
When $t = 2T$; $x(t) = 0.3$	When $t = 6T$; $x(t) = 0.8$
When $t = 3T$; $x(t) = 0.35$	When $t = 7T$; $x(t) = 0.9$

In general, the sampling time instants can be represented as, "nT", where "n" is an integer. When we drop the sampling time "T" , then the samples are functions of the integer variable "n" alone. Therefore, the samples of the continuous time signal will be a discrete time signal, denoted as $x(n)$, which is a function of an integer variable "n" as shown below.

$$x(n) = \{ 0, 0.1, 0.3, 0.35, 0.55, 0.8, 0.8, 0.9 \}$$

Here the discrete signal $x(n)$ is defined for, $n = 0, 1, 2, 3, 4, 5, 6, 7$

$\therefore$ $x(0) = 0$; $x(1) = 0.1$; $x(2) = 0.3$; $x(3) = 0.35$;

$x(4) = 0.55$; $x(5) = 0.8$; $x(6) = 0.8$; $x(7) = 0.9$.

The sample value lies the range of 0 to 1.

Let us choose 3-bit binary to represent the samples in binary code. Now, the possible binary codes are $2^3 = 8$, and so the range can be divided into eight quantization levels, and each sample is assigned, one of the quantization level as shown in the following table.

Quantization level (R = Range = 1)	Binary code	Range represented by quantization level for quantization by truncation
$0 \times \frac{R}{2^3} = 0 \times \frac{1}{8} = 0$	000	$0.000 \le x(n) < 0.125 \Rightarrow 0.000$
$1 \times \frac{R}{2^3} = 1 \times \frac{1}{8} = 0.125$	001	$0.125 \le x(n) < 0.250 \Rightarrow 0.125$
$2 \times \frac{R}{2^3} = 2 \times \frac{1}{8} = 0.25$	010	$0.250 \le x(n) < 0.375 \Rightarrow 0.250$
$3 \times \frac{R}{2^3} = 3 \times \frac{1}{8} = 0.375$	011	$0.375 \le x(n) < 0.500 \Rightarrow 0.375$
$4 \times \frac{R}{2^3} = 4 \times \frac{1}{8} = 0.5$	100	$0.500 \le x(n) < 0.625 \Rightarrow 0.500$
$5 \times \frac{R}{2^3} = 5 \times \frac{1}{8} = 0.625$	101	$0.625 \le x(n) < 0.75 \Rightarrow 0.625$
$6 \times \frac{R}{2^3} = 6 \times \frac{1}{8} = 0.75$	110	$0.750 \le x(n) < 0.875 \Rightarrow 0.750$
$7 \times \frac{R}{2^3} = 7 \times \frac{1}{8} = 0.875$	111	$0.875 \le x(n) \le 1.000 \Rightarrow 0.875$

Let, $x_q(n)$ = Quantized discrete time signal.

$x_c(n)$ = Quantized and coded discrete time signal.

Now, $x_q(n) = \{ 0, 0, 0.25, 0.25, 0.5, 0.75, 0.75, 0.875 \}$

$x_c(n) = \{ 000, 000, 010, 010, 100, 110, 110, 111 \}$

The quantized and coded discrete time signal $x_c(n)$ is called digital signal.

2.2 Discrete Time Signals

2.2.1 Generation of Discrete Time Signals

A discrete time signal can be generated by the following three methods.

The methods 1 and 2 are independent of any time frame but Method 3 depends critically on time.

1. Generate a set of numbers and arrange them as a sequence.

Example :

The numbers 0, 2, 4, ..., 2N form a sequence of even numbers and can be expressed as,

$x(n) = 2n \quad ; \quad 0 \le n \le N$

2. Evaluation of a numerical recursion relation will generate a discrete signal.

Example :

$x(n) = 0.2\,x(n-1)$ with initial condition $x(0) = 1$, gives the sequence, $x(n) = 0.2^n \;;\; 0 \le n < \infty$

When $n = 0 \;;\; x(0) = 1$ (initial condition) $= 0.2^0$

When $n = 1 \;;\; x(1) = 0.2\,x(1-1) = 0.2\,x(0) = 0.2 \quad = 0.2^1$

When $n = 2 \;;\; x(2) = 0.2\,x(2-1) = 0.2\,x(1) = 0.2 \times 0.2 = 0.2^2$

When $n = 3 \;;\; x(3) = 0.2\,x(3-1) = 0.2\,x(2) = 0.2 \times 0.2^2 = 0.2^3$ and so on

$\therefore\; x(n) = 0.2^n \;;\; 0 \le n < \infty$

3. A third method is by uniformly sampling a continuous time signal and using the amplitudes of the samples to form a sequence.

Let, $x(t)$ = Continuous time signal

Now, Discrete signal, $x(nT) = x(t)\big|_{t=nT} \;;\; -\infty < n < \infty$

where, T is the sampling interval

The generation of discrete signal by sampling a continuous time signal is shown in fig 2.1.

2.2.2 Representation of Discrete Time Signals

The discrete time signal can be represented by the following methods.

1. Functional representation

In functional representation, the signal is represented as a mathematical equation, as shown in the following example.

$$x(n) = \begin{cases} -0.5 & ; \; n = -2 \\ 1.0 & ; \; n = -1 \\ -1.0 & ; \; n = 0 \\ 0.6 & ; \; n = 1 \\ 1.2 & ; \; n = 2 \\ 1.5 & ; \; n = 3 \\ 0 & ; \; \text{other } n \end{cases}$$

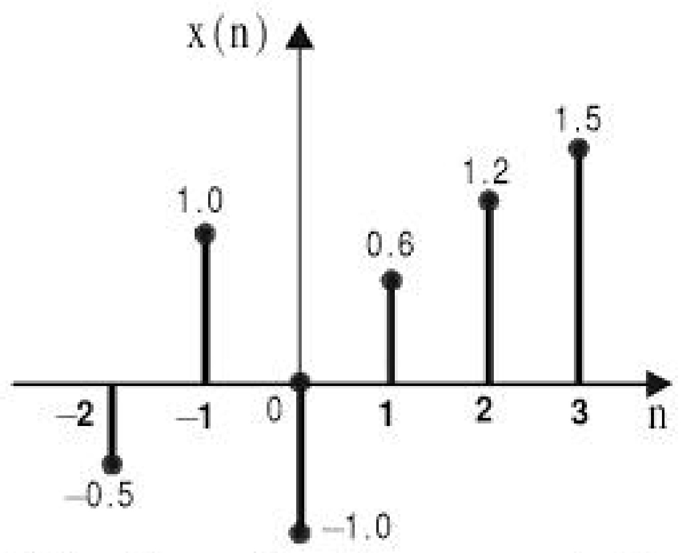


Fig 2.2 : Graphical representation of a discrete time signal.

2. Graphical representation

In graphical representation, the signal is represented in a two-dimensional plane. The independent variable is represented in the horizontal axis and the value of the signal is represented in the vertical axis as shown in fig 2.2.

3. Tabular representation

In tabular representation, two rows of a table are used to represent a discrete time signal. In the first row, the independent variable "n" is tabulated and in the second row the value of the signal for each value of "n" are tabulated as shown in the following table.

n	....	–2	-1	0	1	2	3	....
x(n)	....	–0.5	1.0	–1.0	0.6	1.2	1.5	....

4. Sequence representation

In sequence representation, the discrete time signal is represented as a one-dimensional array as shown in the following examples.

> An infinite duration discrete time signal with the time origin, n = 0, indicated by the symbol - is represented as,
>
> $x(n) = \{ \ldots\ldots -0.5,\ 1.0,\ \underline{-1.0},\ 0.6,\ 1.2,\ 1.5,\ \ldots\ldots \}$
>
> An infinite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,
>
> $x(n) = \{ \underline{-1.0},\ 0.6,\ 1.2,\ 1.5,\ \ldots \}$ or $x(n) = \{-1.0,\ 0.6,\ 1.2,\ 1.5,\ \ldots \}$
>
> A finite duration discrete time signal with the time origin, n = 0, indicated by the symbol - is represented as,
>
> $x(n) = \{ -0.5,\ 1.0,\ \underline{-1.0},\ 0.6,\ 1.2,\ 1.5 \}$
>
> A finite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,
>
> $x(n) = \{ \underline{-1.0},\ -0.6,\ 1.2,\ 1.5 \}$ or $x(n) = \{ -1.0,\ 0.6,\ 1.2,\ 1.5\}$

2.2.3 Standard Discrete Time Signals

1. Digital impulse signal or unit sample sequence

$$\text{Impulse signal, } \delta(n) = 1 \; ; \; n = 0$$
$$= 0 \; ; \; n \neq 0$$

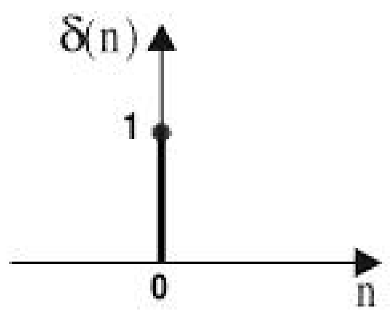


Fig 2.3 : Digital impulse signal.

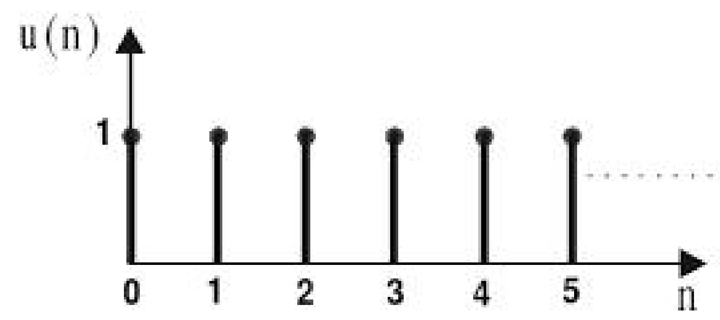


Fig 2.4 : Unit step signal.

2. Unit step signal

$$\text{Unit step signal, } u(n) = 1 \; ; \; n \geq 0$$
$$= 0 \; ; \; n < 0$$

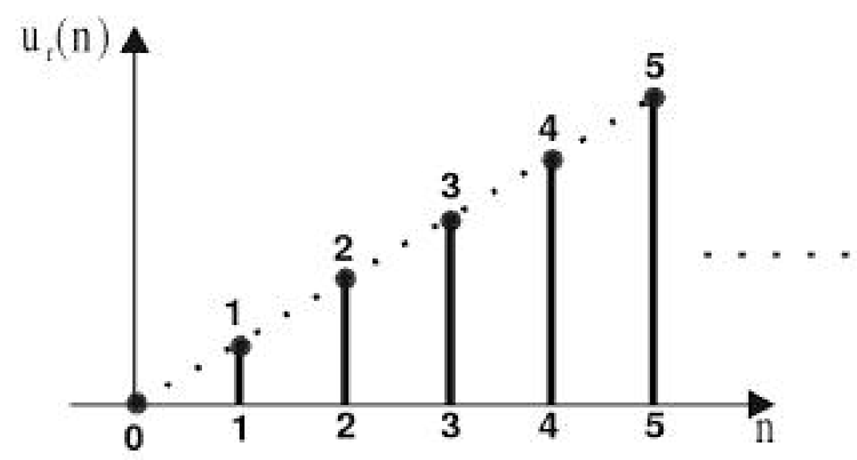


Fig 2.5 : Ramp signal.

3. Ramp signal

$$\text{Ramp signal, } u_r(n) = n \; ; \; n \geq 0$$
$$= 0 \; ; \; n < 0$$

4. Exponential signal

$$\text{Exponential signal, } g(n) = a^n \; ; \; n \geq 0$$
$$= 0 \; ; \; n < 0$$

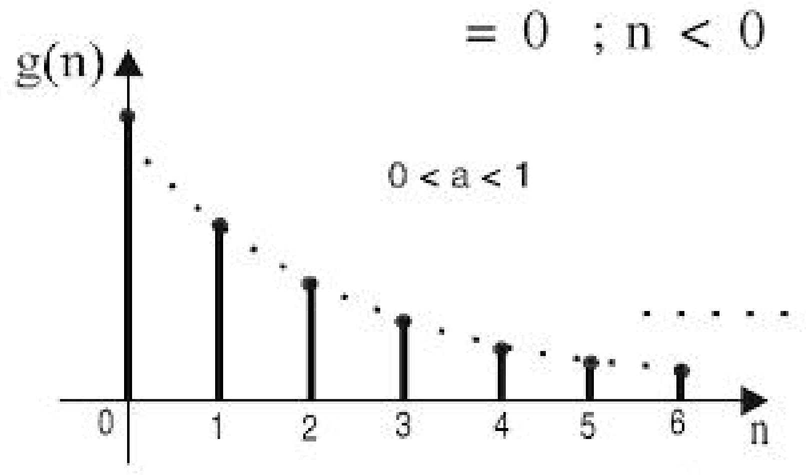


Fig 2.6a : Decreasing exponential signal.

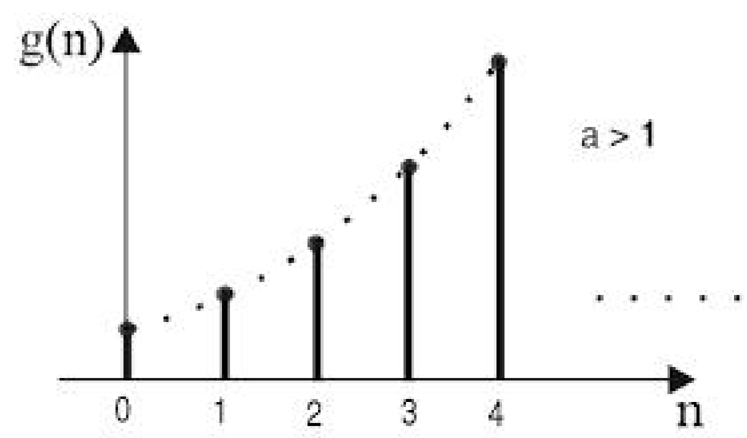


Fig 2.6b : Increasing exponential signal.

Fig 2.6 : Exponential signal.

5. Discrete time sinusoidal signal

The discrete time sinusoidal signal may be expressed as,

$$x(n) = A\cos(\omega_0 n + \theta)\ ;\ \text{for n in the range } -\infty < n < +\infty$$

$$x(n) = A\sin(\omega_0 n + \theta)\ ;\ \text{for n in the range } -\infty < n < +\infty$$

where, ω_0 = Frequency in radians/sample ; θ = Phase in radians

$$f_0 = \frac{\omega_0}{2\pi} = \text{Frequency in cycles/sample}$$

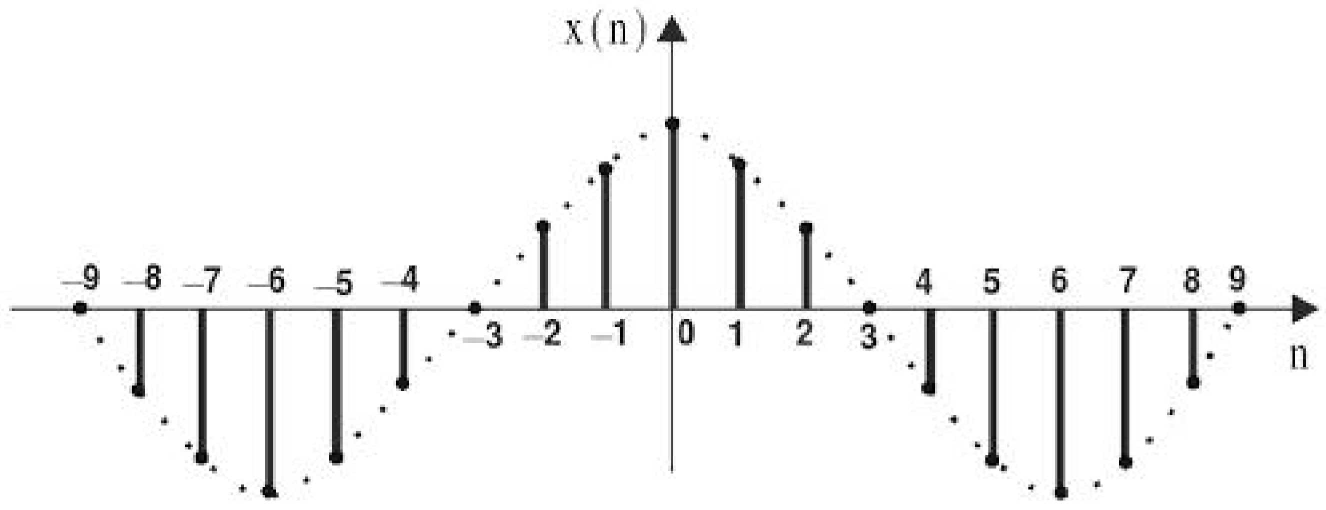


Fig 2.7a : Discrete time sinusoidal signal represented by equation $x(n) = A\cos(\omega_0 n)$.

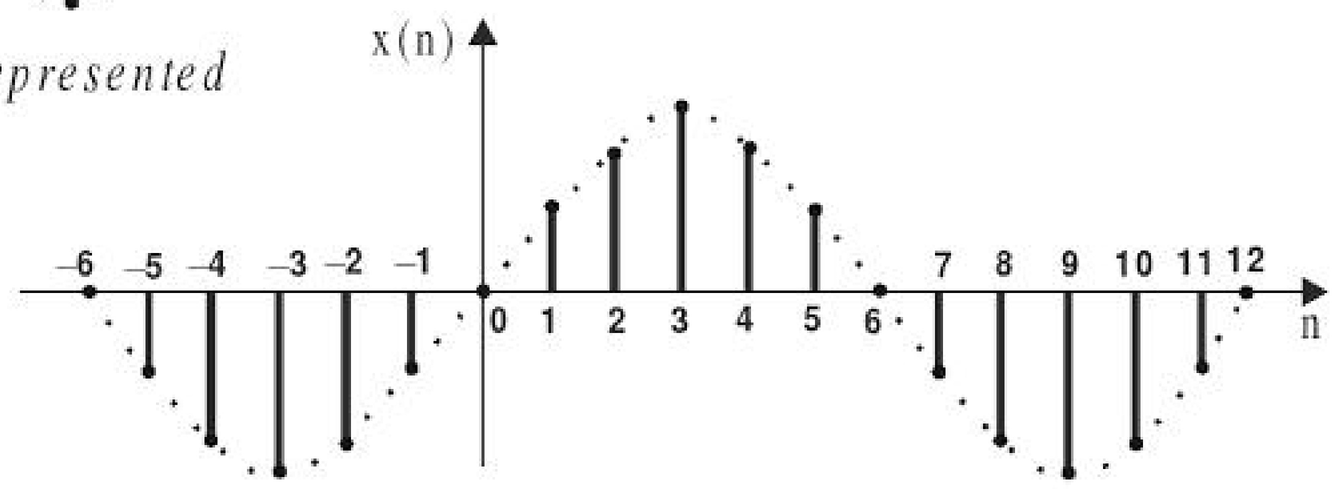


Fig 2.7b : Discrete time sinusoidal signal represented by equation $x(n) = A\sin(\omega_0 n)$.

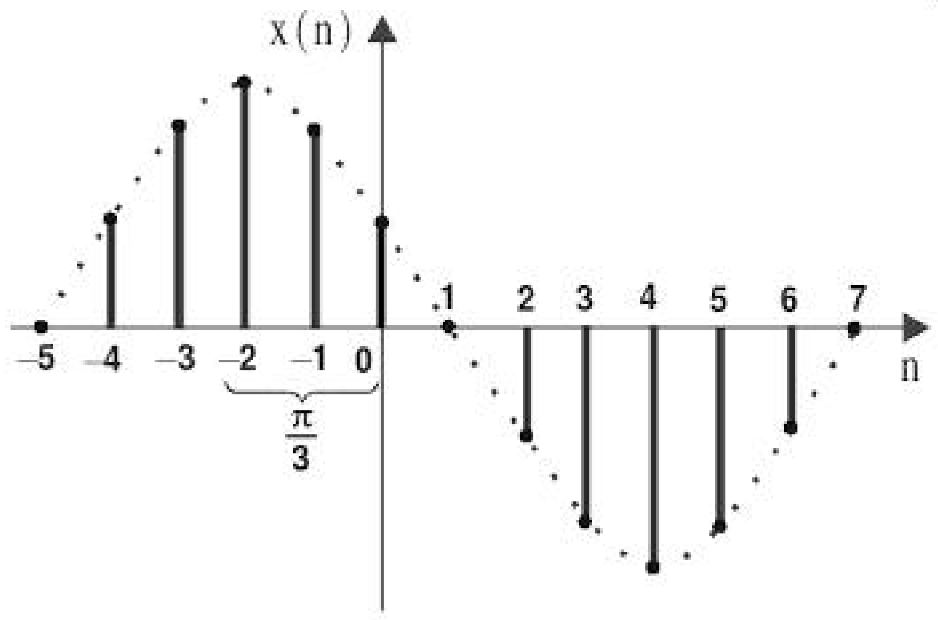


Fig 2.7c : Discrete time sinusoidal signal represented by equation,
$x(n) = A\cos\left(\frac{\pi}{6}n + \frac{\pi}{3}\right);\ \omega_0 = \frac{\pi}{6}\ ;\ \theta = \frac{\pi}{3}$

Fig 2.7 : Discrete time sinusoidal signals.

Properties of Discrete Time Sinusoid

1. A discrete time sinusoid is periodic only if its frequency f_0 is a rational number, (i.e., ratio of two integers).
2. Discrete time sinusoids whose frequencies are separated by integer multiples of 2π are identical.

   $\therefore x(n) = A\cos[(\omega_0 + 2\pi k)\, n + \theta]$, for k = 0,1,2...........are identical in the interval $-\pi \le \omega_0 \le \pi$ and so they are indistinguishable.

> ***Proof :***
>
> $$\cos[(\omega_0 + 2\pi k)\, n + \theta] = \cos(\omega_0 n + 2\pi nk + \theta) = \cos[(\omega_0 n + \theta) + 2\pi nk]$$
>
> $$= \cos(\omega_0 n + \theta)\cos 2\pi nk - \sin(\omega_0 n + \theta)\sin 2\pi nk$$
>
> Since n and k are integers, $\cos 2\pi nk = 1$ and $\sin 2\pi nk = 0$
>
> $$\therefore \cos[(\omega_0 + 2\pi k)\, n + \theta] = \cos(\omega_0 n + \theta), \quad \text{for } k = 0, 1, 2, 3, ...$$

Conclusion

1. The sequences of any two sinusoids with frequencies in the range, $-\pi \le \omega_0 \le \pi$ (or $-1/2 \le f_0 \le 1/2$), are distinct.

$$[-\pi \le \omega \le \pi \xrightarrow{\text{divide by } 2\pi} -1/2 \le f \le 1/2]$$

2. Any discrete time sinusoid with frequency $\omega_0 > |\pi|$ (or $f_0 > |1/2|$) will be identical to another discrete time sinusoid with frequency $\omega_0 < |\pi|$ (or $f_0 < |1/2|$).

6. Discrete time complex exponential signal

The discrete time complex exponential signal is defined as,

$$x(n) = a^n e^{j(\omega_0 n+\theta)} = a^n [\cos(\omega_0 n + \theta) + j \sin(\omega_0 n + \theta)]$$

$$= a^n \cos(\omega_0 n + \theta) + j\, a^n \sin(\omega_0 n + \theta) = x_r(n) + j\, x_i(n)$$

where, $x_r(n) = \text{Real part of } x(n) = a^n \cos(\omega_0 n + \theta)$

$x_i(n) = \text{Imaginary part of } x(n) = a^n \sin(\omega_0 n + \theta)$

The real part of x(n) will give an exponentially increasing cosinusoid sequence for a > 1 and exponentially decreasing cosinusoid sequence for 0 < a < 1.

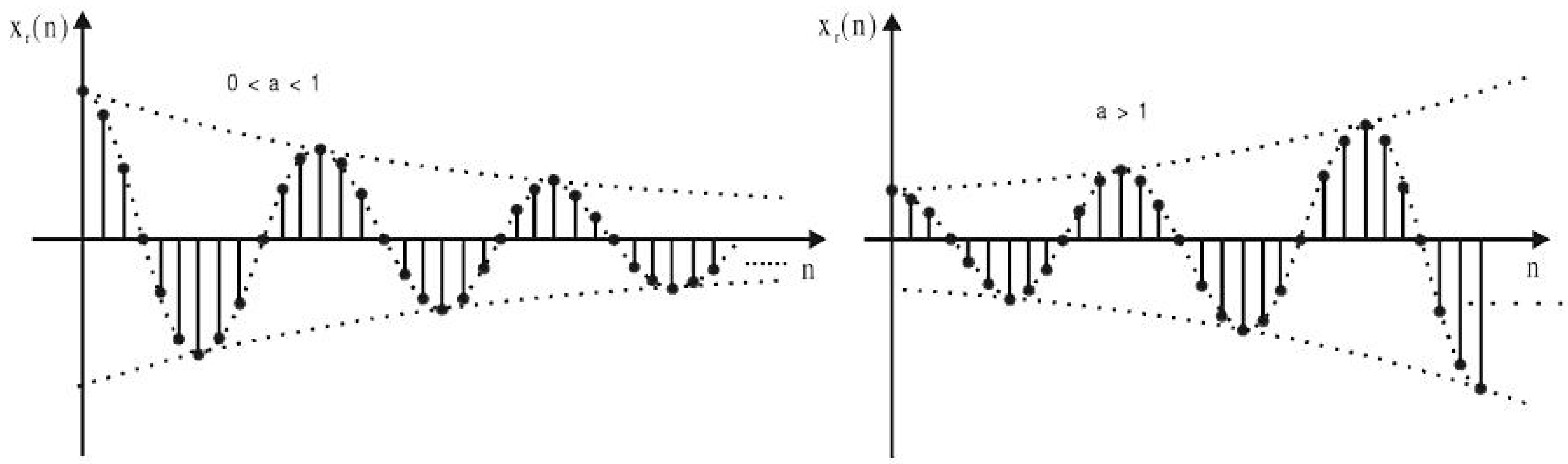


Fig 2.8a : The discrete time sequence represented by the equation, $x_r(n) = a^n \cos \omega_0 n$ for $0 < a < 1$.

Fig 2.8b : The discrete time sequence represented by the equation, $x_r(n) = a^n \cos \omega_0 n$ for $a > 1$.

Fig 2.8 : Real part of complex exponential signal.

The imaginary part of x(n) will give rise to an exponentially increasing sinusoid sequence for a > 1 and exponentially decreasing sinusoid sequence for 0 < a < 1.

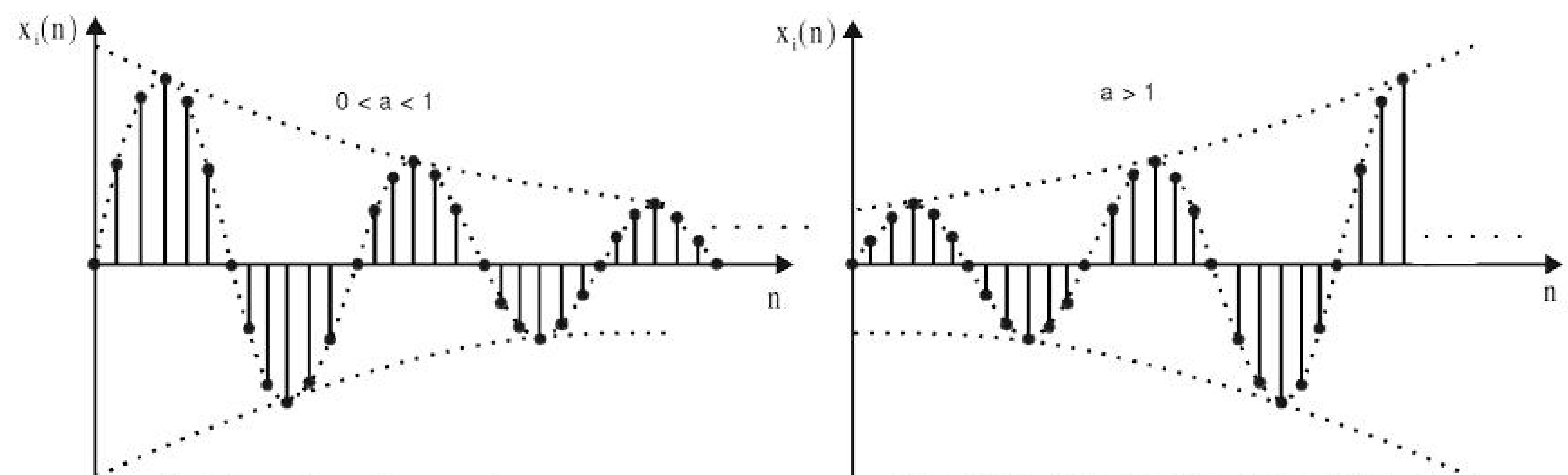


Fig 2.9a : The discrete time sequence represented by the equation, $x_i(n) = a^n \sin \omega_0 n$ for $0 < a < 1$.

Fig 2.9b : The discrete time sequence represented by the equation, $x_i(n) = a^n \sin \omega_0 n$ for $a > 1$.

Fig 2.9 : Imaginary part of complex exponential signal.

2.3 Sampling of Continuous Time (Analog) Signals

The ***sampling*** is the process of conversion of a continuous time signal into a discrete time signal. The sampling is performed by taking samples of continuous time signal at definite intervals of time. Usually, the time interval between two successive samples will be same and such type of sampling is called ***periodic or uniform sampling***.

The time interval between successive samples is called ***sampling time*** (or sampling period or sampling interval), and it is denoted by "T". The unit of sampling period is second (s). [The lower units are millisecond (ms) and microsecond (μs)].

The inverse of sampling period is called ***sampling frequency*** (or sampling rate), and it is denoted by F_s. The unit of sampling frequency is hertz (Hz). (The higher units are kHz and MHz).

Let, $x_a(t)$ = Analog / Continuous time signal.

$x(n)$ = Discrete time signal obtained by sampling $x_a(t)$.

Mathematically, the relation between $x(n)$ and $x_a(t)$ can be expressed as,

$$x(n) = x_a(t)\big|_{t=nT} = x_a(nT) = x_a\left(\frac{n}{F_s}\right) \; ; \qquad \text{for n in the range } -\infty < n < \infty$$

where, T = Sampling period or interval in seconds

$$F_s = \frac{1}{T} = \text{Sampling rate or sampling frequency in hertz}$$

Example : Let, $x_a(t) = A\cos(\Omega_0 t + \theta) = A\cos(2\pi F_0 t + \theta)$

where, Ω_0 = Frequency of analog signal in rad/s

$$F_0 = \frac{\Omega_0}{2\pi} = \text{Frequency of analog signal in Hz}$$

Let $x_a(t)$ be sampled at intervals of T seconds to get x(n), where $T = \frac{1}{F_s}$

$$\therefore\; x(n) = x_a(t)\big|_{t=nT} = A\cos(\Omega_0 t + \theta)\big|_{t=nT}$$

$$= A\cos(\Omega_0 nT + \theta) = A\cos\left(\frac{2\pi F_0}{F_s}n + \theta\right) = A\cos(2\pi f_0 n + \theta) = A\cos(\omega_0 n + \theta)$$

where, $f_0 = \frac{F_0}{F_s}$ = Frequency of discrete sinusoid in cycles/sample

$\omega_0 = 2\pi f_0$ = Frequency of discrete sinusoid in radians/sample

2.3.1 Sampling and Aliasing

In Section 2.2, it is observed that any two sinusoid signals with frequencies in the range $-1/2 \le f \le +1/2$ are distinct and a discrete sinusoid with frequency, $f > |\pm 1/2|$ will be identical to another discrete sinusoid with frequency, $f < |\pm 1/2|$. Therefore, we can conclude that range of frequency of discrete time signal is $-1/2$ to $+1/2$. But the range of frequency of analog signal is $-\infty$ to $+\infty$. While sampling analog signals, the infinite frequency range continuous time signals are mapped (or converted) to finite frequency range discrete time signals.

The relation between frequency of analog and discrete time signal is,

$$f = \frac{F}{F_s} \qquad .....(2.1)$$

The range of frequency of discrete time signal is,

$$-\frac{1}{2} \le f \le \frac{1}{2} \qquad .....(2.2)$$

On substituting for f from equation (2.1) in equation (2.2) we get,

$$-\frac{1}{2} \le \frac{F}{F_s} \le \frac{1}{2} \qquad .....(2.3)$$

On multiplying equation (2.3) by F_s we get,

$$-\frac{F_s}{2} \le F \le \frac{F_s}{2} \qquad .....(2.4)$$

From equation (2.4) we can say that when an analog signal is sampled at a frequency F_s, the highest analog frequency that can be uniquely represented by a discrete time signal will be $F_s/2$. The continuous time signal with frequency above $F_s/2$ will be represented as a signal within the range $+ F_s/2$ to $- F_s/2$. Hence the signal with frequency above $F_s/2$ will have an identical signal with frequency below $F_s/2$ in the discrete form.

Hence infinite number of high frequency continuous time signals will be represented by a single discrete time signal. Such signals are called ***alias.***

The phenomenon of high-frequency component getting the identity of low-frequency component during sampling is called ***aliasing.***

Sampling an analog signal with frequency F by choosing a sampling frequency F_s such that $F_s/2 > F$ will not result in alias. But sampling frequency is selected such that $F_s/2 < F$ that the frequency above $F_s/2$ will have alias with frequency below $F_s/2$. Hence the point of reflection is $F_s/2$, and the frequency $F_s/2$ is called ***folding frequency***.

The discrete time sinusoids, A sin [$(2pf_0 + 2pk)n$], will be alias for integer values of k. It is also observed that, a sinusoidal signal with frequency F_1 will be an alias of sinusoidal signal with frequency F_2 if it is sampled at a frequency $F_s = F_1 - F_2$. In general, if the sampling frequency is any multiple of $F_1 - F_2$, [i.e., $F_s = k(F_1 - F_2)$ where k = 1, 2, 3, ...] the signal with frequency F_2 will be an alias of the signal with frequency F_1.

Let, F_{max} be maximum frequency of analog signal that can be uniquely represented as discrete time signal when sampled at a frequency F_s.

$$\text{Now, } F_{max} = \frac{F_s}{2} \qquad .....(2.5)$$

$$\therefore F_s = 2F_{max} \qquad .....(2.6)$$

The equation (2.6) gives a choice for selecting sampling frequency. From equation (2.6) we can say that for unique representation of analog signal with maximum frequency F_{max}, the sampling frequency should be greater than $2F_{max}$.

i.e., to avoid aliasing F_s ³ $2F_{max}$ ..... (2.7)

When sampling frequency F_s is equal to $2F_{max}$, the sampling rate is called ***Nyquist rate***.

It is observed that a nonshifted sinusoidal signal when sampled at Nyquist rate, will produce zero sample sequence (i.e., discrete sequence with all zeros), (because the sinusoidal signal is sampled at its zero crossings, Refer example 2.3). Hence to avoid zero sampling of sinewave, the sampling frequency F_s should be greater than $2F_{max}$, where F_{max} is the maximum frequency in the analog signal.

A discrete signal obtained by sampling can be reconstructed to an analog signal, only when it is sampled without aliasing. The above concepts of sampling analog signals are summarized as the sampling theorem, given below.

Sampling Theorem :	*A band limited continuous time signal with highest frequency (bandwidth) F_m hertz can be uniquely recovered from its samples provided that the sampling rate F_s is greater than or equal to $2F_m$ samples per second.*

Note : The effects of aliasing in frequency spectrum are discussed in Chapter-4.

Example 2.1

Consider the analog signals, $x_1(t) = 3\cos 2p(20t)$ and $x_2(t) = 3\cos 2p(70t)$.

Find a sampling frequency so that 70 Hz signal is an alias of the 20 Hz signal?

Solution

Let, the sampling frequency, $F_s = 70 - 20 = 50\text{Hz}$.

$$\therefore x_1(n) = x_1(t)\Big|_{t = nT = \frac{n}{F_s}} = 3\cos 2\pi(20t)\Big|_{t = \frac{n}{F_s}} = 3\cos 2\pi\left(\frac{20 \times n}{50}\right) = 3\cos\frac{4\pi}{5}n$$

$$x_2(n) = x_2(t)\Big|_{t = nT = \frac{n}{F_s}} = 3\cos 2\pi(70t)\Big|_{t = \frac{n}{F_s}} = 3\cos 2\pi\left(\frac{70 \times n}{50}\right)$$

$$= 3\cos\frac{14\pi}{5}n = 3\cos\left(2\pi n + \frac{4\pi}{5}n\right) = 3\cos\frac{4\pi}{5}n$$

For integer values of n
$\cos(2pn + q) = \cos q$

From the above analysis, we observe that $x_1(n)$ and $x_2(n)$ are identical, and so $x_2(t)$ is an alias of $x_1(t)$ when sampled at a frequency of 50 Hz.

Example 2.2

Let an analog signal, $x_a(t) = 10\cos 200\,pt$. If the sampling frequency is 150 Hz, find the discrete time signal x(n). Also find an alias frequency corresponding to $F_s = 150\text{ Hz}$.

Solution

$$x(n) = x_a(t)\Big|_{t = nT = \frac{n}{F_s}} = 10\cos 200\pi t\Big|_{t = \frac{n}{F_s}} = 10\cos 200\pi \times \frac{n}{F_s}$$

$$= 10\cos\frac{200\pi \times n}{150} = 10\cos\frac{4\pi}{3}n = 10\cos\left(2\pi - \frac{2\pi}{3}\right)n = 10\cos\frac{2\pi}{3}n = 10\cos 2\pi\frac{1}{3}n$$

We know that the discrete time sinusoids whose frequencies are separated by integer multiples of 2p are identical.

$$\therefore 10\cos\frac{2\pi}{3}n = 10\cos\left(\frac{2\pi}{3} + 2\pi\right)n = 10\cos\frac{8\pi}{3}n = 10\cos 2\pi\frac{4}{3}n$$

Now, $10\cos 2\pi\frac{4}{3}n$ is an alias of $10\cos\frac{2\pi}{3}n$.

Here the frequency of the signal, $10\cos 2\pi\frac{4}{3}n$ is,

$$f = \frac{4}{3} \text{ cycles / sample}$$

We know that, $f = \frac{F}{F_s} \quad \Rightarrow \quad F = f\,F_s = \frac{4}{3} \times 150 = 200\text{Hz}$

$\therefore$ when, $F_s = 150\text{Hz}$, $F = 200\text{Hz}$ is an alias frequency.

Example 2.3

Consider the analog signal, $x_a(t) = 6\cos 50\pi t + 3\sin 200\pi t - 3\cos 100\pi t$.

Determine the minimum sampling frequency and the sampled version of analog signal at this frequency. Sketch the waveform and show the sampling points. Comment on the result.

Solution

The given analog signal can be written as shown below.

$$x_a(t) = 6\cos 50\pi t + 3\sin 200\pi t - 3\cos 100\pi t = 6\cos 2\pi F_1 t + 3\sin 2\pi F_2 t - 3\cos 2\pi F_3 t$$

$$\text{Where, } 2\pi F_1 = 50\pi \quad ; \quad F_1 = 25\,\text{Hz}$$

$$2\pi F_2 = 200\pi \quad ; \quad F_2 = 100\,\text{Hz}$$

$$2\pi F_3 = 100\pi \quad ; \quad F_3 = 50\,\text{Hz}$$

The maximum analog frequency in the signal is 100Hz. The sampling frequency should be twice that of this maximum analog frequency.

i.e., $F_s \geq 2F_{max} \Rightarrow F_s \geq 2 \times 100$

Let, sampling frequency, $F_s = 200\,\text{Hz}$

$$\therefore\ x_a(nT) = x_a(t)\big|_{t=nT} = x_a(t)\Big|_{t=\frac{n}{F_s}}$$

$$= 6\cos\frac{50\pi n}{200} + 3\sin\frac{200\pi n}{200} - 3\cos\frac{100\pi n}{200} = 6\cos\frac{\pi n}{4} + 3\sin\pi n - 3\cos\frac{\pi n}{2}$$

For integer values of n, $\sin\pi n = 0$.

$$\therefore\ x_a(nT) = 6\cos\frac{\pi n}{4} - 3\cos\frac{\pi n}{2}$$

The components of analog waveform and the sampling points are shown in fig1.

<u>Comment</u> : In the sampled version of analog signal $x_a(nT)$, the component $3\sin 200\pi t$ will give always zero samples when sampled at 200Hz for any value of n. This is the drawback in sampling at Nyquist rate (i.e., sampling at $F_s = 2F_{max}$).

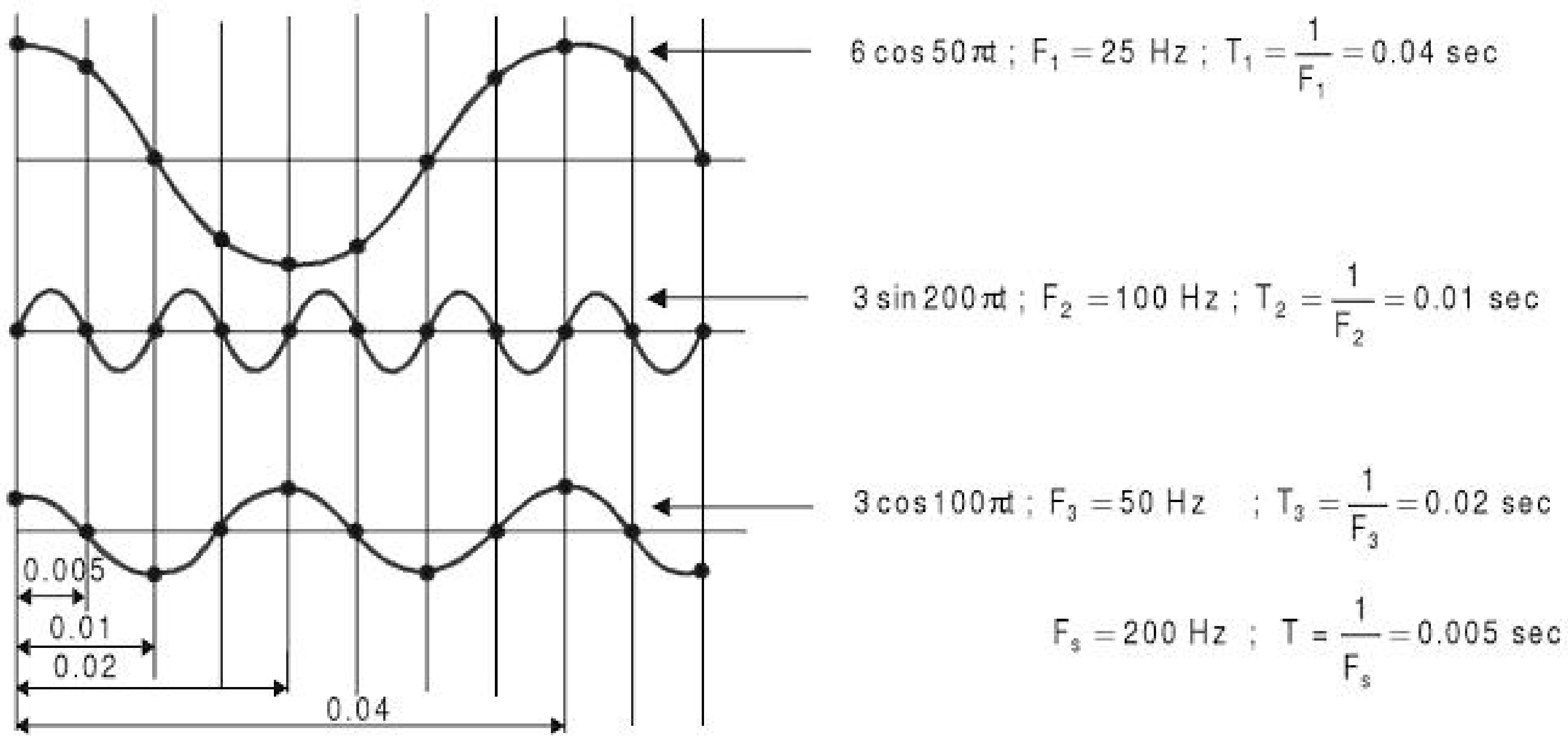


Fig 1 : Sampling points of the components of the signal $x_a(t)$.

2.4 Classification of Discrete Time Signals

The discrete time signals are classified depending on their characteristics. Some ways of classifying discrete time signals are,

1. Deterministic and nondeterministic signals
2. Periodic and aperiodic signals
3. Symmetric and antisymmetric signals
4. Energy and power signals
5. Causal and noncausal signals

2.4.1 Deterministic and Nondeterministic Signals

The signals that can be completely specified by mathematical equations are called ***deterministic signals***.The step, ramp, exponential and sinusoidal signals are examples of deterministic signals.

The signals whose characteristics are random in nature are called ***nondeterministic signals.*** The noise signals from various sources are best examples of nondeterministic signals.

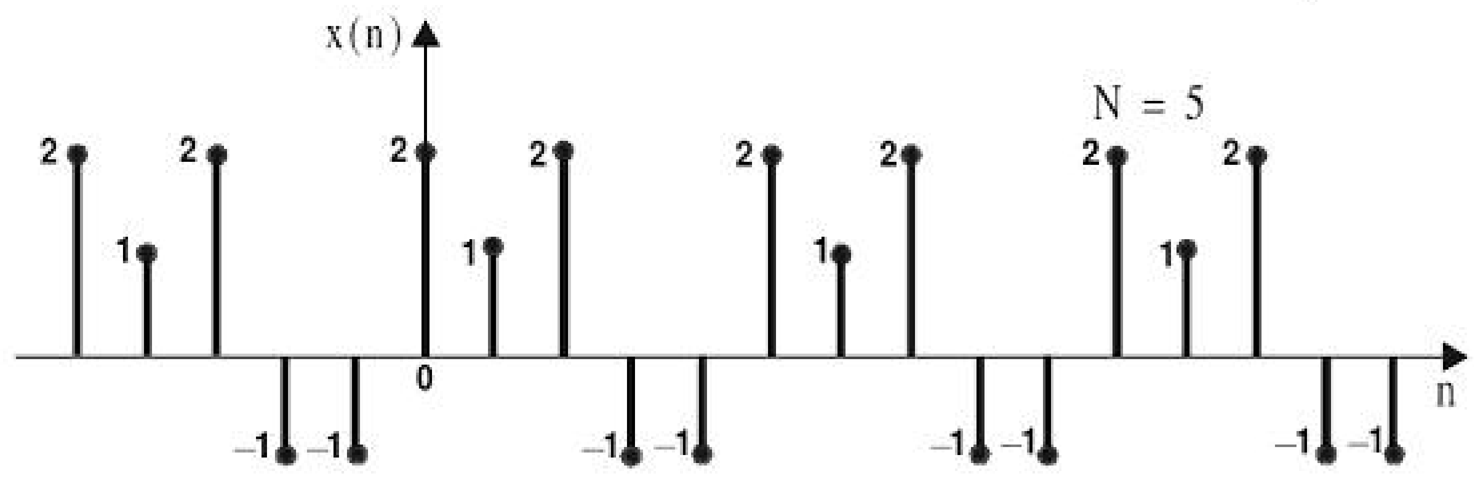
2.4.2 Periodic and Aperiodic Signals

When a discrete time signal x(n), satisfies the condition $\mathbf{x(n+N)=x(n)}$ for integer values of N, then the discrete time signal x(n) is called ***periodic signal***. Here N is the number of samples of a period.

$\boxed{\text{i.e, if, } \mathbf{x(n+N)=x(n)}, \text{ for all n, then x(n) is periodic.}}$

The smallest value of N for which the above equation is true is called ***fundamental period***. If there is no value of N that satisfies the above equation, then x(n) is called ***aperiodic*** or ***nonperiodic*** signal.

When N is the fundamental period, the periodic signals will also satisfy the condition x(n + kN) = x(n), where k is an integer. The periodic signals are power signals. The discrete time sinusoidal and complex exponential signals are periodic signals when their fundamental frequency, f_0 is a rational number.



$$x(n) = \{.......2, 1, \underset{\uparrow}{2}, -1, -1, 2, 1, 2, -1, -1, 2, 1, 2, -1, -1, ...\}$$

Fig 2.10 : Periodic discrete time signal.

When a discrete time signal is a sum or product of two periodic signals with fundamental periods N_1 and N_2, then the discrete time signal will be periodic with period given by LCM of N_1 and N_2.

Example 2.4

Determine whether following signals are periodic or not. If periodic find the fundamental period.

a) $x(n) = \cos\left(\frac{5\pi}{9}n + 1\right)$ **b)** $x(n) = \sin\left(\frac{n}{9} - \pi\right)$ **c)** $x(n) = \sin\frac{\pi}{8}n^2$

d) $x(n) = e^{\frac{j7\pi n}{4}}$ **e)** $x(n) = 2\cos\frac{5\pi n}{3} + 3e^{j\frac{3\pi n}{4}}$

Solution

a) Given that, $\mathbf{x(n) = \cos\left(\frac{5\pi}{9}n + 1\right)}$

Let N and M be two integers.

Now, $x(n+N) = \cos\left(\frac{5\pi}{9}(n+N) + 1\right) = \cos\left(\frac{5\pi n}{9} + 1 + \frac{5\pi}{9}N\right)$

Since, $\cos(q + 2pM) = \cos q$, for periodicity $\frac{5\pi}{9}N$ should be integral multiple of 2π.

Let, $\frac{5\pi}{9}N = M \times 2\pi$

$\therefore\ N = M \times 2\pi \times \frac{9}{5\pi} = \frac{18M}{5}$

Here N is an integer if, M = 5, 10, 15, 20, ...

Let, M = 5 ; \ N = 18

When N = 18 ; $x(n+N) = \cos\left(\frac{5\pi n}{9} + 1 + \frac{5\pi}{9} \times 18\right) = \cos\left(\frac{5\pi n}{9} + 1 + 10\pi\right) = \cos\left(\frac{5\pi n}{9} + 1\right) = x(n)$

Hence x(n) is periodic with fundamental period of 18 samples.

---

b) Given that, $\mathbf{x(n) = \sin\left(\frac{n}{9} - \pi\right)}$

Let N and M be two integers.

Now, $x(n+N) = \sin\left(\frac{n+N}{9} - \pi\right) = \sin\left(\frac{n}{9} + \frac{N}{9} - \pi\right) = \sin\left(\frac{n}{9} - \pi + \frac{N}{9}\right)$

Since, $\sin(q + 2pM) = \sin q$, for periodicity $\frac{N}{9}$ should be equal to integral multiple of $2p$.

Let, $\frac{N}{9} = M \times 2\pi$

\ N = 18 pM

Here N cannot be an integer for any integer value of M and so x(n) will not be periodic.

---

c) Given that, $\mathbf{x(n) = \sin\left(\frac{\pi}{8}n^2\right)}$

$$\therefore\ x(n+N) = \sin\frac{\pi}{8}(n+N)^2 = \sin\frac{\pi}{8}(n^2 + N^2 + 2nN) = \sin\left(\frac{\pi}{8}n^2 + \frac{\pi N^2}{8} + \frac{\pi N}{4}n\right)$$

Let, $\frac{\pi N^2}{8} = 2\pi M_1$

$\therefore\ N = 4\sqrt{M_1}$

Now, N is integer for $M_1 = 1^2, 2^2, 3^2, 4^2$ .....

Let, $\frac{\pi N}{4} = 2\pi M_2$

\ $N = 8\,M_2$

Now, N is integer for $M_2 = 1, 2, 3, 4$ .....

When $M_1 = 2^2$ and $M_2 = 1$, we get a common value for N as, N = 8.

When N = 8 ; $x(n+N) = \sin\left(\frac{\pi}{8}n^2 + \frac{\pi 8^2}{8} + \frac{\pi 8}{4}n\right)$

$= \sin\left(\left(\frac{\pi}{8}n^2 + 2\pi n\right) + 4 \times 2\pi\right) = \sin\left(\frac{\pi}{8}n^2 + 2\pi n\right)$

$= \sin\frac{\pi}{8}n^2 = x(n)$

For interger M, sin (q + 2p M) = sinq

\ x(n) is periodic with fundamental period, N = 18 samples.

d) Given that, $x(n) = e^{\frac{j7\pi n}{4}}$

Let N and M be two integers.

Now, $x(n+N) = e^{\frac{j7\pi(n+N)}{4}} = e^{\frac{j7\pi n}{4}} e^{\frac{j7\pi N}{4}}$

Since, $e^{j2pM} = 1$, for periodicity $\frac{7\pi N}{4}$ should be an integral multiple of 2p.

Let, $\frac{7\pi N}{4} = M \times 2\pi$,

$$\therefore\ N = M \times 2\pi \times \frac{4}{7\pi} = \frac{8M}{7}$$

Here, N is integer, when M = 7, 14, 21, ...

When M = 7 ; N = 8

\ x(n) is periodic with fundamental period of 8 samples.

---

e) Given that, $x(n) = 2\cos\frac{5\pi n}{3} + 3e^{j\frac{3\pi n}{4}}$

Let, $x(n) = x_1(n) + x_2(n)$

where, $x_1(n) = 2\cos\frac{5\pi n}{3}$

$x_2(n) = 3e^{j\frac{3\pi n}{4}}$

Consider, $x_1(n) = 2\cos\frac{5\pi n}{3}$

$$\therefore\ x_1(n+N_1) = 2\cos\frac{5\pi(n+N_1)}{3}$$

$$= 2\cos\left(\frac{5\pi n}{3} + \frac{5\pi N_1}{3}\right) \quad .....(1)$$

Let, $\frac{5\pi N_1}{3} = 2\pi M_1 \Rightarrow N_1 = \frac{6}{5}M_1$

Let, $M_1 = 5$; \ $N_1 = 6$

Substitute $N_1 = 6$ in equation (1),

$$\therefore\ x_1(n+N_1) = 2\cos\left(\frac{5\pi n}{3} + \frac{5\pi}{3}\times 6\right)$$

$$= 2\cos\left(\frac{5\pi n}{3} + 5\times 2\pi\right)$$

$$= 2\cos\frac{5\pi n}{3} = x_1(n)$$

For integer M, cos(q +2pM) = cosq

\ $x_1(n)$ is periodic with fundamental period, $N_1 = 6$ samples.

Consider, $x_2(n) = 3e^{j\frac{3\pi n}{4}}$

$$\therefore\ x_2(n+N_2) = 3e^{j\frac{3\pi(n+N_2)}{4}}$$

$$= 3e^{j\left(\frac{3\pi n}{4} + \frac{3\pi N_2}{4}\right)} \quad .....(2)$$

Let, $\frac{3\pi N_2}{4} = 2\pi M_2 \Rightarrow N_2 = \frac{8}{3}M_2$

Let, $M_2 = 3$; \ $N_2 = 8$

Substitute $N_2 = 8$ in equation (2),

$$\therefore\ x_2(n+N_2) = 3e^{j\left(\frac{3\pi n}{4} + \frac{3\pi\times 8}{4}\right)}$$

$$= 3e^{j\left(\frac{3\pi n}{4} + 3\times 2\pi\right)}$$

$$= 3e^{j\frac{3\pi n}{4}} = x_2(n)$$

For integer M, $e^{j(q + 2pM)} = e^{jq}$

\ $x_2(n)$ is periodic with fundamental period, $N_2 = 8$ samples.

Here, $x(n) = x_1(n) + x_2(n)$, and $x_1(n)$ is periodic with period $N_1 = 6$, and $x_2(n)$ is periodic with period $N_2 = 8$.

Therefore, x(n) is periodic with period N, where N is LCM of N_1 and N_2.

The LCM of 6 and 8 is 24.

\ N = 24

\ x(n) is periodic with fundamental period, N = 24.

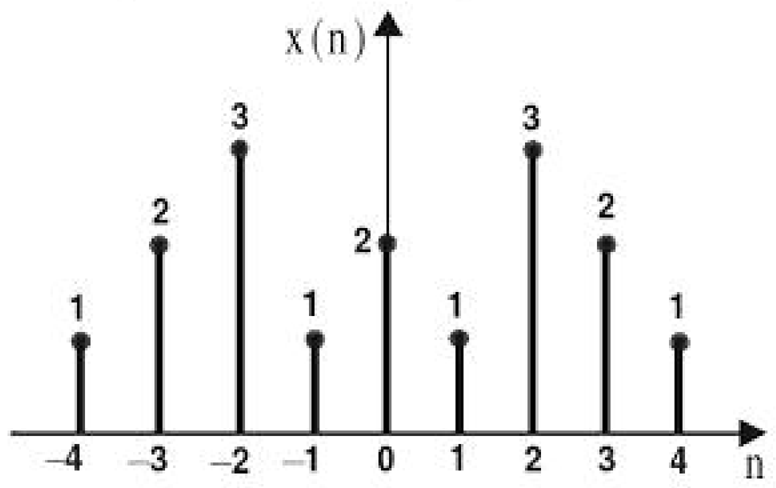
2.4.3 Symmetric (Even) and Antisymmetric (Odd) Signals

The discrete time signals may exhibit symmetry or antisymmetry with respect to n = 0. When a discrete time signal exhibits symmetry with respect to n = 0 then it is called an ***even signal.*** Therefore, the even signal satisfies the condition,

$$\boxed{x(-n) = x(n)}$$

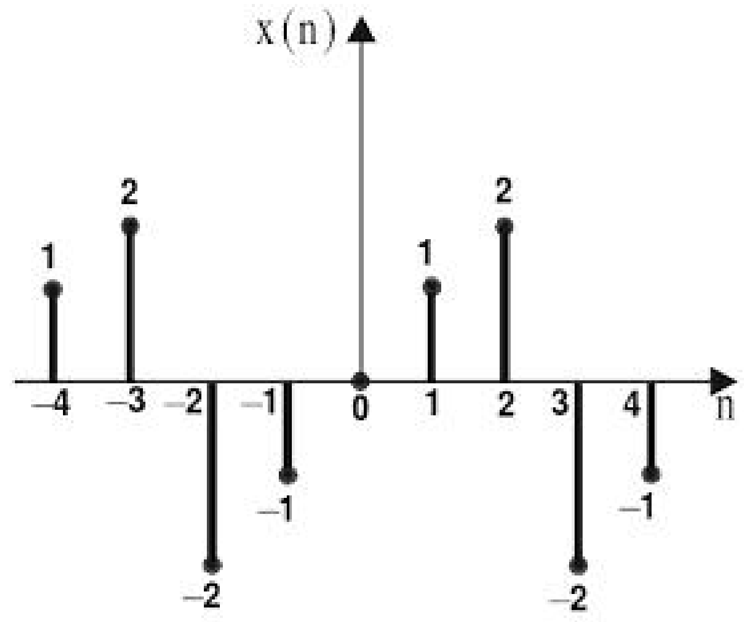
When a discrete time signal exhibits antisymmetry with respect to n = 0, then it is called an ***odd signal***. Therefore the odd signal satisfies the condition,

$$\boxed{x(-n) = -x(n)}$$



$$x(n) = \{1, 2, 3, 1, \underset{\uparrow}{2}, 1, 3, 2, 1\}$$

Fig 2.11a : *Symmetric (or even) signal.*



$$x(n) = \{1, 2, -2, -1, \underset{\uparrow}{0}, 1, 2, -2, -1\}$$

Fig 2.11b : *Antisymmetric (or odd) signal.*

Fig 2.11 : *Symmetric and antisymmetric discrete time signal.*

A discrete time signal x(n) which is neither even nor odd can be expressed as a sum of even and odd signal.

Let, $x(n) = x_e(n) + x_o(n)$

where, $x_e(n)$ = Even part of x(n)

$x_o(n)$ = Odd part of x(n)

Note : If x(n) is even then its odd part will be zero. If x(n) is odd then its even part will be zero.

Now, it can be proved that,

Even part, $x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$

Odd part, $x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$

Proof :

Let, $x(n) = x_e(n) + x_o(n)$ ......(2.8)

On replacing n by –n in equation (2.8) we get,

$$x(-n) = x_e(-n) + x_o(-n) \quad ......(2.9)$$

Since $x_e(n)$ is even, $x_e(-n) = x_e(n)$

Since $x_o(n)$ is odd, $x_o(-n) = -x_o(n)$

Hence the equation (2.9) can be written as,

$$x(-n) = x_e(n) - x_o(n) \quad ......(2.10)$$

On adding equation (2.8) and (2.10) we get,

$$x(n) + x(-n) = 2\,x_e(n)$$

$$\therefore\ x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

On subtracting equation (2.10) from equation (2.8) we get,

$$x(n) - x(-n) = 2\,x_o(n)$$

$$\therefore\ x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

Example 2.5

Determine the even and odd parts of the signals.

a) $x(n) = 3^n$ **b)** $x(n) = 3\,e^{j\frac{\pi}{5}n}$ **c)** $x(n) = \{2,\ -2,\ 6,\ -2\}$

Solution

a) Given that, $x(n) = 3^n$

$$\therefore\ x(-n) = 3^{-n}$$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)] = \frac{1}{2}[3^n + 3^{-n}]$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)] = \frac{1}{2}[3^n - 3^{-n}]$

b) Given that, $x(n) = 3\,e^{j\frac{\pi}{5}n}$

$$x(n) = 3\,e^{j\frac{\pi}{5}n} = 3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n$$

$$\therefore x(-n) = 3\,e^{-j\frac{\pi}{5}n} = 3\cos\frac{\pi}{5}n - j3\sin\frac{\pi}{5}n$$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$$= \frac{1}{2}\left[3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n + 3\cos\frac{\pi}{5}n - j3\sin\frac{\pi}{5}n\right] = \frac{1}{2}\left[6\cos\frac{\pi}{5}n\right] = 3\cos\frac{\pi}{5}n$$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

$$= \frac{1}{2}\left[3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n - 3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n\right]$$

$$= \frac{1}{2}\left[j6\sin\frac{\pi}{5}n\right] = j3\sin\frac{\pi}{5}n$$

c) Given that, $x(n) = \{\underset{\uparrow}{2}, -2, 6, -2\}$

Given that, $x(n) = \{\underset{-}{2}, -2, 6, -2\}$, \ $x(0) = 2$; $x(1) = -2$; $x(2) = 6$; $x(3) = -2$

$x(-n) = \{-2, 6, -2, \underset{-}{2}\}$, \ $x(-3) = -2$; $x(-2) = 6$; $x(-1) = -2$; $x(0) = 2$

Even part, $x_e(n) = \frac{1}{2}\,[x(n) + x(-n)]$

At $n = -3$; $x(n) + x(-n) = 0 + (-2) = -2$

At $n = -2$; $x(n) + x(-n) = 0 + 6 = 6$

At $n = -1$; $x(n) + x(-n) = 0 + (-2) = -2$

At $n = 0$; $x(n) + x(-n) = 2 + 2 = 4$

At $n = 1$; $x(n) + x(-n) = -2 + 0 = -2$

At $n = 2$; $x(n) + x(-n) = 6 + 0 = 6$

At $n = 3$; $x(n) + x(-n) = -2 + 0 = -2$

$\therefore\ x(n) + x(-n) = \{-2, 6, -2, \underset{\uparrow}{4}, -2, 6, -2\}$

$\therefore\ x_e(n) = \frac{1}{2}\,[x(n) + x(-n)]$
$= \{-1, 3, -1, \underset{\uparrow}{2}, -1, 3, -1\}$

Odd part, $x_o(n) = \frac{1}{2}\,[x(n) - x(-n)]$

At $n = -3$; $x(n) - x(-n) = 0 - (-2) = 2$

At $n = -2$; $x(n) - x(-n) = 0 - 6 = -6$

At $n = -1$; $x(n) - x(-n) = 0 - (-2) = 2$

At $n = 0$; $x(n) - x(-n) = 2 - 2 = 0$

At $n = 1$; $x(n) - x(-n) = -2 - 0 = -2$

At $n = 2$; $x(n) - x(-n) = 6 - 0 = 6$

At $n = 3$; $x(n) - x(-n) = -2 - 0 = -2$

\ $x(n) - x(-n) = \{2, -6, 2, \underset{-}{0}, -2, 6, -2\}$

$\therefore\ x_o(n) = \frac{1}{2}\,[x(n) - x(-n)]$
$= \{1, -3, 1, \underset{-}{0}, -1, 3, -1\}$

2.4.4 Energy and Power Signals

The ***energy*** E of a discrete time signal x(n) is defined as,

$$\boxed{\text{Energy, } E = \sum_{n=-\infty}^{\infty} |x(n)|^2} \qquad .....(2.11)$$

The energy of a signal may be finite or infinite, and can be applied to complex valued and real valued signals.

If energy E of a discrete time signal is finite and nonzero, then the discrete time signal is called an ***energy signal.*** The exponential signals are examples of energy signals.

The average ***power*** of a discrete time signal x(n) is defined as,

$$\boxed{\text{Power, } P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2} \qquad .....(2.12)$$

If power P of a discrete time signal is finite and nonzero, then the discrete time signal is called a ***power signal***. The periodic signals are examples of power signals.

For energy signals, the energy will be finite and average power will be zero. For power signals the average power is finite and energy will be infinite.

\ For energy signal, $0 < E < \infty$ and $P = 0$

For power signal, $0 < P < \infty$ and $E = \infty$

Example 2.6

Determine whether the following signals are energy or power signals.

a) $x(n) = \left(\frac{1}{4}\right)^n u(n)$ **b)** $x(n) = \sin\left(\frac{\pi}{3}n\right)$ **c)** $x(n) = u(n)$

Solution

a) Given that, $x(n) = \left(\frac{1}{4}\right)^n u(n)$

Here, $x(n) = \left(\frac{1}{4}\right)^n u(n)$ for all n.

$$\therefore \; x(n) = \left(\frac{1}{4}\right)^n = 0.25^n \quad ; \quad n \geq 0$$

Infinite geometric series sum formula.
$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{\infty} \left|(0.25)^n\right|^2 = \sum_{n=0}^{\infty} (0.25^2)^n$$

$$= \sum_{n=0}^{\infty} (0.0625)^n = \frac{1}{1-0.0625} = 1.067 \text{ Joules}$$

Using infinite geometric series sum formula.

$$\text{Power, } P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left|(0.25)^n\right|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.25^2)^n = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.0625)^n$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \; \frac{(0.0625)^{N+1} - 1}{0.0625 - 1}$$

Using finite geometric series sum formula.

Finite geometric series sum formula.
$$\sum_{n=0}^{N} C^n = \frac{C^{N+1} - 1}{C-1}$$

$$= \frac{1}{\infty} \times \frac{0.0625^{\infty} - 1}{0.0625 - 1} = 0$$

Here E is finite and P is zero and so x(n) is an energy signal.

---

b) Given that, $x(n) = \sin\left(\frac{\pi}{3}n\right)$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} \sin^2\left(\frac{\pi}{3}n\right) = \sum_{n=-\infty}^{+\infty} \frac{1-\cos\frac{2\pi}{3}n}{2}$$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty} \left(1-\cos\frac{2\pi}{3}n\right)\right) = \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty} 1^n - \sum_{n=-\infty}^{+\infty} \cos\frac{2\pi}{3}n\right) = \frac{1}{2}(\infty - 0) = \infty$$

Note : Sum of infinite 1's is infinity. Sum of samples of one period of cosinusoidal signal is zero.

$$\text{Power, } P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \sin^2\frac{\pi n}{3}$$

$$\therefore\ P = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=-N}^{N}\frac{\left(1-\cos\frac{2\pi}{3}n\right)}{2}$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ \frac{1}{2}\left[\sum_{n=-N}^{N}1^n - \sum_{n=-N}^{N}\cos\frac{2\pi}{3}n\right]$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ \frac{1}{2}\left[\underbrace{1+1+\ldots\ldots+1}_{N\ terms}+1+\underbrace{1+\ldots\ldots+1+1}_{N\ terms}-0\right]$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ \frac{1}{2}\left[2N+1\right] = \underset{N\to\infty}{Lt}\ \frac{1}{2} = \frac{1}{2}\ \text{watts}$$

Since P is finite and E is infinite, x(n) is a power signal.

Note: The term $\cos\frac{2\pi}{3}n$ is periodic with periodicity of 3 samples. Samples of $\cos\frac{2\pi}{3}n$ for two periods are given below. It can be observed that sum of samples of a period is zero.

When $n = 0$; $\cos\frac{2\pi}{3}n = 1$, When $n = 1$; $\cos\frac{2\pi}{3}n = -0.5$, When $n = 2$; $\cos\frac{2\pi}{3}n = -0.5$

When $n = 3$; $\cos\frac{2\pi}{3}n = 1$, When $n = 4$; $\cos\frac{2\pi}{3}n = -0.5$, When $n = 5$; $\cos\frac{2\pi}{3}n = -0.5$

c) Given that, x(n) = u(n)

$$E = \sum_{n=-\infty}^{+\infty}|x(n)|^2 = \sum_{n=0}^{+\infty}(u(n))^2$$

$$= \sum_{n=0}^{+\infty}u(n) = 1+1+1\ \ldots\ldots\ldots\infty = \infty$$

$$P = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2 = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=0}^{N}u(n) = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\left(\underbrace{1+1+1+\ldots\ldots\ldots+1}_{N+1\ terms}\right)$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ (N+1) = \underset{N\to\infty}{Lt}\ \frac{N\left(1+\frac{1}{N}\right)}{N\left(2+\frac{1}{N}\right)} = \frac{1+\frac{1}{\infty}}{2+\frac{1}{\infty}} = \frac{1+0}{2+0} = \frac{1}{2}\ \text{watts}$$

Since P is finite and E is infinite, x(n) is a power signal.

2.4.5 Causal, Noncausal and Anticausal signals

A discrete time signal is said to be ***causal***, if it is defined for $n \geq 0$. Therefore if x(n) is causal, then $x(n) = 0$ for $n < 0$.

A discrete time signal is said to be ***noncausal***, if it is defined for either $n \leq 0$, or for both $n \leq 0$ and $n > 0$. Therefore if x(n) is noncausal, then $x(n) \neq 0$ for $n < 0$. A noncausal signal can be converted to causal signal by multiplying the noncausal signal by a unit step signal, u(n).

When a noncausal discrete time signal is defined only for $n \leq 0$, it is called an ***anticausal signal***.

Examples of Causal and Noncausal Signals

$$\left.\begin{array}{l} x(n) = \{\underline{1}, -1, 2, -2, 3, -3\} \\ x(n) = \{\underline{2}, 2, 3, 3, \ldots\ldots\ldots\} \end{array}\right\} \text{Causal signals}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} x(n) = \{1, -1, 2, -2, 3, \underline{-3}\} \\ x(n) = \{\ldots\ldots\ldots, 2, 2, 3, \underline{3}\} \end{array}\right\} \text{Anticausal signals} \\ x(n) = \{2, 3, 4, \underline{5}, 4, 3, 2\} \\ x(n) = \{\ldots\ldots, 2, 3, 4, \underline{5}, 4, 3, 2, \ldots\ldots\} \end{array}\right\} \text{Noncausal signals}$$

2.5 Mathematical Operations on Discrete Time Signals

Some of the mathematical operations that can be performed on discrete time signals are,

1. Scaling : Amplitude scaling and time scaling
2. Folding
3. Shifting : Right shift (or advance) and left shift (or delay)
4. Addition
5. Multiplication

2.5.1. Scaling of Discrete Time Signals

Amplitude Scaling (or Scalar Multiplication)

Amplitude scaling of a discrete time signal by a constant A is accomplished by multiplying the value of every signal sample by the constant A.

Example :

Let y(n) be amplitude scaled signal of x(n), then y(n) = A x(n)

Let, $x(n) =$	10	; $n = 0$	and $A = 0.2$,	When $n = 0$	; $y(0) = A\,x(0) = 0.2 \times 10$	$= 2.0$
$=$	16	; $n = 1$		When $n = 1$	; $y(1) = A\,x(1) = 0.2 \times 16$	$= 3.2$
$=$	20	; $n = 2$		When $n = 2$	; $y(2) = A\,x(2) = 0.2 \times 20$	$= 4.0$

Time Scaling (or Downsampling and Upsampling)

There are two ways of time scaling a discrete time signal. They are downsampling and upsampling.

In a signal x(n), if n is replaced by Dn, where D is an integer, then it is called ***downsampling***.

In a signal x(n), if n is replaced by $\frac{n}{I}$, where I is an integer, then it is called ***upsampling***.

Example :

If $x(n) = b^n$; $n \geq 0$; $0 < b < 1$, then

$x_1(n) = x(2n)$ will be a down sampled version of x(n) and

$x_2(n) = x\left(\frac{n}{2}\right)$ will be an up sampled version on x(n).

When $n = 0$; $x_1(0) = x(0) = b^0$

When $n = 1$; $x_1(1) = x(2) = b^2$

When $n = 2$; $x_1(2) = x(4) = b^4$ and so on.

When $n = 0$; $x_2(0) = x\left(\frac{0}{2}\right) = x(0) = b^0$

When $n = 1$; $x_2(1) = x\left(\frac{1}{2}\right) = 0$

When $n = 2$; $x_2(2) = x\left(\frac{2}{2}\right) = x(1) = b^1$

When $n = 3$; $x_2(3) = x\left(\frac{3}{2}\right) = 0$ and so on.

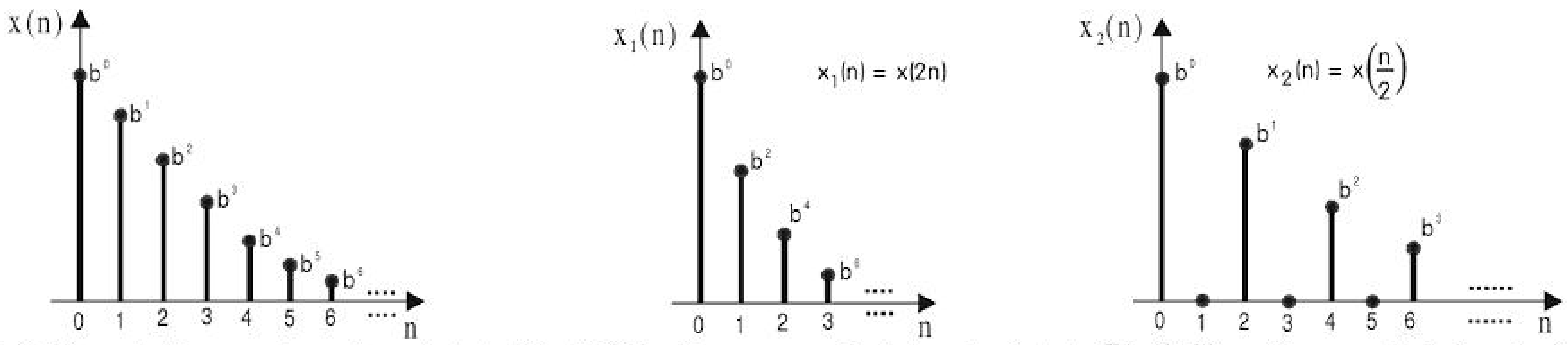


Fig 2.12a : A discrete time signal x(n). *Fig 2.12b : Down sampled signal of x(n).* *Fig 2.12c : Up sampled signal x(n).*

Fig 2.12 : A discrete time signal and its time scaled version.

2.5.2. Folding (or Reflection or Transpose) of Discrete Time Signals

The ***folding*** of a discrete time signal x(n) is performed by changing the sign of the time base n in x(n). The folding operation produces a signal x(–n) which is a mirror image of the signal x(n) with respect to time origin n = 0.

Example :

Let $x(n) = 0.8n$; $-2 \leq n \leq 2$. Now the folded signal, $x_1(n) = x(-n) = -0.8n$; $-2 \leq n \leq 2$

Fig 2.13a : A discrete time signal x(n). *Fig 2.13b : Folded signal of x(n).*

Fig 2.13 : A discrete time signal and its folded version.

2.5.3. Time Shifting of Discrete Time Signals

A signal x(n) may be shifted in time by replacing the independent variable n by n – m, where m is an integer. [i.e, x(n–m) is shifted version of x(n)]. If m is a positive integer, the time shift results in a delay by m units of time. If m is a negative integer, the time shift results in an advance of the signal by |m| units in time. The ***delay*** results in shifting each sample of x(n) to the right. The ***advance*** results in shifting each sample of x(n) to the left.

Example :

Let, x(n) = 3 ; n = 2

= 2 ; n = 3

= 1 ; n = 4

= 0 ; for other n

Let, $x_1(n) = x(n-2)$, where $x_1(n)$ is delayed signal of x(n)

When n = 4 ; $x_1(4) = x(4-2) = x(2) = 3$

When n = 5 ; $x_1(5) = x(5-2) = x(3) = 2$

When n = 6 ; $x_1(6) = x(6-2) = x(4) = 1$

The sample x(2) is available at n = 2 in the original sequence x(n), but the same sample is available at n = 4 in $x_1(n)$. Similarly every sample of x(n) is delayed by two sampling times.

Let, $x_2(n) = x(n+2)$, where $x_2(n)$ is an advanced signal of x(n)

When n = 0 ; $x_2(0) = x(0+2) = x(2) = 3$

When n = 1 ; $x_2(1) = x(1+2) = x(3) = 2$

When n = 2 ; $x_2(2) = x(2+2) = x(4) = 1$

The sample x(2) is available at n = 2 in the original sequence x(n), but the same sample is available at n = 0 in $x_2(n)$. Similarly every sample of x(n) is advanced by two sampling times. Hence the signal $x_2(n)$ is an advanced version of x(n).

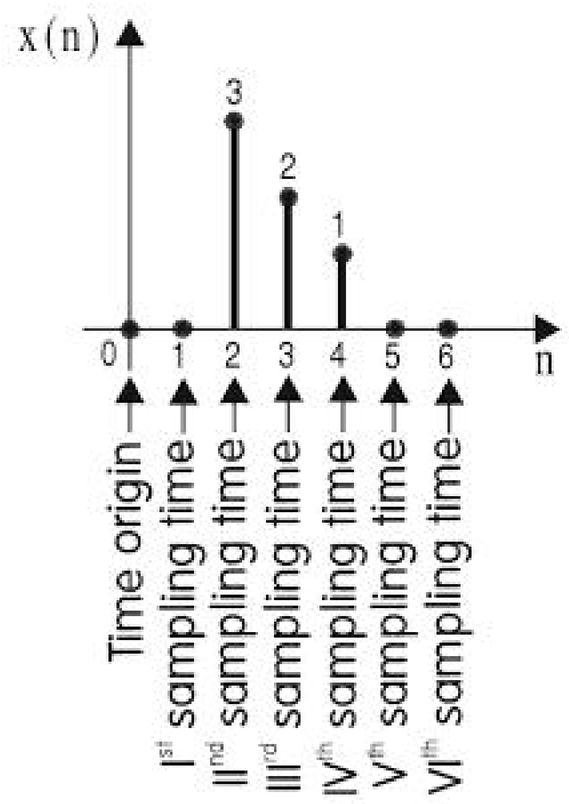


Fig 2.14a : A discrete time signal x(n) .

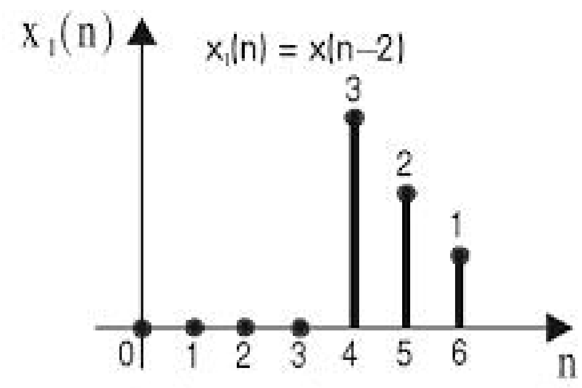


Fig 2.14b : Delayed signal of x(n).

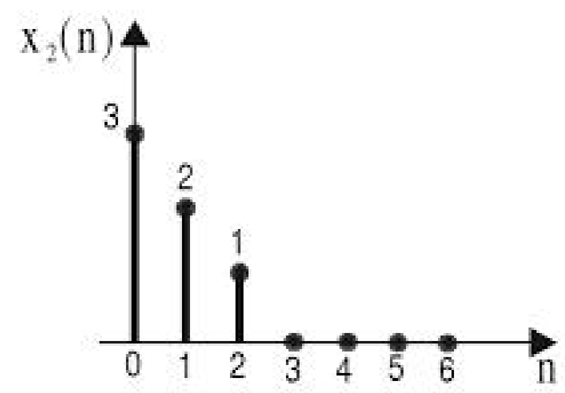


Fig 2.14c : Advanced signal of x(n).

Fig 2.14 : A discrete time signal and its shifted version.

Delayed Unit Impulse Signal

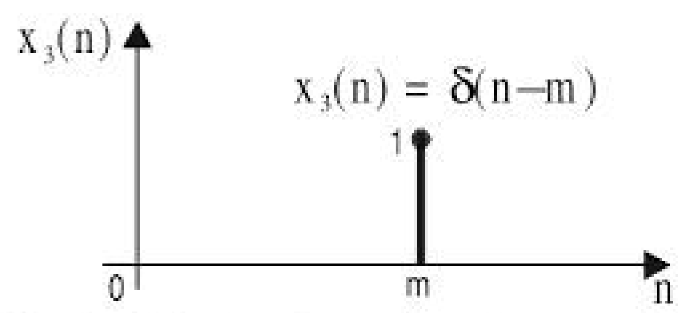


Fig 2.15 : Delayed unit impulse.

The unit impulse signal is defined as,

$\delta(n) = 1$; for $n = 0$

$= 0$; for $n \neq 0$

The unit impulse signal delayed by m units of time is denoted as $\delta(n-m)$.

Now, $\delta(n-m) = 1$; $n = m$

$= 0$; $n \neq m$

Delayed Unit Step Signal

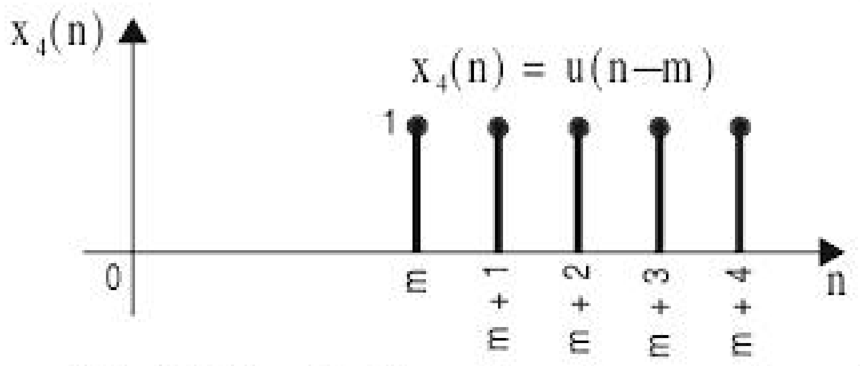


Fig 2.16 : Delayed unit step signal.

The unit step signal is defined as,

$$u(n) = 1 \; ; \text{for } n \geq 0$$
$$= 0 \; ; \text{for } n < 0$$

The unit step signal delayed by m units of time is denoted as $u(n - m)$.

Now, $u(n - m) = 1 \; ; n \geq m$
$= 0 \; ; n < m$

2.5.4. Addition of Discrete Time Signals

The ***addition*** of two discrete time signals is performed on a sample-by-sample basis.

The sum of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the sum of the samples of these two signals at that instant.

i.e., $y(n) = x_1(n) + x_2(n) \; ; \; -\infty < n < \infty$.

Example :

Let, $x_1(n) = \{2, 2, -1\}$ and $x_2(n) = \{-1, 1, 2\}$

When $n = 0$; $y(0) = x_1(0) + x_2(0) = 2 + (-1) = 1$

When $n = 1$; $y(1) = x_1(1) + x_2(1) = 2 + 1 = 3$

When $n = 2$; $y(2) = x_1(2) + x_2(2) = -1 + 2 = 1$

$\therefore \; y(n) = x_1(n) + x_2(n) = \{1, 3, 1\}$

2.5.5. Multiplication of Discrete Time Signals

The ***multiplication*** of two discrete time signals is performed on a sample-by-sample basis.The product of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the product of the samples of these two signals at that instant. The product is also called ***modulation***.

Example :

Let, $x_1(n) = \{ 2, 2, -1 \}$ and $x_2(n) = \{ -1, 1, 2 \}$

When $n = 0$; $y(0) = x_1(0) \times x_2(0) = 2 \times (-1) = -2$

When $n = 1$; $y(1) = x_1(1) \times x_2(1) = 2 \times 1 = 2$

When $n = 2$; $y(2) = x_1(2) \times x_2(2) = -1 \times 2 = -2$

$\therefore \; y(n) = x_1(n) \times x_2(n) = \{-2, 2, -2\}$

2.6 Discrete Time System

A ***discrete time system*** is a device or algorithm that operates on a discrete time signal, called the input or excitation, according to some well-defined rule, to produce another discrete time signal called the output or the response of the system. We can say that the input signal $x(n)$ is transformed by the system into a signal $y(n)$, and the transformation can be expressed mathematically as shown in equation (2.13).The diagrammatic representation of discrete time system is shown in fig 2.17.

Response, $y(n) = \mathcal{H}\{x(n)\}$ .....(2.13)

where, $\mathcal{H}$ denotes the transformation (also called an operator).

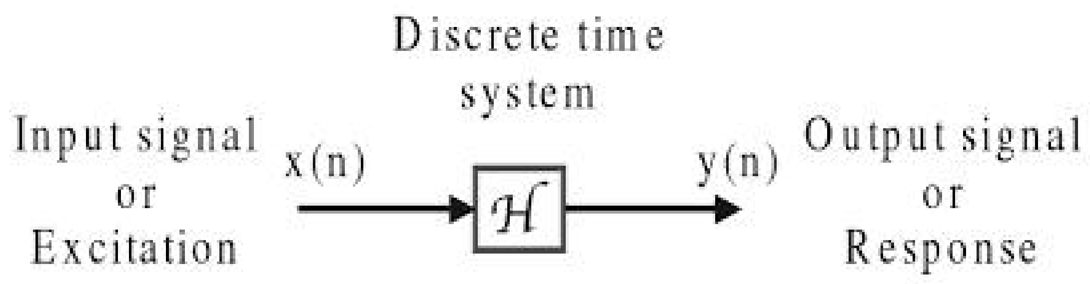


Fig 2.17 : *Representation of discrete time system.*

LTI System

A discrete time system is linear if it obeys the principle of superposition and it is time invariant if its input-output relationship does not change with time. When a discrete time system satisfies the properties of linearity and time invariance then it is called an ***LTI system*** (Linear Time Invariant system).

Impulse Response

When the input to a discrete time system is a unit impulse d(n) then the output is called an ***impulse response*** of the system and is denoted by h(n).

\ Impulse Response, $h(n) = \mathcal{H}\{d(n)\}$ .....(2.14)

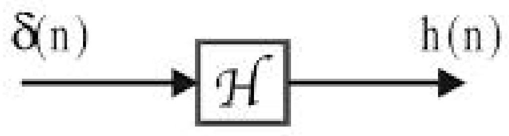


Fig 2.18 : *Discrete time system with impulse input.*

2.6.1 Mathematical Equation Governing Discrete Time System

The mathematical equation governing the discrete time system can be developed as shown below.

The response of a discrete time system at any time instant depends on the present input, past inputs and past outputs.

Let us consider the response at n = 0. Let us assume a relaxed system and so at n = 0, there is no past input or output. Therefore the response at n = 0, is a function of present input alone.

i.e., $y(0) = F[x(0)]$

Let us consider the response at n =1. Now the present input is x(1), the past input is x(0) and past output is y(0). Therefore the response at n = 1, is a function of x(1), x(0), y(0).

i.e., $y(1) = F[y(0), x(1), x(0)]$

Let us consider the response at n = 2. Now the present input is x(2), the past inputs are x(1) and x(0), and past outputs are y(1) and y(0). Therefore the response at n = 2, is a function of x(2), x(1), x(0), y(1), y(0).

i.e., $y(2) = F[y(1), y(0), x(2), x(1), x(0)]$

Similarly, at n = 3, $y(3) = F[y(2),y(1), y(0), x(3), x(2), x(1), x(0)]$

at n = 4, $y(4) = F[y(3),y(2), y(1), y(0), x(4), x(3), x(2), x(1), x(0)]$ and so on.

In general, at any time instant n,

$$y(n) = F[y(n-1), y(n-2), y(n-3), ...y(1), y(0), x(n), x(n-1),$$
$$x(n-2), x(n-3) ..... x(1), x(0)] \quad .....(2.15)$$

For an LTI system, the response y(n) can be expressed as a weighted summation of dependent terms. Therefore the equation (2.15) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \dots\dots\dots$$
$$+\, b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3 x(n-3) + \dots\dots \qquad \dots..(2.16)$$

where, $a_1, a_2, a_3, \dots$ and $b_0, b_1, b_2, b_3, \dots$ are constants.

> *Note : Negative constants are inserted for output signals, because output signals are feedback from output to input. Positive constants are inserted for input signals, because input signals are feed forward from input to output.*

Practically, the response y(n) at any time instant n, may depend on N number of past outputs, present input and M number of past inputs where M £ N. Hence the equation (2.16) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \dots\dots - a_N\, y(n-N)$$
$$+\, b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + \dots\dots + b_M\, x(n-M)$$

$$\boxed{\therefore \quad y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)} \qquad \dots..(2.17)$$

The equation (2.17) is a constant coefficient ***difference equation***, governing the input-output relation of an LTI discrete time system.

In equation (2.17) the value of "N" gives the ***order*** of the system.

If N = 1, the discrete time system is called 1st order system

If N = 2, the discrete time system is called 2nd order system

If N = 3, the discrete time system is called 3rd order system , and so on.

The general difference equation governing 1st order discrete time LTI system is,

$$y(n) = -a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1)$$

The general difference equation governing 2nd order discrete time LTI system is,

$$y(n) = -a_2\, y(n-2) - a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2)$$

2.6.2 Block Diagram and Signal Flow Graph Representation of Discrete Time System

The discrete time system can be represented diagrammatically by ***block diagram*** or ***signal flow graph***. These diagrammatic representations are useful for physical implementation of discrete time system in hardware or software.

The basic elements employed in block diagram or signal flow graph are adder, constant multiplier, unit delay element and unit advance element.

Adder : An adder is used to represent addition of two discrete time signals.

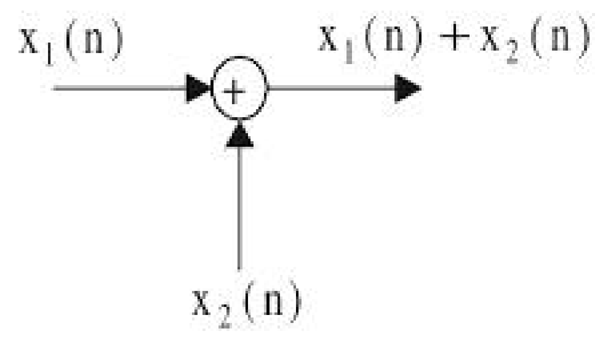
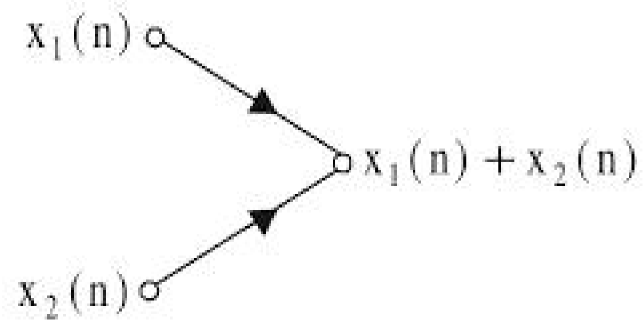
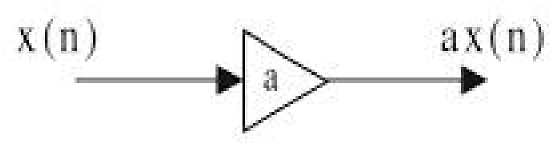
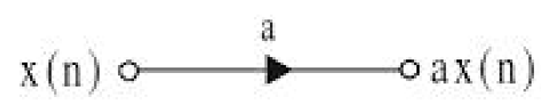
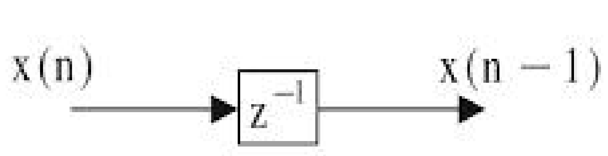
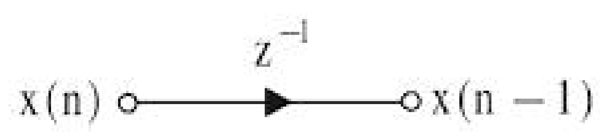
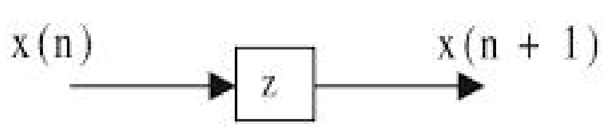
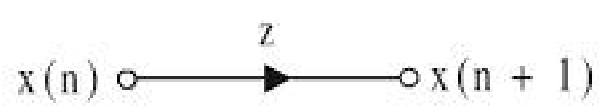
Constant Multiplier : A constant multiplier is used to represent multiplication of a scaling factor (constant) to a discrete time signal.

Unit Delay Element : A unit delay element is used to represent the delay of samples of a discrete time signal by one sampling time.

Unit Advance Element : A unit advance element is used to represent the advance of samples of a discrete time signal by one sampling time.

The symbolic representation of the basic elements of block diagram and signal flow graph are listed in table 2.1.

Table 2.1 : Basic Elements of Block Diagram and Signal Flow Graph

Element	Block diagram representation	Signal flow graph representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

Example 2.7

Construct the block diagram and signal flow graph of the discrete time systems whose input-output relations are described by the following difference equations.

a) $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$

b) $y(n) = 0.4\,y(n-1) + x(n) - 3\,x(n-2)$

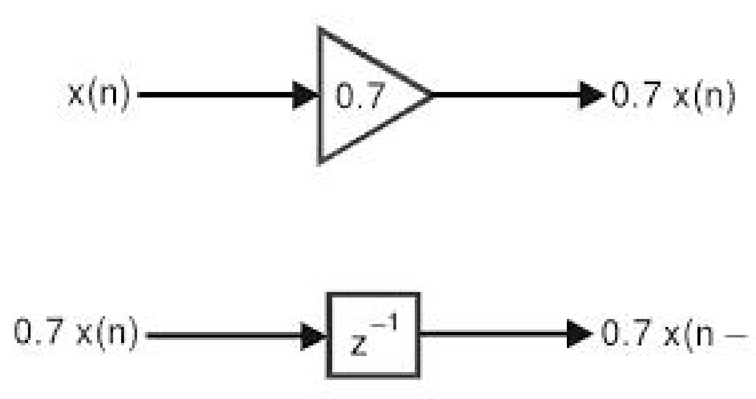
c) $y(n) = 0.2\,y(n-1) + 0.7\,x(n) + 0.9\,x(n-1)$

Solution

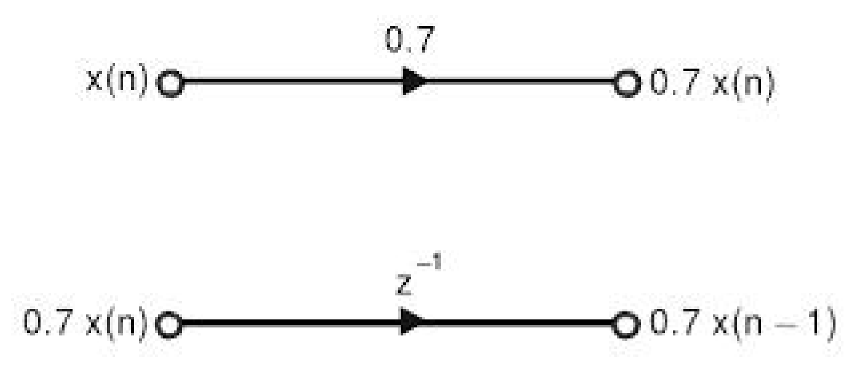
a) Given that, $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$

The individual terms of the given equation are $0.7\,x(n)$ and $0.7\,x(n-1)$. They are represented by basic elements as shown below.

Block diagram representation



Signal flow graph representation



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

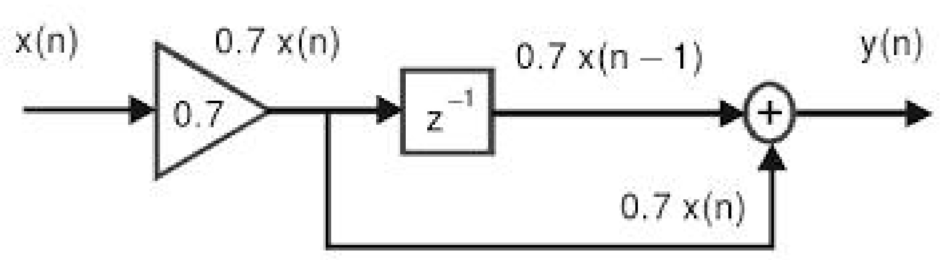


***Fig 1 :** Block diagram of the system*
$y(n) = 0.7\,x(n) + 0.7\,x(n-1).$

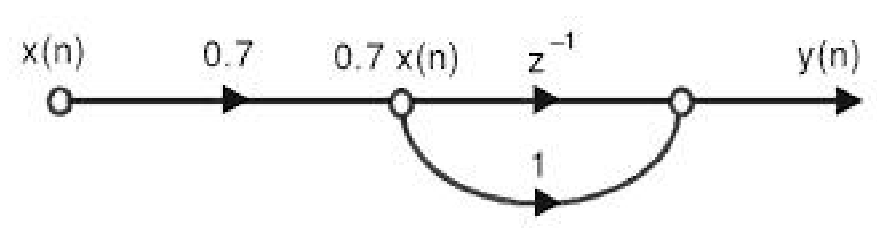
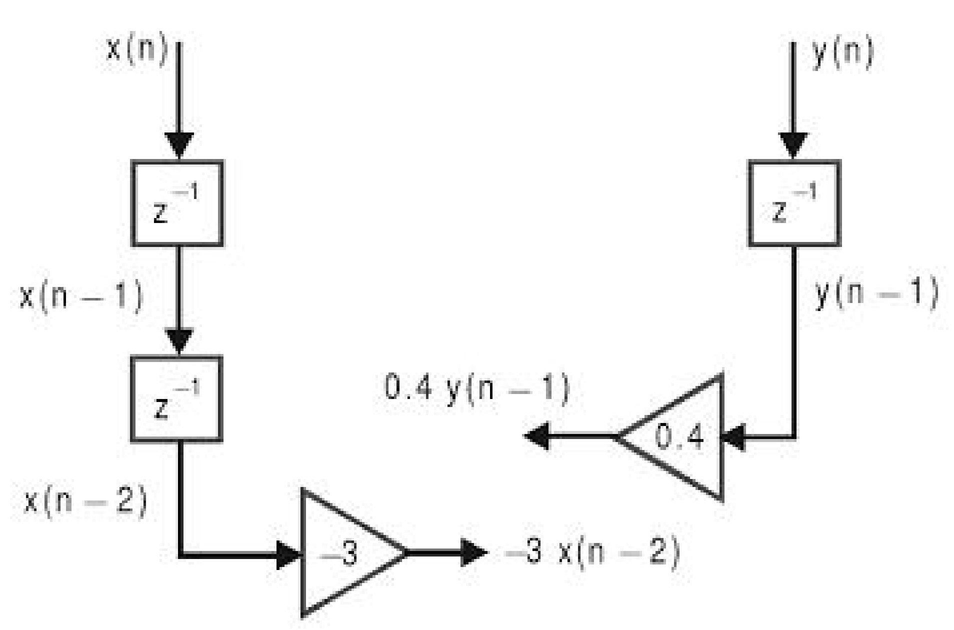


***Fig 2 :** Signal flow graph of the system*
$y(n) = 0.7\,x(n) + 0.7\,x(n-1).$

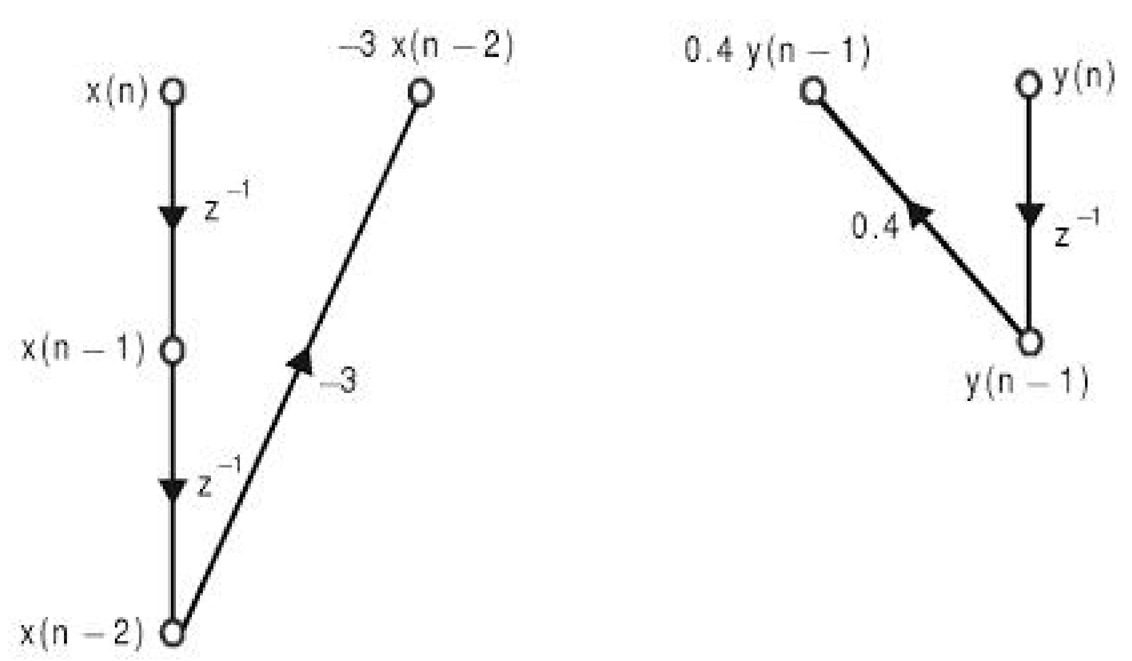
b) Given that, y(n) = 0.4 y(n – 1) + x(n) – 3 x(n – 2)

The individual terms of the given equation are 0.4 y(n – 1) and – 3 x(n – 2). They are represented by basic elements as shown below.

Block diagram representation



Signal flow graph representation



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

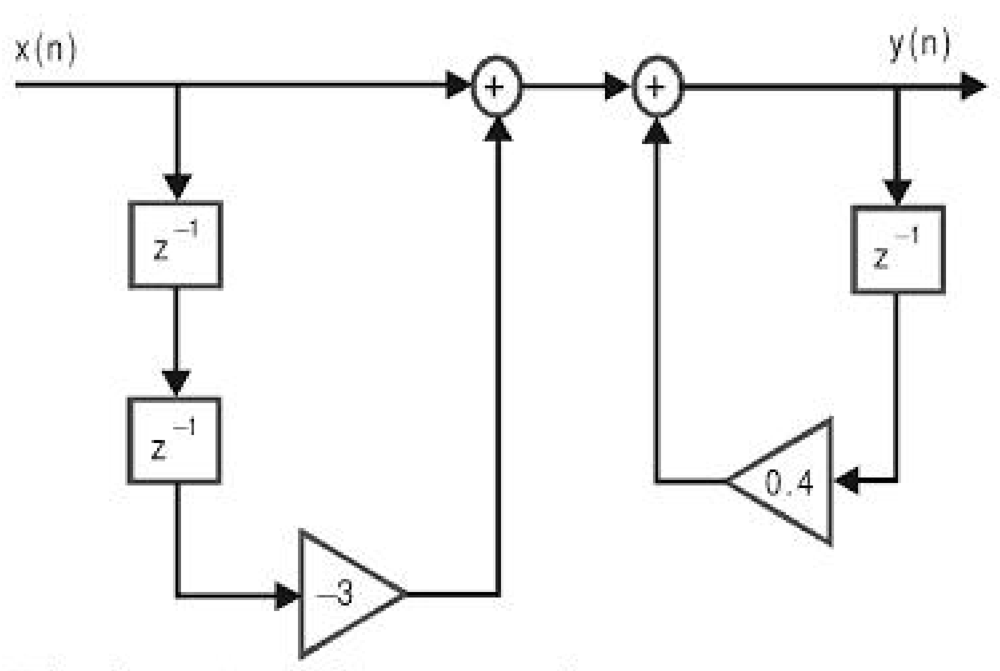


***Fig 3 :** Block diagram of the system described by the equation*
$y(n) = 0.4\,y(n-1) + x(n) - 3\,x(n-2).$

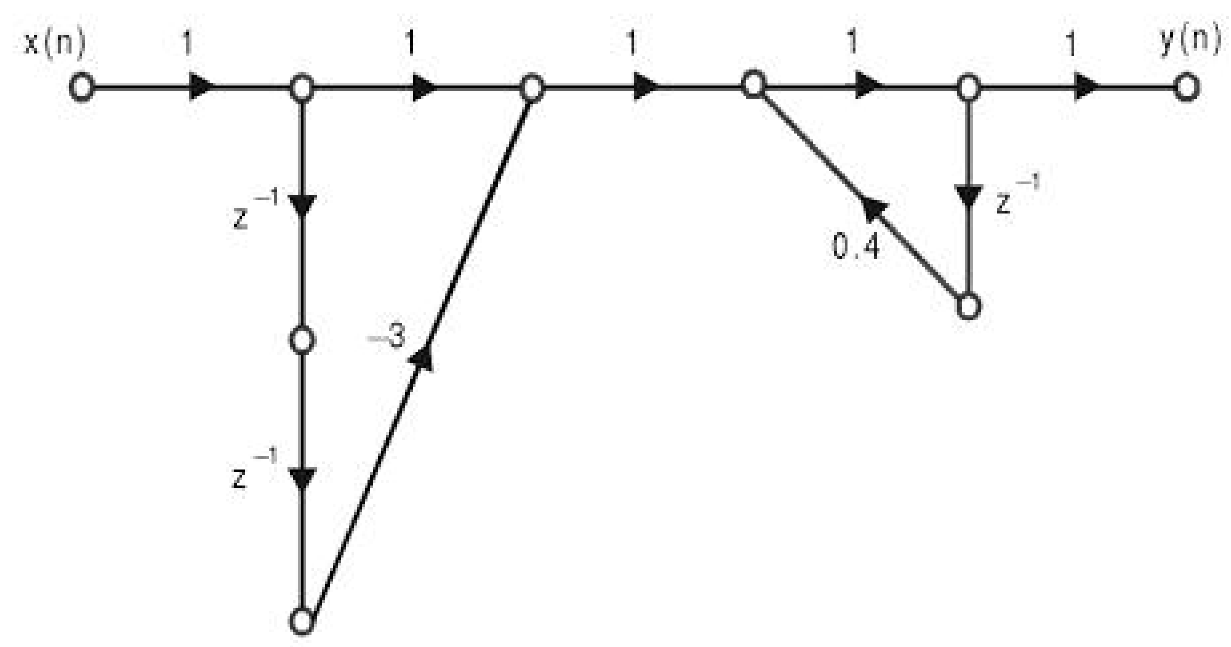
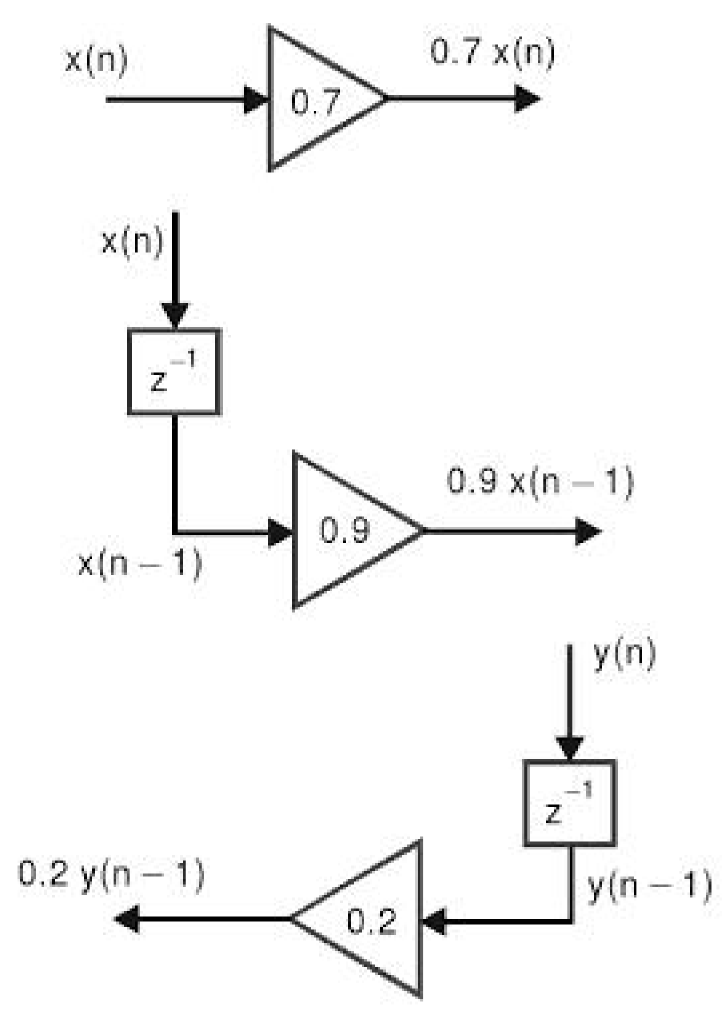


***Fig 4 :** Signal flow graph of the system described by the equation*
$y(n) = 0.4\,y(n-1) + x(n) - 3\,x(n-2).$

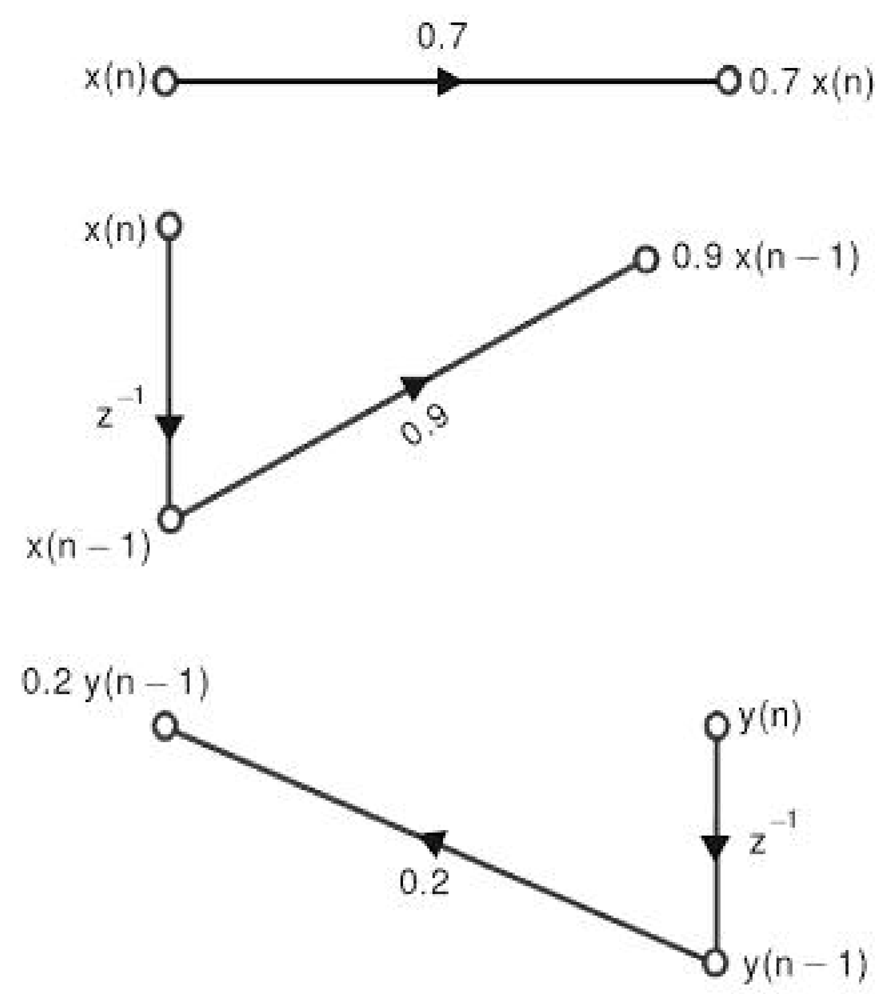
c) Given that, y(n) = 0.2 y(n – 1) + 0.7 x(n) + 0.9 x(n – 1)

The individual terms of the given equation are 0.2 y(n – 1), 0.7 x(n) and 0.9 x(n – 1). They are represented by basic elements as shown below.

Block diagram representation



Signal flow graph representation



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

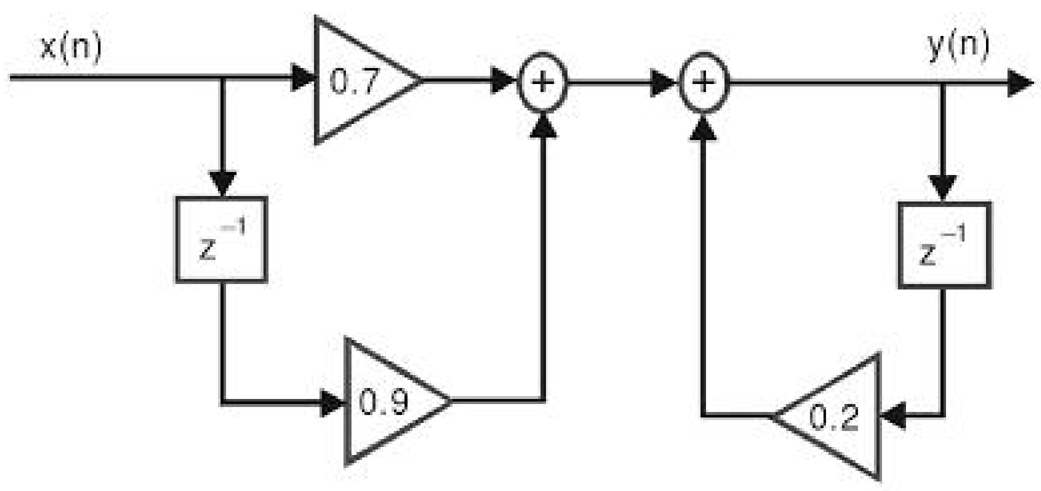


Fig 5 : Block diagram of the system described by the equation y(n) = 0.2 y(n – 1) + 0.7 x(n)+ 0.9 x(n – 1).

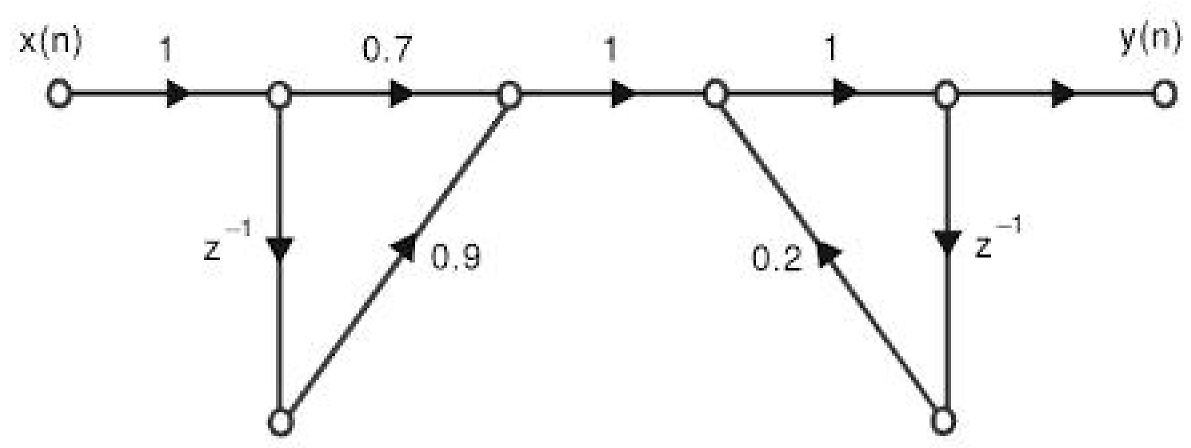


Fig 6 : Signal flow graph of the system described by the equation y(n) = 0.2 y(n – 1) + 0.7 x(n) + 0.9 x(n – 1).

2.7 Response of LTI Discrete Time System in Time Domain

The general equation governing an LTI discrete time system is,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\therefore \quad y(n) + \sum_{m=1}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\text{(or)} \quad \sum_{m=0}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m) \quad \text{with } a_o = 1 \qquad .....(2.18)$$

The solution of the difference equation (2.18) is the ***response*** y(n) of LTI system, which consists of two parts. In mathematics, the two parts of the solution y(n) are homogeneous solution $y_h(n)$ and particular solution $y_p(n)$.

$$\boxed{\therefore\ \text{Response},\ y(n) = y_h(n) + y_p(n)} \qquad .....(2.19)$$

The ***homogeneous solution*** is the response of the system when there is no input.The ***particular solution*** $y_p(n)$ is the solution of difference equation for specific input signal x(n) for $n \geq 0$.

In signals and systems, the two parts of the solution y(n) are called zero-input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

$$\boxed{\therefore\ \text{Response},\ y(n) = y_{zi}(n) + y_{zs}(n)} \qquad .....(2.20)$$

The ***zero-input response*** is mainly due to initial conditions (or initial stored energy) in the system. Hence zero-input response is also called ***free response*** or ***natural response***. The ***zero-input response*** is given by homogeneous solution with constants evaluated using initial conditions.

The ***zero-state response*** is the response of the system due to input signal and with zero initial condition. Hence the zero-state response is called forced response.The ***zero-state response*** or ***forced response*** is given by the sum of homogeneous solution and particular solution with zero initial conditions.

2.7.1 Zero-Input Response or Homogeneous Solution

The ***zero-input response*** is obtained from homogeneous solution $y_h(n)$ with constants evaluated using initial condition.

$$\therefore\ \text{Zero - input response},\ y_{zi}(n) = y_h(n)\Big|_{\text{with constants evaluated using initial conditions}}$$

The ***homogeneous solution*** is obtained when x(n) = 0. Therefore the homogeneous solution is the solution of the equation,

$$\sum_{m=0}^{N} a_m\, y(n-m) = 0 \qquad .....(2.21)$$

Let us assume that the solution of equation (2.21) is in the form of an exponential.

i.e., $y(n) = \lambda^n$

On substituting $y(n) = \lambda^n$ in equation (2.21) we get,

$$\sum_{m=0}^{N} a_m\, \lambda^{n-m} = 0$$

On expanding the above equation (by taking $a_0 = 1$), we get,

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{N-1}\lambda^{n-(N-1)} + a_N\lambda^{n-N} = 0$$

$$\lambda^{n-N}(\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \ldots + a_{N-1}\lambda + a_N) = 0$$

Now, the ***characteristic polynomial*** of the system is given by,

$$\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \ldots + a_{N-1}\lambda + a_N = 0$$

The characteristic polynomial has N roots, which are denoted as $\lambda_1, \lambda_2, \ldots \lambda_N$.

The roots of the characteristic polynomial may be distinct real roots, repeated real roots or complex. The assumed solutions for various types of roots are given below.

Distinct Real Roots

Let the roots $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_N$ be distinct real roots. Now the homogeneous solution will be in the form,

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n + \ldots\ldots + C_N\lambda_N^n$$

where, $C_1, C_2, C_3, \ldots\ldots C_N$ are constants that can be evaluated using initial conditions.

Repeated Real Roots

Let one of the real roots 1_1 repeats p times and the remaining (N – p) roots are distinct real roots. Now, the homogeneous solution is in the form,

$$y_h(n) = (C_1 + C_2 n + C_3 n^2 + ... + C_p n^{p-1})\lambda_1^n + C_{p+1}\lambda_{p+1}^n + ... + C_N \lambda_N^n$$

where, $C_1, C_2, C_3, ...C_N$ are constants that can be evaluated using initial conditions.

Complex Roots

Let the characteristic polynomial has a pair of complex roots 1 and 1^* and the remaining (N – 2) roots be distinct real roots. Now, the homogeneous solution will be in the form,

$$y_h(n) = r^n [C_1 \cos nq + C_2 \sin nq] + C_3 1_3^n + C_4 1_4^n + ... + C_N 1_N^n$$

where, $\lambda = a + jb, \quad \lambda^* = a - jb, \quad r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\dfrac{b}{a}$

$C_1, C_2, C_3 ... C_N$ are constants that can be evaluated using initial conditions.

2.7.2 Particular Solution

The ***particular solution***, $y_p(n)$ is the solution of the difference equation for specific input signal x(n) for n ³ 0. Since the input signal may have different form, the particular solution depends on the form or type of the input signal x(n).

If x(n) is constant, then $y_p(n)$ is also a constant.

> **Example :**
>
> Let, x(n) = u(n) ; now, $y_p(n)$ = K u(n)

If x(n) is exponential, then $y_p(n)$ is also an exponential.

> **Example :**
>
> Let, x(n) = a^n u(n) ; now, $y_p(n) = K\, a^n\, u(n)$

If x(n) is sinusoid, then $y_p(n)$ is also a sinusoid.

> **Example :**
>
> Let, x(n) = A cos w_0n ; now, $y_p(n) = K_1 \cos w_0 n + K_2 \sin w_0 n$

The general form of particular solution for various types of inputs are listed in table 2.2.

Table 2.2 : Particular Solution

Input signal, x(n)	Particular solution, $y_p(n)$
A	K
$A B^n$	$K B^n$
$A n^B$	$K_0 n^B + K_1 n^{(B-1)} + ... + K_B$
$A^n n^B$	$A^n (K_0 n^B + K_1 n^{(B-1)} + ... + K_B)$
$A \cos w_0 n$	$K_1 \cos w_0 n + K_2 \sin w_0 n$
$A \sin w_0 n$	$K_1 \cos w_0 n + K_2 \sin w_0 n$

2.7.3 Zero-State Response

The ***zero-state response*** or ***forced response*** is obtained from the sum of homogeneous solution and particular solution and evaluating the constants with zero initial conditions.

$$\therefore \text{ Zero - state response, } y_{zs}(n) = y_h(n) + y_p(n)\Big|_{\text{with constants } C_1,\, C_2, \ldots.\, C_N \text{ evaluated with zero initial conditions}}$$

2.7.4 Total Response

The total response of discrete time system can be obtained by the following two methods.

Method-1

The ***total response*** is given by sum of homogeneous solution and particular solution.

\ Total response, $y(n) = y_h(n) + y_p(n)$

Procedure to Determine Total Response by Method-1

1. Determine the homogeneous solution $y_h(n)$ with constants C_1, C_2, ...C_N.
2. Determine the particular solution $y_p(n)$ and evaluate the constants K for any value of n ³ 1 so that no term of y(n) vanishes.
3. Now the total response is given by the sum of $y_h(n)$ and $y_p(n)$.

   \ Total response, $y(n) = y_h(n) + y_p(n)$

4. The total response will have N number of constants C_1, C_2, ...C_N. Evaluate the given difference equation for n = 0, 1, 2, ...N – 1 and form one set of N number of equations. Then evaluate the total response for n = 0, 1, 2, ... N – 1 and form another set of N number of equations. Now solve the constants C_1, C_2, ...C_N using the two sets of N number of equations.

Method-2

The ***total response*** is given by sum of zero-input response and zero-state response.

\ Total response, $y(n) = y_{zi}(n) + y_{zs}(n)$

Procedure to Determine Total Response by Method-2

1. Determine the homogeneous solution $y_h(n)$ with constants C_1, C_2, ...C_N.
2. Determine the zero-input response, which is obtained from the homogeneous solution $y_h(n)$ and evaluating the constants C_1, C_2, ...C_N using the initial conditions.
3. Determine the particular solution $y_p(n)$ and evaluate the constants K for any value of n ³ 1 so that no term of y(n) vanishes.
4. Determine the zero-state response, $y_{zs}(n)$ which is given by sum of homogeneous solution and particular solution and evaluating the constants C_1, C_2, ...C_N with zero initial conditions.
5. Now, the total response is given by sum of zero input response and zero state response.

   \ Total response, $y(n) = y_{zi}(n) + y_{zs}(n)$

Example 2.8

Determine the response of first order discrete time system governed by the difference equation,

$$y(n) = -0.8\, y(n-1) + x(n)$$

When the input is unit step, and with initial condition **a)** $y(-1) = 0$ **b)** $y(-1) = 2/9$.

Solution

Given that, $y(n) = -0.8\, y(n-1) + x(n)$

$$\therefore \quad y(n) + 0.8\, y(n-1) = x(n) \qquad \text{.....(1)}$$

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when $x(n) = 0$.

$$\therefore \quad y(n) + 0.8\, y(n-1) = 0 \qquad \text{.....(2)}$$

Put, $y(n) = \lambda^n$ in equation (2).

$$\therefore \quad \lambda^n + 0.8\, \lambda^{(n-1)} = 0$$

$$\lambda^{(n-1)}(\lambda + 0.8) = 0 \quad \Rightarrow \quad \lambda = -0.8$$

The homogeneous solution $y_h(n)$ is given by,

$$y_h(n) = C\, \lambda^n = C\,(-0.8)^n \;; \quad \text{for } n \geq 0 \qquad \text{.....(3)}$$

Particular Solution

Given that the input is unit step and so the particular solution will be in the form,

$$y(n) = K\, u(n) \qquad \text{.....(4)}$$

On substituting for $y(n)$ from equation (4) in equation (1) we get,

$$y(n) + 0.8\, y(n-1) = x(n) \quad \Rightarrow \quad K\, u(n) + 0.8\, K\, u(n-1) = u(n) \qquad \text{.....(5)}$$

In order to determine the value of K, let us evaluate equation (5) for $n = 1$, ($\because$ we have to evaluate equation (5) for any $n \geq 1$, such that none of the term vanishes).

From equation (5) when $n = 1$, we get,

$$K + 0.8\, K = 1 \quad \Rightarrow \quad 1.8\, K = 1 \quad \Rightarrow \quad K = \frac{1}{1.8} = \frac{10}{18} = \frac{5}{9}$$

The particular solution $y_p(n)$ is given by,

$$y_p(n) = K\, u(n) = \frac{5}{9}\, u(n) \;; \quad \text{for all } n$$

$$= \frac{5}{9} \;; \quad \text{for} \quad n \geq 0$$

Total Response

The total response $y(n)$ of the system is given by sum of homogeneous and particular solution.

$\therefore$ Response, $y(n) = y_h(n) + y_p(n)$

$$\therefore \; y(n) = C(-0.8)^n + \frac{5}{9} \;; \quad \text{for } n \geq 0 \qquad \text{.....(6)}$$

When $n = 0$, from equation (1), we get, $y(0) + 0.8\, y(-1) = 1$

$$\therefore \quad y(0) = 1 - 0.8\, y(-1) \qquad \text{.....(7)}$$

When $n = 0$, from equation (6), we get, $y(0) = C + \frac{5}{9}$ .....(8)

On equating (7) and (8) we get, $C + \frac{5}{9} = 1 - 0.8\, y(-1)$

$$\therefore \; C = 1 - 0.8\, y(-1) - \frac{5}{9}$$

$$= \frac{4}{9} - 0.8\, y(-1) \qquad \text{.....(9)}$$

On substituting for C from equation (9) in equation (6) we get,

$$y(n) = \left(\frac{4}{9} - 0.8\ y(-1)\right)(-0.8)^n + \frac{5}{9}$$

a) When y(–1) = 0

$$\therefore\ y(n) = \frac{4}{9}(-0.8)^n + \frac{5}{9}\ ;\quad \text{for } n \geq 0$$

$$= \left[\frac{4}{9}(-0.8)^n + \frac{5}{9}\right] u(n)$$

b) When y(–1) = 2/9

$$\therefore\quad y(n) = \left(\frac{4}{9} - 0.8 \times \frac{2}{9}\right)(-0.8)^n + \frac{5}{9} = \frac{2.4}{9}(-0.8)^n + \frac{5}{9} = \frac{24}{90}(-0.8)^n + \frac{5}{9}$$

$$\therefore\quad y(n) = \frac{5}{9} + \frac{12}{45}(-0.8)^n;\quad \text{for } n \geq 0$$

$$= \left[\frac{5}{9} + \frac{12}{45}(-0.8)^n\right] u(n)$$

Example 2.9

Determine the response y(n), $n \geq 0$ of the system described by the second order difference equation,

$$y(n) - 0.2\ y(n-1) - 0.03\ y(n-2) = x(n) + 0.4\ x(n-1),$$

when the input signal is, $x(n) = 0.2^n\ u(n)$ and with initial conditions $y(-2) = 0$, $y(-1) = 0.5$.

Solution

Given that, y(n) – 0.2 y(n – 1) – 0.03 y(n – 2) = x(n) + 0.4 x(n – 1) .....(1)

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when x(n) = 0.

$$\therefore\ y(n) - 0.2\ y(n-1) - 0.03\ y(n-2) = 0 \quad .....(2)$$

Put $y(n) = \lambda^n$ in equation (2).

$$\therefore\ \lambda^n - 0.2\ \lambda^{n-1} - 0.03\ \lambda^{n-2} = 0$$

$$\lambda^{n-2}(\lambda^2 - 0.2\lambda - 0.03) = 0$$

The characteristic equation is,

$$\lambda^2 - 0.2\lambda - 0.03 = 0 \quad \Rightarrow \quad (\lambda - 0.3)(\lambda + 0.1) = 0$$

$\therefore$ The roots are, $\lambda = 0.3, -0.1$

> The roots of quadratic,
>
> $\lambda^2 - 0.2\lambda - 0.03 = 0$ are,
>
> $$\lambda = \frac{0.2 \pm \sqrt{0.2^2 + 4 \times 0.03}}{2}$$
>
> $$= \frac{0.2 \pm 0.4}{2} = 0.3,\ -0.1$$

The homogeneous solution, $y_h(n)$ is given by,

$$y_h(n) = C_1\ \lambda_1^n + C_2\ \lambda_2^n$$

$$= C_1(0.3)^n + C_2(-0.1)^n\ ;\quad \text{for } n \geq 0 \quad .....(3)$$

Particular Solution

Given that the input is an exponential signal, $0.2^n\ u(n)$ and so the particular solution will be in the form,

$$y(n) = K\ 0.2^n\ u(n) \quad .....(4)$$

On substituting for y(n) from equation (4) in equation (1) we get,

$$K0.2^n u(n) - 0.2\ K\ 0.2^{(n-1)}\ u(n-1) - 0.03\ K\ 0.2^{(n-2)}\ u(n-2) = 0.2^n\ u(n) + 0.4 \times 0.2^{(n-1)}\ u(n) \quad .....(5)$$

In order to determine the value of K, let us evaluate equation (5) for n = 2, ($\because$ we have to evaluate equation (5) for any $n \geq 1$, such that none of the term vanishes).

From equation (5) when n = 2, we get,

$$K\,0.2^2 - 0.2K \times 0.2^1 - 0.03K \times 0.2^0 = 0.2^2 + 0.4 \times 0.2^1$$

$$0.04K - 0.04K - 0.03K = 0.04 + 0.08$$

$$-0.03K = 0.12$$

$$\therefore\ K = -\frac{0.12}{0.03} = -4$$

The particular solution $y_p(n)$ is given by,

$$y_p(n) = K\,0.2^n\,u(n) = (-4)\,0.2^n\,u(n)$$

Total Response

The total response y(n) of the system is given by sum of homogeneous and particular solution.

$$\therefore\ \text{Response, } y(n) = y_h(n) + y_p(n)$$

$$= C_1\,0.3^n + C_2\,(-0.1)^n + (-4)\,0.2^n\,; \quad \text{for } n \geq 0 \qquad .....(6)$$

To find y(0) and y(1)

When n = 0,

From equation (1) we get,

$$y(0) - 0.2\,y(-1) - 0.03\,y(-2) = x(0) + 0.4\,x(-1) \qquad .....(7)$$

Given that, $y(-1) = 0.5$, $y(-2) = 0$

$$x(n) = 0.2^n\,u(n), \qquad \therefore\ x(0) = 0.2^0 = 1$$

$$x(-1) = 0$$

On substituting the above conditions in equation (7) we get,

$$y(0) - 0.2 \times 0.5 - 0.03 \times 0 = 1 + 0$$

$$\therefore\ y(0) = 1.1 \qquad .....(8)$$

When n = 1,

From equation (1) we get,

$$y(1) - 0.2\,y(0) - 0.03\,y(-1) = x(1) + 0.4\,x(0) \qquad .....(9)$$

We know that, $y(0) = 1.1$, $y(-1) = 0.5$, $y(-2) = 0$

Given that, $x(n) = 0.2^n\,u(n)$, $\therefore\ x(0) = 0.2^0 = 1$

$$x(1) = 0.2^1 = 0.2$$

On substituting the above conditions in equation (9) we get,

$$y(1) - 0.2 \times 1.1 - 0.03 \times 0.5 = 0.2 + 0.4 \times 1$$

$$\therefore\ y(1) = 0.6 + 0.235 = 0.835 \qquad .....(10)$$

To solve constants C_1 and C_2

When n = 0,

From equation (6) we get,

$$y(0) = C_1\,0.3^0 + C_2\,(-0.1)^0 + (-4)\,0.2^0 = C_1 + C_2 - 4 \qquad .....(11)$$

From equations (8) and (11) we can write,

$$C_1 + C_2 - 4 = 1.1$$

$$\therefore\ C_1 + C_2 = 5.1 \qquad .....(12)$$

When n = 1,

From equation (6) we get,

$$y(1) = C_1 \times 0.3 + C_2(-0.1) + (-4)\,0.2 = 0.3\,C_1 - 0.1C_2 - 0.8 \qquad .....(13)$$

From equations (10) and (13) we can write,

$$0.3\,C_1 - 0.1C_2 - 0.8 = 0.835$$

$$\therefore\ 0.3\,C_1 - 0.1C_2 = 1.635 \qquad .....(14)$$

Equation (12) $\times$ 0.1 $\Rightarrow$ $0.1C_1 + 0.1C_2 = 0.51$

Equation (13) $\Rightarrow$ $0.3C_1 - 0.1C_2 = 1.635$

Add $0.4C_1 = 2.145$

$$\therefore\ C_1 = \frac{2.145}{0.4} = 5.3625$$

From equation(12),

$$C_2 = 5.1 - C_1 = 5.1 - 5.3625$$
$$= -0.2625$$

Total Response

$$y(n) = [5.3625(0.3)^n - 0.2625(-0.1)^n + (-4)\,0.2^n]\,u(n) \ ; \text{ for all n}$$

2.8 Classification of Discrete Time Systems

The discrete time systems are classified based on their characteristics. Some of the classifications of discrete time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. FIR and IIR systems
7. Recursive and nonrecursive systems

2.8.1 Static and Dynamic Systems

A discrete time system is called ***static*** or ***memoryless*** system if its output at any instant n depends at most on the input sample at the same time but not on the past or future samples of the input. In any other case, the system is said to be ***dynamic*** or to have memory.

Example :

$$\left.\begin{array}{l} y(n) = a\,x(n) \\ y(n) = n\,x(n) + 6\,x^3(n) \end{array}\right\} \text{Static systems}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} y(n) = x(n) + 3\,x(n-1) \\ y(n) = \sum\limits_{m=0}^{N} x(n-m) \end{array}\right\} \text{Finite memory is required} \\ \left.\begin{array}{l} y(n) = \sum\limits_{m=0}^{\infty} x(n-m) \end{array}\right\} \text{Infinite memory is required} \end{array}\right\} \text{Dynamic systems}$$

2.8.2 Time Invariant and Time Variant Systems

A system is said to be ***time invariant*** if its input-output characteristics do not change with time.

Definition : A relaxed system $\mathcal{H}$ is ***time invariant*** or ***shift invariant*** if and only if

$$\mathcal{H}\{x(n)\} = y(n) \text{ implies that, } \mathcal{H}\{x(n-m)\} = y(n-m)$$

for every input signal x(n) and every time shift m.

i.e., in time invariant systems, if $\mathbf{y(n) = \mathcal{H}\{x(n)\}}$ then $\mathbf{y(n-m) = \mathcal{H}\{x(n-m)\}}$.

Alternative Definition for Time Invariance

A system $\mathcal{H}$ is ***time invariant*** if the response to a shifted (or delayed) version of the input is identical to a shifted (or delayed) version of the response based on the unshifted (or undelayed) input.

i.e., In a time invariant system, $\mathbf{\mathcal{H}\{x(n-m)\} = z^{-m}\,\mathcal{H}\{x(n)\}}$; for all values of m .....(2.22)

The operator z^{-m} represents a signal delay of m samples.

The diagrammatic explanation of the above definition of time invariance is shown in fig 2.19.

Procedure to Test for Time Invariance

1. Delay the input signal by m units of time and determine the response of the system for this delayed input signal. Let this response be y(n – m).
2. Delay the response of the system for undelayed input by m units of time. Let this delayed response be $y_d(n)$.
3. Check whether y (n – m) = $y_d(n)$. If they are equal then the system is time invariant.

   Otherwise the system is ***time variant***.

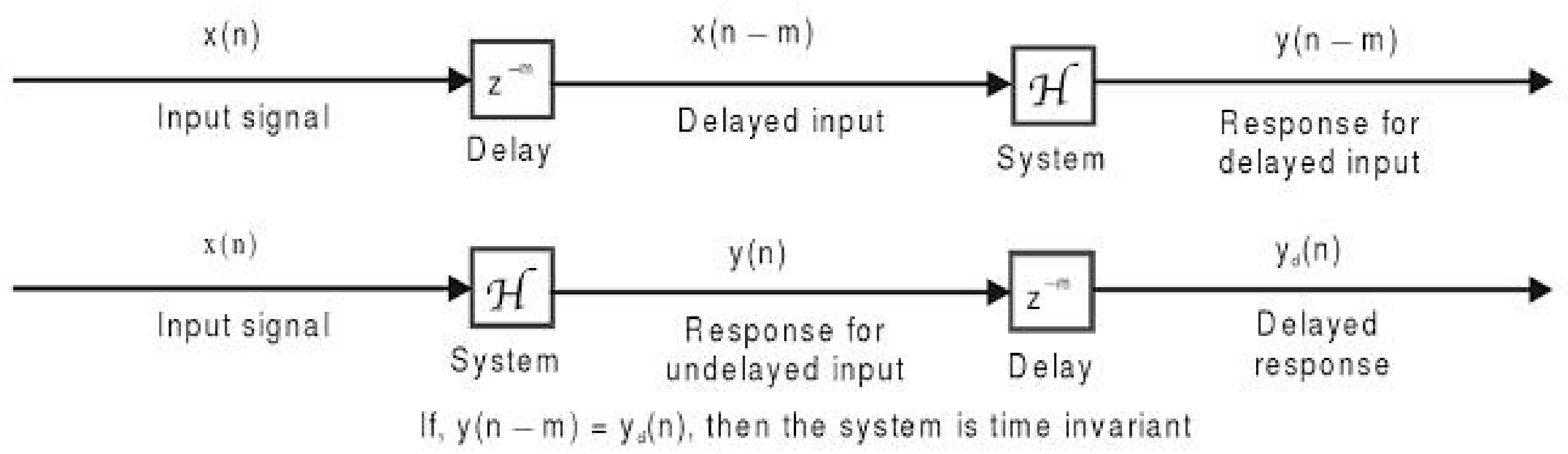


Fig 2.19 : Diagrammatic explanation of time invariance.

Example 2.10

Test the following systems for time invariance.

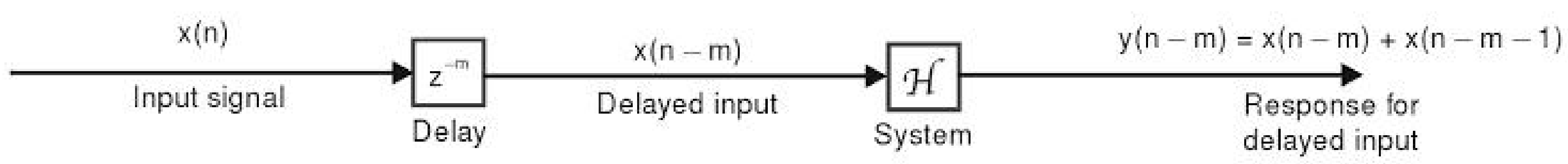
a) y(n) = x(n) + x(n – 1) **b)** y(n) = 2n x(n) **c)** y(n) = x(–n) **d)** y(n) = x(n) – b x(n – 1)

Solution

a) Given that, y(n) = x(n) + x(n – 1)

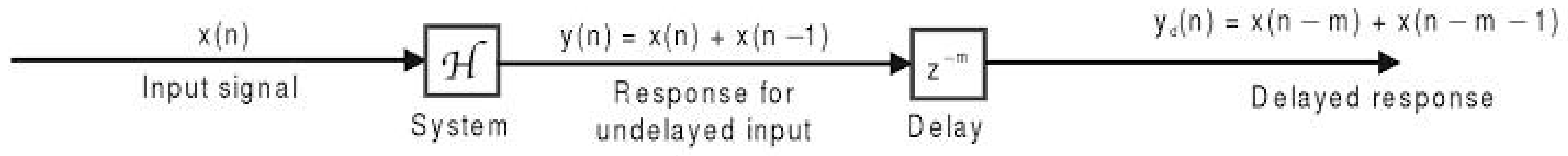
Test 1 : Response for delayed input

Let, y(n – m) = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

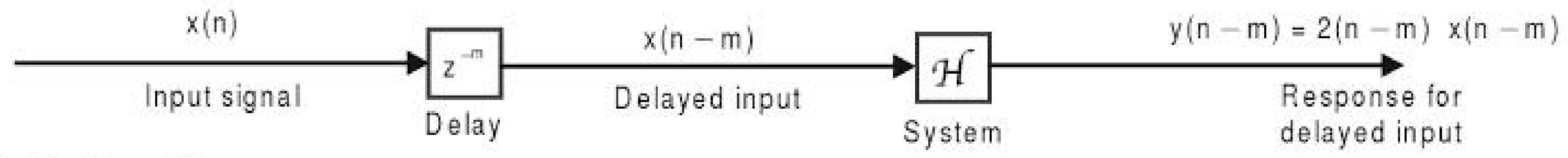


Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

b) Given that, y(n) = 2n x(n)

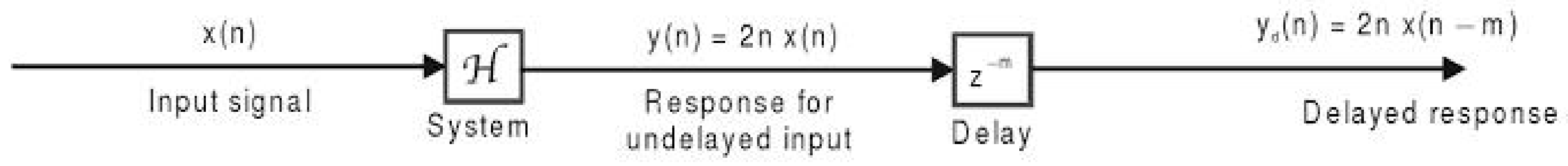
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

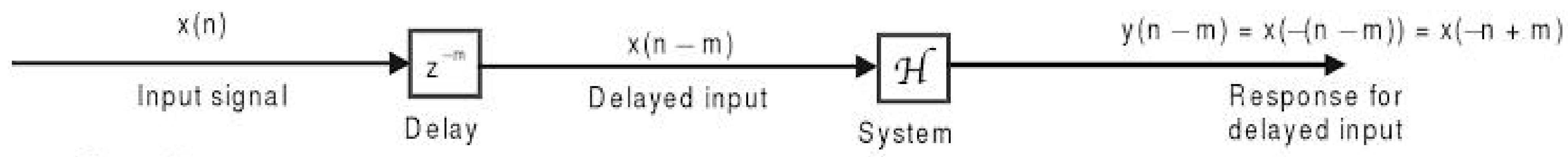


Conclusion : Here, $y(n-m) \neq y_d(n)$, therefore the system is time variant.

c) Given that, y(n) = x(−n)

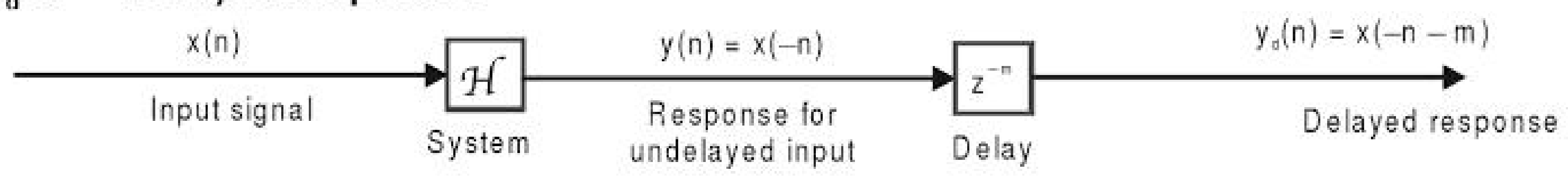
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

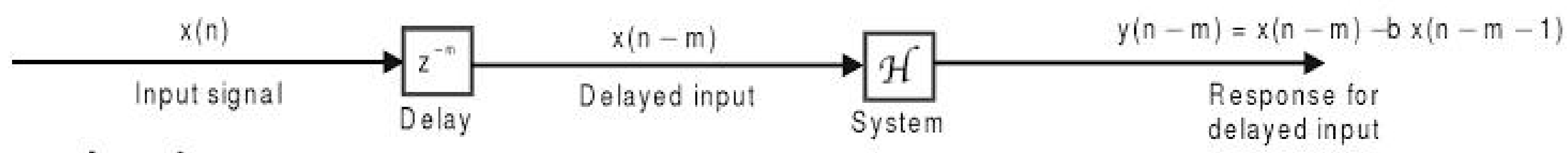


Conclusion : Here, $y(n-m) \neq y_d(n)$, therefore the system is time variant.

d) Given that, y(n) = x(n) – b x(n – 1)

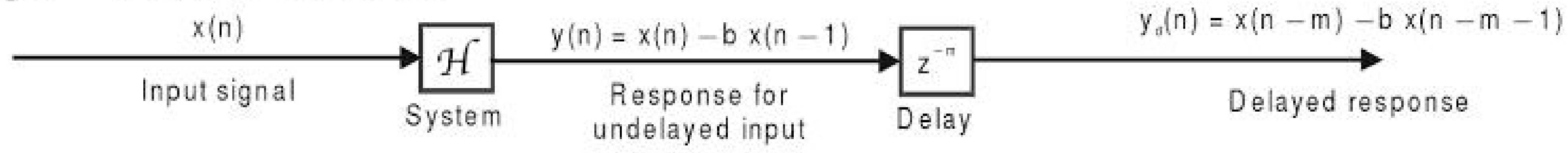
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.



Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

Example 2.11

Test the following systems for time invariance.

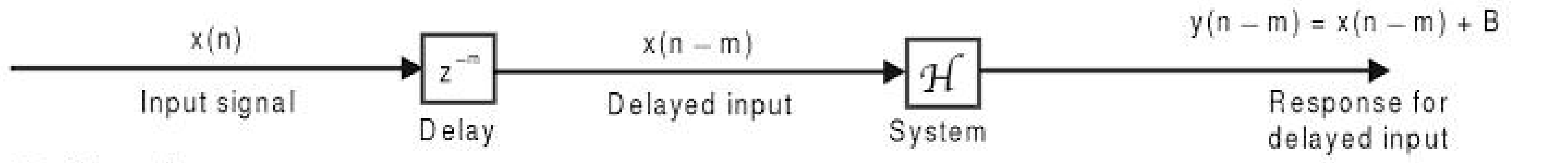
a) $y(n) = x(n) + B$ **b)** $y(n) = n\,x^3(n)$ **c)** $y(n) = b^{x(n)}$ **d)** $y(n) = \sum_{k=0}^{M} b_k\, x(n-k) - \sum_{k=1}^{N} a_k\, y(n-k)$

Solution

a) Given that, $y(n) = x(n) + B$

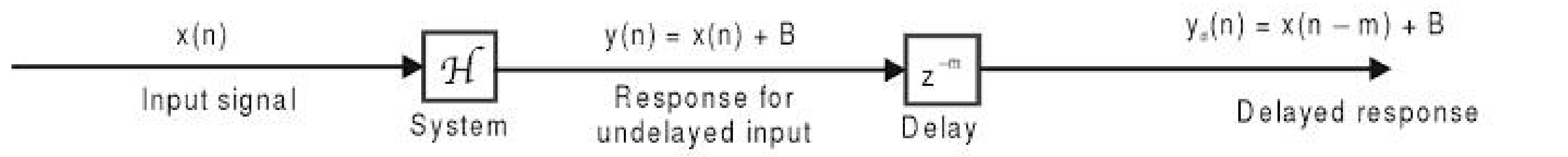
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

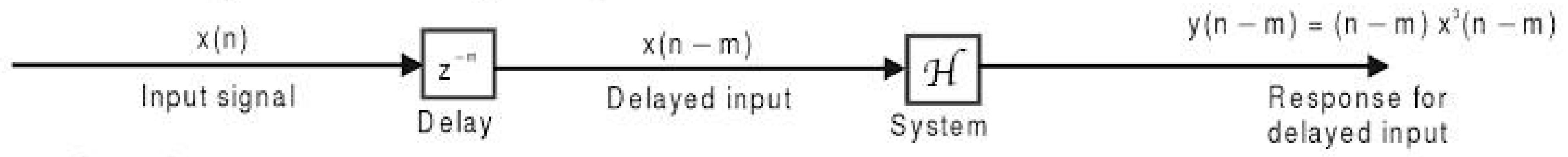


Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

b) Given that, $y(n) = n\,x^3(n)$

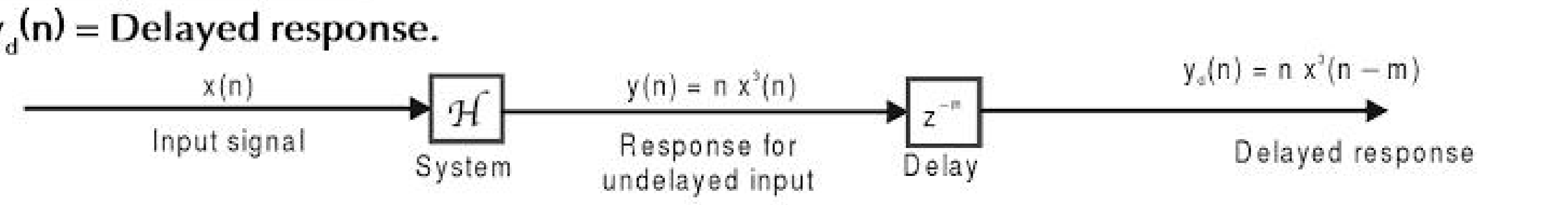
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

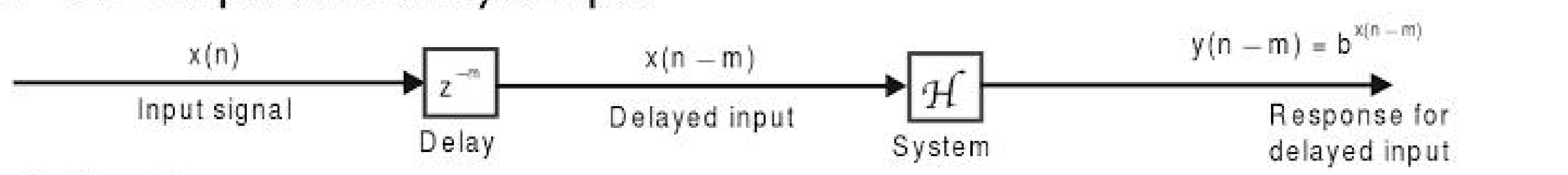


Conclusion : Here, $y(n-m) \neq y_d(n)$, therefore the system is time variant.

c) Given that, $y(n) = b^{x(n)}$

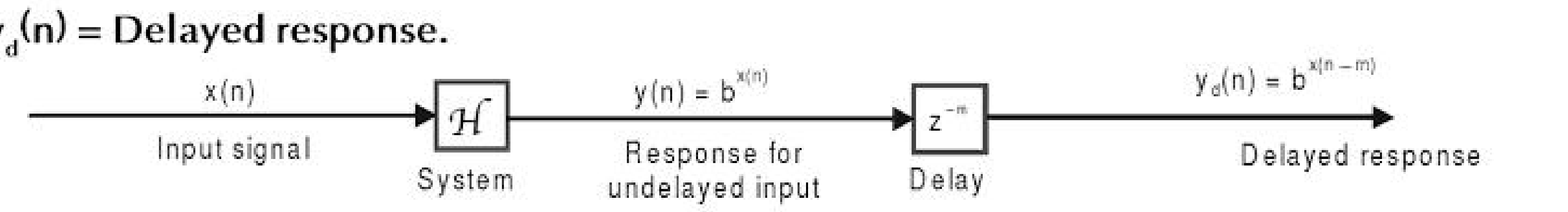
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

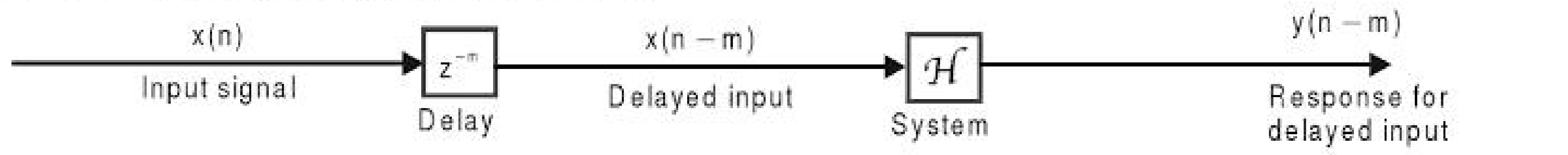


Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

d) Given that, $$y(n) = \sum_{k=0}^{M} b_k\, x(n-k) - \sum_{k=1}^{N} a_k\, y(n-k)$$

Test 1 : Response for delayed input

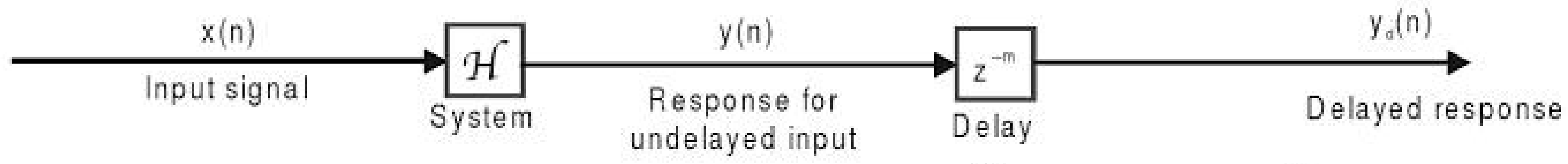
Let, $y(n-m)$ = Response for delayed input.



Response for delayed input, $y(n-m) = H\{x(n-m)\} = \sum_{k=0}^{M} b_k\, x(n-m-k) - \sum_{k=1}^{N} a_k\, y(n-m-k)$

Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.



$$\text{Response for undelayed input} = H\{x(n)\} = y(n) = \sum_{k=0}^{M} b_k\ x(n-k) - \sum_{k=1}^{N} a_k\ y(n-k)$$

$$\text{Delayed response, } y_d(n) = z^{-m}\ H\{x(n)\}$$

$$= z^{-m}\left[\sum_{k=0}^{M} b_k\ x(n-k) - \sum_{k=1}^{N} a_k\ y(n-k)\right]$$

$$= \sum_{k=0}^{M} b_k\ x(n-m-k) - \sum_{k=1}^{N} a_k\ \ y(n-m-k)$$

Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

2.8.3 Linear and Nonlinear Systems

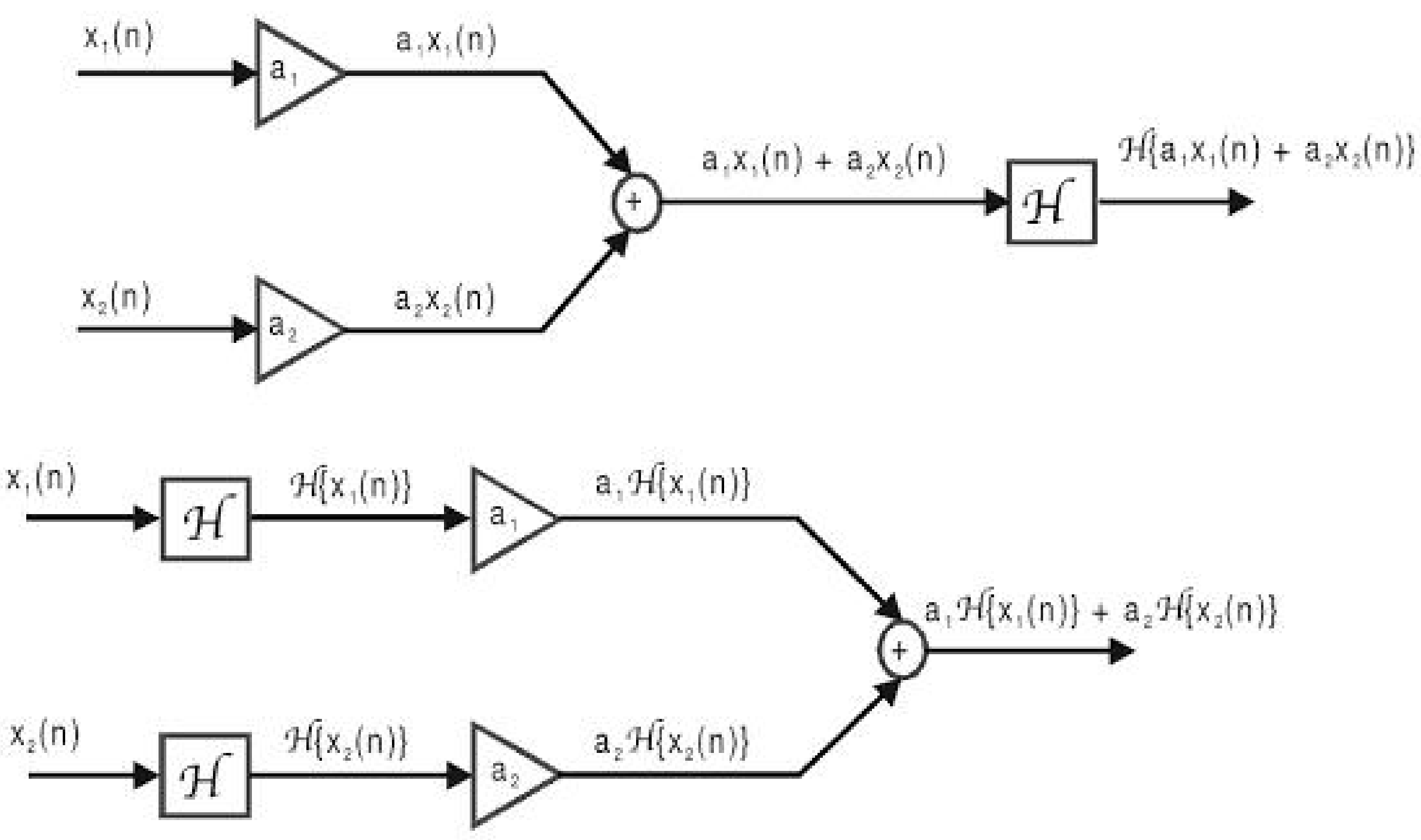
A ***linear system*** is one that satisfies the superposition principle. The ***principle of superposition*** requires that the response of the system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses of the system to each of the individual input signals.

Definition : A relaxed system $\mathcal{H}$ is ***linear*** if

$$\mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, \mathcal{H}\{x_1(n)\} + a_2\, \mathcal{H}\{x_2(n)\} \qquad .....(2.23)$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$ and for any arbitrary constants a_1 and a_2.

If a relaxed system does not satisfy the superposition principle as given by the above definition, then the system is ***nonlinear***.The diagrammatic explanation of linearity is shown in fig 2.20.



The system, $\mathcal{H}$ is linear if and only if, $\mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, \mathcal{H}\{x_1(n)\} + a_2\, \mathcal{H}\{x_2(n)\}$

Fig 2.20 : Diagrammatic explanation of linearity.

Procedure to test for linearity

1. Let $x_1(n)$ and $x_2(n)$ be two inputs to system $\mathcal{H}$, and $y_1(n)$ and $y_2(n)$ be corresponding responses.
2. Consider a signal, $x_3(n) = a_1\, x_1(n) + a_2\, x_2(n)$ which is a weighed sum of $x_1(n)$ and $x_2(n)$.
3. Let $y_3(n)$ be the response for $x_3(n)$.
4. Check whether $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$. If they are equal then the system is linear, otherwise it is nonlinear.

Example 2.12

Test the following systems for linearity.

a) $y(n) = n\,x(n)$ **b)** $y(n) = x(n^2)$ **c)** $y(n) = x^2(n)$ **d)** $y(n) = B\,x(n) + C$

Solution

a) Given that, $y(n) = n\,x(n)$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = n\,x(n)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = n\,x(n)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = n\,x_1(n)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = n\,x_2(n)$$

\ $a_1\,y_1(n) + a_2\,y_2(n) = a_1\,n\,x_1(n) + a_2\,n\,x_2(n)$ .....(1)

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

\ $y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = n[a_1\,x_1(n) + a_2\,x_2(n)] = a_1\,n\,x_1(n) + a_2\,n\,x_2(n)$ .....(2)

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is linear.

---

b) Given that, $y(n) = x(n^2)$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = x(n^2)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = x(n^2)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = x_1(n^2)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = x_2(n^2)$$

\ $a_1\,y_1(n) + a_2\,y_2(n) = a_1\,x_1(n^2) + a_2\,x_2(n^2)$ .....(1)

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, x_1(n^2) + a_2\, x_2(n^2) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is linear.

---

c) Given that, $y(n) = x^2(n)$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = x^2(n)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = x^2(n)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = x_1^2(n)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = x_2^2(n)$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1\, x_1^2(n) + a_2\, x_2^2(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = [a_1\, x_1(n) + a_2\, x_2(n)]^2$$
$$= a_1^2\, x_1^2(n) + a_2^2\, x_2^2(n) + 2\, a_1\, a_2\, x_1(n) x_2(n) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is nonlinear.

---

d) Given that, $y(n) = B\, x(n) + C$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = B\, x(n) + C$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = B\, x(n) + C$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = B\, x_1(n) + C$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = B\, x_2(n) + C$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1\, [B\, x_1(n) + C] + a_2\, [B\, x_2(n) + C]$$
$$= B\, a_1\, x_1(n) + C\, a_1 + B\, a_2\, x_2(n) + C\, a_2 \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = B\, [a_1\, x_1(n) + a_2\, x_2(n)] + C = B\, a_1\, x_1(n) + B\, a_2\, x_2(n) + C \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is nonlinear.

Example 2.13

Test the following systems for linearity.

a) $y(n) = e^{x(n)}$ **b)** $y(n) = b^{x(n)}$ **c)** $y(n) = n\,x^2(n)$

Solution

a) Given that, $y(n) = e^{x(n)}$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = e^{x(n)}$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = e^{x(n)}$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = e^{x_1(n)}$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = e^{x_2(n)}$$

$$\therefore \quad a_1\,y_1(n) + a_2\,y_2(n) = a_1\,e^{x_1(n)} + a_2\,e^{x_2(n)} \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore \quad y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = e^{[a_1x_1(n)+a_2x_2(n)]} = e^{a_1x_1(n)}\,e^{a_2x_2(n)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is nonlinear.

---

b) Given that, $y(n) = b^{x(n)}$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = b^{x(n)}$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = b^{x(n)}$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = b^{x_1(n)}$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = b^{x_2(n)}$$

$$\therefore \quad a_1\,y_1(n) + a_2\,y_2(n) = a_1\,b^{x_1(n)} + a_2\,b^{x_2(n)} \qquad .....(1)$$

Consider a linear combination of inputs, $a_1x_1(n) + a_2x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore\ y_3(n) = \mathcal{H}\{a_1x_1(n) + a_2x_2(n)\} = b^{[a_1x_1(n)+a_2x_2(n)]} = b^{a_1x_1(n)}\ b^{a_2x_2(n)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is nonlinear.

c) Given that, $y(n) = n\, x^2(n)$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = n\, x^2(n)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = n\ x^2(n)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = n\ x_1^2(n)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = n\ x_2^2(n)$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1\, n\, x_1^2(n) + a_2\, n\, x_2^2(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\begin{aligned} \therefore\ y_3(n) &= \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = n[a_1\, x_1(n) + a_2\, x_2(n)]^2 \\ &= n\, a_1^2\, x_1^2(n) + n\, a_2^2\, x_2^2(n) + 2\, n\, a_1\, a_2\, x_1(n)\, x_2(n) \qquad .....(2) \end{aligned}$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is nonlinear.

Example 2.14

Test the following systems for linearity.

a) $y(n) = 3\, x(n) + \dfrac{1}{x(n-2)}$ **b)** $y(n) = x(n) - 2\, x(n-1)$ **c)** $y(n) = \sum\limits_{m=0}^{M} b_m\, x(n-m) - \sum\limits_{m=1}^{N} c_m\, y(n-m)$

Solution

a) Given that, $y(n) = 3\,x(n) + \dfrac{1}{x(n-2)}$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = 3\,x(n) + \dfrac{1}{x(n-2)}$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = 3\,x(n) + \frac{1}{x(n-2)}$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = 3\,x_1(n) + \frac{1}{x_1(n-2)}$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = 3\,x_2(n) + \frac{1}{x_2(n-2)}$$

$$\therefore \; a_1 y_1(n) + a_2 y_2(n) = a_1\left(3x_1(n) + \frac{1}{x_1(n-2)}\right) + a_2\left(3x_2(n) + \frac{1}{x_2(n-2)}\right) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\backslash \; y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = 3[a_1 x_1(n) + a_2 x_2(n)] + \frac{1}{a_1 x_1(n-2) + a_2 x_2(n-2)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1 y_1(n) + a_2 y_2(n)$. Hence the system is nonlinear.

---

b) Given that, $y(n) = x(n) - 2\,x(n-1)$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = x(n) - 2\,x(n-1)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = x(n) - 2x(n-1)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = x_1(n) - 2\,x_1(n-1)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = x_2(n) - 2\,x_2(n-1)$$

$$\backslash \; a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n) - a_1\, 2\,x_1(n-1) + a_2 x_2(n) - a_2\, 2\,x_2(n-1) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\backslash \; y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 x_1(n) + a_2 x_2(n) - 2[a_1 x_1(n-1) + a_2 x_2(n-1)]$$

$$= a_1 x_1(n) - a_1\, 2\,x_1(n-1) + a_2 x_2(n) - a_2\, 2\,x_2(n-1) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$. Hence the system is linear.

---

c) Given that, $y(n) = \sum\limits_{m=0}^{M} b_m\, x(n-m) - \sum\limits_{m=1}^{N} c_m\, y(n-m)$

Let, $\mathcal{H}$ be the system represented by the equation,

$$y(n) = \sum_{m=0}^{M} b_m\, x(n-m) - \sum_{m=1}^{N} c_m\, y(n-m)$$

$$\left.\begin{array}{l}\text{The response of the system} \\ \mathcal{H} \text{ for the input } x(n)\end{array}\right\} = \mathcal{H}\{x(n)\} = y(n) = \sum_{m=0}^{M} b_m\, x(n-m) - \sum_{m=1}^{N} c_m\, y(n-m)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\backslash \; y_1(n) = \mathcal{H}\{x_1(n)\} = \sum_{m=0}^{M} b_m\, x_1(n-m) - \sum_{m=1}^{N} c_m\, y_1(n-m)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = \sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_2(n-m)$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1 \left(\sum_{m=0}^{M} b_m\, x_1(n-m) - \sum_{m=1}^{N} c_m\, y_1(n-m) \right)$$

$$+ a_2 \left(\sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_2(n-m) \right) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for the input $x_3(n)$.

\ $y_3(n) = \mathcal{H}\{x_3(n)\} = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\}$

$$= \sum_{m=0}^{M} b_m\, (a_1\, x_1(n-m) + a_2\, x_2(n-m)) - \sum_{m=1}^{N} c_m\, y_3(n-m)$$

$$= a_1 \sum_{m=0}^{M} b_m\, x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_3(n-m) \qquad .....(2)$$

By time invariant property,

If $y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\}$ then $y_3(n-m) = \mathcal{H}\{a_1\, x_1(n-m) + a_2\, x_2(n-m)\}$

If $y_2(n) = \mathcal{H}\{x_2(n)\}$ then $y_2(n-m) = \mathcal{H}\{x_2(n-m)\}$

If $y_1(n) = \mathcal{H}\{x_1(n)\}$ then $y_1(n-m) = \mathcal{H}\{x_1(n-m)\}$

\ $y_3(n-m) = \mathcal{H}\{a_1\, x_1(n-m) + a_2\, x_2(n-m)\} = a_1\, \mathcal{H}\{x_1(n-m)\} + a_2\, \mathcal{H}\{x_2(n-m)\}$

$$= a_1\, y_1(n-m) + a_2\, y_2(n-m)] \qquad .....(3)$$

Using equation (3), the equation (2) can be written as,

$$y_3(n) = a_1 \sum_{m=0}^{M} b_m\, x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, [a_1\, y_1(n-m) + a_2\, y_2(n-m)]$$

$$= a_1 \sum_{m=0}^{M} b_m\, x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\, x_2(n-m) - a_1 \sum_{m=1}^{N} c_m\, y_1(n-m) - a_2 \sum_{m=1}^{N} c_m\, y_2(n-m)$$

$$= a_1 \left(\sum_{m=0}^{M} b_m\, x_1(n-m) - \sum_{m=1}^{N} c_m\, y_1(n-m) \right) + a_2 \left(\sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_2(n-m) \right) \qquad .....(4)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (4) we can say that the condition for linearity is satisfied. Therefore the system is linear.

2.8.4 Causal and Noncausal Systems

Definition : A system is said to be ***causal*** if the output of the system at any time n depends only on the present input, past inputs and past outputs but does not depend on the future inputs and outputs.

If the system output at any time n depends on future inputs or outputs then the system is called ***noncausal*** system.

The causality refers to a system that is realizable in real time. It can be shown that an LTI system is causal if and only if the impulse response is zero for $n < 0$, (i.e., $h(n) = 0$ for $n < 0$).

Let, $x(n)$ = Present input and $y(n)$ = Present output

\ $x(n-1), x(n-2), ...$, are past inputs

$y(n-1), y(n-2), ...$, are past outputs

In mathematical terms the output of a causal system satisfies the equation of the form,

$$y(n) = F\,[x(n), x(n-1), x(n-2), \ldots, y(n-1), y(n-2) \ldots]$$

where, $F[\cdot]$ is some arbitrary function.

Example 2.15

Test the causality of the following systems.

a) $y(n) = x(n) - x(n-2)$ **b)** $y(n) = \sum_{k=-\infty}^{n} x(k)$ **c)** $y(n) = b\,x(n)$ **d)** $y(n) = n\,x(n)$

Solution

a) Given that, $y(n) = x(n) - x(n-2)$

When $n = 0$, $y(0) = x(0) - x(-2)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past input $x(-2)$

When $n = 1$, $y(1) = x(1) - x(-1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past input $x(-1)$.

From the above analysis we can say that for any value of n, the system output depends on present and past inputs. Hence the system is causal.

b) Given that, $y(n) = \sum_{k=-\infty}^{n} x(k)$

When $n = 0$, $y(0) = \sum_{k=-\infty}^{0} x(k)$

$= \ldots x(-2) + x(-1) + x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past inputs $x(-1)$, $x(-2)$,.....

When $n = 1$, $y(1) = \sum_{k=-\infty}^{1} x(k)$

$= \ldots x(-2) + x(-1) + x(0) + x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past inputs $x(0)$, $x(-1)$, $x(-2)$,..

From the above analysis we can say that for any value of n, the system output depends on present and past inputs. Hence the system is causal.

c) Given that, $y(n) = b\,x(n)$

When $n = 0$, $y(0) = b\,x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$.

When $n = 1$, $y(1) = b\,x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

From the above analysis we can say that the response for any value of n depends on the present input. Hence the system is causal.

d) Given that, $y(n) = n\,x(n)$

When $n = 0$, $y(0) = 0 \times x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$.

When $n = 1$, $y(1) = 1 \times x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

When $n = 2$, $y(2) = 2 \times x(2)$ $\Rightarrow$ The response at $n = 2$, i.e., $y(2)$ depends on the present input $x(2)$.

From the above analysis we can say that the response for any value of n depends on the present input. Hence the system is causal.

Example 2.16

Test the causality of the following systems.

a) $y(n) = x(n) + 2\,x(n+3)$ **b)** $y(n) = x(n^2)$ **c)** $y(n) = x(3n)$ **d)** $y(n) = x(-n)$

Solution

a) Given that, $y(n) = x(n) + 2\,x(n+3)$

When $n = 0$, $y(0) = x(0) + 2\,x(3)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and future input $x(3)$.

When n = 1, y(1) = x(1) + 2 x(4) Þ The response at n = 1, i.e., y(1) depends on the present input x(1) and future input x(4).

From the above analysis we can say that the response for any value of n depends on present and future inputs. Hence the system is noncausal.

b) Given that, $y(n) = x(n^2)$

When n = –1 ; y(–1) = x(1) Þ The response at n = –1, depends on the future input x(1).
When n = 0 ; y(0) = x(0) Þ The response at n = 0, depends on the present input x(0).
When n = 1 ; y(1) = x(1) Þ The response at n = 1, depends on the present input x(1).
When n = 2 ; y(2) = x(4) Þ The response at n = 2, depends on the future input x(4).

From the above analysis we can say that the response for any value of n (except n = 0 and n = 1) depends on future inputs. Hence the system is noncausal.

c) Given that, y(n) = x(3n)

When n = –1 ; y(–1) = x(–3) Þ The response at n = –1, depends on the past input x(–3).
When n = 0 ; y(0) = x(0) Þ The response at n = 0, depends on the present input x(0).
When n = 1 ; y(1) = x(3) Þ The response at n = 1, depends on the future input x(3).

From the above analysis we can say that the response of the system for n > 0, depends on future inputs. Hence the system is noncausal.

d) Given that, y(n) = x(–n)

When n = –2 ; y(–2) = x(2) Þ The response at n = –2, depends on the future input x(2).
When n = –1 ; y(–1) = x(1) Þ The response at n = –1, depends on the future input x(1).
When n = 0 ; y(0) = x(0) Þ The response at n = 0, depends on the present input x(0).
When n = 1 ; y(1) = x(–1) Þ The response at n = 1, depends on the past input x(–1).

From the above analysis we can say that the response of the system for n < 0 depends on future inputs. Hence the system is noncausal.

2.8.5 Stable and Unstable Systems

Definition : An arbitrary relaxed system is said to be ***BIBO stable*** (Bounded Input-Bounded Output stable) if and only if every bounded input produces a bounded output.

Let x(n) be the input of discrete time system and y(n) be the response or output for x(n).The term ***bounded input*** refers to finite value of the input signal x(n) for any value of n. Hence if input x(n) is bounded then there exits a constant M_x such that |x(n)| £ M_x and M_x < ¥, for all n.

Examples of bounded input signal are step signal, decaying exponential signal and impulse signal.
Examples of unbounded input signal are ramp signal and increasing exponential signal.

The term ***bounded output*** refers to finite and predictable output for any value of n. Hence if output y(n) is bounded then there exists a constant M_y such that |y(n)| £ M_y and M_y < ¥, for all n.

In general, the test for stability of the system is performed by applying specific input. On applying a bounded input to a system if the output is bounded then the system is said to be BIBO stable.For LTI (Linear Time Invariant) systems the condition for BIBO stability can be transformed to a condition on impulse response as shown below.

Condition for Stability of LTI System

The condition for stability of an LTI system is,

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty \qquad .....(2.24)$$

i.e., an LTI system is ***stable*** if the impulse response is absolutely summable.

Proof

Let, x(n) = Input to LTI system.

y(n) = Response of LTI system for the input x(n).

Now, by convolution sum formula,

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

Convolution satisfy commutative property.

$$= \sum_{m=-\infty}^{+\infty} h(m)\, x(n-m)$$

$$\therefore \quad |y(n)| = \left| \sum_{m=-\infty}^{+\infty} h(m)\, x(n-m) \right|$$

Taking absolute value on both sides.

$$= \sum_{m=-\infty}^{+\infty} |h(m)\, x(n-m)|$$

For linear system the order summation and absolute value can be interchanged.

$$= \sum_{m=-\infty}^{+\infty} |h(m)|\, |x(n-m)|$$

For linear system the order of multiplication and absolute value can be interchanged.

$$= \sum_{m=-\infty}^{+\infty} |h(m)|\, M_X$$

If input is bounded, then $|x(n-m)|$ = constant = M_x

$$= M_X \sum_{m=-\infty}^{+\infty} |h(m)|$$

M_x is indepentent of summation index m.

$$= M_X \sum_{n=-\infty}^{+\infty} |h(n)|$$

Change index m to n.

In the above equation, if

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty \qquad .....(2.25)$$

then the response y(n) is bounded.

Example 2.17

Test the stability of the following systems.

a) $y(n) = \cos[x(n)]$ **b)** $y(n) = x(-n-3)$ **c)** $y(n) = n\,x(n)$

Solution

a) Given that, $y(n) = \cos[x(n)]$

The given system is a nonlinear system, and so the test for stability should be performed for specific inputs.

The value of cos q lies between –1 to +1 for any value of q. Therefore the output y(n) is bounded for any value of input x(n). Hence the given system is stable.

b) Given that, $y(n) = x(-n-3)$

The given system is a time variant system, and so the test for stability should be performed for specific inputs.

The operations performed by the system on the input signal are folding and shifting. A bounded input signal will remain bounded even after folding and shifting. Therefore in the given system, the output will be bounded as long as input is bounded. Hence the given system is BIBO stable.

c) Given that, $y(n) = n\,x(n)$

The given system is a time variant system, and so the test for stability should be performed for specific inputs.

Case i : If x(n) tends to infinity or constant, as "n" tends to infinity, then y(n) = n x(n) will be infinite as "n" tends to infinity. So the system is unstable.

Case ii : If x(n) tends to zero as "n" tends to infinity, then y(n) = n x(n) will be zero as "n" tends to infinity. So the system is stable.

Example 2.18

Determine the range of values of "p" and "q" for the stability of LTI system with impulse response,

$$h(n) = p^n \quad ; \quad n < 0$$
$$\quad = q^n \quad ; \quad n \geq 0$$

Solution

The condition to be satisfied for the stability of the system is, $\sum\limits_{n=-\infty}^{\infty} |h(n)| < \infty$.

Given that, $h(n) = p^n \quad ; \; n < 0$
$\qquad\qquad = q^n \quad ; \; n \geq 0$

$$\therefore \sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{-1} |p^n| + \sum_{n=0}^{\infty} |q^n| = \sum_{n=1}^{\infty} |p^{-n}| + \sum_{n=0}^{\infty} |q^n|$$

$$= \sum_{n=1}^{\infty} \left|\frac{1}{p^n}\right| + \sum_{n=0}^{\infty} |q^n| = \sum_{n=1}^{\infty} \frac{1}{|p|^n} + \sum_{n=0}^{\infty} |q|^n$$

n is always positive.

$$= \sum_{n=0}^{\infty} \left(\frac{1}{|p|}\right)^n - 1 + \sum_{n=0}^{\infty} |q|^n$$

$|p|^0 = 1$

The summation of infinite terms in the above equation converges if, $0 < 1/|p| < 1$ and $0 < |q| < 1$. Hence by using infinite geometric series formula,

$$\sum_{n=-\infty}^{+\infty} |h(n)| = \frac{1}{1 - \frac{1}{|p|}} - 1 + \frac{1}{1 - |q|}$$

$$= \text{constant}$$

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$
if $0 < |C| < 1$

Therefore, the system is stable if $|p| > 1$ and $|q| < 1$.

Example 2.19

Test the stability of LTI systems, whose impulse responses are,

a) $h(n) = 0.2^n\, u(n)$ **b)** $h(n) = 0.3^n\, u(n) + 2^n\, u(n)$

c) $h(n) = 4^n\, u(-n)$ **d)** $h(n) = 0.2^n\, u(-n) + 3^n\, u(-n)$

Solution

a) $h(n) = 0.2^n\, u(n)$

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} |0.2^n\, u(n)| = \sum_{n=0}^{\infty} 0.2^n$$

$$= \frac{1}{1 - 0.2} = 1.25$$

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$
if $0 < |C| < 1$

Since, $\sum\limits_{n=-\infty}^{+\infty} |h(n)| < \infty$, system is stable.

b) h(n) = 0.3ⁿ u(n) + 2ⁿ u(n)

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|0.3^n u(n) + 2^n u(n)\right|$$

$$= \sum_{n=0}^{\infty} 0.3^n + \sum_{n=0}^{\infty} 2^n u(n) = \frac{1}{1-0.3} + \infty = \infty$$

$$\boxed{\sum_{n=0}^{\infty} C^n = \infty \quad \text{if } C > 1}$$

Since, $\sum_{n=-\infty}^{+\infty} |h(n)| = \infty$, system is unstable.

c) h(n) = 4ⁿ u(-n)

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|4^n u(-n)\right| = \sum_{n=-\infty}^{0} 4^n = \sum_{n=0}^{+\infty} 4^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4^n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \sum_{n=0}^{\infty} 0.25^n = \frac{1}{1-0.25} = 1.3333$$

Since, $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$, system is stable.

d) h(n) = 0.2ⁿ u(-n) + 3ⁿ u(-n)

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|0.2^n u(-n) + 3^n u(-n)\right|$$

$$= \sum_{n=-\infty}^{0} 0.2^n + \sum_{n=-\infty}^{0} 3^n = \sum_{n=0}^{+\infty} 0.2^{-n} + \sum_{n=0}^{+\infty} 3^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{0.2^n} + \sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{1}{0.2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} 5^n + \sum_{n=0}^{\infty} 0.333^n = \infty + \frac{1}{1-0.333} = \infty$$

Since, $\sum_{n=-\infty}^{+\infty} |h(n)| = \infty$, system is unstable.

2.8.6 FIR and IIR Systems

In ***FIR system*** (**F**inite duration **I**mpulse **R**esponse system), the impulse response consists of finite number of samples. The convolution formula for FIR system is given by,

$$y(n) = \sum_{m=0}^{N-1} h(m)\, x(n-m) \qquad .....(2.26)$$

where, $h(n) = 0$; for $n < 0$ and $n \geq N$

From equation (2.26) it can be concluded that the impulse response selects only N samples of the input signal.In effect, the system acts as a window that views only the most recent N input signal samples in forming the ouput. It neglects or simply forgets all prior input samples. Thus a FIR system requires memory of length N. In general, a FIR system is described by the difference equation,

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m) \qquad .....(2.27)$$

where, $b_m = h(m)$; for $m = 0$ to $N - 1$

In ***IIR system*** (**I**nfinite duration **I**mpulse **R**esponse system), the impulse response has infinite number of samples. The convolution formula for IIR systems is given by,

$$y(n) = \sum_{m=0}^{\infty} h(m)\, x(n-m) \qquad .....(2.28)$$

Since this weighted sum involves the present and all the past input sample, we can say that the IIR system requires infinite memory. In general, an IIR system is described by the difference equation,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

2.8.7 Recursive and Nonrecursive Systems

A system whose output y(n) at time n depends on any number of past output values as well as present and past inputs is called a ***recursive system.*** The past outputs are y(n − 1), y(n − 2), y(n − 3), etc.,.

Hence for recursive system, the output y(n) is given by,

$$y(n) = F\,[y(n-1), y(n-2), ...y(n-N), x(n), x(n-1), ...x(n-M)]$$

A system whose output does not depend on past output but depends only on the present and past input is called a ***nonrecursive system.***

Hence for nonrecursive system, the output y(n) is given by,

$$y(n) = F\,[x(n), x(n-1), ..., x(n-M)]$$

In a recursive system, in order to compute $y(n_0)$, we need to compute all the previous values y(0), y(1) ,......., $y(n_0 - 1)$ before calculating $y(n_0)$. Hence the output samples of a recursive system has to be computed in order [i.e., y(0), y(1), y(2), ...]. The IIR systems are recursive systems.

In nonrecursive system, $y(n_0)$ can be computed immediately without having $y(n_0 - 1)$, $y(n_0 - 2)$..... Hence the output samples of nonrecursive system can be computed in any order [i.e. y(50), y(5), y(2), y(100),....]. The FIR systems are nonrecursive systems.

2.9 Discrete or Linear Convolution

The ***Discrete*** or ***Linear convolution*** of two discrete time sequences $x_1(n)$ and $x_2(n)$ is defined as,

$$\boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, x_1(n-m)} \qquad .....(2.29)$$

where, $x_3(n)$ is the sequence obtained by convolving $x_1(n)$ and $x_2(n)$

m is a dummy variable

The convolution relation of equation (2.29) can be symbolically expressed as,

$$x_3(n) = x_1(n) * x_2(n) = x_2(n) * x_1(n) \qquad ..... (2.30)$$

where, the symbol $*$ indicates convolution operation.

In linear convolution, the sequences $x_1(n)$ and $x_2(n)$ are nonperiodic sequences and the sequence $x_3(n)$ obtained by convolution is also nonperiodic. Hence this convolution is also called ***aperiodic convolution.***

Procedure For Evaluating Linear Convolution

Let, $x_1(n)$ = Discrete time sequence with N_1 samples

$x_2(n)$ = Discrete time sequence with N_2 samples

Now, the convolution of $x_1(n)$ and $x_2(n)$ will produce a sequence $x_3(n)$ consisting of N_1+N_2-1 samples. Each sample of $x_3(n)$ can be computed using the equation (2.29). The value of $x_3(n)$ at n = q is obtained by replacing n by q, in equation (2.29).

$$\therefore \quad x_3(q) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(q-m) \qquad .....(2.31)$$

The evaluation of equation (2.31) to determine the value of $x_3(n)$ at $n = q$, involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, to get the sequences $x_1(m)$ and $x_2(m)$.

2. Folding : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.

3. Shifting : Shift $x_2(-m)$ by q to the right if q is positive, shift $x_2(-m)$ by q to the left if q is negative to obtain $x_2(q - m)$.

4. Multiplication : Multiply $x_1(m)$ by $x_2(q - m)$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2(q - m)$.

5. Summation : Sum all the values of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

The above procedure will give the value of $x_3(n)$ at a single time instant say $n = q$. In general, we are interested in evaluating the values of the sequence $x_3(n)$ over all the time instants in the range $-¥ < n < ¥$. Hence the steps 3, 4 and 5 given above must be repeated, for all possible time shifts in the range $-¥ < n < ¥$.

Convolution of finite duration sequences

In convolution of finite duration sequences it is possible to predict the length of resultant sequence.

If the sequence $x_1(n)$ has N_1 samples and sequence $x_2(n)$ has N_2 samples then the output sequence $x_3(n)$ will be a finite duration sequence consisting of "N_1+N_2-1" samples.

i.e., if, Length of $x_1(n) = N_1$

Length of $x_2(n) = N_2$

then, Length of $x_3(n) = N_1 + N_2 - 1$

In the convolution of finite duration sequences it is possible to predict the start and end of the resultant sequence. If $x_1(n)$ starts at $n = n_1$ and $x_2(n)$ starts at $n = n_2$ then, the initial value of n for $x_3(n)$ is "$n = n_1 + n_2$". The value of $x_1(n)$ for $n < n_1$ and the value of $x_2(n)$ for $n < n_2$ are then assumed to be zero.The final value of n for $x_3(n)$ is "$n = (n_1 + n_2) + (N_1 + N_2 - 2)$".

i.e., if, $x_1(n)$ start at $n = n_1$

$x_2(n)$ start at $n = n_2$

then, $x_3(n)$ start at $n = n_1 + n_2$

and $x_3(n)$ end at $n = (n_1 + n_2) + (N_1 + N_2 - 1) - 1$

$= (n_1 + n_2) + (N_1 + N_2 - 2)$

2.9.1 Representation of Discrete Time Signal as Summation of Impulses

A discrete time signal can be expressed as summation of impulses and this concept will be useful to prove that the response of discrete time LTI system can be determined using discrete convolution.

Let, $x(n)$ = Discrete time signal

$d(n)$ = Unit impulse signal

$d(n - m)$ = Delayed impulse signal

We know that, $\delta(n) = 1$; at $n = 0$

$= 0$; when $n \neq 0$

and, $\delta(n - m) = 1$; at $n = m$

$= 0$; when $n \neq m$

If we multiply the signal x(n) with the delayed impulse $\delta(n - m)$ then the product is nonzero only at $n = m$ and zero for all other values of n. Also at $n = m$, the value of product signal is m^{th} sample x(m) of the signal x(n).

$$\therefore\ x(n)\,\delta(n - m) = x(m)$$

Each multiplication of the signal x(n) by an unit impulse at some delay m, in essence picks out the single value x(m) of the signal x(n) at $n = m$, where the unit impulse is nonzero. Consequently if we repeat this multiplication for all possible delays in the range $-\infty < m < \infty$ and add all the product sequences, the result will be a sequence that is equal to the sequence x(n).

$$
\begin{aligned}
\text{For example, } x(n)\,\delta(n-(-2)) &= x(-2)\\
x(n)\,\delta(n-(-1)) &= x(-1)\\
x(n)\,\delta(n) &= x(0)\\
x(n)\,\delta(n-1) &= x(1)\\
x(n)\,\delta(n-2) &= x(2)
\end{aligned}
$$

From the above products we can say that each sample of x(n) can be expressed as a product of the sample and delayed impulse, as shown below.

$$
\begin{aligned}
\therefore\ x(-2) &= x(-2)\,\delta(n-(-2))\\
x(-1) &= x(-1)\,\delta(n-(-1))\\
x(0) &= x(0)\,\delta(n)\\
x(1) &= x(1)\,\delta(n-1)\\
x(2) &= x(2)\,\delta(n-2)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ x(n) &= \cdots + x(-2) + x(-1) + x(0) + x(1) + x(2) + \cdots\cdots\\
&= \cdots + x(-2)\,\delta(n-(-2)) + x(-1)\,\delta(n-(-1)) + x(0)\,\delta(n) + x(1)\,\delta(n-1)\\
&\quad + x(2)\,\delta(n-2) + \cdots\cdots\\
&= \sum_{m=-\infty}^{+\infty} x(m)\,\delta(n-m) \qquad \text{.....(2.32)}
\end{aligned}
$$

In equation (2.32) each product $x(m)\,\delta(n - m)$ is an impulse and the summation of impulses gives the sequence x(n).

2.9.2 Response of LTI Discrete Time System Using Discrete Convolution

In an LTI system, the response y(n) of the system for an arbitrary input x(n) is given by convolution of input x(n) with impulse response h(n) of the system. It is expressed as,

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m) \qquad \text{.....(2.33)}$$

where, the symbol $*$ represents convolution operation.

<u>*Proof :*</u>

Let y(n) be the response of system $\mathcal{H}$ for an input x(n)

$$\backslash\ y(n) = \mathcal{H}\{x(n)\} \qquad .....(2.34)$$

From equation (2.32) we know that the signal x(n) can be expressed as a summation of impulses,

$$\text{i.e., } x(n) = \sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m) \qquad .....(2.35)$$

where, d(n – m) is the delayed unit impulse signal.

From equations (2.34) and (2.35) we get,

$$y(n) = \mathcal{H}\left\{\sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m)\right\} \qquad .....(2.36)$$

The system $\mathcal{H}$ is a function of n and not a function of m. Hence by linearity property the equation (2.36) can be written as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\ \mathcal{H}\{\delta(n-m)\} \qquad .....(2.37)$$

Let the response of the LTI system to the unit impulse input d(n) be denoted by h(n),

$$\backslash\ h(n) = \mathcal{H}\{d(n)\}$$

Then by time invariance property the response of the system to the delayed unit impulse input d(n – m) is given by,

$$h(n-m) = \mathcal{H}\{d(n-m)\} \qquad .....(2.38)$$

Using equation (2.38), the equation (2.37) can be expressed as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\ h(n-m)$$

The above equation represents the convolution of input x(n) with the impulse response h(n) to yield the output y(n). Hence it is proved that the response y(n) of LTI discrete time system for an arbitrary input x(n) is given by convolution of input x(n) with impulse response h(n) of the system.

2.9.3 Properties of Linear Convolution

The Discrete or Linear convolution will satisfy the following properties.

Commutative property : $x_1(n) * x_2(n) = x_2(n) * x_1(n)$

Associative property : $[x_1(n) * x_2(n)] * x_3(n) = x_1(n) * [x_2(n) * x_3(n)]$

Distributive property : $x_1(n) * [x_2(n) + x_3(n)] = [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$

<u>*Proof of Commutative Property :*</u>

Consider convolution of $x_1(n)$ and $x_2(n)$.

By commutative property we can write,

$$\underset{\text{(LHS)}}{x_1(n) * x_2(n)} = \underset{\text{(RHS)}}{x_2(n) * x_1(n)}$$

$$\text{LHS} = x_1(n) * x_2(n)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\ x_2(n-m) \qquad .....(2.39)$$

where, m is a dummy variable used for convolution operation.

Let, $n - m = p$ | when $m = -\infty$, $p = n - m = n + \infty = +\infty$

$\therefore\ m = n - p$ | when $m = +\infty$, $p = n - m = n - \infty = -\infty$

On replacing m by (n – p) and (n – m) by p in equation (2.39) we get,

$$\text{LHS} = \sum_{p=-\infty}^{+\infty} x_1(n-p)\, x_2(p) = \sum_{p=-\infty}^{+\infty} x_2(p)\, x_1(n-p)$$

$$= x_2(n) * x_1(n)$$

p is a dummy variable used for convolution operation.

$$= \text{RHS}$$

Proof of Associative Property :

Consider the discrete time signals $x_1(n)$, $x_2(n)$ and $x_3(n)$.

Let, $y_1(n) = x_1(n) * x_2(n)$ .....(2.40)

Let us replace n by p

$$\therefore\ y_1(p) = x_1(p) * x_2(p)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m) \qquad .....(2.41)$$

Let, $y_2(n) = x_2(n) * x_3(n)$ .....(2.42)

$$\therefore\ y_2(n) = \sum_{q=-\infty}^{+\infty} x_1(q)\, x_2(n-q)$$

$$\therefore\ y_2(n-m) = \sum_{q=-\infty}^{+\infty} x_1(q)\, x_2(n-q-m) \qquad .....(2.43)$$

where p, m and q are dummy variables used for convolution operation.

By associative property we can write,

$$\underbrace{[x_1(n) * x_2(n)] * x_3(n)}_{\text{LHS}} = \underbrace{x_1(n) * [x_2(n) * x_3(n)]}_{\text{RHS}}$$

$$\text{LHS} = [x_1(n) * x_2(n)] * x_3(n)$$

$$= y_1(n) * x_3(n) \qquad \boxed{\text{Using equation (2.40)}}$$

$$= \sum_{p=-\infty}^{+\infty} y_1(p)\, x_3(n-p)$$

$$= \sum_{p=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m)\, x_3(n-p) \qquad \boxed{\text{Using equation (2.41)}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m) \sum_{p=-\infty}^{+\infty} x_2(p-m)\, x_3(n-p) \qquad .....(2.44)$$

Let, $p - m = q$ | when $p = -\infty$, $q = p - m = -\infty - m = -\infty$

$\therefore\ p = q + m$ | when $p = +\infty$, $q = p - m = +\infty - m = +\infty$

On replacing (p – m) by q, and p by (q + m) in the equation (2.44) we get,

$$\text{LHS} = \sum_{m=-\infty}^{+\infty} x_1(m) \sum_{q=-\infty}^{+\infty} x_2(q)\, x_3(n-q-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, y_2(n-m) \qquad \boxed{\text{Using equation (2.43)}}$$

$$= x_1(n) * y_2(n)$$

$$= x_1(n) * [x_2(n) * x_3(n)] \qquad \boxed{\text{Using equation (2.42)}}$$

$$= \text{RHS}$$

Proof of Distributive Property :

Consider the discrete time signals $x_1(n)$, $x_2(n)$ and $x_3(n)$. By distributive property we can write,

$$\underbrace{x_1(n) * [x_2(n) + x_3(n)]}_{\text{LHS}} = \underbrace{[x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]}_{\text{RHS}}$$

$$\text{LHS} = x_1(n) * [x_2(n) + x_3(n)]$$

$$= x_1(n) * x_4(n) \qquad \boxed{x_4(n) = x_2(n) + x_3(n)}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_4(n-m) \qquad \boxed{\text{m is a dummy variable used for convolution operation.}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, [x_2(n-m) + x_3(n-m)] \qquad \boxed{x_4(n-m) = x_2(n-m) + x_3(n-m)}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) + \sum_{m=-\infty}^{+\infty} x_1(m)\, x_3(n-m)$$

$$= [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$$

$$= \text{RHS}$$

2.9.4 Interconnections of Discrete Time Systems

Smaller discrete time systems may be interconnected to form larger systems. Two possible basic ways of interconnection are ***cascade connection*** and ***parallel connection***. The cascade and parallel connections of two discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ are shown in fig 2.21.

Fig 2.21a : Cascade connection.   *Fig 2.21b : Parallel connection.*

Fig 2.21 : Interconnection of discrete time systems.

Cascade Connected Discrete Time Systems

Two cascade connected discrete time systems with impulse response $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by convolution of individual impulse responses.

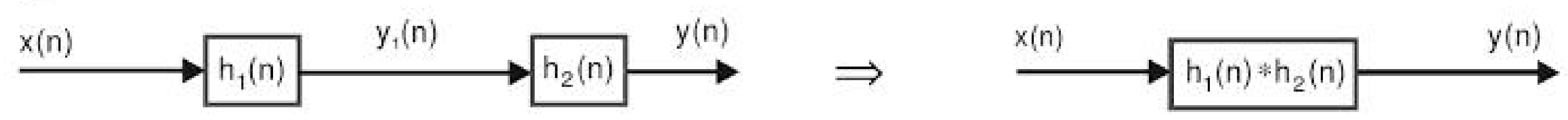


Fig 2.22 : Cascade connected discrete time system and their equivalent.

Proof:

With reference to fig 2.22 we can write,

$$y_1(n) = x(n) * h_1(n) \qquad .....(2.45)$$

$$y(n) = y_1(n) * h_2(n) \qquad .....(2.46)$$

Using equation (2.45), the equation (2.46) can be written as,

$$y(n) = x(n) * h_1(n) * h_2(n)$$

$$= x(n) * [h_1(n) * h_2(n)]$$

$$= x(n) * h(n) \qquad .....(2.47)$$

where, $h(n) = h_1(n) * h_2(n)$

From equation (2.47) we can say that the overall impulse response of two cascaded discrete time systems is given by convolution of individual impulse responses.

Parallel Connected Discrete Time Systems

Two parallel connected discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by sum of individual impulse responses.

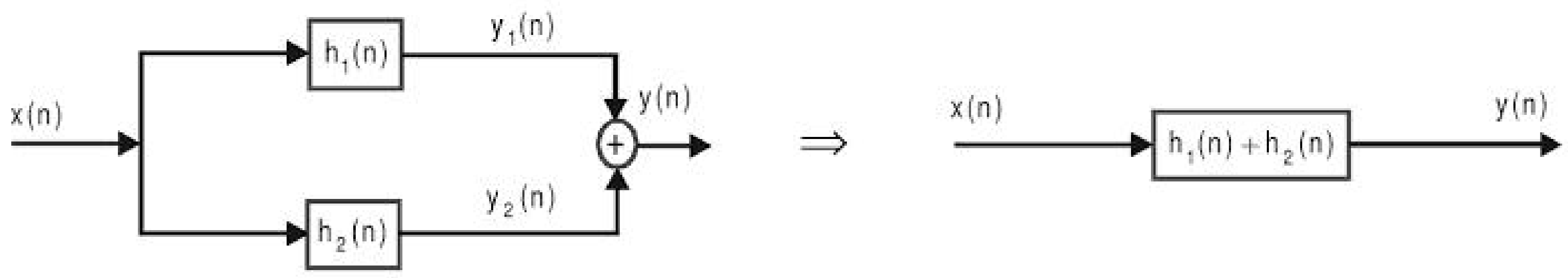


Fig 2.23 : Parallel connected discrete time systems and their equivalent.

Proof:

With reference to fig 2.23 we can write,

$$y_1(n) = x(n) * h_1(n) \quad .....(2.48)$$

$$y_2(n) = x(n) * h_2(n) \quad .....(2.49)$$

$$y(n) = y_1(n) + y_2(n) \quad .....(2.50)$$

On substituting for $y_1(n)$ and $y_2(n)$ from equations (2.48) and (2.49) in equation (2.50) we get,

$$y(n) = [\,x(n) * h_1(n)] + [\,x(n) * h_2(n)] \quad .....(2.51)$$

By using distributive property of convolution, the equation (2.51) can be written as shown below,

$$y(n) = x(n) * [h_1(n) + h_2(n)]$$

$$= x(n) * h(n) \quad .....(2.52)$$

where, $h(n) = h_1(n) + h_2(n)$

From equation (2.52) we can say that the overall impulse response of two parallel connected discrete time systems is given by sum of individual impulse responses.

Example 2.20

Determine the impulse response for the cascade of two LTI systems having impulse responses,

$h_1(n) = \left(\frac{2}{5}\right)^n u(n)$ and $h_2(n) = \left(\frac{1}{5}\right)^n u(n)$.

Solution

Let h(n) be the impulse response of cascade system. Now h(n) is given by convolution of $h_1(n)$ and $h_2(n)$.

$$\therefore\ h(n) = h_1(n) * h_2(n) = \sum_{m=-\infty}^{+\infty} h_1(m)\, h_2(n-m)$$

where, m is a dummy variable used for convolution operation

$$h_1(m) = \left(\frac{2}{5}\right)^m \ ; \quad h_2(m) = \left(\frac{1}{5}\right)^m \ ; \quad h_2(n-m) = \left(\frac{1}{5}\right)^{n-m}$$

The product $h_1(m)\, h_2(n-m)$ will be nonzero in the range 0 £ m £ n. Therefore the summation index in the above equation is changed to m = 0 to n.

$$\therefore\ h(n) = \sum_{m=0}^{n} h_1(m)\, h_2(n-m) = \sum_{m=0}^{n} \left(\frac{2}{5}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{2}{5}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m} = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{2}{5}\right)^m 5^m$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{2 \times 5}{5}\right)^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} 2^m$$

$$= \left(\frac{1}{5}\right)^n \left(\frac{2^{n+1}-1}{2-1}\right) = \left(\frac{1}{5}\right)^n (2^{n+1}-1) \ ; \quad \text{for } n \geq 0$$

$$= \left(\frac{1}{5}\right)^n (2^{n+1}-1)\, u(n) \ ; \quad \text{for all } n$$

Finite geometric series sum formula

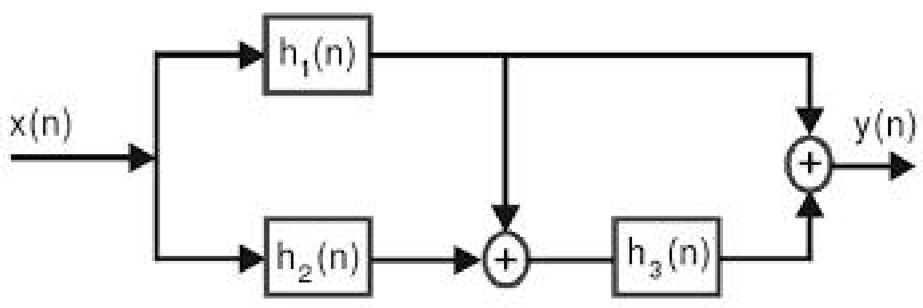
$$\sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1}$$

Using finite geometric series sum formula.

Example 2.21

Determine the overall impulse response of the interconnected discrete time systems shown below,

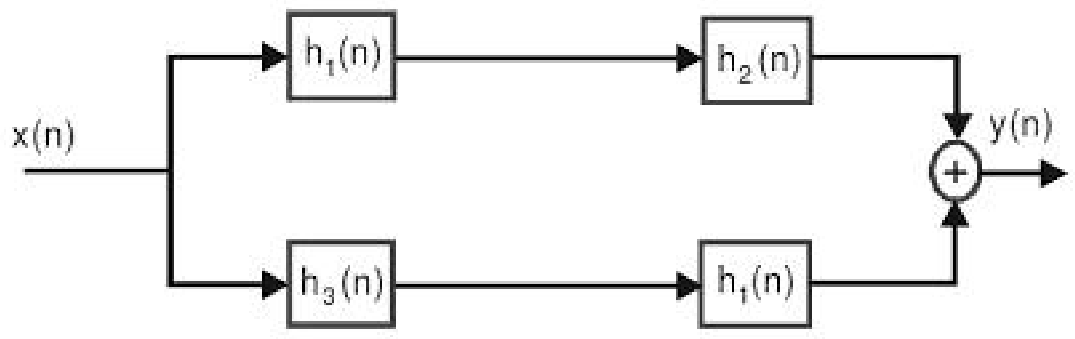
a)



$$h_1(n) = \left(\frac{1}{3}\right)^n u(n); \quad h_2(n) = \left(\frac{1}{2}\right)^n u(n);$$

$$h_3(n) = \left(\frac{1}{5}\right)^n u(n)$$

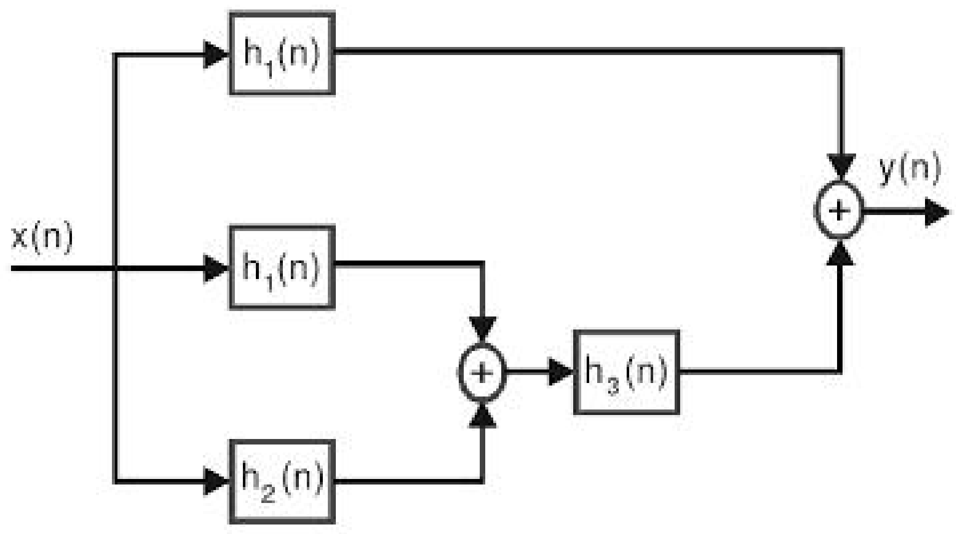
b)



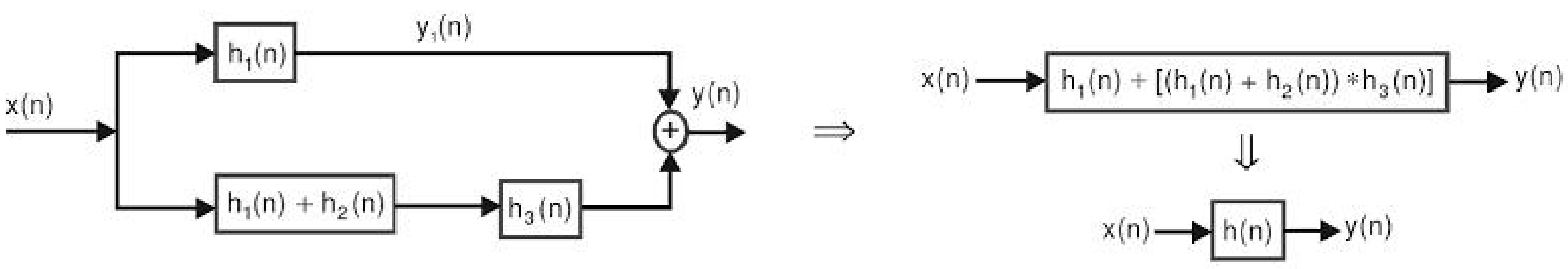
$$h_1(n) = a^n u(n) \quad ; \quad h_2(n) = \delta(n-1) \quad ; \quad h_3(n) = \delta(n-2)$$

Solution

a) The given system can be redrawn as shown below.



The above system can be reduced to single equivalent system as shown below.



Here, $h(n) = h_1(n) + [(h_1(n) + h_2(n)) * h_3(n)]$

$$= h_1(n) + [h_1(n) * h_3(n)] + [h_2(n) * h_3(n)]$$

Using distributive property.

Let us evaluate the convolution of $h_1(n)$ and $h_3(n)$.

$$h_1(n) * h_3(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_3(n-m)$$

The product of $h_1(m)\, h_3(n-m)$ will be nonzero in the range $0 \le m \le n$. Therefore the summation index in the above equation can be changed to m = 0 to n.

$$\therefore \; h_1(n) * h_3(n) = \sum_{m=0}^{n} h_1(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{3}\right)^m$$

$$\therefore\ h_1(n) * h_3(n) = \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^{n+1} - 1}{\frac{5}{3} - 1}$$

Using finite geometric series sum formula.

Finite geometric series sum formula
$$\sum_{m=0}^{N} C^m = \frac{C^{N+1} - 1}{C - 1}$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^n \frac{5}{3} - 1}{\frac{5-3}{3}} = \left(\frac{1}{5}\right)^n \left[\frac{3}{2}\left(\frac{5}{3}\right)^n \frac{5}{3} - \frac{3}{2}\right]$$

$$= \frac{5}{2}\left(\frac{1}{5}\right)^n \left(\frac{5}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n = \frac{5}{2}\left(\frac{1}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n \ ; \text{ for } n \geq 0$$

$$= \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) \ ; \text{ for all } n$$

Let us evaluate the convolution of $h_2(n)$ and $h_3(n)$.

$$h_2(n) * h_3(n) = \sum_{m=-\infty}^{+\infty} h_2(m)\, h_3(n-m)$$

The product of $h_2(m)$ and $h_3(n-m)$ will be nonzero in the range 0 £ m £ n. Therefore the summation index in the above equation can be change to m = 0 to n.

$$\therefore\ h_2(n) * h_3(n) = \sum_{m=0}^{n} h_2(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

Finite geometric series sum formula
$$\sum_{m=0}^{N} C^m = \frac{C^{N+1} - 1}{C - 1}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{2}\right)^m$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^{n+1} - 1}{\frac{5}{2} - 1}$$

Using finite geometric series sum formula.

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^n \frac{5}{2} - 1}{\frac{5-2}{2}} = \left(\frac{1}{5}\right)^n \left[\frac{2}{3}\left(\frac{5}{2}\right)^n \frac{5}{2} - \frac{2}{3}\right]$$

$$= \frac{5}{3}\left(\frac{1}{5}\right)^n \left(\frac{5}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n = \frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n \text{ for } n \geq 0$$

$$= \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n) \text{ for all } n$$

Now, the overall impulse response h(n) is given by,

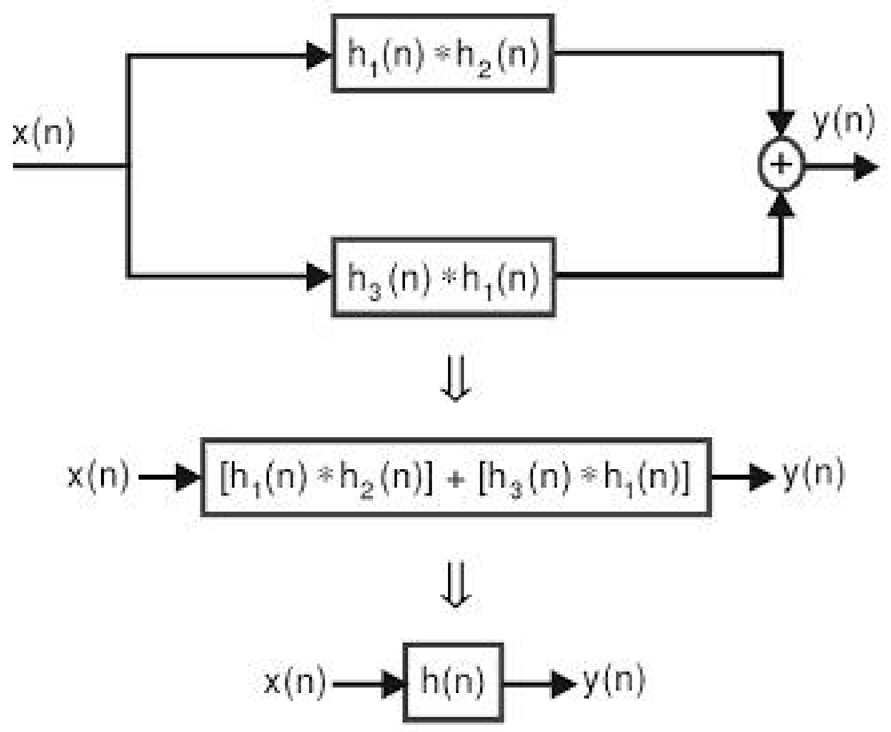
$$h(n) = h_1(n) + \left[h_1(n) * h_3(n)\right] + \left[h_2(n) * h_3(n)\right]$$

$$= \left(\frac{1}{3}\right)^n u(n) + \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n)$$

$$= \left(1 + \frac{5}{2}\right)\left(\frac{1}{3}\right)^n u(n) - \left(\frac{3}{2} + \frac{2}{3}\right)\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n)$$

$$= \left[\frac{7}{2}\left(\frac{1}{3}\right)^n - \frac{13}{6}\left(\frac{1}{5}\right)^n + \frac{5}{3}\left(\frac{1}{2}\right)^n\right] u(n)$$

b) The given system can be reduced to single equivalent system as shown below.



Here, $h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$

Let us evaluate the convolution of $h_1(n)$ and $h_2(n)$.

$$h_1(n) * h_2(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_2(n-m)$$

$$= \sum_{m=-\infty}^{\infty} h_2(m)\, h_1(n-m)$$ Using commutative property.

$$= \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{-m}$$

The product of $\delta(m-1)$ and a^{-m} in the above equation will be nonzero only when $m = 1$.

$$\therefore\ h_1(n) * h_2(n) = a^n a^{-1} = a^{n-1} \quad ; \text{ for } n \geq 1$$

$$= a^{n-1} u(n-1) \quad ; \text{ for all n.}$$

Let us evaluate the convolution of $h_3(n)$ and $h_1(n)$.

$$h_3(n) * h_1(n) = \sum_{m=-\infty}^{\infty} h_3(m)\, h_1(n-m)$$

$$= \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{-m}$$

The product of $\delta(m-2)$ and a^{-m} in the above equation will be nonzero only when $m = 2$.

$$\therefore\ h_1(n) * h_2(n) = a^n a^{-2} = a^{n-2} \quad ; \text{ for } n \geq 2$$

$$= a^{n-2} u(n-2) \quad ; \text{ for all n}$$

Now, the overall impulse response $h(n)$ is given by,

$$h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$$

$$= a^{(n-1)} u(n-1) + a^{(n-2)} u(n-2)$$

2.9.5 Methods of Performing Linear Convolution

Method 1: Graphical Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence.

1. Change the index "n" of input sequences to "m" to get $x_1(m)$ and $x_2(m)$.
2. Sketch the graphical representation of the input sequences $x_1(m)$ and $x_2(m)$.
3. Let us fold $x_2(m)$ to get $x_2(-m)$. Sketch the graphical representation of the folded sequence $x_2(-m)$.
4. Shift the folded sequence $x_2(-m)$ to the left graphically so that the product of $x_1(m)$ and shifted $x_2(-m)$ gives only one nonzero sample. Now multiply $x_1(m)$ and shifted $x_2(-m)$ to get a product sequence, and then sum up the samples of product sequence, which is the first sample of output sequence.
5. To get the next sample of output sequence, shift $x_2(-m)$ of previous step to one position right and multiply the shifted sequence with $x_1(m)$ to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
2. To get subsequent samples of output sequence, the step 5 is repeated until we get a nonzero product sequence.

Method 2: Tabular Method

The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence, folded and shifted sequence is represented by a row in a table.

Method 3: Matrix Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence. In matrix method one of the sequences is represented as a row and the other as a column as shown below.

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence $x_3(n)$. (The sum of the diagonal elements are shown below for reference).

	....	$x_2(0)$	$x_2(1)$	$x_2(2)$	$x_2(3)$	....
⋮		⋮	⋮	⋮	⋮	
$x_1(0)$	....	$x_1(0)\,x_2(0)$	$x_1(0)\,x_2(1)$	$x_1(0)\,x_2(2)$	$x_1(0)\,x_2(3)$	....
$x_1(1)$	....	$x_1(1)\,x_2(0)$	$x_1(1)\,x_2(1)$	$x_1(1)\,x_2(2)$	$x_1(1)\,x_2(3)$	....
$x_1(2)$	....	$x_1(2)\,x_2(0)$	$x_1(2)\,x_2(1)$	$x_1(2)\,x_2(2)$	$x_1(2)\,x_2(3)$	....
$x_1(3)$	....	$x_1(3)\,x_2(0)$	$x_1(3)\,x_2(1)$	$x_1(3)\,x_2(2)$	$x_1(3)\,x_2(3)$	....
⋮		⋮	⋮	⋮	⋮	

$$\vdots$$

$$x_3(0) = \cdots + x_1(0)\,x_2(0) + \cdots$$

$$x_3(1) = \cdots + x_1(1)\,x_2(0) + x_1(0)\,x_2(1) + \cdots$$

$$x_3(2) = \cdots + x_1(2)\,x_2(0) + x_1(1)\,x_2(1) + x_1(0)\,x_2(2) + \cdots$$

$$x_3(3) = \cdots + x_1(3)\,x_2(0) + x_1(2)\,x_2(1) + x_1(1)\,x_2(2) + x_1(0)\,x_2(3) + \cdots$$

$$\vdots$$

$$\backslash \;\; x_3(n) = \{\ldots\ x_3(0), x_3(1), x_3(2), x_3(3), \ldots\}$$

Example 2.22

Determine the response of the LTI system whose input x(n) and impulse response h(n) are given by,

$x(n) = \{\underline{1}, 2, 0.5, 1\}$ and $h(n) = \{1, \underline{2}, 1, -1\}$

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m)$$

In this example the convolution operation is performed by three methods.

The Input sequence starts at n = 0 and the impulse response sequence starts at n = –1. Therefore the output sequence starts at n = 0 + (–1) = –1.

The input and impulse response consists of 4 samples, so the output consists of 4 + 4 – 1 = 7 samples.

Method 1 : Graphical Method

The graphical representation of x(n) and h(n) after replacing n by m are shown below. The sequence h(m) is folded with respect to m = 0 to obtain h(–m).

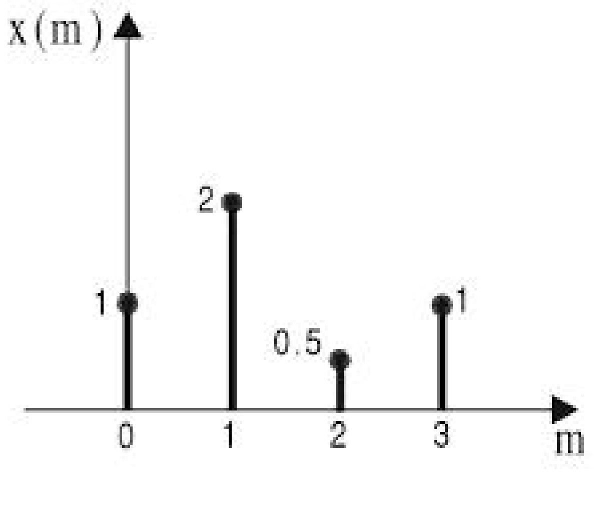


Fig 1 : Input sequence.

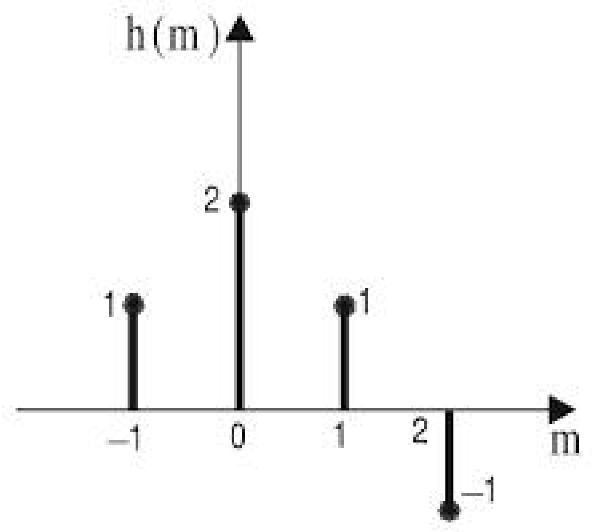


Fig 2 : Impulse response.

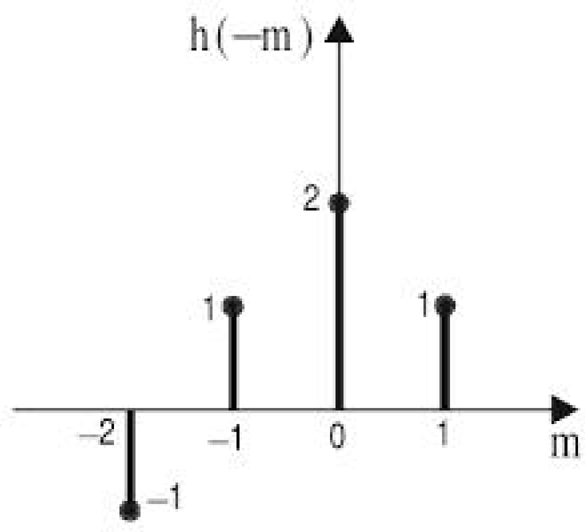


Fig 3 : Folded impulse response.

The samples of y(n) are computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\,h_n(m)\ ;\ \text{where } h_n(m) = h(n-m)$$

The computation of each sample using the above equation are graphically shown in fig 4 to fig 10. The graphical representation of output sequence is shown in fig 11.

When n = –1 ; $y(-1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(-1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_{-1}(m) = \sum_{m=-\infty}^{+\infty} v_{-1}(m)$

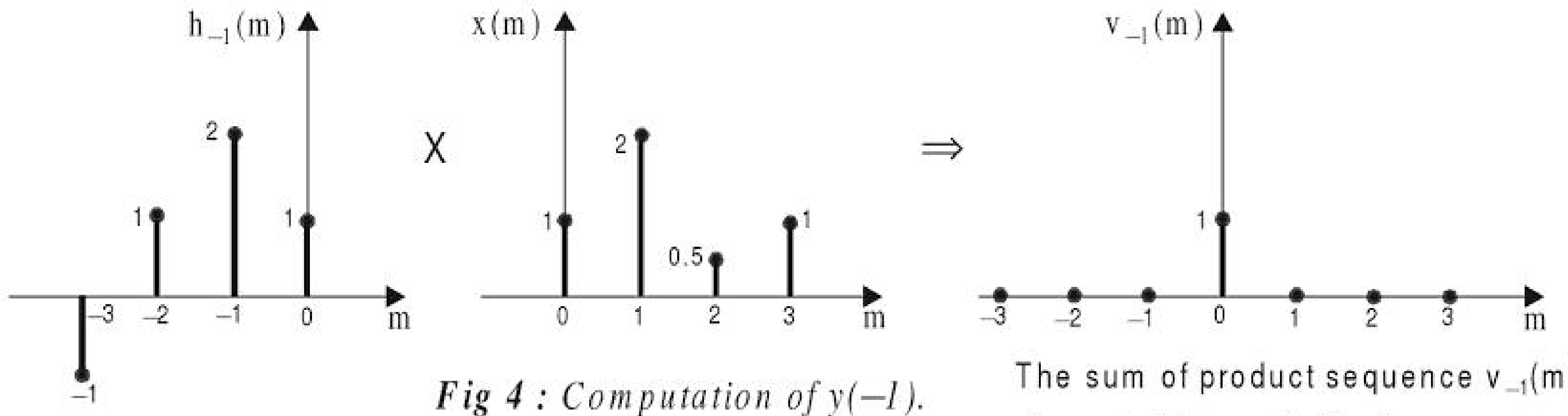


Fig 4 : Computation of y(–1).

The sum of product sequence $v_{-1}(m)$ gives $y(-1)$. $\therefore\ y(-1) = 1$

When n = 0 ; $y(0) = \sum_{m=-\infty}^{+\infty} x(m)\, h(0-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_0(m) = \sum_{m=-\infty}^{+\infty} v_0(m)$

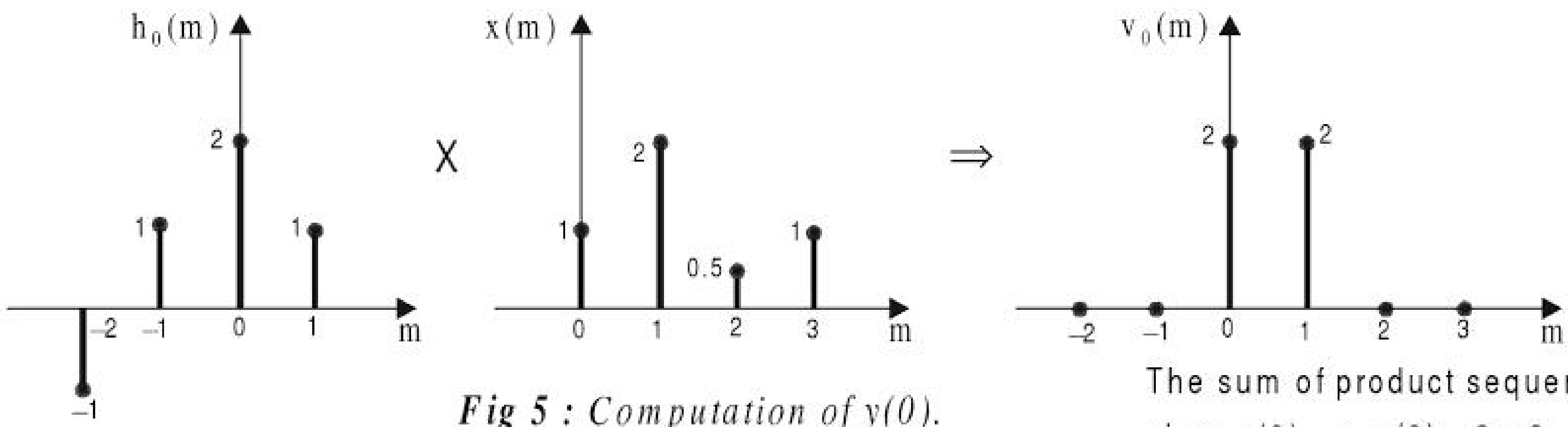


Fig 5 : Computation of y(0).

The sum of product sequence $v_0(m)$ gives $y(0)$. $\therefore\ y(0) = 2 + 2 = 4$

When n = 1 ; $y(1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_1(m) = \sum_{m=-\infty}^{+\infty} v_1(m)$

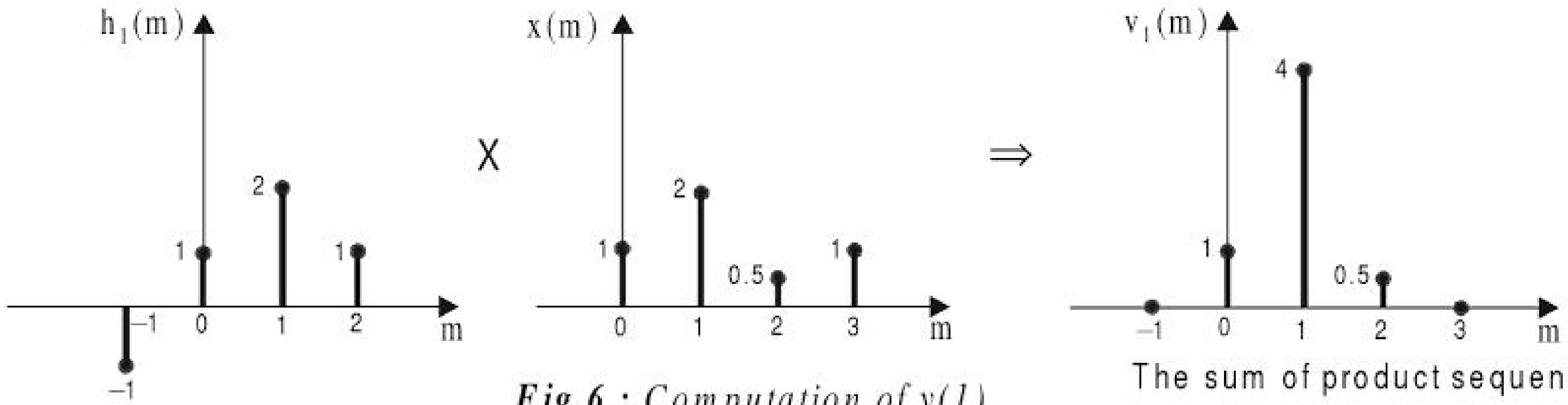


Fig 6 : Computation of y(1).

The sum of product sequence $v_1(m)$ gives $y(1)$. $\therefore\ y(1) = 1 + 4 + 0.5 = 5.5$

When n = 2 ; $y(2) = \sum_{m=-\infty}^{+\infty} x(m)\, h(2-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_2(m) = \sum_{m=-\infty}^{+\infty} v_2(m)$

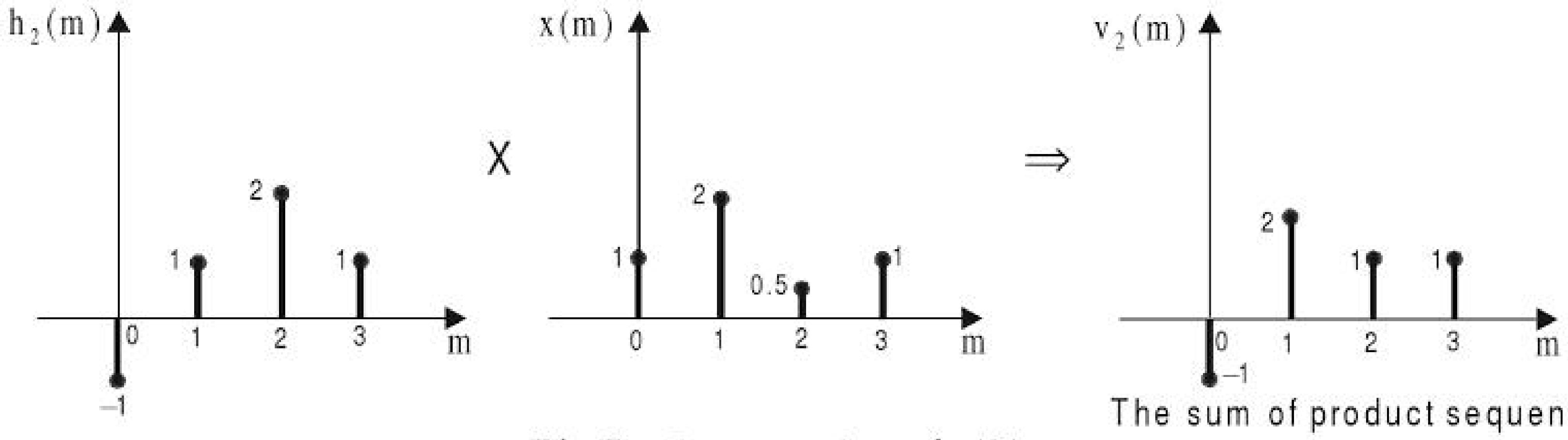


Fig 7 : Computation of y(2).

The sum of product sequence $v_2(m)$ gives $y(2)$. $\therefore\ y(2) = -1 + 2 + 1 + 1 = 3$

When n = 3 ; $y(3) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h(3-m) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h_3(m) = \sum\limits_{m=-\infty}^{+\infty} v_3(m)$

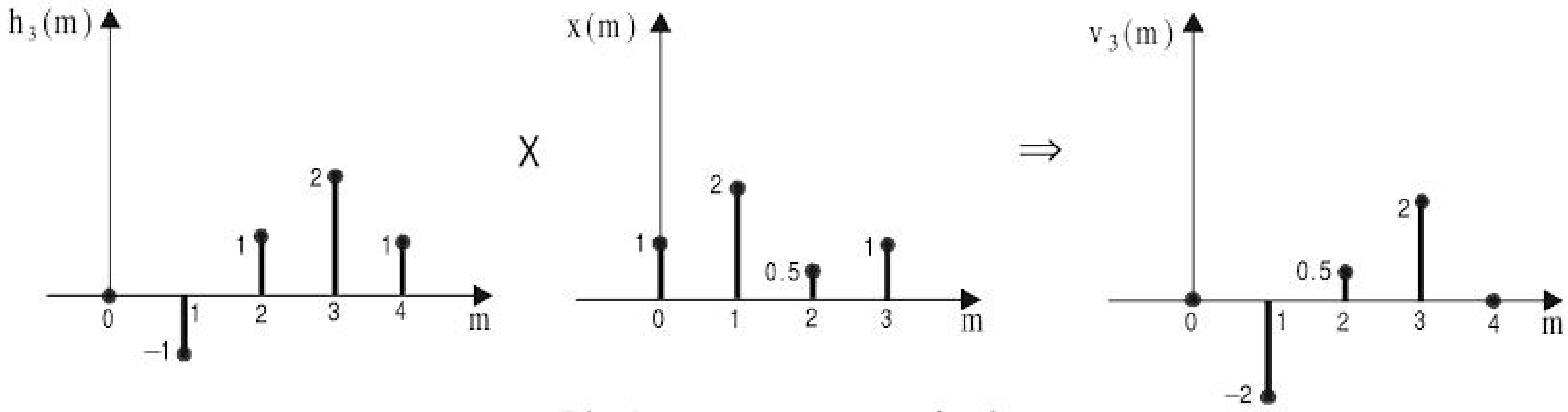


Fig 8 : Computation of y(3).

The sum of product sequence $v_3(m)$ gives y(3). $\therefore$ y(3) = –2 + 0.5 + 2 = 0.5

When n = 4 ; $y(4) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h(4-m) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h_4(m) = \sum\limits_{m=-\infty}^{+\infty} v_4(m)$

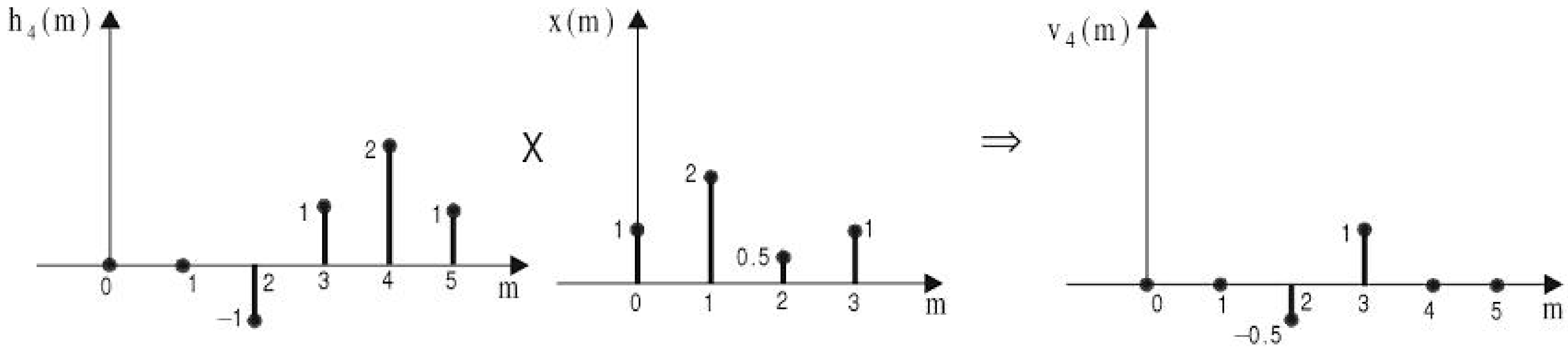


Fig 9 : Computation of y(4).

The sum of product sequence $v_4(m)$ gives y(4). $\therefore$ y(4) = –0.5 + 1 = 0.5

When n = 5 ; $y(5) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h(5-m) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h_5(m) = \sum\limits_{m=-\infty}^{+\infty} v_5(m)$

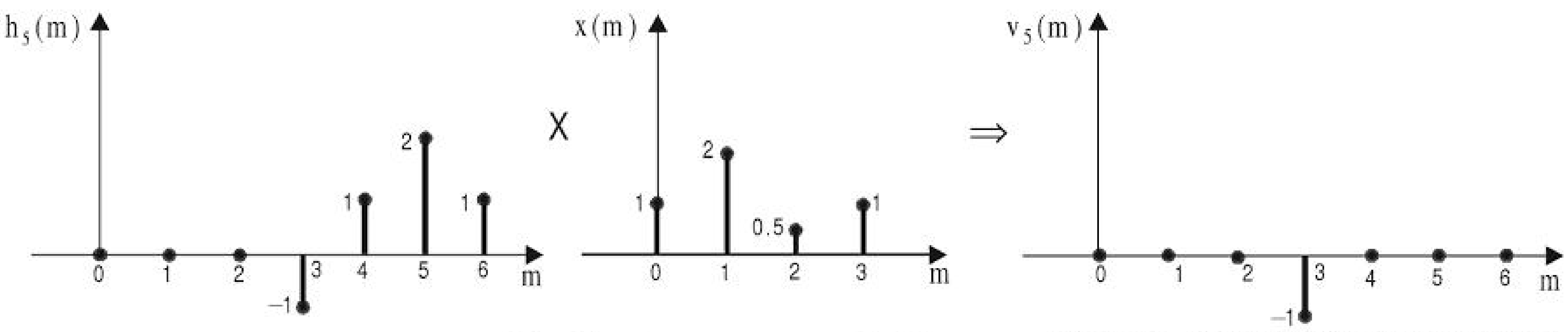


Fig 10 : Computation of y(5).

The sum of product sequence $v_5(m)$ gives y(5). $\therefore$ y(5) = –1

The output sequence, $y(n) = \{1, \underset{\uparrow}{4}, 5.5, 3, 0.5, 0.5, -1\}$

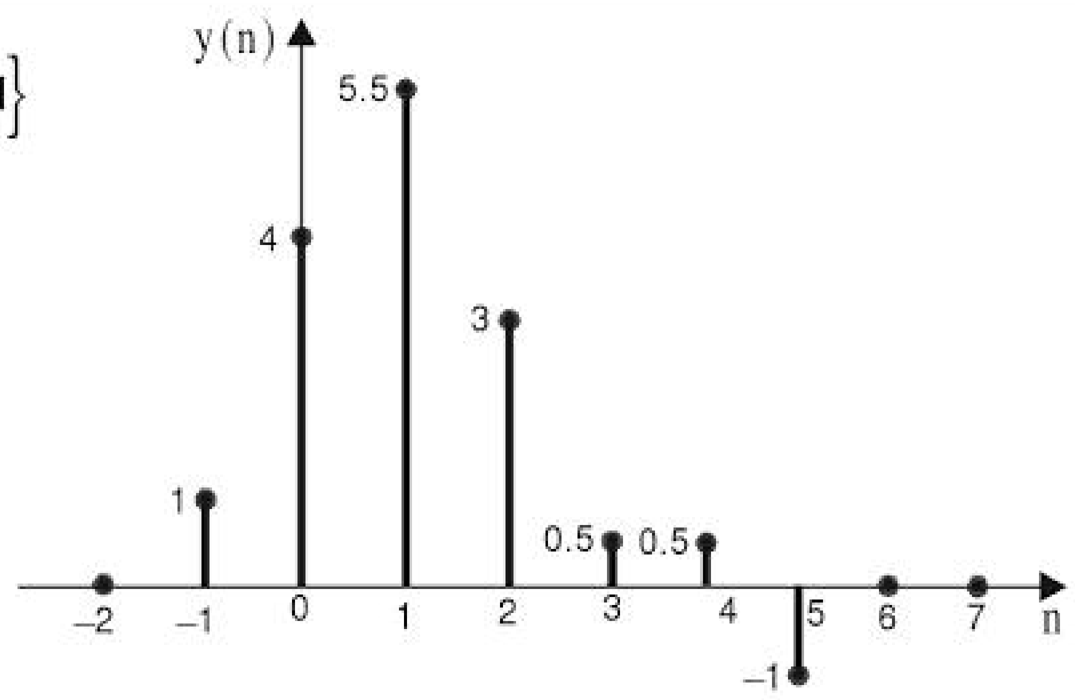


Fig 11 : Graphical representation of y(n).

Method 2 : Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

Note : The unfilled boxes in the table are considered as zeros.

m	–3	–2	–1	0	1	2	3	4	5	6
x(m)				1	2	0.5	1			
h(m)			1	2	1	–1				
h(–m)		–1	1	2	1					
$h(-1-m) = h_{-1}(m)$	–1	1	2	1						
$h(0-m) = h_0(m)$		–1	1	2	1					
$h(1-m) = h_1(m)$			–1	1	2	1				
$h(2-m) = h_2(m)$				–1	1	2	1			
$h(3-m) = h_3(m)$					–1	1	2	1		
$h(4-m) = h_4(m)$						–1	1	2	1	
$h(5-m) = h_5(m)$							–1	1	2	1

Each sample of y(n) is computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\,h_n(m), \quad \text{where } h_n(m) = h(n-m)$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence (i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$). The sum of all the samples of the product sequence gives y(q).

When n = –1 ; $y(-1) = \sum_{m=-3}^{3} x(m)\,h_{-1}(m)$ $\boxed{\because \text{ The product is valid only for } m = -3 \text{ to } +3.}$

$$= x(-3)\,h_{-1}(-3) + x(-2)h_{-1}(-2) + x(-1)h_{-1}(-1) + x(0)\,h_{-1}(0) + x(1)\,h_{-1}(1) + x(2)\,h_{-1}(2) + x(3)\,h_{-1}(3)$$

$$= 0 + 0 + 0 + 1 + 0 + 0 + 0 = 1$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

When n = 0 ; $y(0) = \sum_{m=-2}^{3} x(m)\,h_0(m) = 0 + 0 + 2 + 2 + 0 + 0 = 4$

When n = 1 ; $y(1) = \sum_{m=-1}^{3} x(m)\,h_1(m) = 0 + 1 + 4 + 0.5 + 0 = 5.5$

When n = 2 ; $y(2) = \sum_{m=0}^{3} x(m)\,h_2(m) = -1 + 2 + 1 + 1 = 3$

When n = 3 ; $y(3) = \sum_{m=0}^{4} x(m)\,h_3(m) = 0 - 2 + 0.5 + 2 + 0 = 0.5$

When n = 4 ; $y(4) = \sum_{m=0}^{5} x(m)\,h_4(m) = 0 + 0 - 0.5 + 1 + 0 + 0 = 0.5$

When n = 5 ; $y(5) = \sum_{m=0}^{6} x(m)\,h_5(m) = 0 + 0 + 0 - 1 + 0 + 0 + 0 = -1$

The output sequence, $y(n) = \{1, \underset{\uparrow}{4}, 5.5, 3, 0.5, 0.5, -1\}$

Method 3 : Matrix Method

The input sequence x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two-dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).

x(n) ↓ \ h(n) →	1	2	1	–1
1	1×1	1×2	1×1	$1 \times (-1)$
2	2×1	2×2	2×1	$2 \times (-1)$
0.5	0.5×1	0.5×2	0.5×1	$0.5 \times (-1)$
1	1×1	1×2	1×1	$1 \times (-1)$

$\Rightarrow$

x(n) ↓ \ h(n) →	1	2	1	–1
1	1	2	1	–1
2	2	4	2	–2
0.5	0.5	1	0.5	–0.5
1	1	2	1	–1

$y(-1) = 1$

$y(0) = 2 + 2 = 4$

$y(1) = 0.5 + 4 + 1 = 5.5$

$y(2) = 1 + 1 + 2 + (-1) = 3$

$y(3) = 2 + 0.5 + (-2) = 0.5$

$y(4) = 1 + (-0.5) = 0.5$

$y(5) = -1$

\ $y(n) = \{1, 4, 5.5, 3, 0.5, 0.5, -1\}$

Example 2.23

Determine the output y(n) of a relaxed LTI system with impulse response,

$h(n) = a^n\, u(n)$; where $|a| < 1$ and

When input is a unit step sequence, i.e., x(n) = u(n).

Solution

The graphical representation of x(n) and h(n) after replacing n by m are shown below. Also the sequence x(m) is folded to get x(–m).

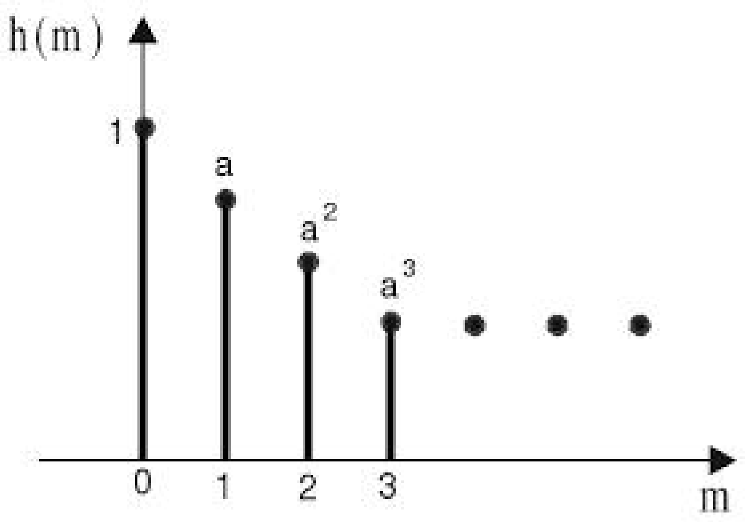


Fig 1 : Impulse response.

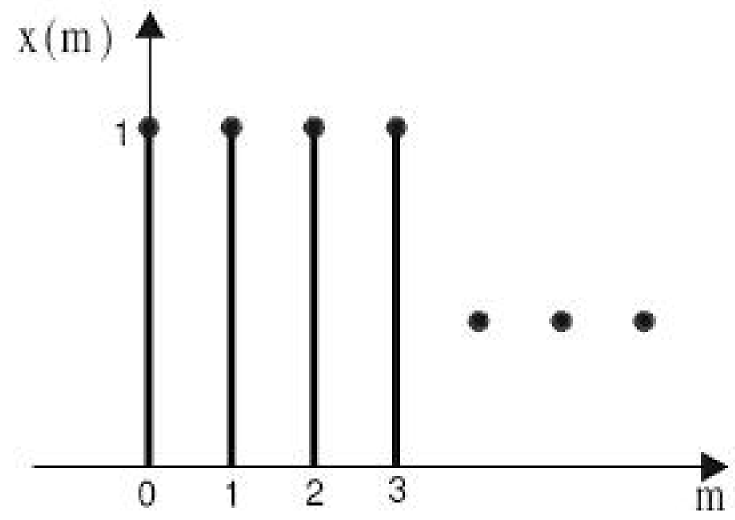


Fig 2 : Impulse sequence.

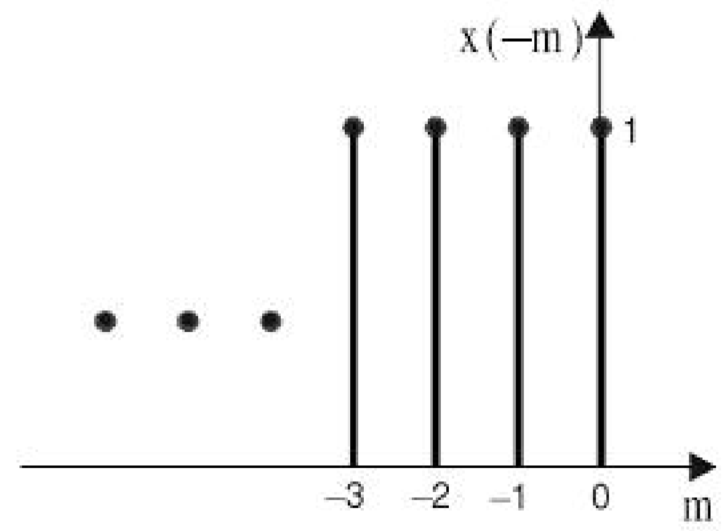


Fig 3 : Folded input sequence.

Here both h(m) and x(m) are infinite duration sequences starting at n = 0. Hence the output sequence y(n) will also be an infinite duration sequence starting at n = 0.

By convolution formula,

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) = \sum_{m=0}^{\infty} h(m)\, x_n(m) \; ; \quad \text{where } x_n(m) = x(n-m)$$

The computation of some samples of y(n) using the above equation are graphically shown below.

When n = 0 ; $y(0) = \sum\limits_{m=0}^{\infty} h(m)\, x(0-m) = \sum\limits_{m=0}^{\infty} h(m)\, x_0(m) = \sum\limits_{m=0}^{\infty} v_0(m)$

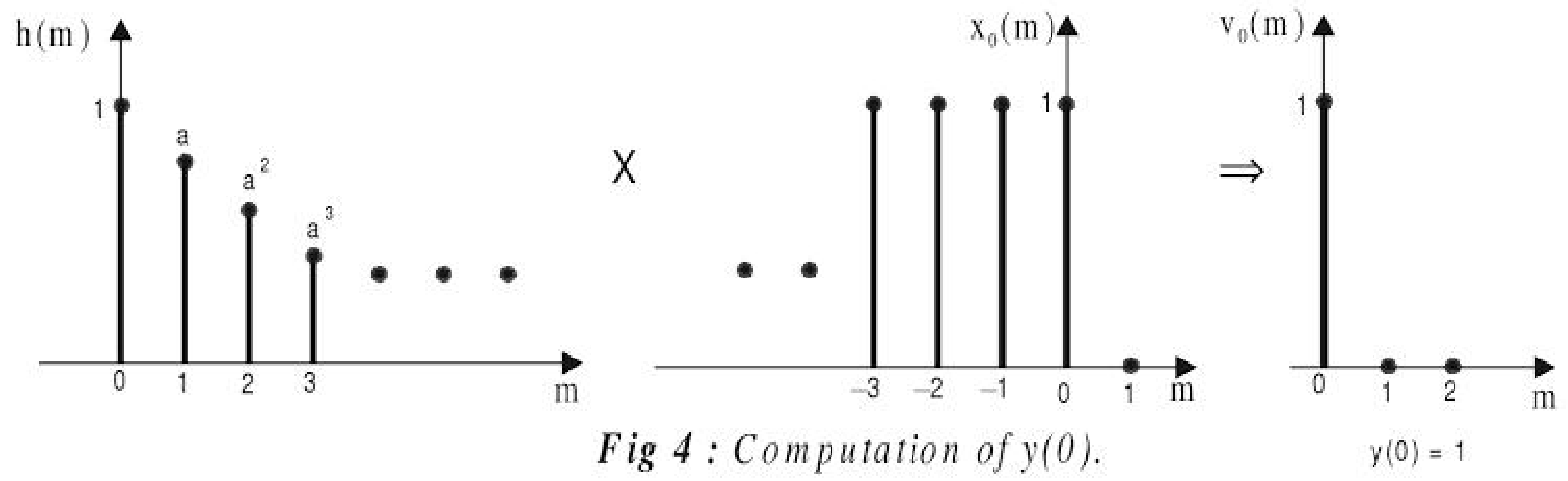


Fig 4 : Computation of y(0).

When n = 1 ; $y(1) = \sum\limits_{m=0}^{\infty} h(m)\, x(1-m) = \sum\limits_{m=0}^{\infty} h(m)\, x_1(m) = \sum\limits_{m=0}^{\infty} v_1(m)$

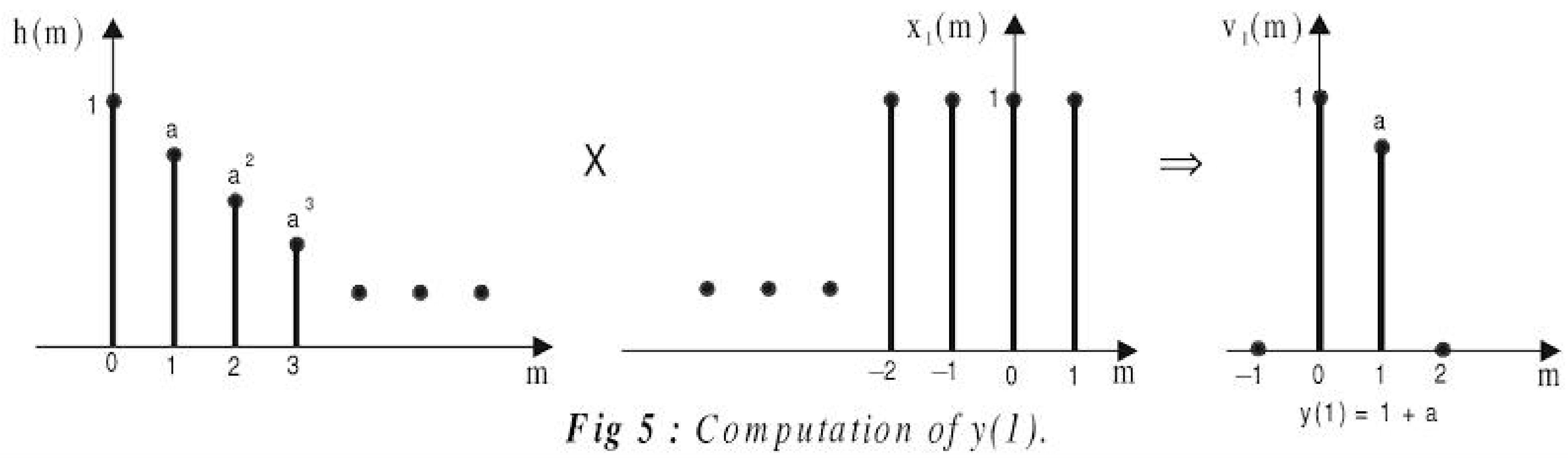


Fig 5 : Computation of y(1).

When n = 2 ; $y(2) = \sum\limits_{m=0}^{\infty} h(m)\, x(2-m) = \sum\limits_{m=0}^{\infty} h(m)\, x_2(m) = \sum\limits_{m=0}^{\infty} v_2(m)$

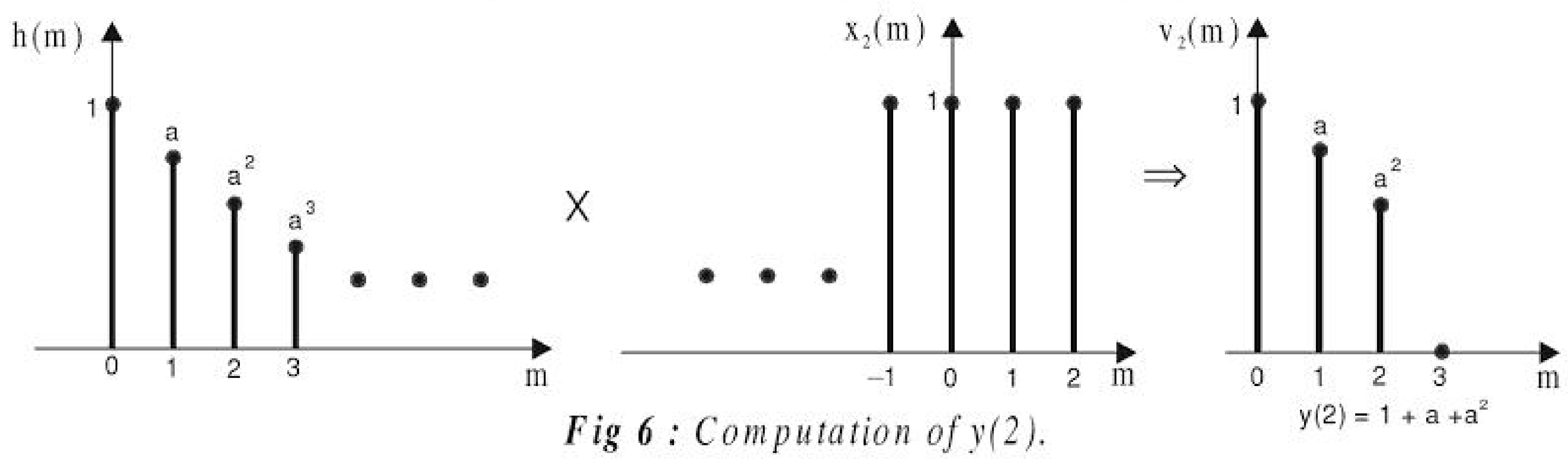


Fig 6 : Computation of y(2).

Solving similarly for other values of n, we can write y(n) for any value of n as shown below.

$$y(n) = 1 + a + a^2 + \ldots\ldots + a^n = \sum_{p=0}^{n} a^p \quad ; \quad \text{for } n \geq 0$$

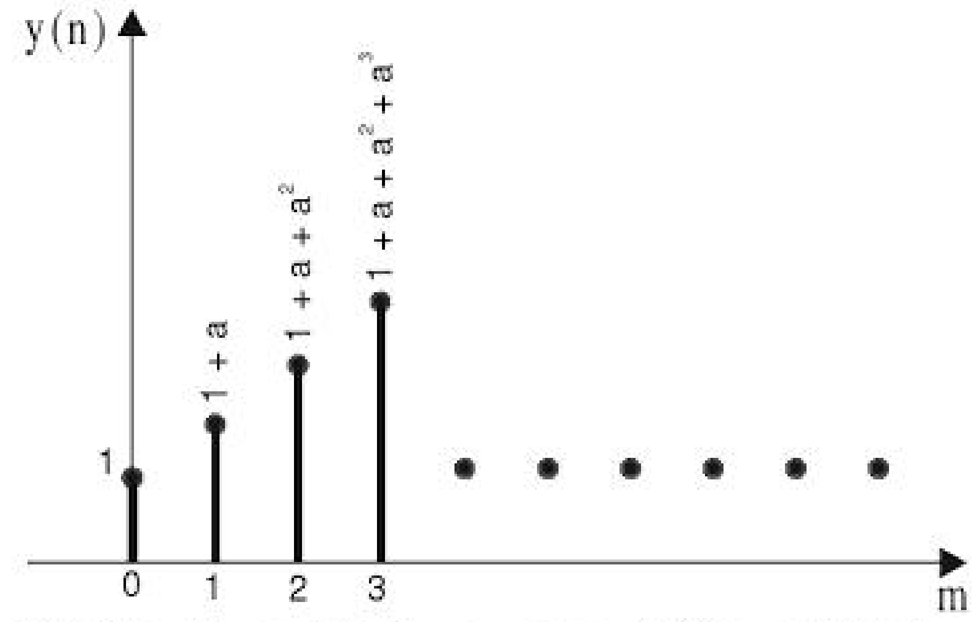


Fig 7 : Graphical representation of y(n).

2.10 Circular Convolution

2.10.1 Circular Representation and Circular Shift of Discrete Time Signal

Consider a finite duration sequence x(n) and its periodic extension $x_p(n)$. The periodic extension of x(n) can be expressed as $x_p(n) = x(n + N)$, where N is the periodicity. Let N = 4. The sequence x(n) and its periodic extension are shown in fig 2.24.

Let, x(n) = 1 ; n = 0
= 2 ; n = 1
= 3 ; n = 2
= 4 ; n = 3

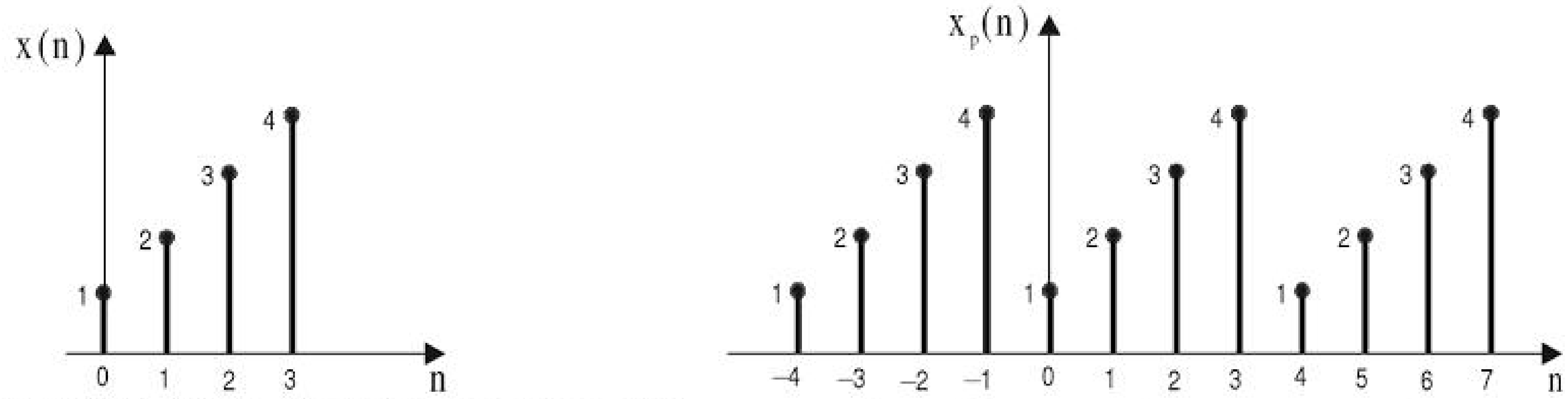


Fig 2.24a : *Finite duration sequence x(n).* **Fig 2.24b :** *Periodic extension of x(n).*

Fig 2.24 : *A finite duration sequence and its periodic extension.*

Let us delay the periodic sequence $x_p(n)$ by two units of time as shown in fig 2.25(a). (For delay the sequence is shifted right). Let us denote one period of this delayed sequence by $x_1(n)$. One period of the delayed sequence is shown in fig 2.25(b).

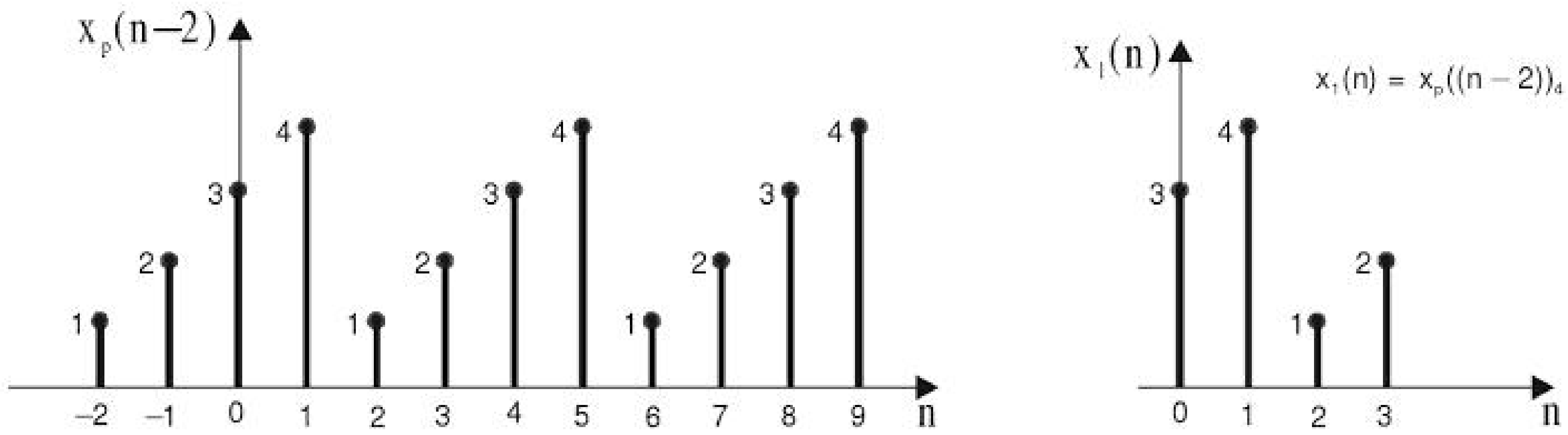


Fig 2.25a: *$x_p(n)$ delayed by two units of time.* **Fig 2.25b:** *One period of $x_p(n-2)$.*

Fig 2.25: *Delayed version of $x_p(n)$.*

The sequence $x_1(n)$ can be represented by $x_p(n-2, (\text{mod } 4))$, or $x_p((n-2))_4$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence x(n) and one period of the delayed sequence $x_1(n)$ are shown below.

$$x_1(n) = x_p(n - 2, (\text{mod } 4)) = x_p((n-2))_4$$

\ When n = 0; $x_1(0) = x_p((0-2))_4 = x_p((-2))_4 = x(2) = 3$

When n = 1; $x_1(1) = x_p((1-2))_4 = x_p((-1))_4 = x(3) = 4$

When n = 2; $x_1(2) = x_p((2-2))_4 = x_p((0))_4 = x(0) = 1$

When n = 3; $x_1(3) = x_p((3-2))_4 = x_p((1))_4 = x(1) = 2$

The periodic sequences $x_p(n)$ and $x_1(n)$ can be represented as points on a circle as shown in fig 2.26. From fig 2.26 we can say that, $x_1(n)$ is simply $x_p(n)$ shifted circularly by two units in time, where the counter clockwise (anticlockwise) direction has been arbitrarily selected for right shift or delay.

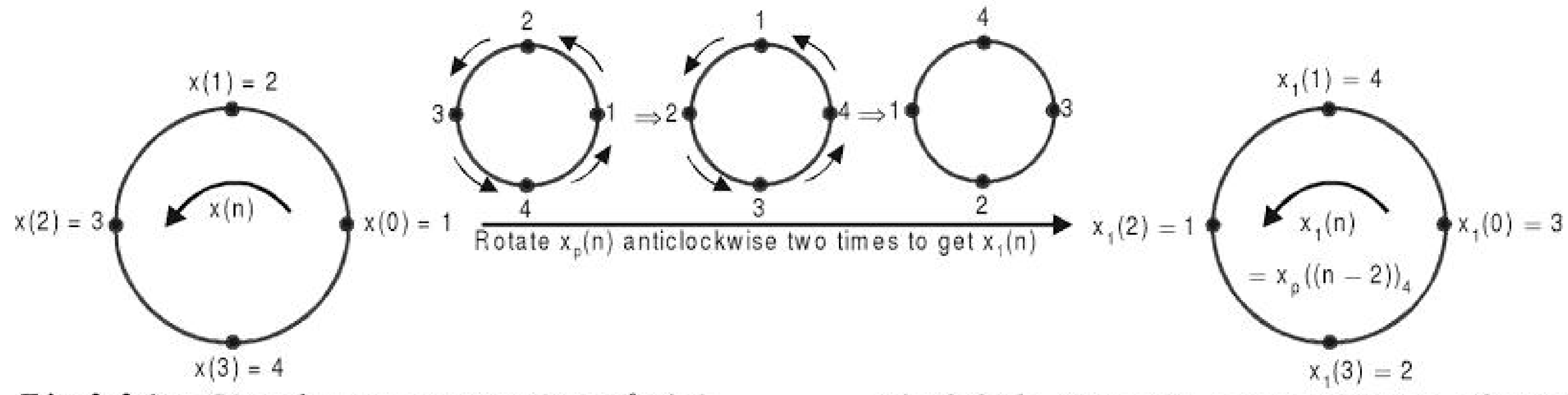


Fig 2.26a: *Circular representation of x(n).* ***Fig 2.26b:*** *Circular representation of $x_1(n)$.*

Fig 2.26: *Circular representation of a signal and its delayed version.*

Let us advance the periodic sequence $x_p(n)$ by three units of time as shown in fig 2.27(a). Let us denote one period of this advanced sequence by $x_2(n)$. One period of the advanced sequence is shown in fig 2.27(b).

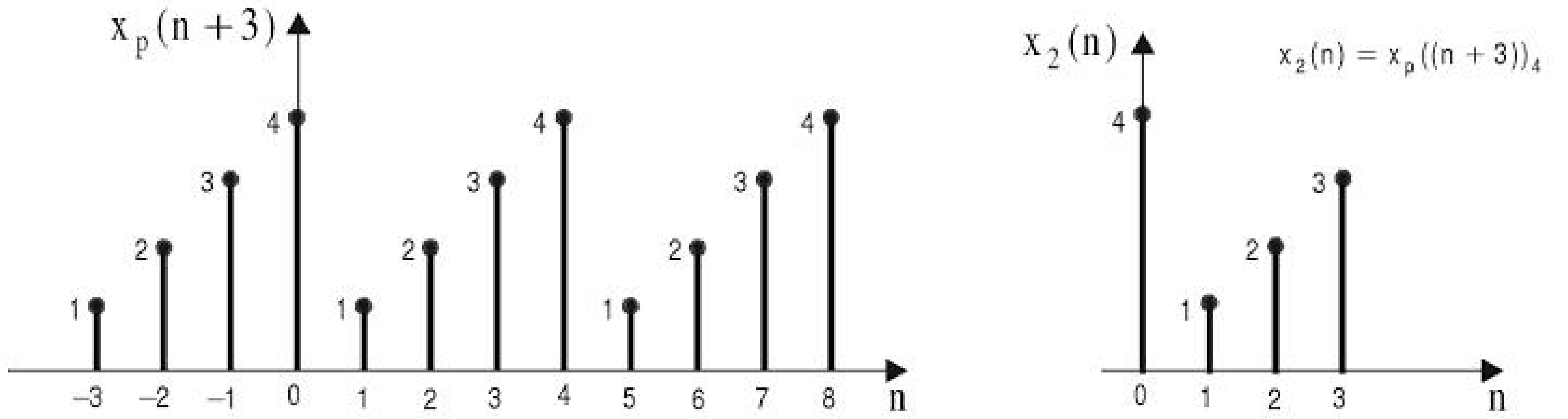


Fig 2.27a: *$x_p(n)$ advanced by three units of time.* ***Fig 2.27b:*** *One period of $x_p(n+3)$.*

Fig 2.27: *Advanced version of $x_p(n)$.*

The sequence $x_2(n)$ can be represented by $x_p(n+3, (\text{mod } 4))$ or $x_p((n+3))_4$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence x(n) and one period of the advanced sequence $x_2(n)$ are shown below.

$$x_2(n) = x_p(n+3, (\text{mod } 4)) = x_p((n+3))_4$$

$$\therefore \text{ When } n = 0;\ x_2(0) = x_p((0+3))_4 = x_p((3))_4 = x(3) = 4$$

$$\text{When } n = 1;\ x_2(1) = x_p((1+3))_4 = x_p((4))_4 = x(0) = 1$$

$$\text{When } n = 2;\ x_2(2) = x_p((2+3))_4 = x_p((5))_4 = x(1) = 2$$

$$\text{When } n = 3;\ x_2(3) = x_p((3+3))_4 = x_p((6))_4 = x(2) = 3$$

The periodic sequences $x_p(n)$ and $x_2(n)$ can be represented as points on a circle as shown in fig 2.28. From fig 2.28 we can say that $x_2(n)$ is simply $x_p(n)$ shifted circularly by three units in time where clockwise direction has been selected for left shift or advance.

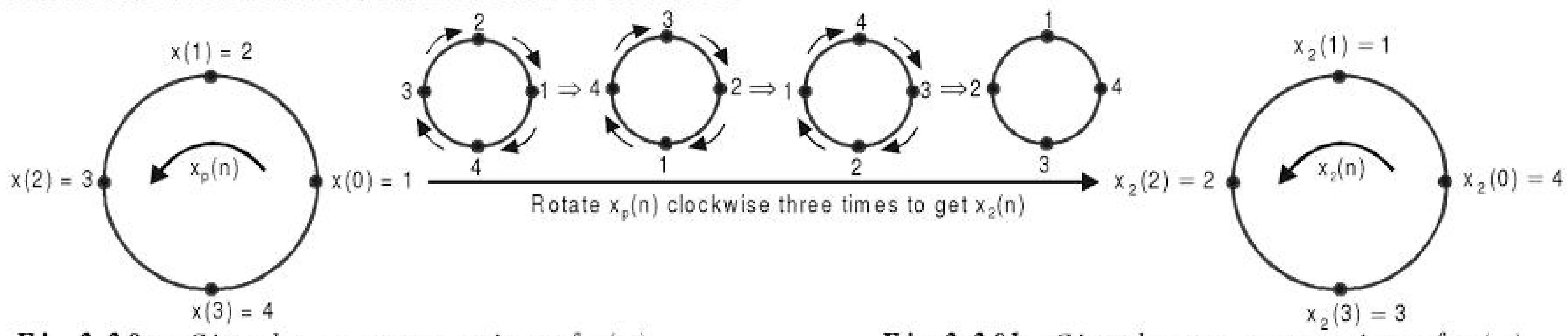


Fig 2.28a: *Circular representation of x(n).* ***Fig 2.28b:*** *Circular representation of $x_2(n)$.*

Fig 2.28: *Circular representation of a signal and its advanced version.*

Thus we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension and viceversa. If a nonperiodic N-point sequence is represented on the circumference of a circle then it becomes a periodic sequence of periodicity N. When the sequence is shifted circularly, the samples repeat after N shifts. This is similar to modulo-N operation. Hence, in general, the circular shift may be represented by the index mod-N. Let x(n) be an N-point sequence represented on a circle and $x'(n)$ be its ***circularly shifted sequence*** by m units of time.

Now, $x'(n) = x(n-m, \text{mod } N) \equiv x((n-m))_N$ ..... (2.53)

When m is positive, the equation (2.53) represents delayed sequence and when m is negative, the equation (2.53) represents advanced sequence.

2.10.2 Circular Symmetries of Discrete Time Signal

The circular representation of a sequence and the resulting periodicity gives rise to new definitions for even symmetry, odd symmetry and the time reversal of the sequence.

An N-point sequence is called even if it is symmetric about the point zero on the circle. This implies that,

$$x(N-n) = x(n) \quad ; \text{ for } \quad 0 \le n \le N-1 \qquad \text{...... (2.54)}$$

An N-point sequence is called odd if it is antisymmetric about the point zero on the circle. This implies that,

$$x(N-n) = -\,x(n) \quad ; \text{ for } \quad 0 \le n \le N-1 \qquad \text{...... (2.55)}$$

The time reversal of a N-point sequence is obtained by reversing its sample about the point zero on the circle. Thus the sequence x(–n, (mod N)) is simply written as,

$$x\,(-n, (\text{mod } N)) = x(N-n) \quad ; \text{ for } \quad 0 \le n \le N-1 \qquad \text{...... (2.56)}$$

This time reversal is equivalent to plotting x(n) in a clockwise direction on a circle, as shown in fig 2.29.

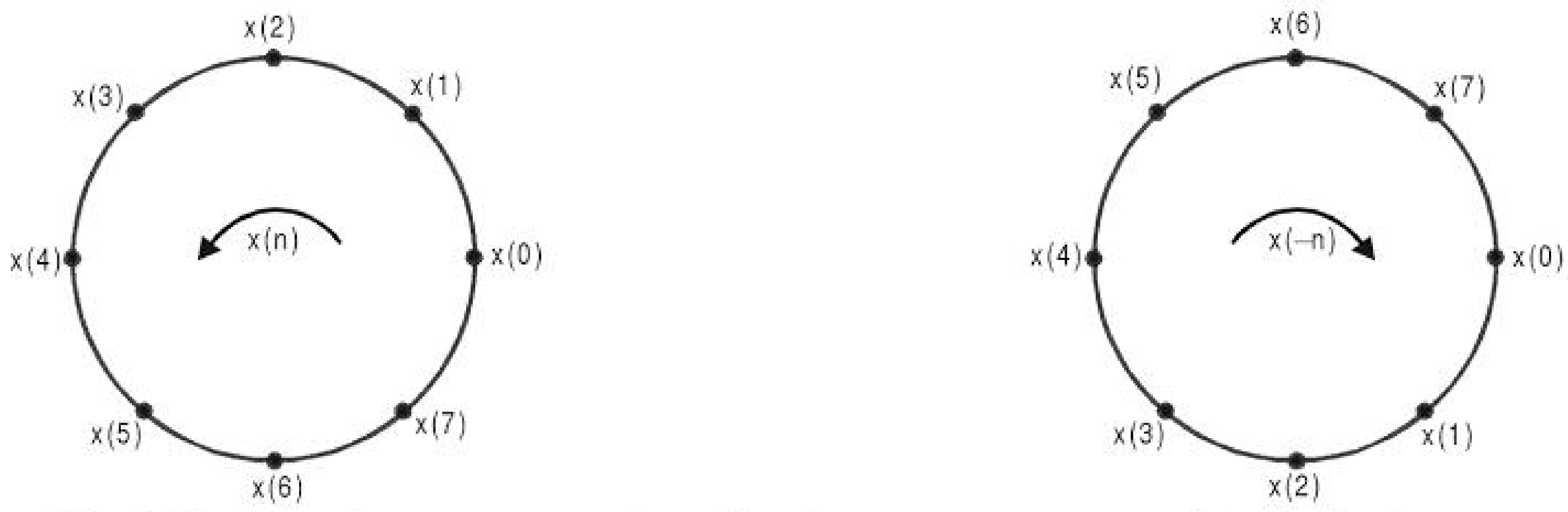


Fig 2.29: *Circular representation of an 8-point sequence and its folded sequence.*

2.10.3 Definition of Circular Convolution

The ***circular convolution*** of two periodic discrete time sequences $x_1(n)$ and $x_2(n)$ with periodicity of N samples is defined as,

$$\boxed{x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=0}^{N-1} x_2(m)\, x_1((n-m))_N} \qquad \text{.....(2.57)}$$

where, $x_3(n)$ is the sequence obtained by circular convolution,

$x_1((n-m))_N$ represents circular shift of $x_1(n)$

$x_2((n-m))_N$ represents circular shift of $x_2(n)$

m is a dummy variable.

The output sequence $x_3(n)$ obtained by circular convolution is also a periodic sequence with periodicity of N samples. Hence this convolution is also called ***periodic convolution***.

The convolution relation of equation (2.57) can be symbolically expressed as

$$x_3(n) = x_1(n) \circledast x_2(n) = x_2(n) \circledast x_1(n) \qquad .....(2.58)$$

where, the symbol $\circledast$ indicates circular convolution operation.

The circular convolution is defined for periodic sequences. But circular convolution can be performed with nonperiodic sequences by periodically extending them.The circular convolution of two sequences requires that, at least one of the sequences should be periodic. Hence it is sufficient if one of the sequences is periodically extended in order to perform circular convolution.

The circular convolution of finite duration sequences can be performed only if both the sequences consist of the same number of samples. If the sequences have different number of samples, then convert the smaller size sequence to the length of larger size sequence by appending zeros.

Circular convolution basically involves the same four steps as that for linear convolution, namely, folding one sequence, shifting the folded sequence, multiplying the two sequences and finally summing the values of the product sequence. Like linear convolution, any one of the sequence is folded and rotated in circular convolution.

The difference between the two is that in circular convolution the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences by modulo-N operation. In linear convolution there is no modulo-N operation.

2.10.4 Procedure for Evaluating Circular Convolution

Let, $x_1(n)$ and $x_2(n)$ be periodic discrete time sequences with periodicity of N-samples. If $x_1(n)$ and $x_2(n)$ are non-periodic then convert the sequences to N-sample sequences and periodically extend the sequence $x_2(n)$ with periodicity of N-samples.

Now the circular convolution of $x_1(n)$ and $x_2(n)$ will produce a periodic sequence $x_3(n)$ with periodicity of N-samples. The samples of one period of $x_3(n)$ can be computed using the equation (2.57). The value of $x_3(n)$ at $n = q$ is obtained by replacing n by q, in equation (2.57).

$$\therefore\ x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N \qquad .....(2.59)$$

The evaluation of equation (2.59) to determine the value of $x_3(n)$ at $n = q$ involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, in order to get the sequences $x_1(m)$ and $x_2(m)$. Represent the samples of one period of the sequences on circles.

2. Folding : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.

3. Rotation : Rotate $x_2(-m)$ by q times in anti-clockwise if q is positive, rotate $x_2(-m)$ by q times in clockwise if q is negative to obtain $x_2((q-m))_N$.

4. Multiplication : Multiply $x_1(m)$ by $x_2((q-m))_N$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2((q-m))_N$.

5. Summation : Sum up the samples of one period of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

The above procedure will give the value of $x_3(n)$ at a single time instant say $n = q$. In general we are interested in evaluating the values of the sequence $x_3(n)$ in the range $0 < n < N-1$ Hence the steps 3 , 4 and 5 given above must be repeated, for all possible time shifts in the range $0 < n < N-1$

2.10.5 Linear Convolution via Circular Convolution

When two numbers of N-point sequences are circularly convolved, it produces another N-point sequence. For circular convolution, one of the sequence should be periodically extended. Also the resultant sequence is periodic with period N.

The linear convolution of two sequences of length N_1 and N_2 produces an output sequence of length $N_1 + N_2 - 1$. To perform linear convolution via circular convolution both the sequences should be converted to $N_1 + N_2 - 1$ point sequences by padding with zeros. Then perform circular convolution of $N_1 + N_2 - 1$ point sequences. The resultant sequence will be same as that of linear convolution of N_1 and N_2 point sequences.

2.10.6 Methods of Computing Circular Convolution

Method 1 : Graphical Method

In graphical method, the given sequences are converted to same size and represented on circles. In case of periodic sequences, the samples of one period are represented on circles. One of the sequence is folded and shifted circularly. Let $x_1(n)$ and $x_2(n)$ be the given sequences. Let $x_3(n)$ be the sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to get a sample of $x_3(n)$ at $n = q$.

1. Change the index n in the sequences $x_1(n)$ and $x_2(n)$ to get $x_1(m)$ and $x_2(m)$ and then represent the sequences on circles.
2. Fold one of the sequence. Let us fold $x_2(m)$ to get $x_2(-m)$.
3. Rotate (or shift) the sequence $x_2(-m)$, q times to get the sequence $x_2((q-m))_N$. If q is positive then rotate (or shift) the sequence in anticlockwise direction and if q is negative then rotate (or shift) the sequence in clockwise direction.
4. The sample of $x_3(q)$ at $n = q$ is given by,

$$x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,q}(m)$$

where, $x_{2,q}(m) = x_2((q-m))_N$

Determine the product sequence $x_1(m)\, x_{2,q}(m)$ for one period.

5. The sum of all the samples of the product sequence gives the sample $x_3(q)$ [i.e., $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Method 2 : Tabular Method

Let $x_1(n)$ and $x_2(n)$ be the given N-point sequences. Let $x_3(n)$ be the N-point sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$.

1. Change the index n in the sequences $x_1(n)$ and $x_2(n)$ to get $x_1(m)$ and $x_2(m)$ and then represent the sequences as two rows of tabular array.
2. Fold one of the sequence. Let us fold $x_2(m)$ to get $x_2(-m)$.
3. Periodically extend $x_2(-m)$. Here the periodicity is N, where N is the length of the given sequences.
4. Shift the sequence $x_2(-m)$, q times to get the sequence $x_2((q-m))_N$. If q is positive then shift the sequence to the right and if q is negative then shift the sequence to the left.

5. The sample of $x_3(q)$ at n = q is given by, $x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,q}(m)$

   where $x_{2,q}(m) = x_2((q-m))_N$

   Determine the product sequence $x_1(m)\, x_{2,q}(m)$ for one period.

6. The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e., $x_3(n)$ at n = q]. The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Method 3: Matrix Method

Let $x_1(n)$ and $x_2(n)$ be the given N-point sequences. The circular convolution of $x_1(n)$ and $x_2(n)$ yields another N-point sequence $x_3(n)$.

In this method an (N ´ N) matrix is formed using one of the sequence as shown below. Another sequence is arranged as a column vector (column matrix) of order (N ´ 1). The product of the two matrices gives the resultant sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \cdots & x_2(4) & x_2(3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & \cdots & x_2(0) & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(1) & x_2(0) \end{bmatrix} \times \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ \vdots \\ x_1(N-2) \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix}$$

Example 2.24

Perform circular convolution of the two sequences, $x_1(n) = \{2, 1, 2, -1\}$ and $x_2(n) = \{1, 2, 3, 4\}$

Solution

Method 1: Graphical Method of Computing Circular Convolution

Let $x_3(n)$ be the sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$.

The circular convolution of $x_1(n)$ and $x_2(n)$ is given by,

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,n}(m)$$

where $x_{2,n}(m) = x_2((n-m))_N$ and m is the dummy variable used for convolution.

The index n in the given sequences are changed to m and each sequence is represented as points on a circle as shown below. The folded sequence $x_2(-m)$ and circularly shifted sequences $x_2(n-m)$ are also represented on the circle.

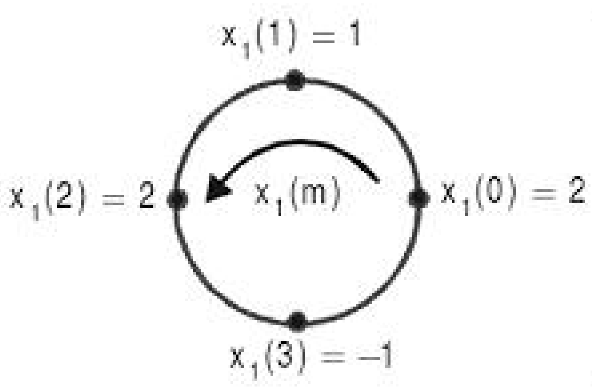


Fig 1.

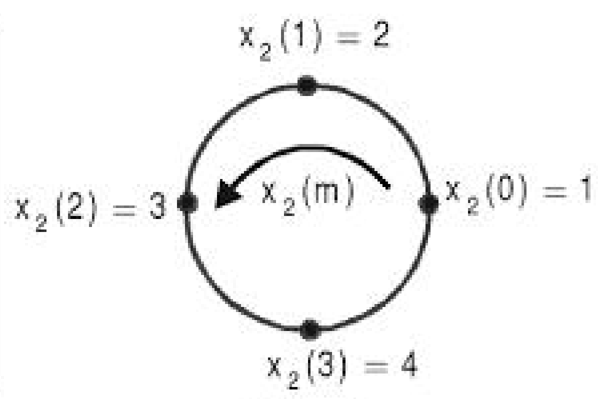


Fig 2.

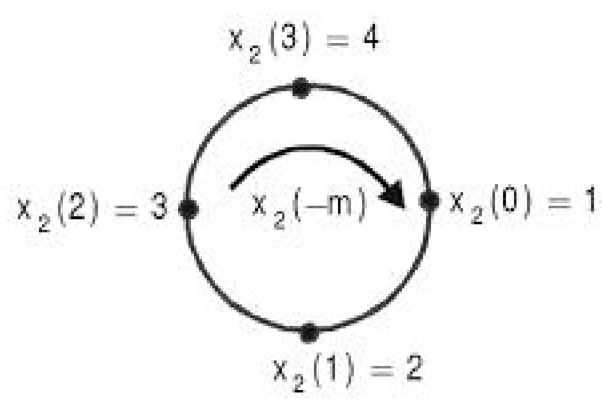


Fig 3.

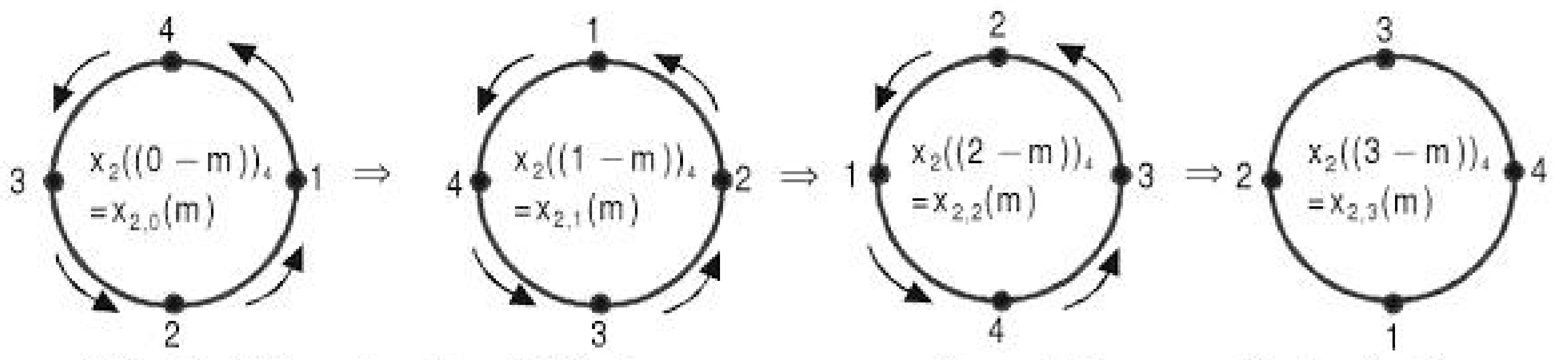


Fig 4: Circularly shifted sequences $x_2(-m)$ for n = 0, 1, 2, 3.

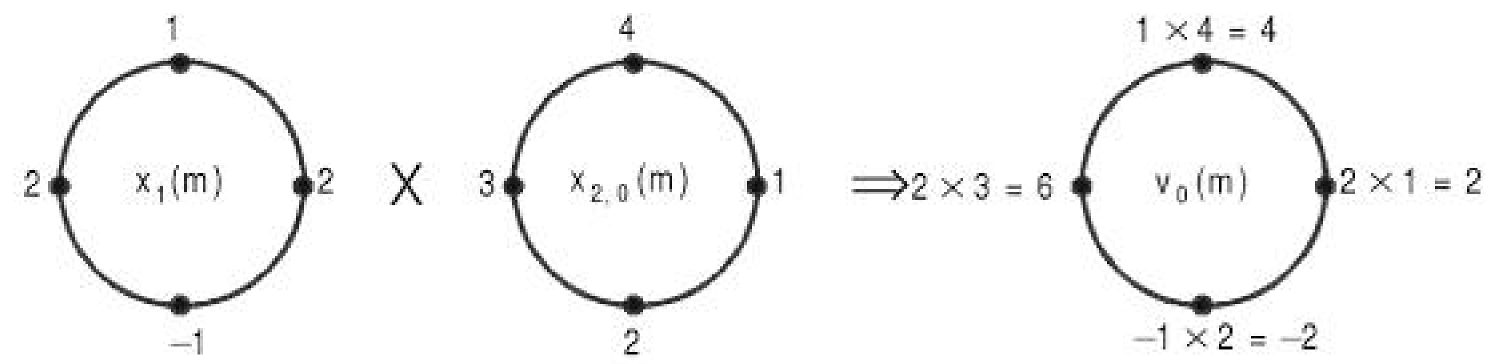
The given sequences are 4-point sequences . ∴ N = 4.

Each sample of $x_3(n)$ is given by sum of the samples of product sequence defined by the equation,

$$x_3(n) = \sum_{m=0}^{3} x_1(m)\ x_{2,n}(m) = \sum_{m=0}^{3} v_n(m) \quad ; \quad \text{where } v_n(m) = x_1(m)\ x_{2,n}(m) \qquad .....(1)$$

Using the above equation (1), graphical method of computing each sample of $x_3(n)$ are shown in fig 5 to fig 8.

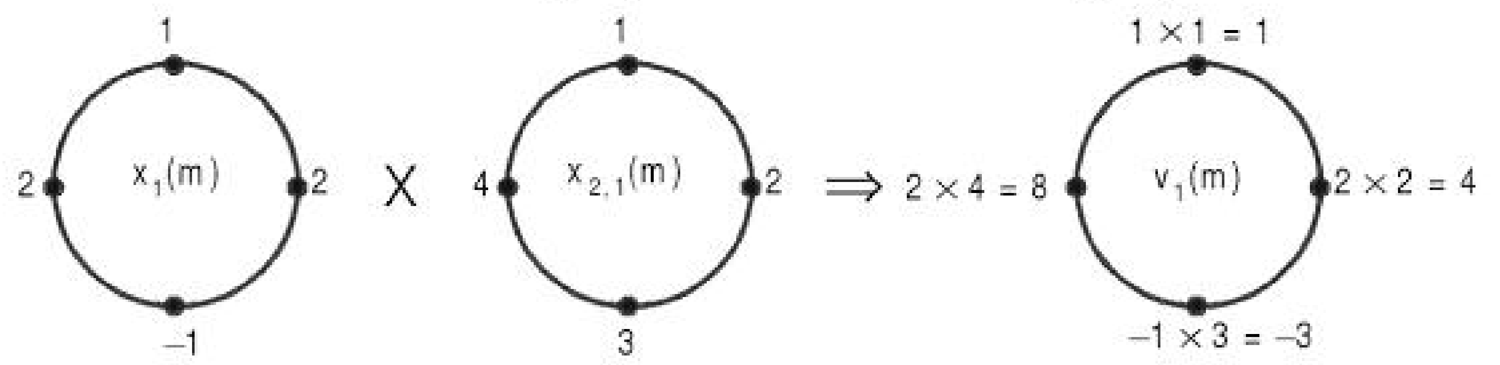
When n = 0 ; $x_3(0) = \sum_{m=0}^{3} x_1(m)\ x_2((0-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,0}(m) = \sum_{m=0}^{3} v_0(m)$



The sum of samples of $v_0(m)$ gives $x_3(0)$

Fig 5: Computation of $x_3(0)$. $\therefore\ x_3(0) = 2 + 4 + 6 - 2 = 10$

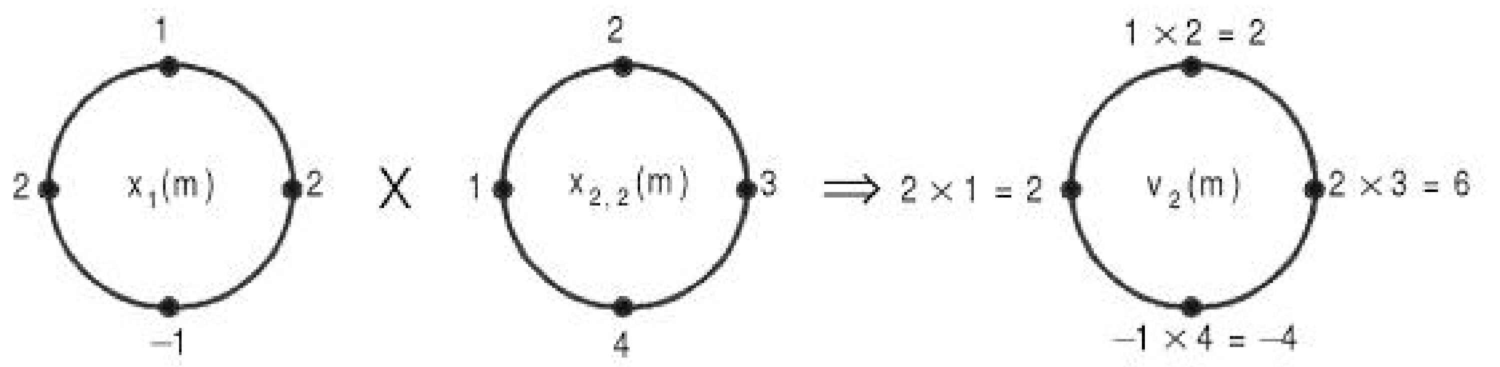
When n = 1 ; $x_3(1) = \sum_{m=0}^{3} x_1(m)\ x_2((1-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,1}(m) = \sum_{m=0}^{3} v_1(m)$



The sum of samples of $v_1(m)$ gives $x_3(1)$

Fig 6: Computation of $x_3(1)$. $\therefore\ x_3(1) = 4 + 1 + 8 - 3 = 10$

When n = 2 ; $x_3(2) = \sum_{m=0}^{3} x_1(m)\ x_2((2-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,2}(m) = \sum_{m=0}^{3} v_2(m)$



The sum of samples of $v_2(m)$ gives $x_3(2)$

Fig 7: Computation of $x_3(2)$. $\therefore\ x_3(2) = 6 + 2 + 2 - 4 = 6$

When n = 3 ; $x_3(3) = \sum_{m=0}^{3} x_1(m)\ x_2((3-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,3}(m) = \sum_{m=0}^{3} v_3(m)$

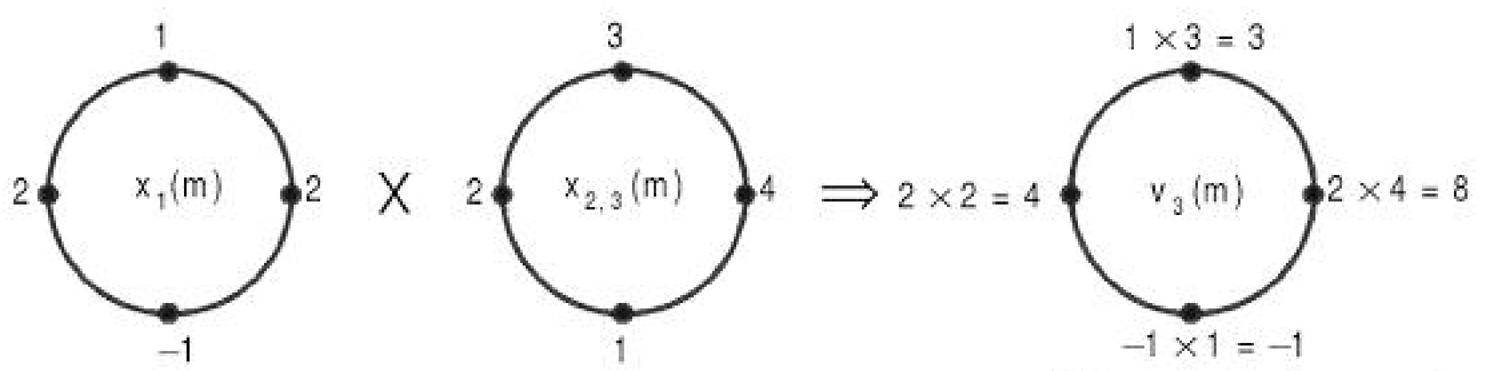


Fig 8: Computation of $x_3(3)$. The sum of samples of $v_3(m)$ gives $x_3(3)$

$\therefore\ x_3(3) = 8 + 3 + 4 - 1 = 14$

∴ $x_3(n) = \{10, 10, 6, 14\}$

Method 2 : Circular Convolution Using Tabular Array

The index n in the given sequences are changed to m and then, the given sequences can be represented in the tabular array as shown below. Here the shifted sequences $x_{2,\,n}(m)$ are periodically extended with a periodicity of N = 4. Let $x_3(n)$ be the sequence obtained by convolution of $x_1(n)$ and $x_2(n)$. Each sample of $x_3(n)$ is given by the equation,

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,n}(m), \text{ where } x_{2,n}(m) = x_2((n-m))_N$$

Note : The boldfaced numbers are samples obtained by periodic extension.

m	–3	–2	–1	0	1	2	3
$x_1(m)$				2	1	2	–1
$x_2(m)$				1	2	3	4
$x_2((-m))_4 = x_{2,0}(m)$	4	3	2	1	**4**	**3**	**2**
$x_2((1-m))_4 = x_{2,1}(m)$		4	3	2	1	**4**	**3**
$x_2((2-m))_4 = x_{2,2}(m)$			4	3	2	1	**4**
$x_2((3-m))_4 = x_{2,3}(m)$				4	3	2	1

To determine a sample of $x_3(n)$ at n = q, multiply the sequence, $x_1(m)$ and $x_{2,q}(m)$, to get a product sequence $x_1(m)\, x_{2,q}(m)$. [i.e., multiply the corresponding elements of the row $x_1(m)$ and $x_{2,\,q}(m)$]. The sum of all the samples of the product sequence gives $x_3(q)$.

$$\text{When } n = 0\ ;\ x_3(0) = \sum_{m=0}^{3} x_1(m)\, x_{2,0}(m)$$
$$= x_1(0)\, x_{2,0}(0) + x_1(1)\, x_{2,0}(1) + x_1(2)\, x_{2,0}(2) + x_1(3)\, x_{2,0}(3)$$
$$= 2\times 1 + 1\times 4 + 2\times 3 + (-1)\times 2 = 2 + 4 + 6 - 2 = 10$$

The samples of $x_3(n)$ for other values of n are calculated as shown for n = 0.

$$\text{When } n = 1;\ x_3(1) = \sum_{m=0}^{3} x_1(m)\, x_{2,1}(m) = 4 + 1 + 8 - 3 = 10$$

$$\text{When } n = 2;\ x_3(2) = \sum_{m=0}^{3} x_1(m)\, x_{2,2}(m) = 6 + 2 + 2 - 4 = 6$$

$$\text{When } n = 3;\ x_3(3) = \sum_{m=0}^{3} x_1(m)\, x_{2,3}(m) = 8 + 3 + 4 - 1 = 14$$

$$\therefore\quad x_3(n) = \{\underset{\uparrow}{10},\ 10,\ 6,\ 14\}$$

Method 3 : Circular Convolution Using Matrices

The sequence $x_1(n)$ can be arranged as a column vector of order N ´ 1 and using the samples of $x_2(n)$ the N ´ N matrix is formed as shown below. The product of the two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\times 2+4\times 1+3\times 2+2\times -1 \\ 2\times 2+1\times 1+4\times 2+3\times -1 \\ 3\times 2+2\times 1+1\times 2+4\times -1 \\ 4\times 2+3\times 1+2\times 2+1\times -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 6 \\ 14 \end{bmatrix}$$

\ $x_3(n) = \{10, 10, 6, 14\}$

Example 2.25

Perform the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$, where,

$$x_1(n) = \{0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6\}$$

$$x_2(n) = \{0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5\}$$

Solution

Let $x_3(n)$ be the result of the circular convolution of $x_1(n)$ and $x_2(n)$. The given sequences consists of eight samples. Then $x_3(n)$ will also have 8 samples.

The sequences are represented in the tabular array as shown below after replacing n by m. The sequence $x_2(m)$ is folded and shifted.

The shifted sequences $x_{2,n}(m)$ are periodically extended with a periodicity of N = 8.

Note : The boldfaced numbers are samples obtained by periodic extension

m	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
$x_1(m)$								0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$x_2(m)$								0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
$x_2((-m))_8 = x_{2,0}(m)$	1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**	**0.5**	**0.3**
$x_2((1-m))_8 = x_{2,1}(m)$		1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**	**0.5**
$x_2((2-m))_8 = x_{2,2}(m)$			1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**
$x_2((3-m))_8 = x_{2,3}(m)$				1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**
$x_2((4-m))_8 = x_{2,4}(m)$					1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**
$x_2((5-m))_8 = x_{2,5}(m)$						1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**
$x_2((6-m))_8 = x_{2,6}(m)$							1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5
$x_2((7-m))_8 = x_{2,7}(m)$								1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1

Each sample of $x_3(n)$ is given by the equation,

$$x_3(n) = \sum_{m=0}^{7} x_1(m)\, x_2((n-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,n}(m) \; ; \quad \text{where } x_{2,n}(m) = x_2((n-m))_8$$

The samples of $x_3(0)$ are calculated as shown below.

$$\text{When } n = 0\ ;\ x_3(n) = \sum_{m=0}^{7} x_1(m)\ x_2((0-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,0}(m)$$

$$= x_1(0)\ x_{2,0}(0) + x_1(1)\ x_{2,0}(1) + x_1(2)\ x_{2,0}(2) + x_1(3)\ x_{2,0}(3)$$

$$+ x_1(4)\ x_{2,0}(4) + x_1(5)\ x_{2,0}(5) + x_1(6)\ x_{2,0}(6) + x_1(7)\ x_{2,0}(7)$$

$$= 0.02 + 0.6 + 0.78 + 0.88 + 0.9 + 0.84 + 0.7 + 0.48 = 5.20$$

The samples of $x_3(n)$ for other values of n are calculated as shown for n = 0.

$$\text{When } n = 1;\ x_3(1) = \sum_{m=0}^{7} x_1(m)\ x_2((1-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,1}(m) = 6.00$$

$$\text{When } n = 2;\ x_3(2) = \sum_{m=0}^{7} x_1(m)\ x_2((2-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,2}(m) = 6.48$$

$$\text{When } n = 3;\ x_3(3) = \sum_{m=0}^{7} x_1(m)\ x_2((3-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,3}(m) = 6.64$$

$$\text{When } n = 4;\ x_3(4) = \sum_{m=0}^{7} x_1(m)\ x_2((4-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,4}(m) = 6.48$$

$$\text{When } n = 5;\ x_3(5) = \sum_{m=0}^{7} x_1(m)\ x_2((5-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,5}(m) = 6.00$$

$$\text{When } n = 6;\ x_3(6) = \sum_{m=0}^{7} x_1(m)\ x_2((6-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,6}(m) = 5.20$$

$$\text{When } n = 7;\ x_3(7) = \sum_{m=0}^{7} x_1(m)\ x_2((7-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,7}(m) = 4.08$$

$$\therefore x_3(n) = \{\underset{\uparrow}{5.20},\ 6.00,\ 6.48,\ 6.64,\ 6.48,\ 6.00,\ 5.20,\ 4.08\}$$

Example 2.26

Find the linear and circular convolution of the sequences, $x(n) = \{\underset{\uparrow}{1},\ 0.5\}$ and $h(n) = \{\underset{\uparrow}{0.5},\ 1\}$.

Solution

Linear Convolution by Tabular Array

Let, $y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m)\ h(n-m)$; where m is a dummy variable for convolution.

Since both x(n) and h(n) starts at n = 0, the output sequence y(n) will also start at n = 0.

Since the length of x(n) and h(n) is 2, the length of y(n) is 2 + 2 – 1 = 3.

Let us change the index n to m in x(n) and h(n). The sequences x(m) and h(m) are represented in the tabular array as shown below.

Note : The unfilled boxes in the table are considered as zeros.

m	–1	0	1	2
x(m)		1	0.5	
h(m)		0.5	1	
$h(-m) = h_0(m)$	1	0.5		
$h(1-m) = h_1(m)$		1	0.5	
$h(2-m) = h_2(m)$			1	0.5

Each sample of y(n) is given by the relation,

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\ h(n-m) = \sum_{m=-\infty}^{\infty} x(m)\ h_n(m) \quad ; \quad \text{where } h_n(m) = h(n-m)$$

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-\infty}^{\infty} x(m)\ h(-m) = \sum_{m=-1}^{1} x(m)\ h_0(m) = x(-1)\ h_0(-1) + x(0)\ h_0(0) + x(1)\ h_0(1)$$

$$= 0 \times 1 + 1 \times 0.5 + 0.5 \times 0 = 0 + 0.5 + 0 = 0.5$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-\infty}^{\infty} x(m)\ h(1-m) = \sum_{m=0}^{1} x(m)\ h_1(m) = 1 + 0.25 = 1.25$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=-\infty}^{\infty} x(m)\ h(2-m) = \sum_{m=0}^{2} x(m)\ h_2(m) = 0 + 0.5 + 0 = 0.5$$

$$\therefore y(n) = \{\underset{\uparrow}{0.5},\ 1.25,\ 0.5\}$$

Circular Convolution by Tabular Array

$$\text{Let, } y(n) = x(n) \circledast h(n) = \sum_{m=0}^{N-1} x(m)\ h((n-m))_N \quad ; \quad \text{where m is a dummy variable for convolution.}$$

The index n in the sequences are changed to m and the sequences are represented in the tabular array as shown below. The shifted sequence $h_n(m)$ is periodically extended with periodicity N = 2.

Note : The boldfaced number is the sample obtained by periodic extension.

m	–1	0	1
x(m)		1	0.5
h(m)		0.5	1
$h((-m))_2 = h_0(m)$	1	0.5	**1**
$h((1-m))_2 = h_1(m)$		1	0.5

Each sample of y(n) is given by the equation,

$$y(n) = \sum_{m=0}^{N-1} x(m)\ h((n-m))_N = \sum_{m=0}^{N-1} x(m)\ h_n(m); \ \text{where } h_n(m) = h((n-m))_N$$

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=0}^{N-1} x(m)\ h((0-m))_2 = \sum_{m=0}^{1} x(m)\ h_0(m)$$

$$= x(0)\ h_0(0) + x(1)\ h_0(1) = 1 \times 0.5 + 0.5 \times 1 = 0.5 + 0.5 = 1.0$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=0}^{N-1} x(m)\ h((1-m))_2 = \sum_{m=0}^{1} x(m)\ h_1(m)$$

$$= x(0)\ h_1(0) + x(1)\ h_1(1) = 1 \times 1 + 0.5 \times 0.5 = 1 + 0.25 = 1.25$$

$$\therefore y(n) = \{\underset{\uparrow}{1.0},\ 1.25\}$$

Example 2.27

The input x(n) and impulse response h(n) of a LTI system are given by,

$$x(n) = \{\underset{\uparrow}{-1},\ 1,\ 2,\ -2\} \ ;\ h(n) = \{0.5,\ \underset{\uparrow}{1},\ -1,\ 2,\ 0.75\}$$

Determine the response of the system **a)** using linear convolution and **b)** using circular convolution.

Solution

a) Response of LTI system using linear convolution

Let y(n) be the response of LTI system. By convolution sum formula,

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m) \quad ; \text{ where m is a dummy variable used for convolution.}$$

The sequence x(n) starts at n = 0 and h(n) starts at n = –1. Hence y(n) will start at n = 0 + (–1) = –1. The length of x(n) is 4 and the length of h(n) is 5. Hence the length of y(n) is (4 + 5 – 1) = 8. Also y(n) ends at n = 0 + (–1) + (4 + 5 –2) = 6.

Let us change the index n to m in x(n) and h(n). The sequences x(m) and h(m) are represented on the tabular array as shown below. Let us fold h(m) to get h(–m) and shift h(–m) to perform convolution operation.

Note : The unfilled boxes in the table are considered as zeros.

m	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x(m)					–1	1	2	–2				
h(m)				0.5	1	–1	2	0.75				
h(–m)		0.75	2	–1	1	0.5						
$h(-1-m) = h_{-1}(m)$	0.75	2	–1	1	0.5							
$h(0-m) = h_0(m)$		0.75	2	–1	1	0.5						
$h(1-m) = h_1(m)$			0.75	2	–1	1	0.5					
$h(2-m) = h_2(m)$				0.75	2	–1	1	0.5				
$h(3-m) = h_3(m)$					0.75	2	–1	1	0.5			
$h(4-m) = h_4(m)$						0.75	2	–1	1	0.5		
$h(5-m) = h_5(m)$							0.75	2	–1	1	0.5	
$h(6-m) = h_6(m)$								0.75	2	–1	1	0.5

Each sample of y(n) is given by summation of the product sequence, x(m) h(n – m). To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence [i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$]. The sum of all the samples of the product sequence gives y(q).

$$\text{i.e., } y(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\,h_n(m)$$

$$\text{When } n = -1\ ;\ y(-1) = \sum_{m=-4}^{3} x(m)\,h_{-1}(m)$$

$$= x(-4)\,h_{-1}(-4) + x(-3)\,h_{-1}(-3) + x(-2)\,h_{-1}(-2) + x(-1)\,h_{-1}(-1) + x(0)\,h_{-1}(0) + x(1)\,h_{-1}(1) + x(2)\,h_{-1}(2) + x(3)\,h_{-1}(3)$$

$$= 0 + 0 + 0 + 0 + (-0.5) + 0 + 0 + 0 = -0.5$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-3}^{3} x(m)\,h_0(m) = 0 + 0 + 0 + (-1) + 0.5 + 0 + 0 = -0.5$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-2}^{3} x(m)\,h_1(m) = 0 + 0 + 1 + 1 + 1 + 0 = 3$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=-1}^{3} x(m)\,h_2(m) = 0 + (-2) + (-1) + 2 + (-1) = -2$$

When n = 3 ; $y(3) = \sum_{m=0}^{4} x(m)\, h_3(m) = -0.75 + 2 + (-2) + (-2) + 0 = -2.75$

When n = 4 ; $y(4) = \sum_{m=0}^{5} x(m)\, h_4(m) = 0 + 0.75 + 4 + 2 + 0 + 0 = 6.75$

When n = 5 ; $y(5) = \sum_{m=0}^{6} x(m)\, h_5(m) = 0 + 0 + 1.5 + (-4) + 0 + 0 + 0 = -2.5$

When n = 6 ; $y(6) = \sum_{m=0}^{7} x(m)\, h_6(m) = 0 + 0 + 0 + (-1.5) + 0 + 0 + 0 + 0 = -1.5$

The response of LTI system y(n) is,

$$y(n) = \{-0.5, \underline{-0.5}, 3, -2, -2.75, 6.75, -2.5, -1.5\}$$

---

b) Response of LTI System Using Circular Convolution

The response of LTI system is given by linear convolution of x(n) and h(n). Let y(n) be the response sequence of LTI system. To get the result of linear convolution from circular convolution, both the sequences should be converted to the size of y(n) and perform circular convolution of the converted sequences. Also the converted sequences should start and end at the same value of n as that of y(n).

The length of x(n) is 4 and the length of h(n) is 5. Hence the length of y(n) is (4 + 5 – 1) = 8. Therefore both the sequences should be converted to 8-point sequences.

The x(n) starts at n = 0 and h(n) starts at n = –1. Hence y(n) will start at n = 0 + (–1) = –1. The y(n) will end at n = [0 +(–1)] + (4 + 5 – 2) = 6. Therefore the converted sequences should start at n = –1 and end at n = 6.

$$\therefore\ x(n) = \{0, \underline{-1}, 1, 2, -2, 0, 0, 0\} \text{ and } h(n) = \{0.5, \underline{1}, -1, 2, 0.75, 0, 0, 0\}$$

The converted sequences x(n) and h(n) are represented on the tabular array after replacing the index n by m as shown below. The sequence h(m) is folded and shifted.

The shifted sequences $h_n(m)$ are periodically extended with a periodicity of N = 8.

Note : The boldfaced numbers are samples obtained by periodic extension of the sequences.

m	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x(m)							0	–1	1	2	–2	0	0	0	
h(m)							0.5	1	–1	2	0.75	0	0	0	
h(–m)		0	0	0	0.75	2	–1	1	0.5						
$h((-1-m))_8 = h_{-1}(m)$	0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**	**–1**	**1**
$h((0-m))_8 = h_0(m)$		0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**	**–1**
$h((1-m))_8 = h_1(m)$			0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**
$h((2-m))_8 = h_2(m)$				0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**
$h((3-m))_8 = h_3(m)$					0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**
$h((4-m))_8 = h_4(m)$						0	0	0	0.75	2	–1	1	0.5	**0**	**0**
$h((5-m))_8 = h_5(m)$							0	0	0	0.75	2	–1	1	0.5	**0**
$h((6-m))_8 = h_6(m)$	**0**	**0**	**0.75**	**2**	**–1**	**1**	**0.5**	0	0	0	0.75	2	–1	1	0.5

Let y(n) be the sequence obtained by circular convolution of x(n) and h(n).

Now, each sample of y(n) is given by,

$$y(n) = \sum_{m=-1}^{6} x(m)\, h((n-m))_8 = \sum_{m=-1}^{6} x(m)\, h_n(m) \quad ; \quad \text{where } h_n(m) = h((n-m))_8$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence x(m) $h_q(m)$, [i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$]. The sum of all the samples of the product sequence gives y(q).

$$\text{When } n = -1\ ;\ y(-1) = \sum_{m=-1}^{6} x(m)\, h_{-1}(n) = x(-1)\, h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2)$$

$$+\ x(3)\, h_{-1}(3) + x(4)\, h_{-1}(4) + x(5)\, h_{-1}(5) + x(6)\, h_{-1}(6)$$

$$= 0 + (-0.5) + 0 + 0 + 0 + 0 + 0 + 0 = -0.5$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-1}^{6} x(m)\, h_0 m = 0 + (-1) + 0.5 + 0 + 0 + 0 + 0 + 0 = -0.5$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-1}^{6} x(m)\, h_1 m = 0 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 3$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=-1}^{6} x(m)\, h_2 m = 0 + (-2) + (-1) + 2 + (-1) + 0 + 0 + 0 = -2$$

$$\text{When } n = 3\ ;\ y(3) = \sum_{m=-1}^{6} x(m)\, h_3 m = 0 + (-0.75) + 2 + (-2) + (-2) + 0 + 0 + 0 = -2.75$$

$$\text{When } n = 4\ ;\ y(4) = \sum_{m=-1}^{6} x(m)\, h_4 m = 0 + 0 + 0.75 + 4 + 2 + 0 + 0 + 0 = 6.75$$

$$\text{When } n = 5\ ;\ y(5) = \sum_{m=-1}^{6} x(m)\, h_5 m = 0 + 0 + 0 + 1.5 + (-4) + 0 + 0 + 0 = -2.5$$

$$\text{When } n = 6\ ;\ y(6) = \sum_{m=-1}^{6} x(m)\, h_6 m = 0 + 0 + 0 + 0 + (-1.5) + 0 + 0 + 0 = -1.5$$

The response of LTI system y(n) is,

$$y(n) = \{-0.5, \underset{\uparrow}{-0.5}, 3, -2, -2.75, 6.75, -2.5, -1.5\}$$

> *Note : 1. Since circular convolution is periodic, the convolution is performed for any one period.*
> *2. It can be observed that the results of both the methods are same.*

2.11 Sectioned Convolution

The response of an LTI system for any arbitrary input is given by linear convolution of the input and the impulse response of the system. If one of the sequences (either the input sequence or impulse response sequence) is very much larger than the other, then it is very difficult to compute the linear convolution for the following reasons.

1. The entire sequence should be available before convolution can be carried out. This makes long delay in getting the output.

2. Large amounts of memory is required to store the sequences.

The above problems can be overcome in the sectioned convolutions. In this technique the larger sequence is sectioned (or splitted) into the size of smaller sequence. Then the linear convolution of each section of longer sequence and the smaller sequence is performed. The output sequences obtained from the convolutions of all the sections are combined to get the overall output sequence. There are two methods of sectioned convolutions. They are overlap add method and overlap save method.

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Convolution of Section 1

m	–1	0	1	2
$x_1(m)$		1	–1	
$h(m)$		–1	1	
$h(-m) = h_o(m)$	1	–1		
$h(1-m) = h_1(m)$		1	–1	
$h(2-m) = h_2(m)$			1	–1

$$y_1(n) = x_1(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, h_n(m)\ ; \quad n = 0, 1, 2$$

where $h_n(m) = h(n-m)$

When $n = 0$; $y_1(0) = \sum x_1(m)\, h_0(m) = 0 - 1 + 0 = -1$

When $n = 1$; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 = 2$

When $n = 2$; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 + 0 = -1$

Convolution of Section 2

m	–1	0	1	2	3	4
$x_2(m)$				2	–2	
$h(m)$		–1	1			
$h(-m)$	1	–1				
$h(2-m) = h_2(m)$			1	–1		
$h(3-m) = h_3(m)$				1	–1	
$h(4-m) = h_4(m)$					1	–1

$$y_2(n) = x_2(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_2(m)\, h_n(m)\ ; \quad n = 2, 3, 4$$

where $h_n(m) = h(n - m)$

When $n = 2$; $y_2(2) = \sum x_2(m)\, h_2(m) = 0 - 2 + 0 = -2$

When $n = 3$; $y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 = 4$

When $n = 4$; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 - 2 + 0 = -2$

Convolution of Section 3

m	–1	0	1	2	3	4	5	6
$x_3(m)$						3	–3	
$h(m)$		–1	1					
$h(-m)$	1	–1						
$h(4-m) = h_4(m)$					1	–1		
$h(5-m) = h_5(m)$						1	–1	
$h(6-m) = h_6(m)$							1	–1

$$y_3(n) = x_3(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_3(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x_3(m)\, h_n(m)\ ; \quad n = 4, 5, 6$$

where $h_n(m) = h(n-m)$

When $n = 4$; $y_3(4) = \sum x_3(m)\, h_4(m) = 0 - 3 + 0 = -3$

When $n = 5$; $y_3(5) = \sum x_3(m)\, h_5(m) = 3 + 3 = 6$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 - 3 + 0 = -3$

Convolution of Section 4

m	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$								4	–4	
$h(m)$		–1	1							
$h(-m)$	1	–1								
$h(6-m) = h_6(m)$							1	–1		
$h(7-m) = h_7(m)$								1	–1	
$h(8-m) = h_8(m)$									1	–1

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Method 2

In method 2, the overlapping samples are placed at the end of the section.Each section of longer sequence is converted to 3-sample sequence, using the samples of original longer sequence as shown below. It can be observed that the last sample of $x_1(n)$ is placed as overlapping sample at the end of $x_2(n)$. The last sample of $x_2(n)$ is placed as overlapping sample at the end of $x_3(n)$. The last sample of $x_3(n)$ is placed as overlapping sample at the end of $x_4(n)$. Since there is no previous section for $x_1(n)$, the overlapping sample of $x_1(n)$ is taken as zero.

$x_1(n) = 1$; $n = 0$	$x_2(n) = 2$; $n = 2$	$x_3(n) = 3$; $n = 4$	$x_4(n) = 4$; $n = 6$
$= -1$; $n = 1$	$= -2$; $n = 3$	$= -3$; $n = 5$	$= -4$; $n = 7$
$= 0$; $n = 2$	$= -1$; $n = 4$	$= -2$; $n = 6$	$= -3$; $n = 8$

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have three samples. The circular convolution of each section is performed by tabular method as shown below.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $\therefore$ $y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

> *Note :* 1. *Here* $N_1 = 8$, $N_2 = 2$, $N_3 = 2$. $\therefore$ $(N_2 - 1) = 2 - 1 = 1$ *and* $(N_2 + N_3 - 1) = 2 + 2 - 1 = 3$
> 2. *The boldfaced numbers in the tables are obtained by periodic extension.*
> 3. *For convenience of convolution the index n is replaced by m in* $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ *and* $h(n)$.

Convolution of Section 1

m	–2	–1	0	1	2
$x_1(m)$			1	–1	0
h(m)			–1	1	0
$h((-m))_3 = h_0(m)$	0	1	–1	**0**	**1**
$h((1-m))_3 = h_1(m)$		0	1	–1	**0**
$h((2-m))_3 = h_2(m)$			0	1	–1

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m = m_i}^{m_f} x_1(m)\, h((n-m))_N$$

$$= \sum_{m=0}^{2} x_1(m)\, h_n(m) \ ; \quad n = 0,\ 1,\ 2$$

where $h_n(m) = h((n-m))_N$

When $n = 0$; $y_1(0) = \sum x_1(m)\, h_0(m) = -1 + 0 + 0 = -1$

When $n = 1$; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 + 0 = 2$

When $n = 2$; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 + 0 = -1$

Convolution of Section 2

m	–2	–1	0	1	2	3	4
$x_2(m)$					2	–2	–1
h(m)			–1	1	0		
h(–m)	0	1	–1				
$h((2-m))_3 = h_2(m)$			0	1	–1	**0**	**1**
$h((3-m))_3 = h_3(m)$				0	1	–1	**0**
$h((4-m))_3 = h_4(m)$					0	1	–1

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Convolution of Section 3

m	−2	−1	0	1	2	3	4	5	6	7	8	9	10
$x_3(m)$									4	5	6		
$h(m)$			2	1	−1								
$h(-m) = h_0(m)$	−1	1	2										
$h(6-m) = h_6(m)$							−1	1	2				
$h(7-m) = h_7(m)$								−1	1	2			
$h(8-m) = h_8(m)$									−1	1	2		
$h(9-m) = h_9(m)$										−1	1	2	
$h(10-m) = h_{10}(m)$											−1	1	2

$$y_3(n) = x_3(n) * h(n) = \sum_{m=-\infty}^{\infty} x_3(m)\, h(n-m) = \sum_{m=-\infty}^{\infty} x_3(m)\, h_n(m)\ ; \quad n = 6,\ 7,\ 8,\ 9,\ 10$$

$$\text{where } h_n(m) = h(n-m)$$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 + 0 + 8 + 0 + 0 = 8$

When $n = 7$; $y_3(7) = \sum x_3(m)\, h_7(m) = 0 + 4 + 10 + 0 = 14$

When $n = 8$; $y_3(8) = \sum x_3(m)\, h_8(m) = -4 + 5 + 12 = 13$

When $n = 9$; $y_3(9) = \sum x_3(m)\, h_9(m) = 0 - 5 + 6 + 0 = 1$

When $n = 10$; $y_3(10) = \sum x_3(m)\, h_{10}(m) = 0 + 0 - 6 + 0 + 0 = -6$

To Combine the Output of the Convolution of Each Section

It can be observed that the last $N_2 - 1$ sample in an output sequence overlaps with the first $N_2 - 1$ sample of next output sequence. In this method, the overall output is obtained by combining the outputs of the convolution of all sections. The overlapped portions (or samples) are added while combining the output.

The output of all sections can be represented in a table as shown below. Then the samples corresponding to same value of n are added to get the overall output.

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	2	5	7	1	−3						
$y_2(n)$				−2	−5	−7	−1	3			
$y_3(n)$							8	14	13	1	− 6
$y(n)$	2	5	7	−1	−8	−7	7	17	13	1	− 6

$\therefore\ y(n) = x(n) * h(n) = \{2, 5, 7, -1, -8, -7, 7, 17, 13, 1, -6\}$

b) Overlap Save Method

In this method the longer sequence is sectioned into sequences of size equal to smaller sequence. The number of samples that will be obtained in the output of linear convolution of each section is determined. Then each section of longer sequence is converted to the size of output sequence using the samples of original longer sequences. The smaller sequence is also converted to the size of output sequence by appending with zeros. Then the circular convolution of each section is performed.

Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n). Given that $x(n) = \{1, 2, 3, -1, -2, -3, 4, 5, 6\}$.

Let x(n) be sectioned into three sequences each consisting of three samples as shown below.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

$N_3 = N_2$ = Length of each section of longer sequence.

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Now it can be proved that,

$$h(n) * h'(n) = \delta(n) \qquad .....(2.60)$$

Therefore the cascade of a system and its inverse is identity system.

Proof :

With reference to fig 2.36 we can write,

$$y(n) = x(n) * h(n) \qquad .....(2.61)$$

$$w(n) = y(n) * h'(n) \qquad .....(2.62)$$

On substituting for y(n) from equation (2.61) in equation (2.62) we get,

$$w(n) = x(n) * h(n) * h'(n) \qquad .....(2.63)$$

In equation (2.63),

if, $h(n) * h'(n) = \delta(n)$, then, $x(n) * \delta(n) = x(n)$

In a inverse system, w(n) = x(n), and so,

$h(n) * h'(n) = \delta(n)$. Hence proved.

2.12.2 Deconvolution

In an LTI system the response y(n) is given by convolution of input x(n) and impulse response h(n).

i.e., $y(n) = x(n) * h(n)$

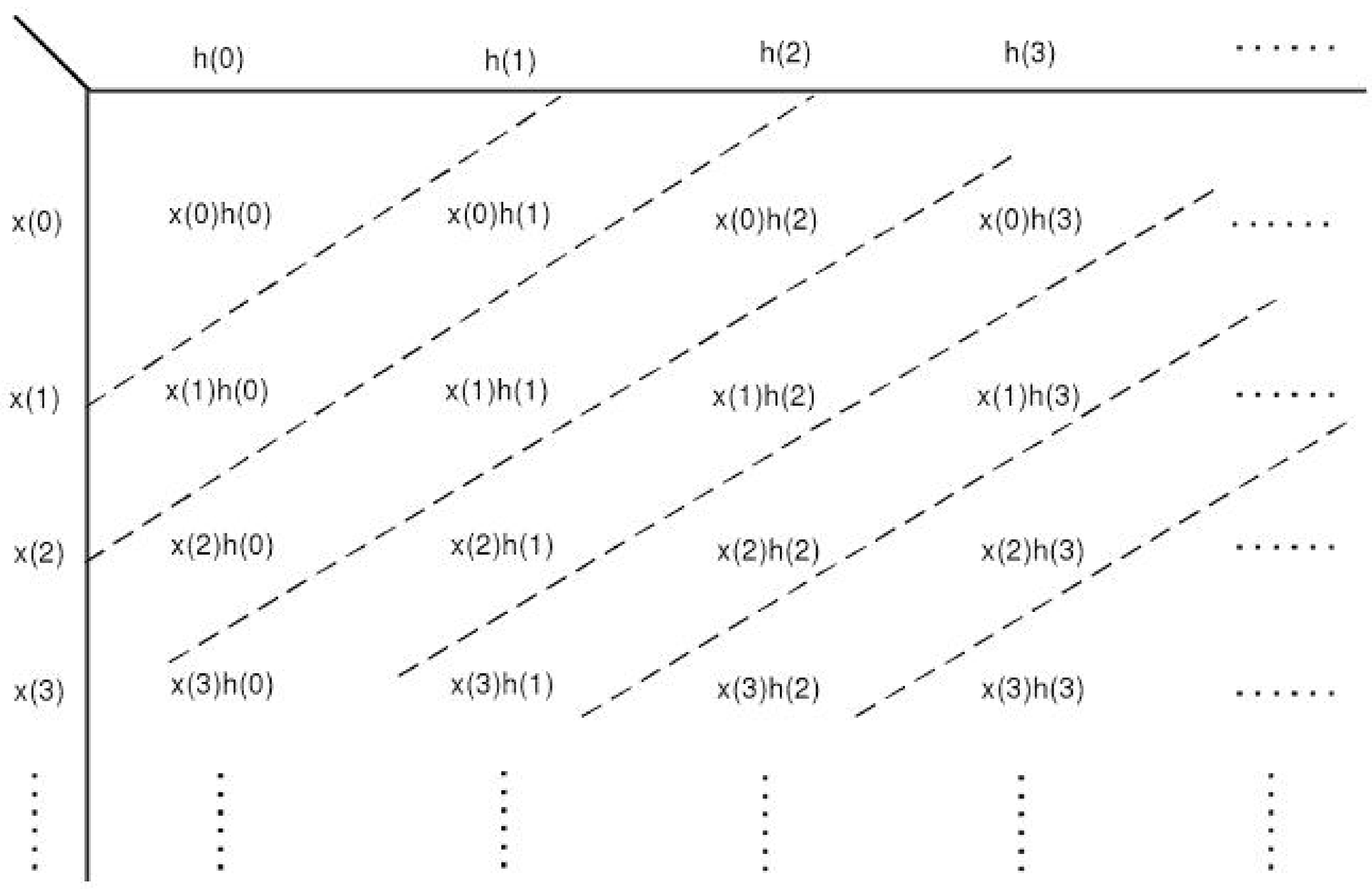
The process of recovering the input from the response of a system is called ***deconvolution***. (or the process of recovering x(n) from x(n) * h(n) is called ***deconvolution***).

When the response y(n) and impulse response h(n) are available, then the input x(n) can be computed using the equation (2.64).

$$x(n) = \frac{1}{h(0)}\left[y(n) - \sum_{m=0}^{n-1} x(m)\ h(n-m)\right] \qquad .....(2.64)$$

Proof :

Let x(n) and h(n) be finite duration sequences starting from n = 0. Consider the matrix method of convolution of x(n) and h(n) shown below.



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Properties of Correlation

1. The crosscorrelation sequence $r_{xy}(m)$ is simply a folded version of $r_{yx}(m)$,

   i.e., $r_{xy}(m) = r_{yx}(-m)$

   Similarly for autocorrelation sequence,

   $r_{xx}(m) = r_{xx}(-m)$

   Hence autocorrelation is an even function.

2. The crosscorrelation sequence satisfies the condition,

   $$\left|r_{xy}(m)\right| \le \sqrt{r_{xx}(0)\, r_{yy}(0)} = \sqrt{E_x\, E_y}$$

   where, E_x and E_y are energy of x(n) and y(n) respectively.

   On applying the above condition to autocorrelation sequence we get,

   $$\left|r_{xx}(m)\right| \le r_{xx}(0) = E_x$$

   From the above equations we infer that the crosscorrelation sequence and autocorrelation sequences attain their respective maximum values at zero shift/lag.

3. Using the maximum value of crosscorrelation sequence, the normalized crosscorrelation sequence is defined as,

   $$\rho_{xy}(m) \le \frac{r_{xy}(m)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}}$$

   Using the maximum value of autocorrelation sequence, the normalized autocorrelation sequence is defined as,

   $$\rho_{xx}(m) \le \frac{r_{xx}(m)}{r_{xx}(0)}$$

Methods of Computing Correlation

Method 1: Graphical Method

Let x(n) and y(n) be the input sequences and $r_{xy}(m)$ be the output sequence.

1. Sketch the graphical representation of the input sequences x(n) and y(n).
2. Shift the sequence y(n) to the left graphically so that the product of x(n) and shifted y(n) gives only one nonzero sample. Now multiply x(n) and shifted y(n) to get a product sequence, and then sum up the samples of product sequence, which is the first sample of output sequence.
3. To get the next sample of output sequence, shift y(n) of previous step to one position right and multiply the shifted sequence with x(n) to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
4. To get subsequent samples of output sequence, the step 3 is repeated until we get a nonzero product sequence.

Method 2: Tabular Method

The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence and shifted sequence is represented on a row in a table.

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The crosscorrelation sequence, $r_{xy}(m) = \{1, 1.5, 3.5, 4, 3, 2\}$

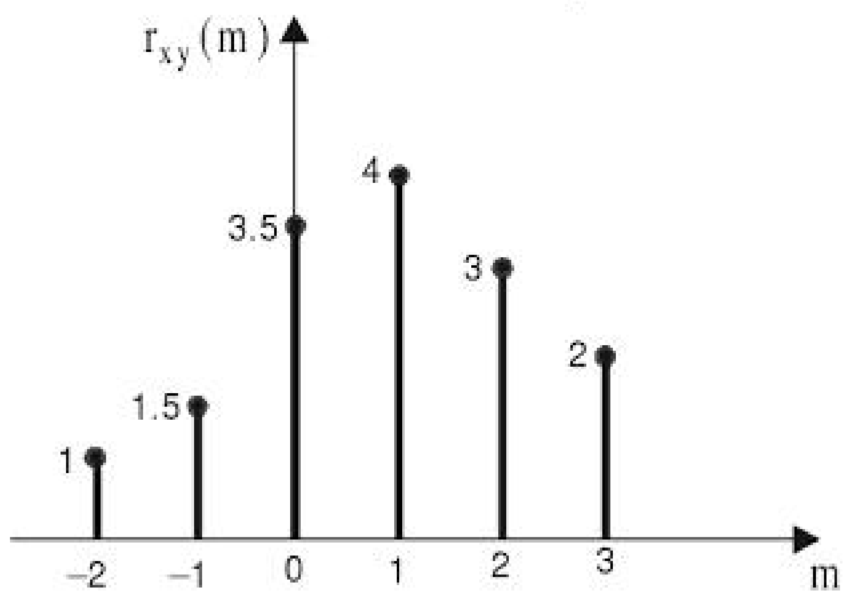

***Fig 9 :** Graphical representation of $r_{xy}(m)$.*

Method 2: Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

n	−2	−1	0	1	2	3	4	5
x(n)			1	1	2	2		
y(n)			1	0.5	1			
$y(n-(-2)) = y_{-2}(n)$	1	0.5	1					
$y(n-(-1)) = y_{-1}(n)$		1	0.5	1				
$y(n) = y_0(n)$			1	0.5	1			
$y(n-1) = y_1(n)$				1	0.5	1		
$y(n-2) = y_2(n)$					1	0.5	1	
$y(n-3) = y_3(n)$						1	0.5	1

Note: The unfilled boxes in the table are considered as zeros.

Each sample of $r_{xy}(m)$ is given by,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, y_m(n)\ ; \text{ where } y_m(n) = y(n-m)$$

To determine a sample of $r_{xy}(m)$ at $m = q$, multiply the sequence $x(n)$ and $y_q(n)$ to get a product sequence [i.e., multiply the corresponding elements of the row $x(n)$ and $y_q(n)$]. The sum of all the samples of the product sequence gives $r_{xy}(q)$.

When $m = -2$; $r_{xy}(-2) = \sum_{n=-2}^{3} x(n)\, y_{-2}(n) = 0 + 0 + 1 + 0 + 0 + 0 = 1$

When $m = -1$; $r_{xy}(-1) = \sum_{n=-1}^{3} x(n)\, y_{-1}(n) = 0 + 0.5 + 1 + 0 + 0 = 1.5$

When $m = 0$; $r_{xy}(0) = \sum_{n=0}^{3} x(n)\, y_0(n) = 1 + 0.5 + 2 + 0 = 3.5$

When $m = 1$; $r_{xy}(1) = \sum_{n=0}^{3} x(n)\, y_1(n) = 0 + 1 + 1 + 2 = 4$

When $m = 2$; $r_{xy}(2) = \sum_{n=0}^{4} x(n)\, y_2(n) = 0 + 0 + 2 + 1 + 0 = 3$

When $m = 3$; $r_{xy}(3) = \sum_{n=0}^{5} x(n)\, y_3(n) = 0 + 0 + 0 + 2 + 0 + 0 = 2$

$\therefore$ Crosscorrelation sequence, $r_{xy}(m) = \{1,\ 1.5,\ \underset{\uparrow}{3.5},\ 4,\ 3,\ 2\}$

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2.14.2 Methods of Computing Circular Correlation

Method 1 : Graphical Method

In graphical method the given sequences are converted to same size and represented on circles. In case of periodic sequences, the samples of one period are represented on circles. Let x(n) and y(n) be the given real sequences. Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n). The following procedure can be used to get a sample of $\bar{r}_{xy}(m)$ at $m = q$.

1. Represent the sequences x(n) and y(n) on circles.
2. Rotate (or shift) the sequence y(n), q times to get the sequence $y((n-q))_N$. If q is positive then rotate (or shift) the sequence in anticlockwise direction and if q is negative then rotate (or shift) the sequence in clockwise direction.
3. The sample of $\bar{r}_{xy}(q)$ at m = q is given by,

$$\bar{r}_{xy}(q) = \sum_{n=0}^{N-1} x(n)\, y((n-q))_N = \sum_{n=0}^{N-1} x(n)\, y_q(n)$$

where, $y_q(n) = y((n-q))_N$

Determine the product sequence $x(n)y_q(n)$ for one period.

4. The sum of all the samples of the product sequence gives the sample $\bar{r}_{xy}(q)$ [i.e., $\bar{r}_{xy}(m)$ at m = q].

The above procedure is repeated for all possible values of m to get the sequence $\bar{r}_{xy}(m)$.

Method 2 : Using Tabular Array

Let x(n) and y(n) be the given real sequences. Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n). The following procedure can be used to get a sample of $\bar{r}_{xy}(m)$ at $m = q$.

1. Represent the sequences x(n) and y(n) as two rows of tabular array.
2. Periodically extend y(n). Here the periodicity is N, where N is the length of the given sequences.
3. Shift the sequence y(n), q times to get the sequence $y((n-q))_N$. If q is positive then shift the sequence to the right and if q is negative then shift the sequence to the left.
4. The sample of $\bar{r}_{xy}(q)$ at m = q is given by,

$$\bar{r}_{xy}(q) = \sum_{n=0}^{N-1} x(n)\, y((n-q))_N = \sum_{n=0}^{N-1} x(n)\, y_q(n)$$

where, $y_q(n) = y((n-q))_N$

Determine the product sequence $x(n)y_q(n)$ for one period.

5. The sum of all the samples of the product sequence gives the sample $\bar{r}_{xy}(q)$ [i.e., $\bar{r}_{xy}(m)$ at m = q].

The above procedure is repeated for all possible values of m to get the sequence $\bar{r}_{xy}(m)$.

Method 3: Using Matrices

Let x(n) and y(n) be the given N-point sequences. The circular correlation of x(n) and y(n) yields another N-point sequence $\bar{r}_{xy}(m)$.

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2.15. Summary of Important Concepts

1. The discrete signal is a function of a discrete independent variable.
2. In a discrete time signal, the value of discrete time signal and the independent variable time are discrete.
3. The digital signal is same as discrete signal except that the magnitude of the signal is quantized.
4. A discrete time sinusoid is periodic only if its frequency is a rational number.
5. Discrete time sinusoids whose frequencies are separated by an integer multiple of 2p are identical.
6. The sampling is the process of conversion of continuous time signal into discrete time signal.
7. The time interval between successive samples is called sampling time or sampling period.
8. The inverse of sampling period is called sampling frequency.
9. The phenomenon of high frequency component getting the identity of low-frequency component during sampling is called aliasing.
10. For analog signal with maximum frequency F_{max}, the sampling frequency should be greater than $2F_{max}$.
11. When sampling frequency F_s is equal to $2F_{max}$, the sampling rate is called Nyquist rate.
12. The signals that can be completely specified by mathematical equations are called deterministic signals.
13. The signals whose characteristics are random in nature are called nondeterministic signals.
14. A signal x(n) is periodic with periodicity of N samples if $x(n + N) = x(n)$.
15. When a signal exhibits symmetry with respect to $n = 0$ then it is called an even signal.
16. When a signal exhibits antisymmetry with respect to $n = 0$, then it is called an odd signal.
17. When the energy E of a signal is finite and nonzero, the signal is called energy signal.
18. When the power P of a signal is finite and nonzero, the signal is called power signal.
19. For energy signals, the energy will be finite and average power will be zero.
20. For power signals the average power is finite and energy will be infinite.
21. A signal is said to be causal, if it is defined for n ³ 0.
22. A signal is said to be noncausal, if it is defined for both $n \leq 0$ and $n > 0$.
23. A signal is said to be anticausal, if it is defined for $n \leq 0$.
24. A discrete time system is a device or algorithm that operates on a discrete time signal.
25. When a system satisfies the properties of linearity and time invariance, it is called an LTI system.
26. When the input to a discrete time system is unit impulse d(n), the output is called impulse response, h(n).
27. In a static or memoryless system, the output at any instant n depends on input at the same time.
28. A system is said to be time invariant if its input-output characteristics do not change with time.
29. A linear system is one that satisfies the superposition principle.
30. A system is said to be causal if the output does not depends on future inputs/outputs.
31. When a system output at any time n depends on future inputs/outputs, it is called a noncausal system.
32. System is said to be BIBO stable if and only if every bounded input produces a bounded output.
33. When a system output at any time n depends on past outputs, it is called a recursive system.
34. A system whose output does not depends on past outputs is called a nonrecursive system.
35. The convolution of N_1 and N_2 sample sequences produce a sequence consisting of N_1+N_2-1 samples.
36. In an LTI system, response for an arbitrary input is given by convolution of input with impulse response.
37. The output sequence of circular convolution is also periodic sequence with periodicity of N samples.
38. The inverse system is used to recover the input from the response of a system.
39. The process of recovering the input from the response of a system is called deconvolution.
40. The correlation of two different sequences is called crosscorrelation.
41. The correlation of a sequence with itself is called autocorrelation.

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Differences

1. Correlation operation does not involve change of index and folding of one of the input sequence.
2. The convolution operation is commutative, [i.e., $x(n) * y(n) = y(n) * x(n)$], whereas in correlation operation in order to satisfy commutative property, while performing correlation of y(n) and x(n), the shifting has to performed in opposite direction to that of performing correlation of x(n) and y(n).

Q2.15 ***Let $r_{xy}(m)$ be the correlation sequence obtained by correlation of x(n) and y(n), how will you determine the start and end point of $r_{xy}(m)$? What will be the length of $r_{xy}(m)$?***

Let, length of x(n) be N_1 and starts at $n = n_1$. Let length of y(n) be N_2 and starts at $n = n_2$.

Now, $r_{xy}(m)$ will start at $m_i = n_1 - (n_2 + N_2 - 1)$

$r_{xy}(m)$ will end at $m_f = m_i + (N_1 + N_2 - 2)$

The length of $r_{xy}(m)$ is $N_1 + N_2 - 1$.

Q2.16 ***What are the differences between crosscorrelation and autocorrelation?***

1. Crosscorrelation operation is correlation of two different sequences, whereas autocorrelation is correlation of a sequence with itself.
2. Autocorrelation operation is an even function, whereas crosscorrelation is not an even function.

Q2.17 ***Perform the correlation of the two sequences, x(n) = {1, 2, 3} and y(n) = {2, 4, 1}.***

Solution

Given that, $x(n) = \{\underset{\uparrow}{1}, 2, 3\}$ and $y(n) = \{\underset{\uparrow}{2}, 4, 1\}$. $\therefore$ $y(-n) = \{1, 4, \underset{\uparrow}{2}\}$

The sequence x(n) is arranged as a column and the folded sequence y(−n) is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with column element. The sum of the diagonal elements gives the samples of the crosscorrelation sequence, $r_{xy}(m)$.

x(n) ↓ \ y(−n) →	1	4	2
1	1 × 1	1 × 4	1 × 2
2	2 × 1	2 × 4	2 × 2
3	3 × 1	3 × 4	3 × 2

⇒

x(n) ↓ \ y(−n) →	1	4	2
1	1	4	2
2	2	8	4
3	3	12	6

$r_{xy}(-2) = 1$; $r_{xy}(-1) = 2 + 4 = 6$; $r_{xy}(0) = 3 + 8 + 2 = 13$; $r_{xy}(1) = 12 + 4 = 16$; $r_{xy}(2) = 6$;

$\therefore$ $r_{xy}(m) = \{1, 6, \underset{\uparrow}{13}, 16, 6\}$

Q2.18 ***Perform the circular correlation of the two sequences, x(n) = {1, 2, 3} and y(n) = {2, 4, 1}.***

Solution

Let $\bar{r}_{xy}(m)$ be the sequence obtained from circular correlation of x(n) and y(n). The sequence x(n) can be arranged as a column vector of order 3×1 and using the samples of y(n) a 3×3 matrix is formed as shown below. The product of two matrices gives the sequence $\bar{r}_{xy}(m)$.

$$\begin{bmatrix} y(0) & y(1) & y(2) \\ y(2) & y(0) & y(1) \\ y(1) & y(2) & y(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} = \begin{bmatrix} \bar{r}_{xy}(0) \\ \bar{r}_{xy}(1) \\ \bar{r}_{xy}(2) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 4 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 17 \\ 12 \end{bmatrix}$$

$\therefore$ $\bar{r}_{xy}(m) = \{13, 17, 12\}$

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Note: The above program should be stored as a separate file in the current working directory

Program to perform amplitude scaling and time shift on y(n)

```
%To Perform Amplitude scaling and Time shift on signal x(n)=1+n;
%for n= 0 to 2
%include y.m file in current work directory which declare given signal as
%function y(n)

n=-5:1:5;           %specify range of n

y0 =y(n);           %assign the given signal as y0
y1 =1.5*y(n);       %compute the amplified version of x(n)
y2 =0.5*y(n);       %compute the attenuated version of x(n)
y3 =y(n-2);         %compute the delayed version of x(n)
y4 =y(n+2);         %compute the advanced version of x(n)

%plot the given signal and amplitude scaled signal
subplot(2,3,1);stem(n,y0);
xlabel('n');ylabel('x(n)');title('Signal x(n)');
subplot(2,3,2);stem(n,y1);
xlabel('n');ylabel('x1(n)');title('Amplified signal 1.5x(n)');
subplot(2,3,3);stem(n,y2);
xlabel('n');ylabel('x2(n)');title('Attenuated signal 0.5x(n)');

%plot the given signal and time shifted signal
subplot(2,3,4);stem(n,y0);
xlabel('n');ylabel('x(n)');title('Signal x(n)');
subplot(2,3,5);stem(n,y3);
xlabel('n');ylabel('x3(n)');title('Delayed signal x(n-2)');
subplot(2,3,6);stem(n,y4);
xlabel('n');ylabel('x4(n)');title('Advanced signal x(n+2)');
```

OUTPUT

The input and output waveforms of program 2.4 are shown in fig P2.4.

Program 2.5

Write a MATLAB program to perform convolution of the following two discrete time signals.

x1(n)=1; 1<n<10 x2(n)=1; 2<n<10

```
%*******************Program to perform convolution of two signals
%*****************x1(N)=1; n= 1 to 10 and x2(n)=1; n= 2 to 10

n = 0 : 1 : 15;             %specify range of n

x1=1.*(n>=1 & n<=10);       %generate signal x1(n)
x2=1.*(n>=2 & n<=10);       %generate signal x2(n)
N1=length(x1);
N2=length(x2);
x3=conv(x1,x2);             %perform convolution of signals x1(n) and x2(n)
n1=0 : 1 : N1+N2-2;         %specify range of n for x3(n)
```

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10. The zero input response (or) natural response is mainly due to,

a) Initial stored energy in the system **b)** Initial conditions in the system
c) Specific input signal **d)** both a and b

11. If $x(n) = a^n\, u(n)$ is the input signal, then the particular solution $y_p(n)$ will be,

a) $K^n\, a^n\, u(n)$ **b)** $K\, a^n\, u(n)$
c) $K_1\, a^n\, u(n) + K_2\, a^n\, u(n)$ **d)** $K\, a^{-n}\, u(n)$

12. The discrete time system, $y(n) = x(n-3) - 4x(n-10)$ is a,

a) dynamic system **b)** memoryless system **c)** time varying system **d)** none of the above

13. An LTI discrete time system is causal if and only if,

a) h(n) ¹ 0 for n < 0 **b)** $h(n) = 0$ for $n < 0$ **c)** h(n) ¹ ¥ for n < 0 **d)** h(n) ¹ 0 for n > 0

14. Which of the following system is causal?

a) $h(n) = n\left(\frac{1}{2}\right)^n u(n+1)$ **b)** $y(n) = x^2(n) - x(n+1)$ **c)** $y(n) = x(-n) + x(2n-1)$ **d)** $h(n) = n\left(\frac{1}{2}\right)^n u(n)$

15. An LTI system is stable, if the impulse response is,

a) $\sum_{n=-\infty}^{\infty} |h(n)| = 0$ **b)** $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$ **c)** $\sum_{n=-\infty}^{\infty} |h(n)| \neq 0$ **d)** either a or b

16. The system $y(n) = \sin[x(n)]$ is,

a) stable **b)** BIBO stable **c)** unstable **d)** none of the above

17. Two parallel connected discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system with impulse response,

a) $h_1(n) * h_2(n)$ **b)** $h_1(n) + h_2(n)$ **c)** $h_1(n) - h_2(n)$ **d)** $h_1(n) * [h_1(n) + h_2(n)]$

18. Sectioned convolution is performed if one of the sequence is very much larger than the other in order to overcome,

a) long delay in getting output **b)** larger memory space requirment
c) both a and b **d)** none of the above

19. In overlap save method, the convolution of various sections are performed by,

a) zero padding **b)** linear convolution **c)** circular convolution **d)** both b and c

20. If $x(n)$ is N_1-point sequence,if $y(n)$ is N_2-point sequence, if $r_{xy}(m)$ is the correlation sequence starts at $m = m_i$, then the value of m corresponding to last sample of $r_{xy}(m)$ is,

a) $m_f = m_i + (N_1 + N_2 - 2)$ **b)** $m_f = m_i + (2N - 2)$ **c)** $m_f = m_i + (N_1 + N_2 - 1)$ **d)** $m_f = m_i + (2N + 1)$

21. For a system, $y(n) = nx(n)$, the inverse system will be,

a) $y\left(\frac{1}{n}\right)$ **b)** $\frac{1}{n}\, y(n)$ **c)** $ny(n)$ **d)** $n^{-1}y(n)$

22. For a system $y(n) = x(n-3)$ the impulse response of the system and the inverse system will be ——— and ——— respectively.

a) h(n) = d(n + 3), x(n) = y(n – 3) **b)** $h(n) = \delta(3n),\ x(n) = y\left(\frac{n}{3}\right)$
c) h(n) = d(n – 3), x(n) = y(n + 3) **d)** h(n) = d(n + 3), x(n) = y(3n)

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E2.17 Perform crosscorrelation of the sequences,

$$x(n)=\{\underset{\uparrow}{-1}, \; 2 \; 3, \; -4,\} \; ; \qquad h(n)=\{2, \; \underset{\uparrow}{-1}, \; -3,\}$$

E2.18 Determine the autocorrelation sequence for $x(n)=\{1, \; \underset{\uparrow}{4}, \; 3, \; -5, \; 2\}$.

E2.19 Find the inverse system for the following discrete time system,

$$y(n)=\sum_{p=0}^{n} c^{p}\, x(p-2) \; ; \quad \text{for} \quad n \geq 0$$

E2.20 A discrete time system is excited by an input x(n), and the response is, $y(n)=\{\underset{\uparrow}{4}, \; 3, \; 6, \; 7.5, \; 3, \; 30, \; -8\}$.

If the impulse response of the system is $h(n)=\{\underset{\uparrow}{2}, \; 4, \; -2\}$, ***then what will be the input to the system?***

E2.21 Perform circular correlation of the sequence, $x(n)=\{-1, \; 1, \; 2, \; 6\}$ ***and*** $y(n)=\{4, \; -2, \; -1, \; 2\}$.

Answers

E2.1 **a)** periodic; N=16 **b)** nonperiodic **c)** periodic; N=6 **d)** periodic; N=32 **e)** nonperiodic. **f)** periodic; N=8

E2.2 **a)** $x_e(n)=\frac{1}{2}\left[a^{-2n}+a^{2n}\right]$ **b)** $x_e(n)=8\cos\frac{\pi}{6}n$ **c)** $x_e(n)=\{1, 1, 2, \underset{\uparrow}{6}, 2, 1, 1\}$

$x_o(n)=\frac{1}{2}\left[a^{-2n}-a^{2n}\right]$ $x_o(n)=-j8\sin\frac{\pi}{6}n$ $x_o(n)=\{-1, -1, -2, \underset{\uparrow}{0}, 2, 1, 1\}$

E2.3 **a)** $x(n)=2\sin\frac{4\pi n}{3}$; Alias frequency = 100Hz   **b)** $F_s = 25\text{Hz}$

c) $F_{s,min}=100\,\text{Hz}$; $x(nT)=3\sin\frac{2\pi n}{5}+2\cos\frac{\pi n}{2}$ $(\sin \pi n = 0, \text{for integer n})$

The component sin100pt will give always zero samples when sampled at 100Hz for any value of n (Refer fig E2.3c).

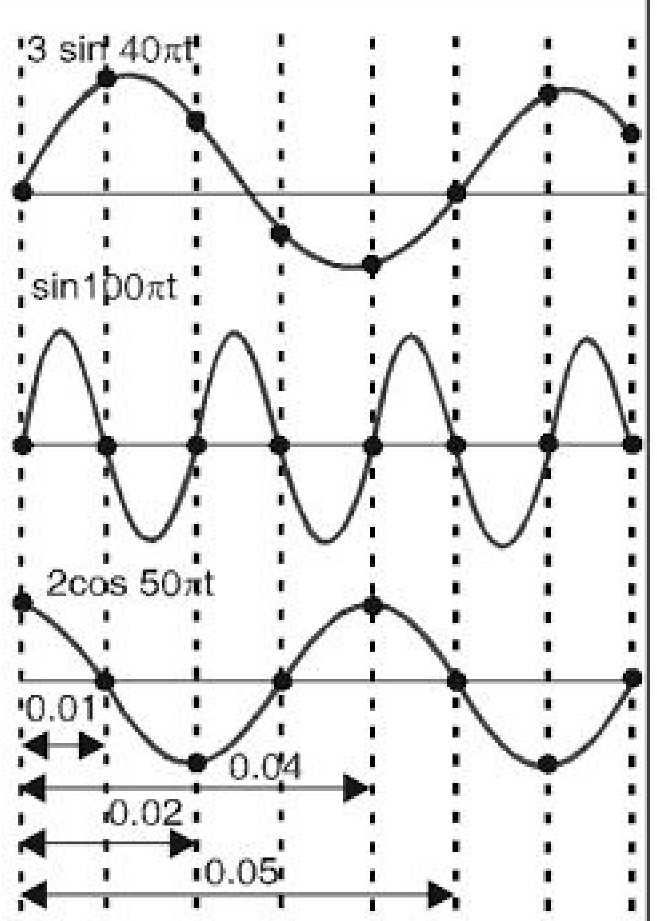

Fig E2.3c : Sampling points.

E2.4 **a)** E = 1.435J ; P = 0 ; Energy signal.

b) E = ¥ ; P = 0.5W ; Power signal.

c) E = ¥ ; P = 0.25W ; Power signal.

d) E = ¥ ; P = 2 W ; Power signal.

E2.5 **a)**

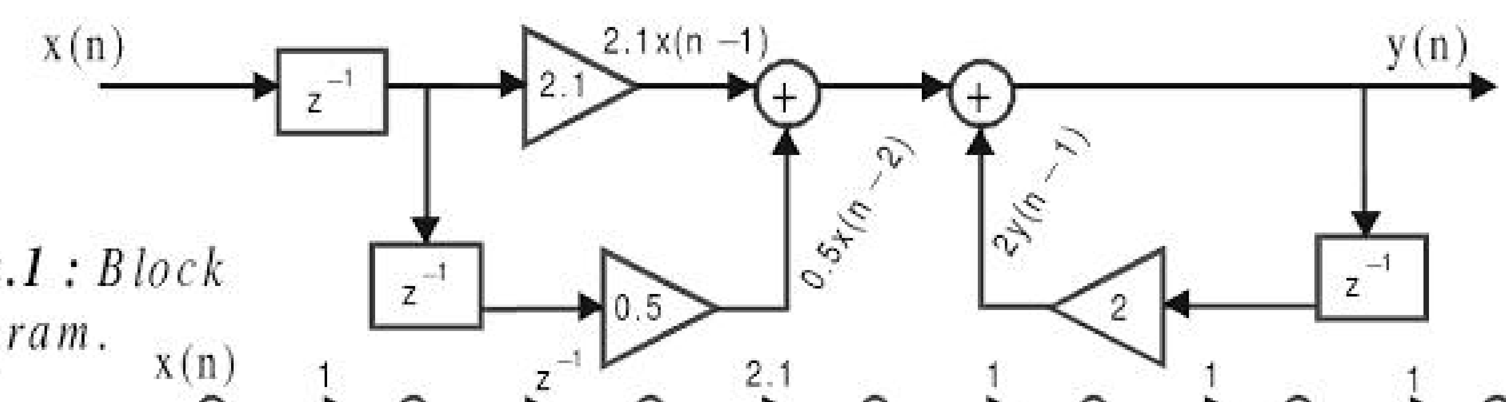

Fig E2.5a.1 : Block diagram.

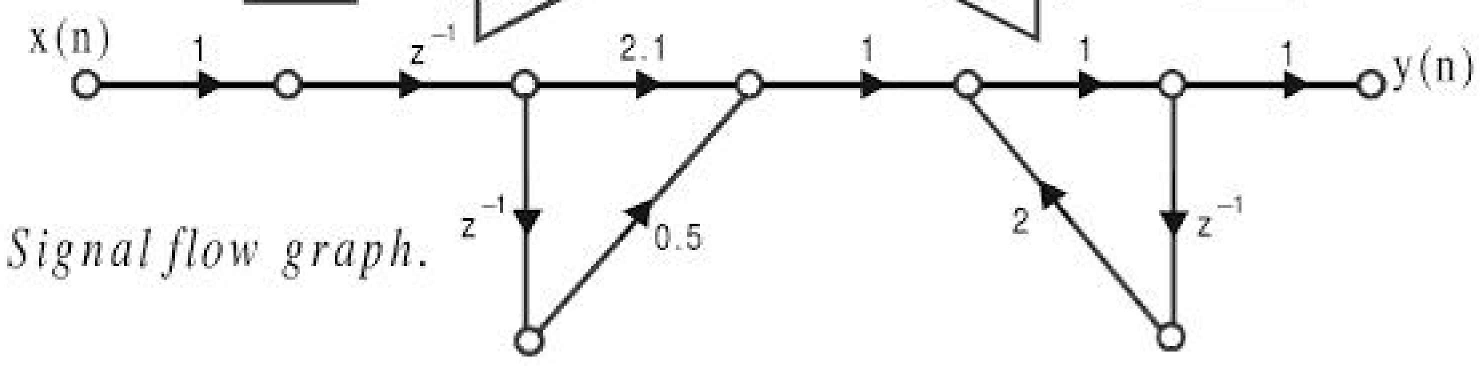

Fig E2.5a.2 : Signal flow graph.

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Proof :

Consider the definition of Z-transform of x(n),

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{k=-\infty}^{+\infty} x(k)\, z^{-k}$$
Let n → k

$$X(z)\, z^{n-1} = \sum_{k=-\infty}^{+\infty} x(k)\, z^{-k}\, z^{n-1}$$
Multiply both sides by z^{n-1}

Let us integrate the above equation on both sides over a closed contour "C" within the ROC of X(z) which encloses the origin.

$$\therefore \oint_c X(z)\, z^{n-1}\, dz = \oint_c \sum_{k=-\infty}^{+\infty} x(k)\, z^{n-1-k} dz$$

$$= \sum_{k=-\infty}^{+\infty} x(k) \oint_c z^{n-1-k} dz$$

Interchanging the order of summation and integration.

Multiply and divide by 2pj.

$$= 2\pi j \sum_{k=-\infty}^{+\infty} x(k)\, \frac{1}{2\pi j} \oint_c z^{n-1-k} dz \qquad .....(3.4)$$

By Cauchy integral theorem,

$$\frac{1}{2\pi j}\oint_c z^{n-1-k}\, dz = 1 \quad ; \quad k = n$$
$$= 0 \quad ; \quad k \neq n$$

On applying Cauchy integral theorem the equation (3.4) reduces to,

$$\left. \sum_{k=-\infty}^{+\infty} x(k) \right|_{n=k} = x(n)$$

$$\oint_c X(z)\, z^{n-1}\, dz = 2\pi j\, x(n)$$

$$\therefore\ x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz$$

Geometric Series

The Z-transform of a discrete time signal involves convergence of geometric series. Hence the following two geometric series sum formula will be useful in evaluating Z-transform.

1. Infinite geometric series sum formula.

If C is a complex constant and $0 < |C| < 1$, then,

$$\boxed{\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}} \qquad .....(3.5)$$

2. Finite geometric series sum formula.

If C is a complex constant and,

$$\text{When } C \neq 1,\ \sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C} = \frac{C^N-1}{C-1} \quad \text{or} \quad \sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1} \qquad .....(3.6)$$

$$\text{When } C = 1,\ \sum_{n=0}^{N-1} C^n = N \quad \text{or} \quad \sum_{n=0}^{N} C^n = N+1 \qquad .....(3.7)$$

Note : The infinite geometric series sum formula requires that the magnitude of C be strictly less than unity, but the finite geometric series sum formula is valid for any value of C.

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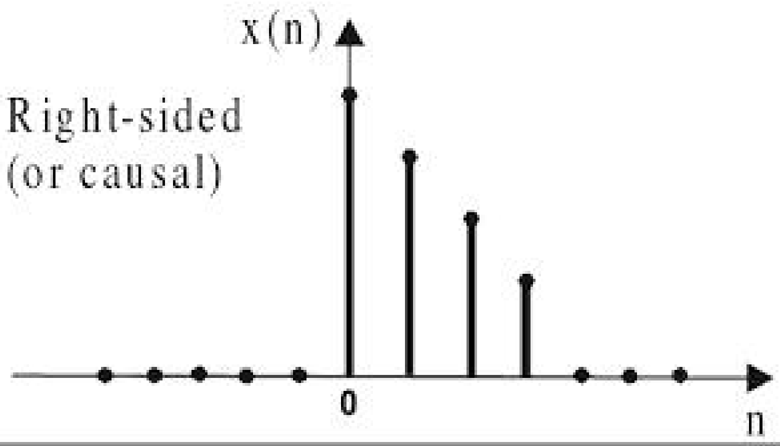
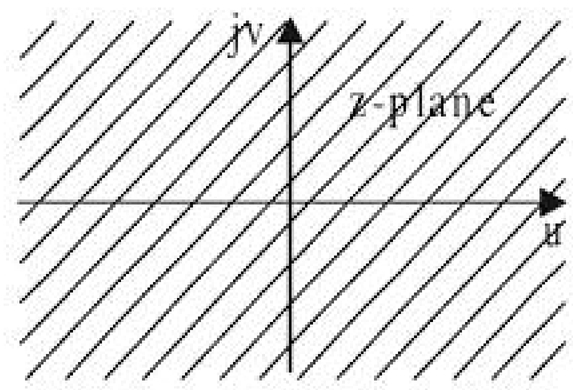
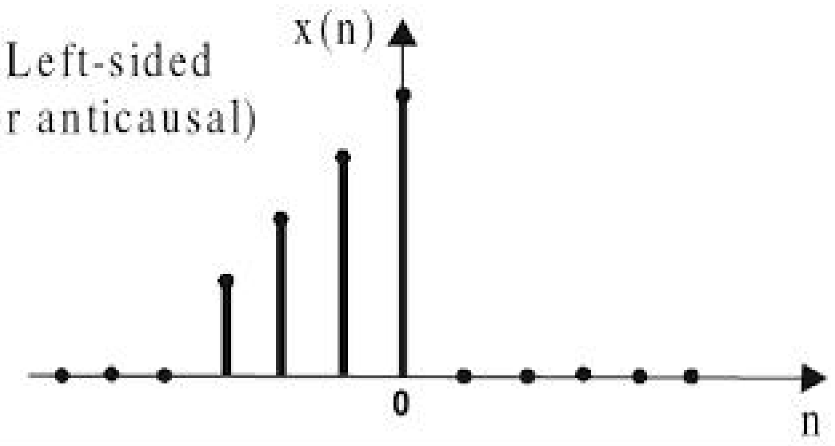
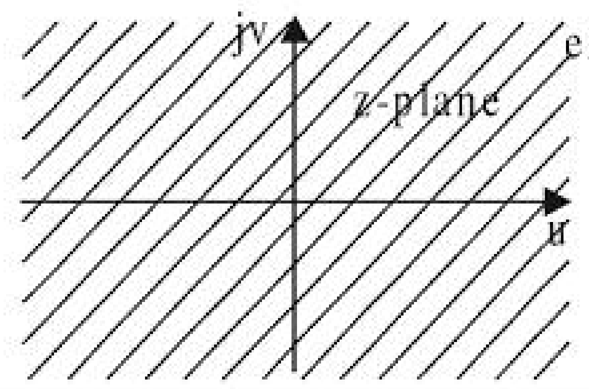
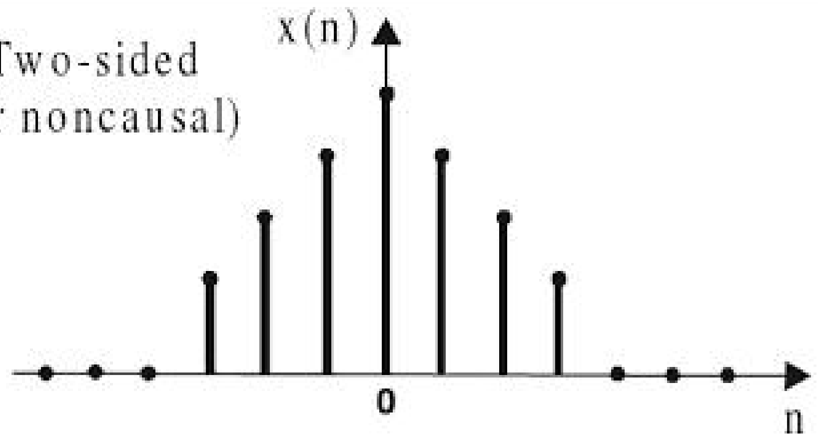
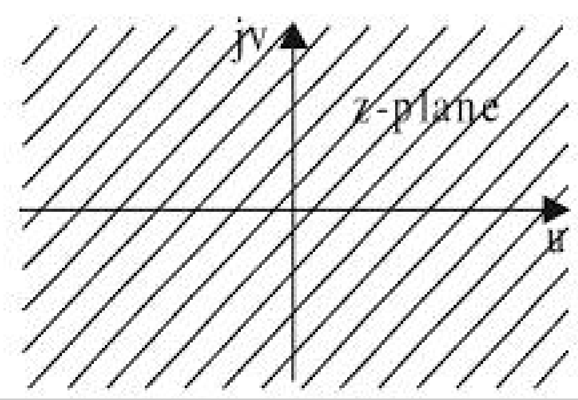
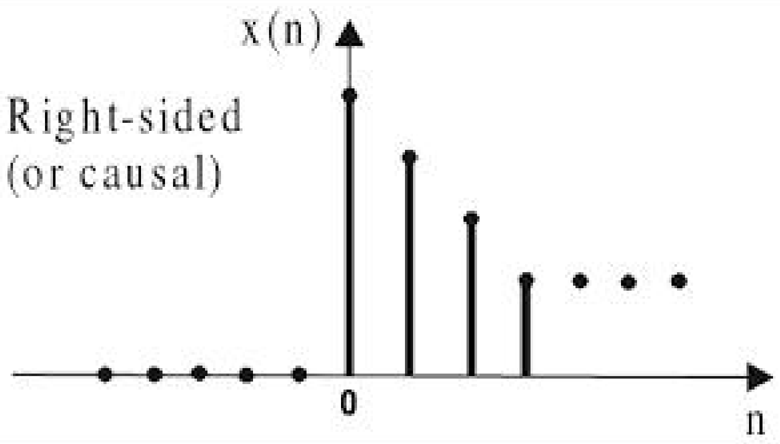
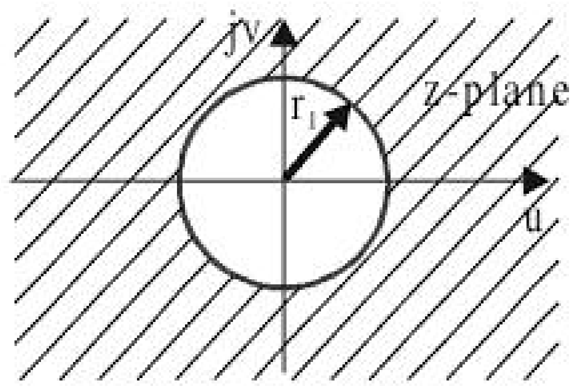
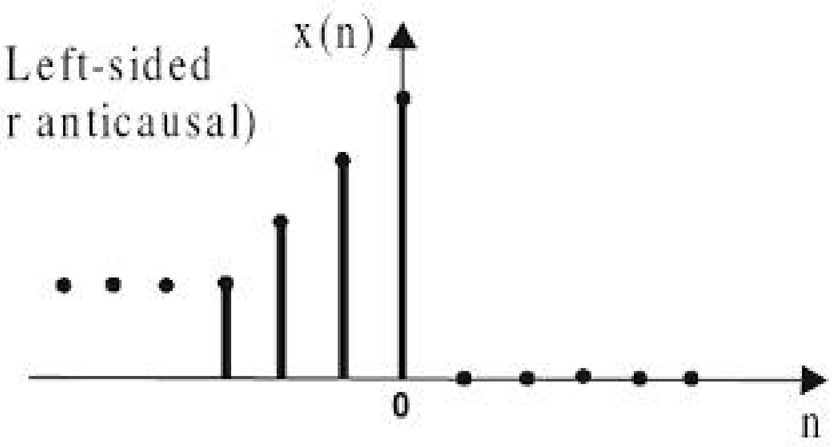
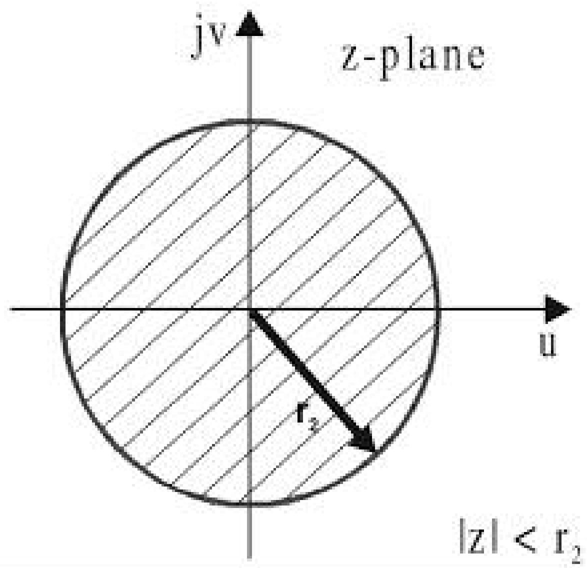
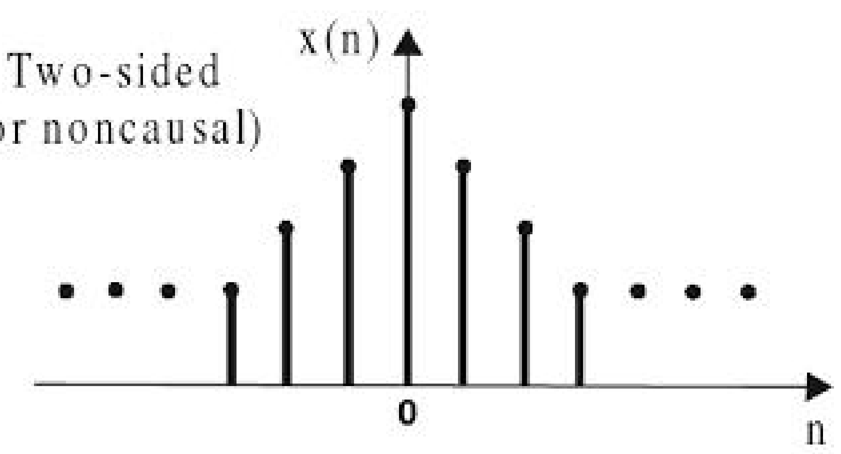
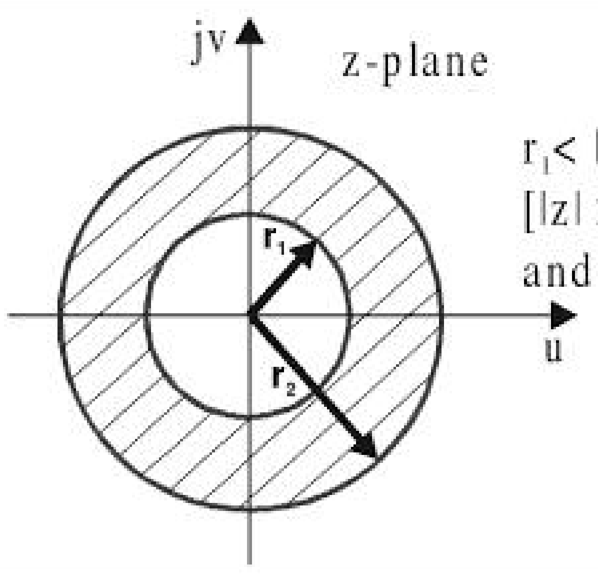
Table 3.2 : Characteristic Families of Signals and Corresponding ROC

Signal	ROC in z-plane
Finite Duration Signals	
Right-sided (or causal) 	 Entire z-plane except z = 0
Left-sided (or anticausal) 	 Entire z-plane except $z = \infty$
Two-sided (or noncausal) 	 Entire z-plane except z = 0 and $z = \infty$
Infinite Duration Signals	
Right-sided (or causal) 	 $\|z\|>r_1$
Left-sided (or anticausal) 	 $\|z\| < r_2$
Two-sided (or noncausal) 	 $r_1 < \|z\| < r_2$ [$\|z\| > r_1$ and $\|z\| < r_2$]

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3.3 Properties of Ƶ-Transform

1. Linearity property

The linearity property of Ƶ-transform states that the Ƶ-transform of linear weighted combination of discrete time signals is equal to similar linear weighted combination of Ƶ-transform of individual discrete time signals.

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$ then by linearity property,

$$\mathcal{Z}\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(z) + a_2X_2(z) \qquad ; \quad \text{where, } a_1 \text{ and } a_2 \text{ are constants.}$$

Proof :

By definition of Ƶ-transform,

$$X_1(z) = \mathcal{Z}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} \qquad .....(3.8)$$

$$X_2(z) = \mathcal{Z}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(3.9)$$

$$\therefore\ \mathcal{Z}\{a_1x_1(n)+a_2x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[a_1x_1(n)+a_2x_2(n)\right] z^{-n} = \sum_{n=-\infty}^{+\infty} \left[a_1x_1(n)\, z^{-n} + a_2x_2(n)\, z^{-n}\right]$$

$$= \sum_{n=-\infty}^{+\infty} a_1x_1(n)\, z^{-n} + \sum_{n=-\infty}^{+\infty} a_2x_2(n) z^{-n} = a_1 \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} + a_2 \sum_{n=-\infty}^{+\infty} x_2(n) z^{-n}$$

$$= a_1\, X_1(z) + a_2X_2(z) \qquad \boxed{\text{Using equations (3.8) and (3.9).}}$$

2. Shifting property

Case i: Two-sided Ƶ-transform

The shifting property of Ƶ-transform states that, Ƶ-transform of a shifted signal shifted by m-units of time is obtained by multiplying z^m to Ƶ-transform of unshifted signal.

Let, $\mathcal{Z}\{x(n)\} = X(z)$

Now, by shifting property,

$$\mathcal{Z}\{x(n-m)\} = z^{-m}\, X(z)$$

$$\mathcal{Z}\{x(n+m)\} = z^{m}\, X(z)$$

Proof :

By definition of Ƶ-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(3.10)$$

$$\therefore\quad \mathcal{Z}\{x(n-m)\} = \sum_{n=-\infty}^{+\infty} x(n-m)\, z^{-n}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-(m+p)} \qquad \boxed{\begin{array}{l}\text{Let, } n-m=p,\ \therefore\ n=p+m \\ \text{when } n \to -\infty,\ p \to -\infty \\ \text{when } n \to +\infty,\ p \to +\infty\end{array}}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-m}\ z^{-p}$$

$$= z^{-m} \sum_{p=-\infty}^{+\infty} x(p)\, z^{-p} = z^{-m} \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad \boxed{\text{Let, } p \to n}$$

$$= z^{-m}\, X(z) \qquad \boxed{\text{Using equation (3.10).}}$$

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7. Convolution theorem

If $\mathcal{Z}\{x_1(n)\} = X_1(z)$

and $\mathcal{Z}\{x_2(n)\} = X_2(z)$

then $\mathcal{Z}\{x_1(n) * x_2(n)\} = X_1(z)\, X_2(z)$

$$\text{where, } x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) \qquad .....(3.21)$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X_1(z) = \mathcal{Z}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} \qquad .....(3.22)$$

$$X_2(z) = \mathcal{Z}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(3.23)$$

$$\therefore\ \mathcal{Z}\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[x_1(n) * x_2(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\right] z^{-n} \qquad \boxed{\text{Using equation (3.21).}}$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\, z^{-n}\, z^{-m}\, z^{m} \qquad \boxed{\text{Multiply by } z^{m} \text{ and } z^{-m}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, z^{-m} \sum_{n=-\infty}^{+\infty} x_2(n-m)\, z^{-(n-m)}$$

Let, n – m = p
when n ® -¥, p ® -¥
when n ® +¥, p ® +¥

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, z^{-m} \sum_{p=-\infty}^{+\infty} x_2(p)\, z^{-p}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n}\right]\left[\sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n}\right] \qquad \boxed{\begin{array}{l}\text{Let } m = n, \text{ in first summation.}\\ \text{Let } p = n, \text{ in second summation.}\end{array}}$$

$$= X_1(z)\, X_2(z) \qquad \boxed{\text{Using equations (3.22) and (3.23).}}$$

8. Correlation property

If $\mathcal{Z}\{x(n)\} = X(z)$ and $\mathcal{Z}\{y(n)\} = Y(z)$

then $\mathcal{Z}\{r_{xy}(m)\} = X(z)\, Y(z^{-1})$

$$\text{where, } r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) \qquad .....(3.24)$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(3.25)$$

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n} \qquad .....(3.26)$$

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Proof :

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Now, by definition of inverse $\mathcal{Z}$-transform,

$$x_1(n) = \frac{1}{2\pi j} \oint_C X_1(z)\, z^{n-1}\, dz = \frac{1}{2\pi j} \oint_C X_1(v)\, v^{n-1}\, dv \qquad \boxed{\text{let, } z = v} \qquad .....(3.32)$$

Now, by definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\left\{x_2(n)\right\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(3.33)$$

Using the definition of $\mathcal{Z}$ - transform, the $\mathcal{Z}\left\{x_1(n)\, x_2^*(n)\right\}$ can be written as,

$$\mathcal{Z}\left\{x_1(n)\, x_2^*(n)\right\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} \qquad .....(3.34)$$

On substituting for $x_1(n)$ from equation (3.32) in equation (3.34) we can write,

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2\pi j} \oint_C X_1(v)\, v^{n-1}\, dv\right] x_2^*(n)\, z^{-n}$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{+\infty} x_2^*(n)\, z^{-n}\, v^{n}\right] v^{-1}\, dv \qquad \boxed{\begin{array}{l}\text{Interchanging the}\\ \text{order of summation}\\ \text{and integration.}\end{array}}$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{+\infty} x_2^*(n) \left(\frac{z}{v}\right)^{-n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{+\infty} x_2(n) \left(\frac{z^*}{v^*}\right)^{-n}\right]^* v^{-1}\, dv$$

$$= \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{z^*}{v^*}\right) v^{-1}\, dv \qquad \boxed{\text{using equation (3.33).}}$$

Let us take limit z ® 1 in the above equation,

$$\therefore \lim_{z \to 1} \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} = \lim_{z \to 1} \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{z^*}{v^*}\right) v^{-1}\, dv$$

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

$$\therefore \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(z)\, X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz \qquad \boxed{\text{let } v = z}$$

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c) Given that, $x(t) = \cos W_0 t$

The discrete time signal is generated by replacing t by nT, where T is the sampling time period.7

$\backslash \quad x(n) = \cos(W_0 nT) = \cos wn$; where $w = W_0 T$

By the definition of one-sided Z-transform,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \cos\omega n \times z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n} + e^{-j\omega n}}{2} z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n$$

Using infinite geometric series sum formula.

$$= \frac{1}{2}\frac{1}{1 - e^{j\omega} z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-j\omega} z^{-1}}$$

$$= \frac{1}{2}\frac{z}{z - e^{j\omega}} + \frac{1}{2}\frac{z}{z - e^{-j\omega}}$$

$$= \frac{z(z - e^{-j\omega}) + z(z - e^{j\omega})}{2(z - e^{j\omega})(z - e^{-j\omega})} = \frac{z^2 - z e^{-j\omega} + z^2 - z e^{j\omega}}{2(z^2 - z e^{-j\omega} - z e^{j\omega} + e^{j\omega} e^{-j\omega})}$$

$$= \frac{2z^2 - z(e^{j\omega} + e^{-j\omega})}{2[z^2 - z(e^{j\omega} + e^{-j\omega}) + 1]} = \frac{z^2 - z(e^{j\omega} + e^{-j\omega})/2}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1}$$

$$= \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1} \quad ; \quad \text{where } \omega = \Omega_0 T$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Example 3.5

Find the one-sided Z-transform of the discrete time signals generated by mathematically sampling the following continuous time signals.

a) $e^{-at} \cos W_0 t$ **b)** $e^{-at} \sin W_0 t$

Solution

a) Given that, $x(t) = e^{-at} \cos\Omega_0 t$

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

$\backslash \quad x(n) = e^{-anT} \cos W_0 nT = e^{-anT} \cos wn$; where $w = W_0 T$

By the definition of one-sided Z-transform we get,

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT} \cos\omega n\, z^{-n} = \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right) z^{-n}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT} e^{j\omega} z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT} e^{-j\omega} z^{-1}\right)^n$$

Using infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$$

$$= \frac{1}{2}\frac{1}{1 - e^{-aT} e^{j\omega} z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-aT} e^{-j\omega} z^{-1}}$$

$$= \frac{1}{2}\frac{1}{1 - e^{j\omega}/z e^{aT}} + \frac{1}{2}\frac{1}{1 - e^{-j\omega}/z e^{aT}}$$

$$= \frac{1}{2}\left[\frac{z e^{aT}}{z e^{aT} - e^{j\omega}} + \frac{z e^{aT}}{z e^{aT} - e^{-j\omega}}\right]$$

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3.4 Poles and Zeros of Rational Function of z

Let, X(z) be Z-transform of x(n). When X(z) is expressed as a ratio of two polynomials in z or z^{-1}, then X(z) is called a ***rational function*** of z.

Let X(z) be expressed as a ratio of two polynomials in z, as shown below.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_N z^{-N}} \quad .....(3.35)$$

where, N(z) = Numerator polynomial of X(z)

D(z) = Denominator polynomial of X(z)

In equation (3.35) let us scale the coefficients of numerator polynomial by b_0 and that of denominator polynomial by a_0, and then convert the polynomials to positive power of z as shown below.

$$X(z) = \frac{b_0\left(1 + \frac{b_1}{b_0} z^{-1} + \frac{b_2}{b_0} z^{-2} + \frac{b_3}{b_0} z^{-3} + ... + \frac{b_M}{b_0} z^{-M}\right)}{a_0\left(1 + \frac{a_1}{a_0} z^{-1} + \frac{a_2}{a_0} z^{-2} + \frac{a_3}{a_0} z^{-3} + ... + \frac{a_N}{a_0} z^{-N}\right)}$$

$$= G\,\frac{z^{-M}\left(z^M + \frac{b_1}{b_0} z^{M-1} + \frac{b_2}{b_0} z^{M-2} + \frac{b_3}{b_0} z^{M-3} + ... \frac{b_M}{b_0}\right)}{z^{-N}\left(z^N + \frac{a_1}{a_0} z^{N-1} + \frac{a_2}{a_0} z^{N-2} + \frac{a_3}{a_0} z^{N-3} + ... \frac{a_N}{a_0}\right)} \qquad \boxed{\text{Let, } M = N}$$

$$= G\,\frac{(z - z_1)(z - z_2)(z - z_3)\,.....\,(z - z_N)}{(z - p_1)(z - p_2)(z - p_3)\,.....\,(z - p_N)} \quad .....(3.36)$$

where, $z_1, z_2, z_3, ...z_N$ are roots of numerator polynomial

$p_1, p_2, p_3, ...p_N$ are roots of denominator polynomial

G is a scaling factor.

In equation (3.36) if the value of z is equal to one of the roots of the numerator polynomial, then the function X(z) will become zero.

Therefore the roots of numerator polynomial $z_1, z_2, z_3, ...z_N$ are called zeros of X(z). Hence the ***zeros*** are defined as values z at which the function X(z) become zero.

In equation (3.36) if the value of z is equal to one of the roots of the denominator polynomial then the funcion X(z) will become infinite. Therefore the roots of denominator polynomial $p_1, p_2, p_3, ...p_N$ are called poles of X(z). Hence the ***poles*** are defined as values of z at which the function X(z) become infinite.

Since the function X(z) attains infinite values at poles, the ROC of X(z) does not include poles.

In a realizable system, the number of zeros will be less than or equal to number of poles. Also for every zero, we can associate one pole (the missing zeros are assumed to exist at infinity).

Let z_i be the zero associated with the pole p_i. If we evaluate |X(z)| for various values of z, then |X(z)| will be zero for $z = z_i$ and infnite for $z = p_i$. Hence the plot of |X(z)| in a three-dimensional plane will look like a pole (or pillar-like structure) and so the point $z = p_i$ is called a pole.

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3.5 Inverse Z-Transform

Let X(z) be Z-transform of the discrete time signal x(n).The inverse Z-transform is the process of recovering the discrete time signal x(n) from its Z-transform X(z). The signal x(n) can be uniquely determined from X(z) and its ROC.

The inverse Z-transform can be determined by the following three methods.

1. Direct evaluation by contour integration (or residue method).
2. Partial fraction expansion method.
3. Power series expansion method.

3.5.1 Inverse Z-Transform by Contour Integration or Residue Method

Let, X(z) be Z-transform of x(n).

Now by definition of inverse Z-transform,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1} dz \qquad \text{.....(3.38)}$$

Using partial fraction expansion technique the function $X(z)\, z^{n-1}$ can be expressed as shown below.

$$X(z)\, z^{n-1} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \text{.....} + \frac{A_N}{z-p_N} \qquad \text{.....(3.39)}$$

where, $p_1, p_2, p_3, \text{.....}\ p_N$ are poles of $X(z)\, z^{n-1}$ and $A_1, A_2, A_3, \text{.....}\ A_N$ are residues.

The residue A_1 is obtained by multiplying the equation (3.39) by $(z - p_1)$ and letting $z = p_1$.

Similarly other residues are evaluated.

$$\therefore\ A_1 = (z-p_1)\, X(z)\, z^{n-1}\Big|_{z=p_1} \qquad \text{.....(3.40.1)}$$

$$A_2 = (z-p_2)\, X(z)\, z^{n-1}\Big|_{z=p_2} \qquad \text{.....(3.40.2)}$$

$$A_3 = (z-p_3)\, X(z)\, z^{n-1}\Big|_{z=p_3} \qquad \text{.....(3.40.3)}$$

$$\vdots$$

$$A_N = (z-p_N)\, X(z)\, z^{n-1}\Big|_{z=p_N} \qquad \text{.....(3.40.N)}$$

Using equation (3.39) the equation (3.38) can be written as,

$$x(n) = \frac{1}{2\pi j} \oint_C \left[\frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \text{.....} + \frac{A_N}{z-p_N} \right] dz$$

$$= \frac{1}{2\pi j} \left[A_1 \oint_C \frac{dz}{z-p_1} + A_2 \oint_C \frac{dz}{z-p_2} + A_3 \oint_C \frac{dz}{z-p_3} + \text{.....} + A_N \oint_C \frac{dz}{z-p_N} \right]$$

.....(3.41)

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$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)......(z^2+az+b)......(z-p_N)}$$

$$= \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + ... + \frac{A_x}{z-(x+jy)} + \frac{A_x^*}{z-(x-jy)} + ... + \frac{A_N}{z-p_N}$$

The residues of real and nonrepeated roots are evaluated as explained in case i.

The residue A_x is evaluated as that of case i and the residue A_x^* is the conjugate of A_x.

3.5.3 Inverse Z-Transform by Power Series Expansion Method

Let X(z) be Z-transform of x(n), and X(z) be a rational function of z as shown below.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_N z^{-N}}$$

On dividing the numerator polynomial N(z) by denominator polynomial D(z) we can express X(z) as a power series of z. It is possible to express X(z) as positive power of z or as negative power of z or with both positive and negative power of z as shown below.

Case i **:** $X(z) = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + ...$ .....(3.50.1)

Case ii **:** $X(z) = \frac{N(z)}{D(z)} = d_0 + d_1 z^{1} + d_2 z^{2} + d_3 z^{3} + ...$ .....(3.50.2)

Case iii **:** $X(z) = \frac{N(z)}{D(z)} = ... + e_{-3} z^{3} + e_{-2} z^{2} + e_{-1} z + e_0$

$$+ e_1 z^{-1} + e_2 z^{-2} + e_3 z^{-3} + ...$$ .....(3.50.3)

The case-i power series of z is obtained when the ROC is exterior of a circle of radius r in z-plane (i.e., ROC is $|z| > r$).

The case-ii power series of z is obtained when the ROC is interior of a circle of radius r in z-plane (i.e., ROC is $|z| < r$).

The case-iii power series of z is obtained when the ROC is in between two circles of radius r_1 and r_2 in z-plane (i.e., ROC is $r_1 < |z| < r_2$).

By the definition of Z-transform, we get,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

On expanding the summation we get,

$$X(z) = ... x(-3)\, z^{3} + x(-2)\, z^{2} + x(-1)\, z^{1} + x(0)\, z^{0}$$

$$+ x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + ...$$ .....(3.51)

On comparing the coefficients of z of equations (3.50) and (3.51), the samples of x(n) are determined. [i.e., the coefficient of z^i is the i^{th} sample, x(i) of the signal x(n)].

Note : The different methods of evaluation of inverse Z-transform of a function X(z) will result in different type of mathematical expressions. But the inverse Z-transform is unique for a specified ROC and so on evaluating the expressions for each value of n, we may get a same signal.

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When $n = 0$, $x(0) = 3 + 0 + 0 = 3$

When $n = 1$, $x(1) = 0 + 1 - 11 = -10$

When $n = 2$, $x(2) = 0 - 1 + 33 = 32$

When $n = 3$, $x(3) = 0 + 1 - 99 = -98$

When $n = 4$, $x(4) = 0 - 1 + 297 = 296$

$\therefore x(n) = \{3, -10, 32, -98, 296, ...\}$

Alternate Method

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 + 4z + 3}$$

$$\therefore \frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z^2 + 4z + 3)} = \frac{3z^2 + 2z + 1}{z(z + 1)(z + 3)}$$

By partial fraction expansion technique $\frac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z + 1)(z + 3)} = \frac{A_1}{z} + \frac{A_2}{z + 1} + \frac{A_3}{z + 3}$$

$$A_1 = z\frac{X(z)}{z}\bigg|_{z=0} = \cancel{z}\frac{3z^2 + 2z + 1}{\cancel{z}(z + 1)(z + 3)}\bigg|_{z=0} = \frac{0 + 0 + 1}{(0 + 1)(0 + 3)} = \frac{1}{3}$$

$$A_2 = (z + 1)\frac{X(z)}{z}\bigg|_{z=-1} = \cancel{(z + 1)}\frac{3z^2 + 2z + 1}{z\cancel{(z + 1)}(z + 3)}\bigg|_{z=-1} = \frac{3(-1)^2 + 2(-1) + 1}{-1 \times (-1 + 3)} = -1$$

$$A_3 = (z + 3)\frac{X(z)}{z}\bigg|_{z=-3} = \cancel{(z + 3)}\frac{3z^2 + 2z + 1}{z(z + 1)\cancel{(z + 3)}}\bigg|_{z=-3} = \frac{3(-3)^2 + 2(-3) + 1}{-3 \times (-3 + 1)} = \frac{22}{6} = \frac{11}{3}$$

$$\therefore \frac{X(z)}{z} = \frac{1}{3}\frac{1}{z} - \frac{1}{z + 1} + \frac{11}{3}\frac{1}{z + 3}$$

$$\therefore X(z) = \frac{1}{3} - \frac{z}{z + 1} + \frac{11}{3}\frac{z}{z + 3}$$

$$= \frac{1}{3} - \frac{z}{z - (-1)} + \frac{11}{3}\frac{z}{z - (-3)}$$

$\mathcal{Z}\{\delta(n)\} = 1$
$\mathcal{Z}\{a^n u(n)\} = \frac{z}{z - a}$

On taking inverse $\mathcal{Z}$-transform of $X(z)$ we get,

$$x(n) = \frac{1}{3}\,\delta(n) - (-1)^n u(n) + \frac{11}{3}(-3)^n u(n) = \frac{1}{3}\,\delta(n) + [-(-1)^n + \frac{11}{3}(-3)^n]\,u(n)$$

When $n = 0$, $x(0) = \frac{1}{3} - 1 + \frac{11}{3} = 3$

When $n = 1$, $x(1) = 0 + 1 + \frac{11}{3} \times -3 = -10$

When $n = 2$, $x(2) = 0 - 1 + \frac{11}{3} \times (-3)^2 = 32$

When $n = 3$, $x(3) = 0 + 1 + \frac{11}{3} \times (-3)^3 = -98$

When $n = 4$, $x(4) = 0 - 1 + \frac{11}{3} \times (-3)^4 = 296$

$\therefore x(n) = \{3, -10, 32, -98, 296, ...\}$

Note: The closed form expression of x(n) in the two methods look different, but on evaluating x(n) for various values of n we get same signal x(n).

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c) Given that, $X(z) = \dfrac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5\,z^{-2}} = \frac{z^{-1}(z + 1)}{z^{-2}(z^2 - z + 0.5)}$$

$$= \frac{z\,(z + 1)}{(z^2 - z + 0.5)} = \frac{z\,(z + 1)}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}$$

The roots of the quadratic $z^2 - z + 0.5 = 0$ are,

$$z = \frac{1 \pm \sqrt{1 - 4 \times 0.5}}{2}$$

$$= 0.5 \pm j0.5$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{(z + 1)}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)} = \frac{A}{z - 0.5 - j0.5} + \frac{A^*}{z - 0.5 + j0.5}$$

$$A = (z - 0.5 - j0.5)\left.\frac{X(z)}{z}\right|_{z\,=\,0.5\,+\,j0.5}$$

$$= \cancel{(z - 0.5 - j0.5)}\left.\frac{(z + 1)}{\cancel{(z - 0.5 - j0.5)}\,(z - 0.5 + j0.5)}\right|_{z\,=\,0.5\,+\,j0.5}$$

$$= \frac{0.5 + j0.5 + 1}{0.5 + j0.5 - 0.5 + j0.5} = \frac{1.5 + j0.5}{j1} = -j(j0.5 + 1.5) = 0.5 - j1.5$$

$$A^* = (0.5 - j1.5)^* = 0.5 + j1.5$$

$$\therefore \quad \frac{X(z)}{z} = \frac{0.5 - j1.5}{z - 0.5 - j0.5} + \frac{0.5 + j1.5}{z - 0.5 + j0.5}$$

$$\mathcal{Z}\,\{a^n\,u(n)\} = \frac{z}{z - a}$$

$$X(z) = (0.5 - j1.5)\,\frac{z}{z - (0.5 + j0.5)} + (0.5 + j1.5)\,\frac{z}{z - (0.5 - j0.5)}$$

On taking inverse $\mathcal{Z}$-transform of $X(z)$ we get,

$$x(n) = (0.5 - j1.5)\,(0.5 + j0.5)^n\,u(n) + (0.5 + j1.5)\,(0.5 - j0.5)^n\,u(n)$$

Alternatively the above result can be expressed as shown below.

$180^o = \pi\,\text{rad}$; $\therefore 1^o = \dfrac{\pi}{180}$ rad

$\therefore 71.6^o = \dfrac{71.6}{180}\pi = 0.4\,\pi\,\text{rad}$

$\therefore 45^o = \dfrac{45}{180}\pi = 0.25\pi\,\text{rad}$

Here, $0.5 - j1.5 = 1.581\angle{-71.6^o} = 1.581\angle{-0.4\pi}$

$0.5 + j1.5 = 1.581\angle 71.6^o = 1.581\angle 0.4\pi$

$0.5 + j0.5 = 0.707\angle 45^o = 0.707\angle 0.25\pi$

$0.5 - j0.5 = 0.707\angle{-45^o} = 0.707\angle{-0.25\pi}$

$$\therefore\ x(n) = [1.581\angle{-0.4\pi}]\,[0.707\angle 0.25\pi]^n\,u(n) + [1.581\angle 0.4\pi]\,[0.707\angle{-0.25\pi}]^n\,u(n)$$

$$= [1.581\angle{-0.4\pi}]\,[0.707^n\angle 0.25\pi n]\,u(n) + [1.581\angle 0.4\pi]\,[0.707^n\angle{-0.25\pi n}]\,u(n)$$

$$= 1.581\ (0.707)^n\,[1\angle \pi(0.25n - 0.4) + 1\angle{-\pi(0.25n - 0.4)}]\,u(n)$$

$$= 1.581\,(0.707)^n\,[\cos(\pi(0.25n - 0.4)) + j\sin(\pi(0.25n - 0.4)) + \cos(\pi(0.25n - 0.4))$$
$$- j\sin(\pi(0.25n - 0.4))]\,u(n)$$

$$= 1.581\,(0.707)^n\,2\cos(\pi(0.25\,n - 0.4))\,u(n)$$

$$= 3.162\,(0.707)^n\cos(\pi(0.25n - 0.4))\,u(n)$$

---

d) Given that, $X(z) = \dfrac{2}{(1 + z^{-1})\,(1 - z^{-1})^2}$

$$X(z) = \frac{2}{(1 + z^{-1})\,(1 - z^{-1})^2} = \frac{2}{z^{-1}(z + 1)\,z^{-2}(z - 1)^2} = \frac{2z^3}{(z + 1)\,(z - 1)^2}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{2z^2}{(z + 1)\,(z - 1)^2}$$

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b) ROC is |z| < 0.2

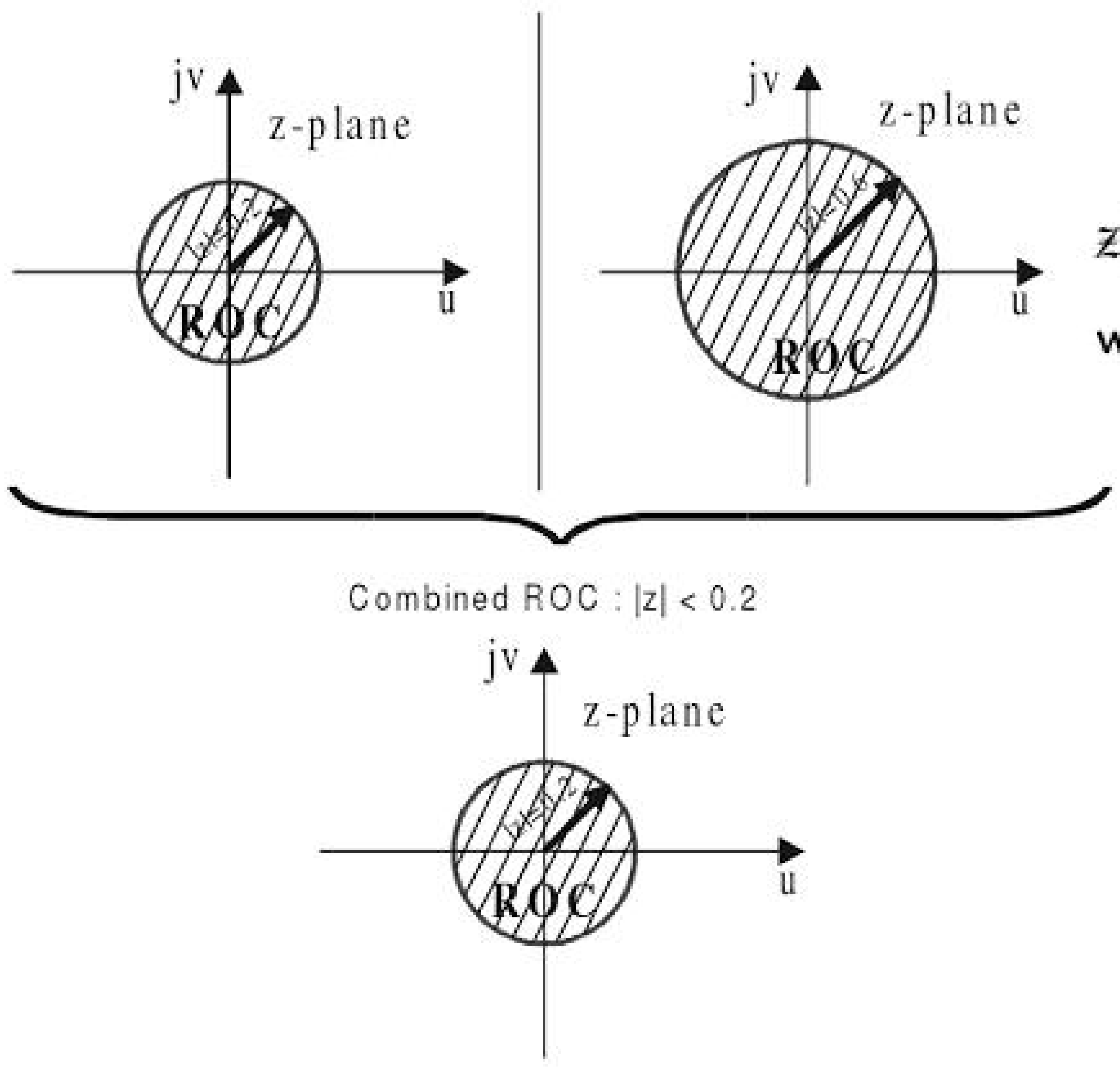
The specified ROC is interior of the circle whose radius corresponds to the smallest pole, hence x(n) will be an anticausal (or left-sided) signal. (Refer section 3.4.2).

$$\therefore\ x(n) = 1.5(-(0.6)^n\, u(-n-1)) - 0.5\,[-(0.2)^n\, u(-n-1)]$$
$$= -1.5\,(0.6)^n\, u(-n-1) + 0.5\,(0.2)^n\, u(-n-1)$$

$$\boxed{\mathcal{Z}\,\{-a^n\, u(-n-1)\} = \frac{z}{z-a}\ ;\ \text{ROC } |z|<|a|}$$

$$\mathcal{Z}\left\{-0.2^n\, u(-n-1)\right\} = \frac{z}{z-0.2}$$
with ROC : |z| < 0.2

$$\mathcal{Z}\left\{-0.6^n\, u(-n-1)\right\} = \frac{z}{z-0.6}$$
with ROC : |z| < 0.6



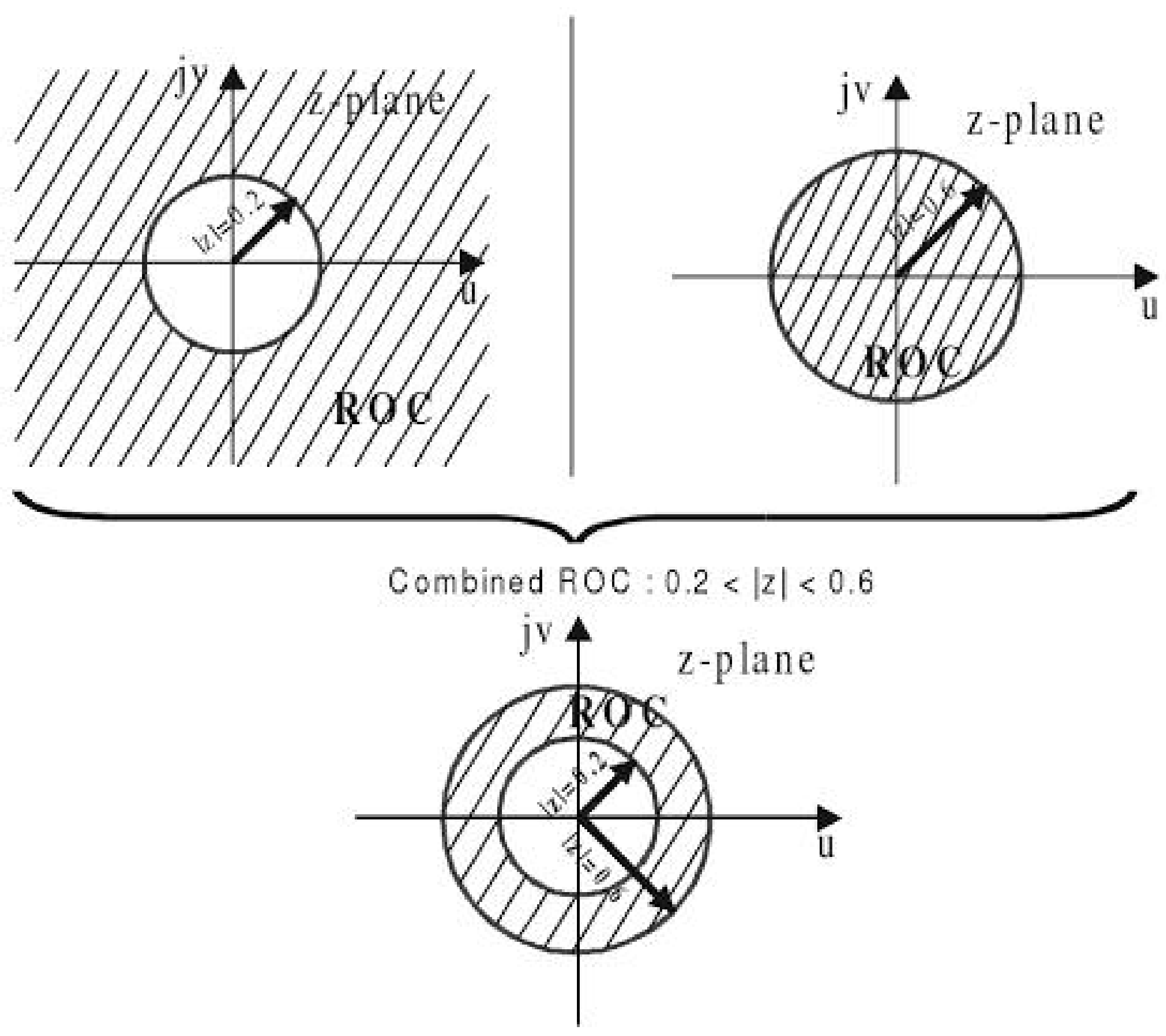
c) ROC is 0.2 < |z| < 0.6

The specified ROC is the region in between two circles of radius 0.2 and 0.6. Hence the term corresponds to the pole, $p_1 = 0.6$ will be anticausal signal (because |z| < 0.6) and the term corresponds to the pole, $p_2 = 0.2$, will be a causal signal (because |z| > 0.2). (Refer section 3.4.2).

$$\therefore\ x(n) = 1.5(-(0.6)^n\, u(-n-1)) - 0.5\,(0.2)^n\, u(n)$$
$$= -1.5(0.6)^n\, u(-n-1)) - 0.5\,(0.2)^n\, u(n)$$

$$\mathcal{Z}\left\{0.2^n\, u(n)\right\} = \frac{z}{z-0.2}$$
with ROC : |z| > 0.2

$$\mathcal{Z}\left\{-0.6^n\, u(-n-1)\right\} = \frac{z}{z-0.6}$$
with ROC : |z| < 0.6



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Deconvolution

The ***deconvolution*** operation is performed to extract the input x(n) of an LTI system from the response y(n) of the system.

From equation (3.59) get,

$$X(z) = \frac{Y(z)}{H(z)}$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

$$\text{Input, } x(n) = \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\left\{\frac{Y(z)}{H(z)}\right\}$$

Procedure : 1. Take $\mathcal{Z}$-transform of y(n) to get Y(z).

2. Take $\mathcal{Z}$-transform of h(n) to get H(z).
3. Divide Y(z) by H(z) to get X(z), [i.e., X(z) = Y(z) / H(z)].
4. Take inverse $\mathcal{Z}$-transform of X(z) to get x(n).

3.6.5 Stability in z-Domain

Location of Poles for Stability

Let, h(n) be the impulse response of an LTI discrete time system. Now, if h(n) satisfies the condition,

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty \qquad .....(3.60)$$

then the LTI discrete time system is stable. [Refer Chapter 2, equation (2.24)].

The stability condition of equation (3.60) can be transformed as a condition on location of poles of transfer function of the LTI discrete time system in z-plane.

Let, h(n) = a^n u(n)

$$\text{Now, } \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} |a^n\, u(n)| = \sum_{n=0}^{\infty} a^n$$

If |a| is such that, 0 < |a| < 1, then $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ = constant, and so the system is stable.

If |a| > 1, then $\sum_{n=0}^{\infty} a^n = \infty$ and so the system is unstable.

$$\text{Now, } H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\{a^n\, u(n)\} = \frac{z}{z-a}$$

Here H(z) has pole at z = a.

If |a| < 1, then the pole will lie inside the unit circle and if |a| > 1, then the pole will lie outside the unit circle. Therefore we can say that, ***for a stable discrete time system the poles should lie inside the unit circle.*** The various types of impulse response of LTI discrete time system and their transfer functions and the locations of poles are summarized in table 3.5.

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Table 3.5 : Continued....

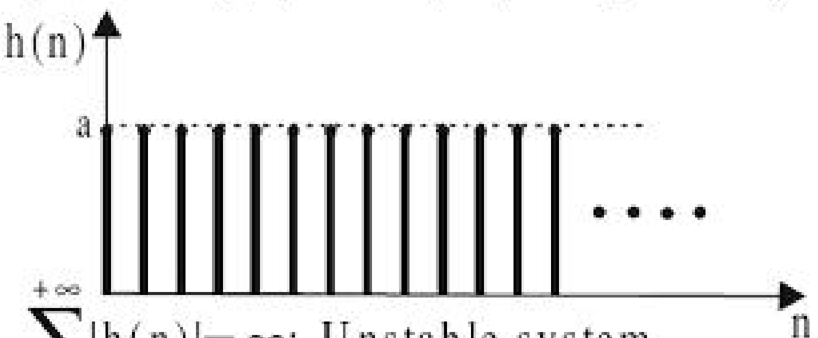
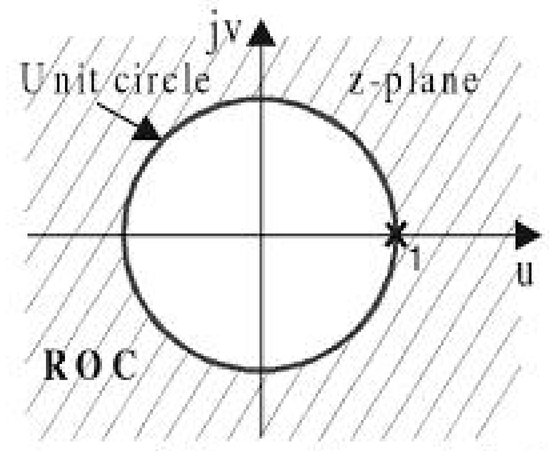
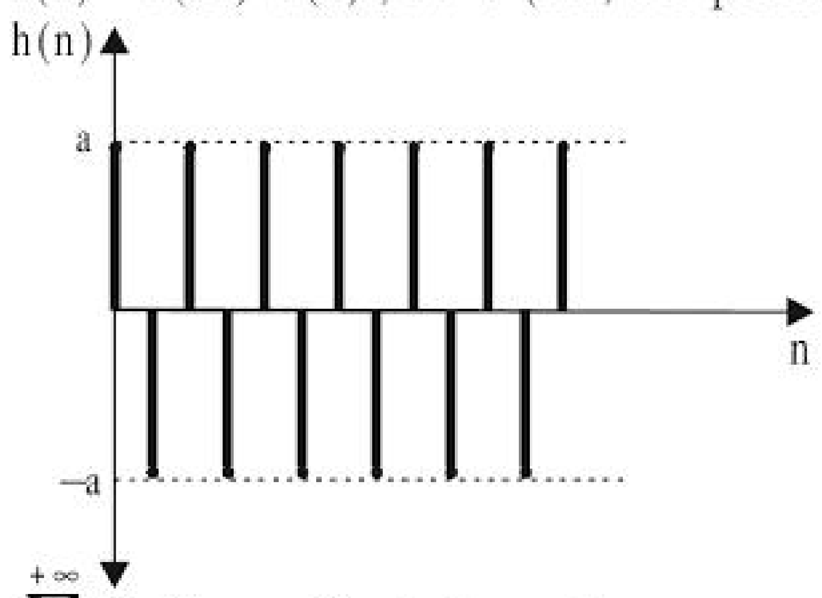
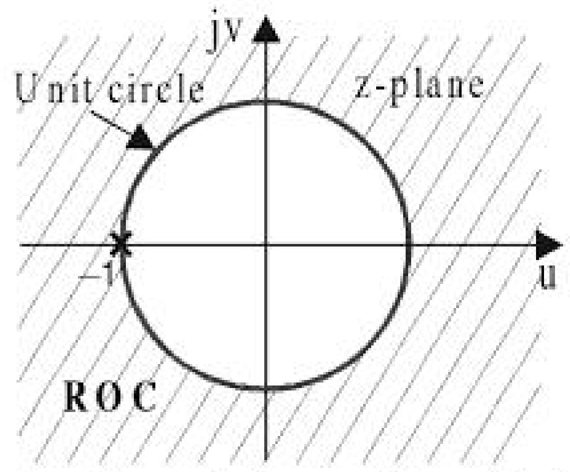
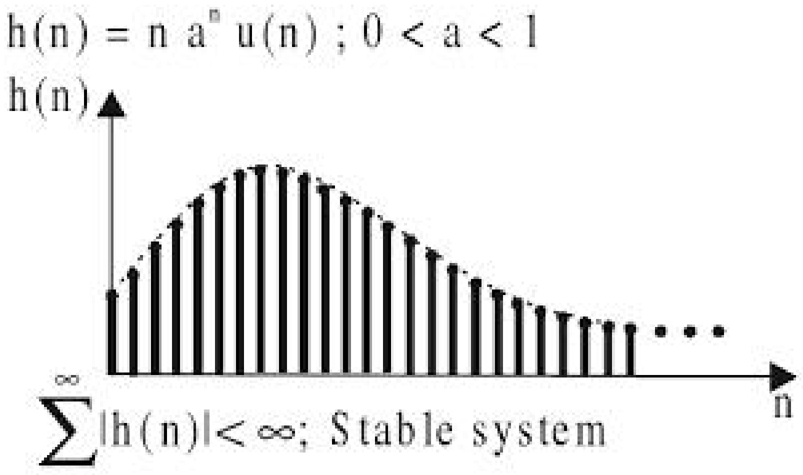
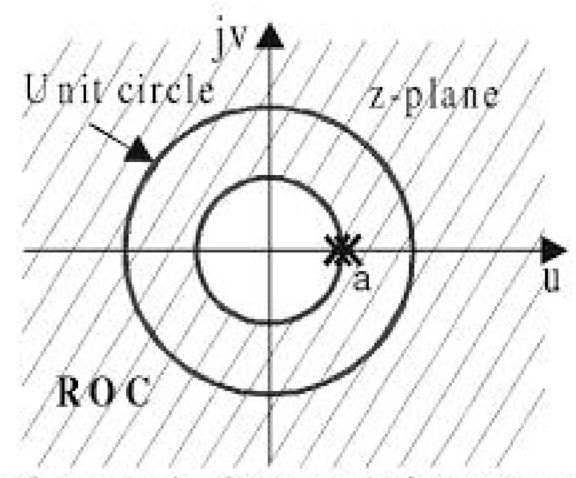
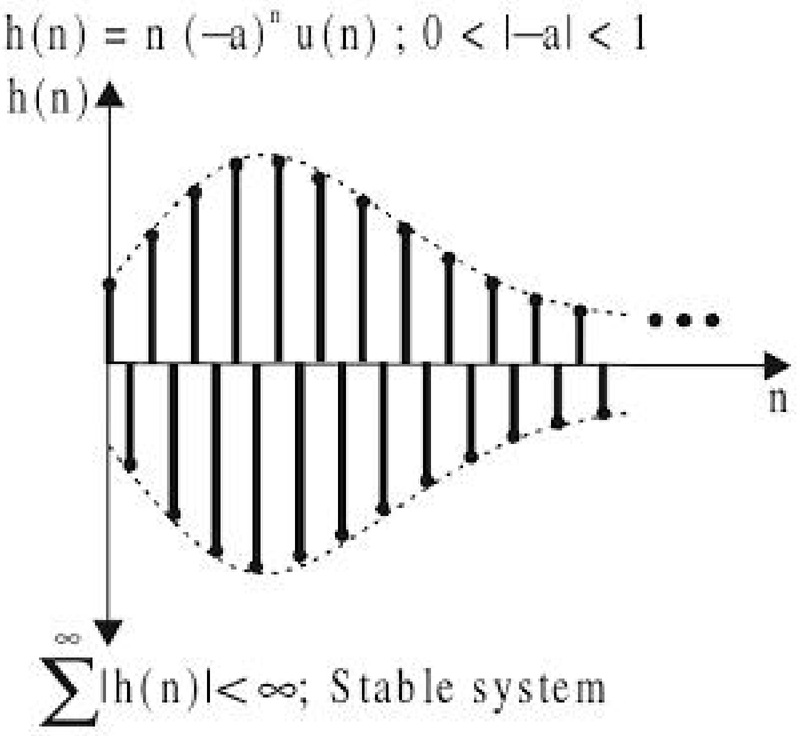
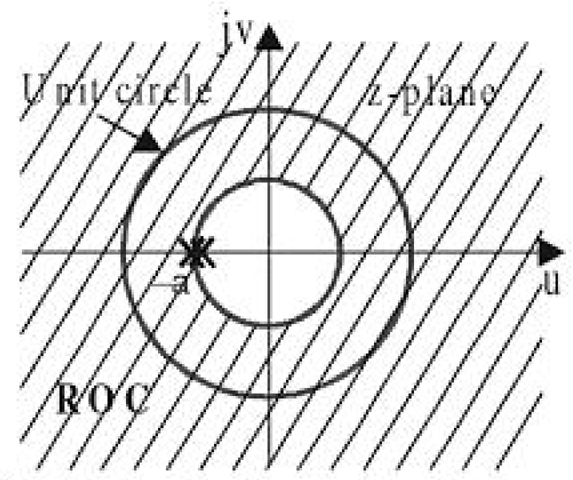
Impulse response h(n)	Transfer function H(z)	Location of poles in z-plane and ROC
$h(n) = a\,u(n)$; $a > 0$ (i.e., a is positive)  $\sum_{n=0}^{+\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \frac{az}{z-1}$ ROC is $\lvert z\rvert > 1$ pole at $z = 1$	 The pole $z = 1$ lies on the unit circle. The ROC does not contain the unit circle.
$h(n) = a(-1)^n u(n)$; $a > 0$ (i.e., a is positive)  $\sum_{n=0}^{+\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \frac{az}{z+1}$ ROC is $\lvert z\rvert > 1$ pole at $z = -1$	 The pole at $z = -1$ lies on the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,a^n u(n)$; $0 < a < 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \frac{az}{(z-a)^2}$ ROC is $\lvert z\rvert > a$ Two poles at $z = a$	 Since $0 < a < 1$, the two poles at $z = a$ lie inside the unit circle. The ROC contains the unit circle.
$h(n) = n(-a)^n u(n)$; $0 < \lvert -a\rvert < 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \frac{az}{(z+a)^2}$ ROC is $\lvert z\rvert > a$ Two poles at $z = -a$	 Since $0 < \lvert -a\rvert < 1$, the two poles at $z = -a$ lie inside the unit circle. The ROC contains the unit circle.

Table 3.5 : Continued....

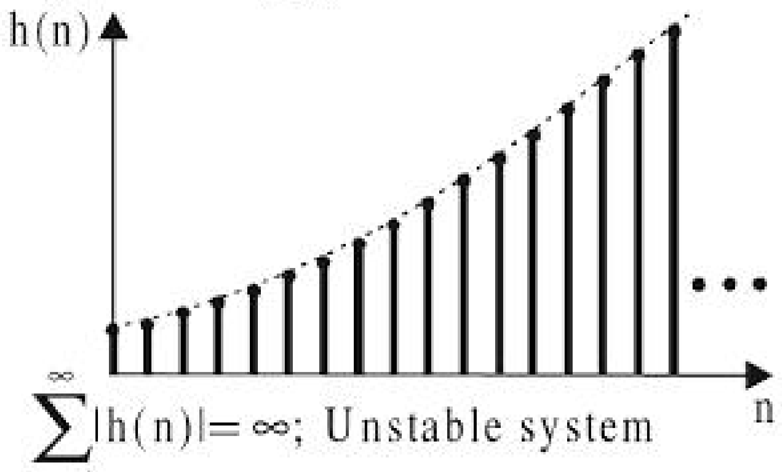
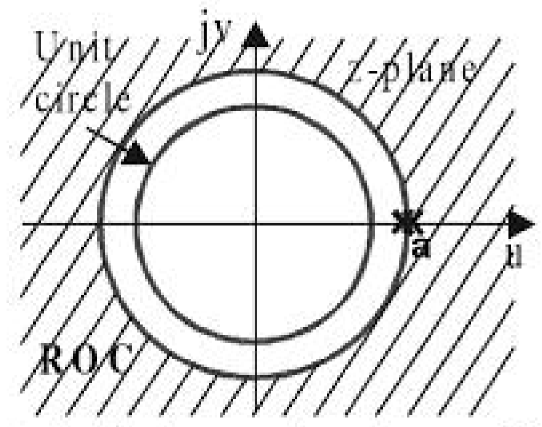
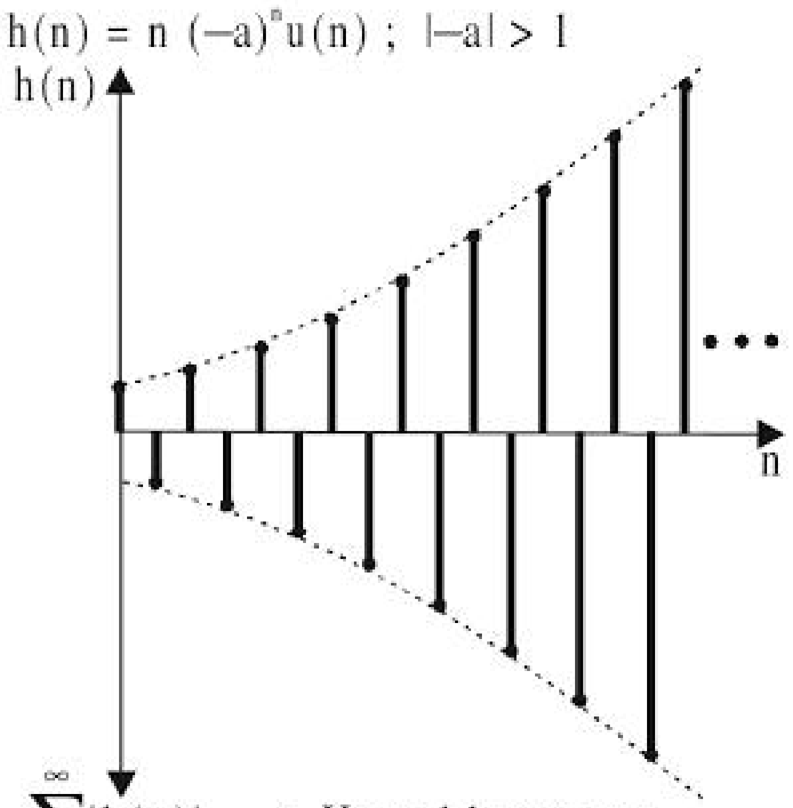
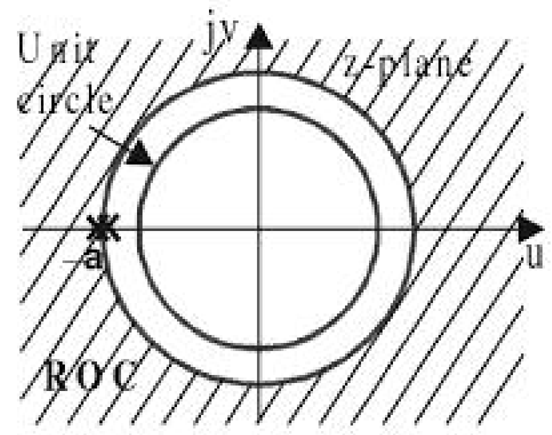
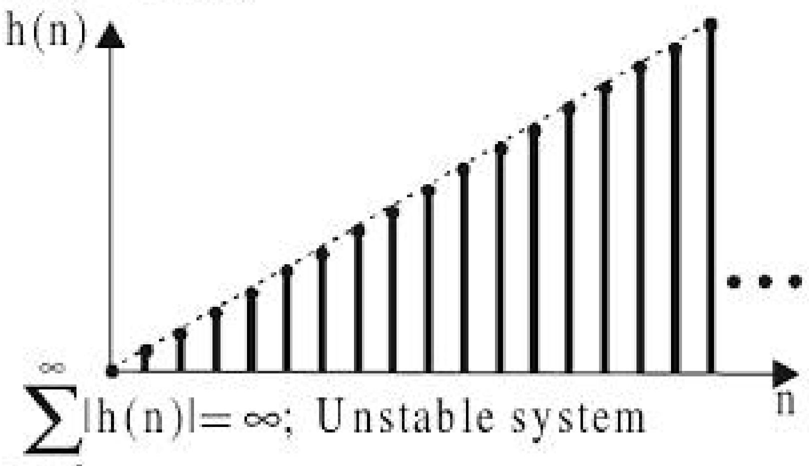
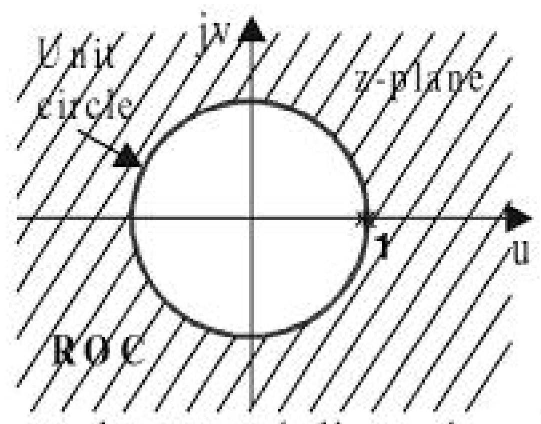
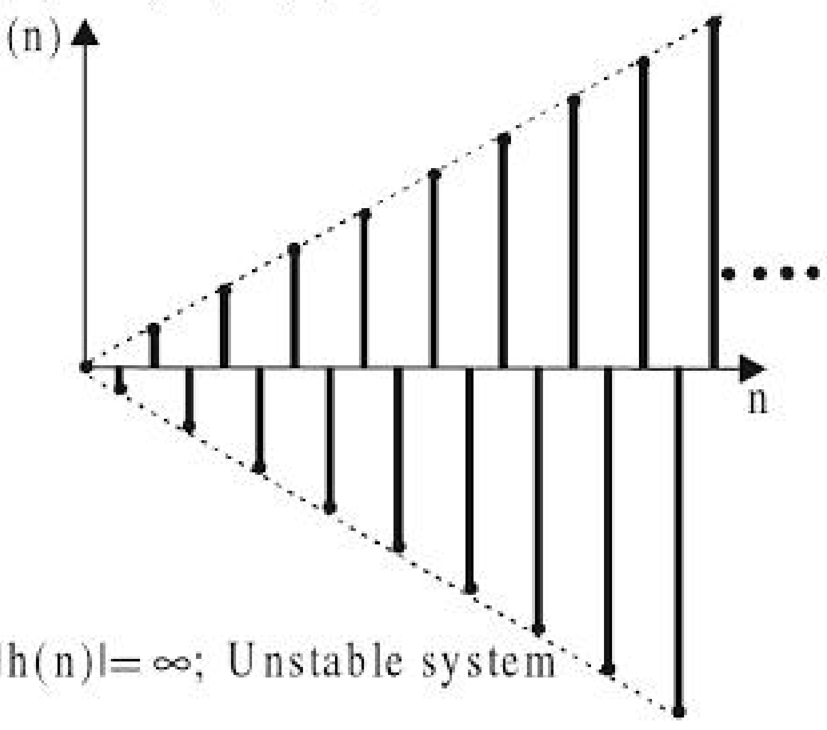
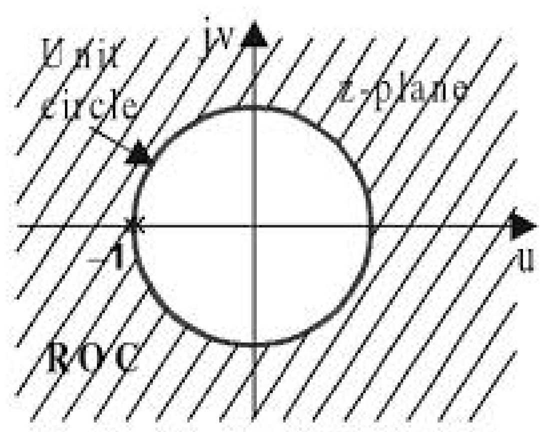
Impulse response h(n)	**Transfer function H(z)**	**Location of poles in z-plane and ROC**
$h(n) = n\,a^n u(n)\ ;\ a > 1$  $\sum_{n=0}^{\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{(z - a)^2}$ ROC is $\lvert z \rvert > a$ Two poles at $z = a$	 Since $a > 1$, the two poles at $z = a$ lie outside the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,(-a)^n u(n)\ ;\ \lvert -a \rvert > 1$  $\sum_{n=0}^{\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{(z + a)^2}$ ROC is $\lvert z \rvert > \lvert -a \rvert$ Two poles at $z = -a$	 Since $\lvert -a \rvert > 1$, the two poles at $z = -a$ lie outside the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,u(n)$  $\sum_{n=0}^{\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{(z - 1)^2}$ ROC is $\lvert z \rvert > 1$ Two poles at $z = 1$	 The two poles at $z = 1$, lie on the unit circle. The ROC does not contain the unit circle.
$h(n) = n(-1)^n u(n)\ ;\ \lvert -a \rvert > 1$  $\sum_{n=0}^{\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{(z + 1)^2}$ ROC is $\lvert z \rvert > 1$ Two poles at $z = -1$	 The two poles at $z = -1$, lie on the unit circle. The ROC does not contain the unit circle.

image
not
available

ROC of a Stable System

Let, H(z) be Z-transform of h(n). Now, by definition of Z-transform we get,

$$H(z) = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n}$$

Let us evaluate H(z) for z = 1.

$$\therefore\ H(z) = \sum_{n=-\infty}^{+\infty} h(n)$$

On taking absolute value on both sides we get,

$$|H(z)| = \left|\sum_{n=-\infty}^{+\infty} h(n)\right| \quad \Rightarrow \quad |H(z)| = \sum_{n=-\infty}^{+\infty} |h(n)|$$

For a stable LTI discrete time system,

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty \quad \Rightarrow \quad |H(z)| < \infty$$

Therefore, we can conclude that z = 1 will be a point in the ROC of a stable system. Hence *for a stable discrete time system the ROC of impulse response should include the unit circle.*

General Condition for Stability in z-plane

On combining the condition for location of poles and the ROC we can say that *for a stable LTI discrete time system the poles should lie inside the unit circle and the unit circle should be included in ROC of impulse response of the system.*

3.7 Relation Between Laplace Transform and Z-Transform

3.7.1 Impulse Train Sampling of Continuous Time Signal

Consider a periodic impulse train p(t) shown in fig 3.12a, with period T. The pulse train can be mathematically expressed as shown in equation (3.61).

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad \text{..... (3.61)}$$

When a continuous time signal x(t) is multiplied by the impulse train p(t), the product signal will have impulses. A continuous time signal x(t) and the product of x(t) and p(t) are shown in fig 3.12b and fig 3.12c respectively. In fig 3.12c, the magnitudes of the impulses are equal to magnitude of x(t), and so the product signal is impulse sampled version of x(t), with sampling period T. Let us denote the product signal as $x_p(t)$ and it is mathematically expressed as shown in equation (3.62).

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT) \qquad \text{..... (3.62)}$$

where, x(nT) are samples of x(t) at t = nT

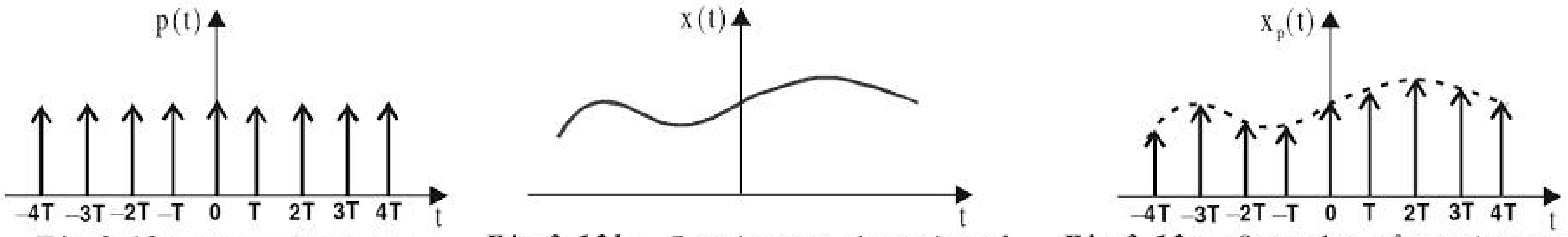


Fig 3.12a : Impulse train. *Fig 3.12b : Continuous time signal.* *Fig 3.12c : Samples of continuous time signal.*

Fig 3.12 : Impulse sampling of continuous time signal.

3.7.2 Transformation From Laplace Transform to Z-Transform

Let x(t) be a continuous time signal, and $x_p(t)$ be its impulse sampled version of discrete time signal. From equation (3.62) we get,

$\mathcal{L}\{\delta(t)\} = 1$
If $\mathcal{L}\{x(t)\} = X(s)$ then by time shifting property $\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{x_p(t)\} = X_p(s) = \mathcal{L}\left\{\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)\right\} = \sum_{n=-\infty}^{\infty} x(nT)\,\mathcal{L}\{\delta(t-nT)\}$$

$$\therefore X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\,e^{-nsT} = \sum_{n=-\infty}^{\infty} x(nT)\left(e^{sT}\right)^{-n} \qquad \text{..... (3.63)}$$

where $X_p(s)$ is Laplace transform of $x_p(t)$.

Let us take a transformation, $e^{sT} = z$.

On substituting, $e^{sT} = z$, in equation (3.63) we get,

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\,z^{-n} \qquad \text{.... (3.64)}$$

The Z-transform of x(nT), using the definition of Z-transform is given by,

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)\,z^{-n} \qquad \text{.... (3.65)}$$

On comparing equations (3.64) and (3.65) we can say that, if a discrete time signal x(nT) is a sampled version of x(t), then *Z-transform of the discrete time signal can be obtained from Laplace transform of sampled version of x(t), by choosing the transformation, $e^{sT} = z$.* This transformation is also called ***impulse invariant transformation***.

3.7.3 Relation Between s-Plane and z-Plane

Consider a point s_1 in s-plane as shown in fig 3.13. Now the transformation,

$$e^{s_1T} = z_1 \qquad \text{..... (3.66)}$$

will transform the point s_1 to a corresponding point z_1 in z-plane.

Let the coordinates of s_1 be s_1 and W_1 as shown in fig 3.13.

$$\therefore\ s_1 = \sigma_1 + j\Omega_1 \qquad \text{..... (3.67)}$$

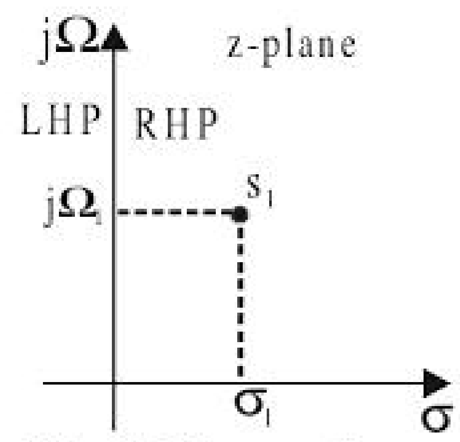


***Fig 3.13** : s-plane.*

Using equation (3.67) the equation (3.66) can be written as,

$$z_1 = e^{(\sigma_1 + j\Omega_1)T} = e^{\sigma_1T}\,e^{j\Omega_1T} \qquad \text{..... (3.68)}$$

On separating the magnitude and phase of equation (3.68) we get,

$$|z_1| = e^{\sigma_1T}\ ; \quad \angle z_1 = j\Omega_1T \qquad \text{..... (3.69)}$$

From equation (3.69) the following observations can be made.

1. If $s_1 < 0$ (i.e., s_1 is negative), then the point-s_1 lies on Left Half (LHP) of s-plane.
   In this case, $|z_1| < 1$, hence the corresponding point-z_1 will lie inside the unit circle in z-plane.

2. If $s_1 = 0$ (i.e., real part is zero), then the point-s_1 lies on imaginary axis of s-plane.
   In this case, $|z_1| = 1$, hence the corresponding point-z_1 will lie on the unit circle in z-plane.

3. If s$_1$ > 0 (i.e., s$_1$ is positive), then the point-s$_1$ lies on the Right Half (RHP) of s-plane.
   In this case $|z_1| > 1$, hence the corresponding point-z_1 will lie outside the unit circle in z-plane.

The above discussions are applicable for mapping of any point on s-plane to z-plane.

In general all points of s-plane, described by the equation,

$$s_1 = \sigma_1 + j\Omega_1 + j\frac{2\pi k}{T}, \quad \text{for } k = 0, \ \pm 1, \ \pm 2 \ldots.. \qquad \ldots..(3.70)$$

map as a single point in the z-plane described by equation,

$\boxed{e^{\pm j2\pi k} = 1 \ ; \ \text{for integer k}}$

$$z_1 = e^{\left(\sigma_1 + j\Omega_1 + \frac{j2\pi k}{T}\right)T} = e^{\sigma_1 T}\, e^{j\Omega_1 T}\, e^{j2\pi k} = e^{\sigma_1 T}\, e^{j\Omega_1 T} \qquad \ldots..(3.71)$$

The equation (3.70) represents a strip of width 2p/T in the s-plane for values of imaginary part of s in the range –p/T £ W £ +p/T is mapped into the entire z-plane. Similarly the strip of width 2p/T in the s-plane for values of imaginary part of s in the range p/T £ W £ 3p/T is also mapped into the entire z-plane. Likewise the strip of width 2p/T in the s-plane for values of imaginary part of s in the range -3p/T £ W £ -p/T is also mapped into the entire z-plane.

In general any strip of width 2p/T in the s-plane for values of imaginary part of s in the range (2k – 1)p/T £ W £ (2k + 1) p/T, where k is an integer, is mapped into the entire z-plane. Therefore we can say that the transformation, $e^{sT} = z$, leads to many-to-one mapping, (and does not provide one-to-one mapping).

In this mapping, *the left half portion of each strip in s-plane maps into the interior of the unit circle in z-plane, right half portion of each strip in s-plane maps into the exterior of the unit circle in z-plane and the imaginary axis of each strip in s-plane maps into the unit circle in z-plane* as shown in fig 3.14.

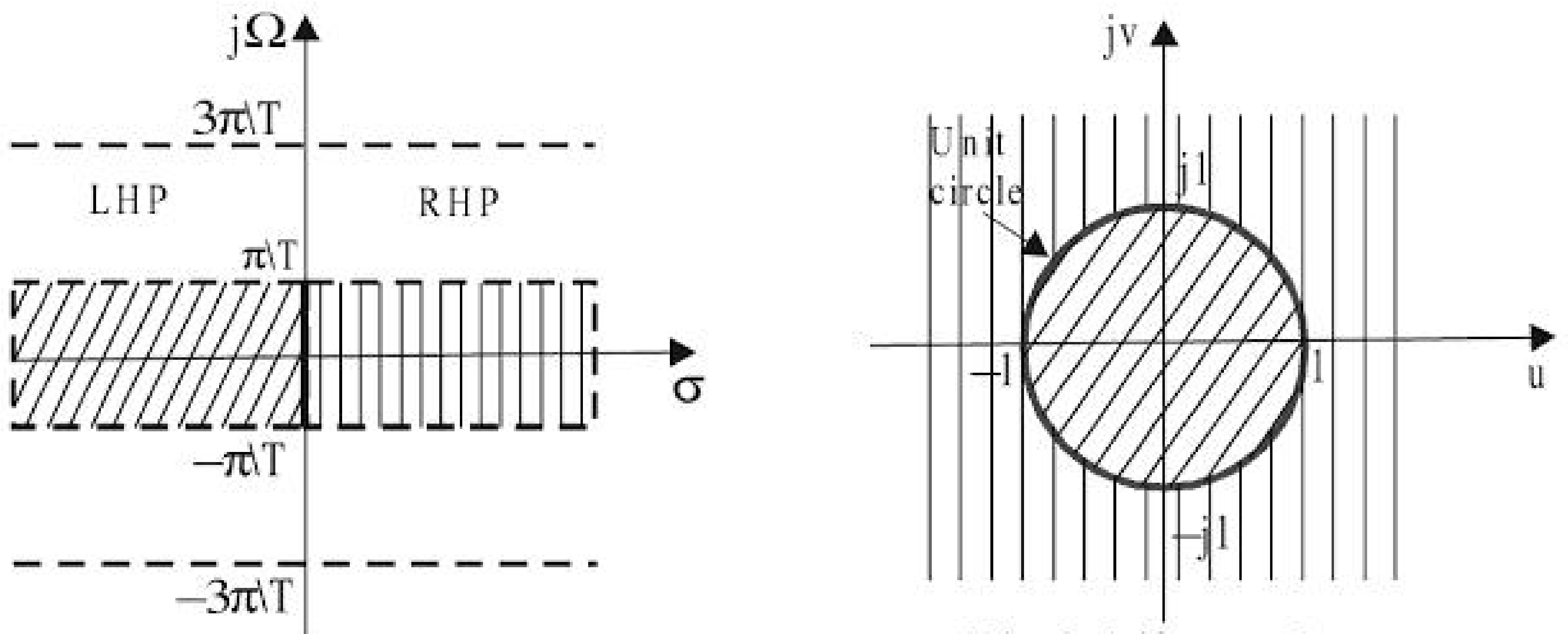


***Fig 3.14a** : s-plane.*   ***Fig 3.14b** : z-plane.*

***Fig 3.14** : Mapping of s-plane into z-plane.*

Relation Between Frequency of Continuous Time and Discrete Time Signal

Let, W = Frequency of continuous time signal in rad/sec.

w = Frequency of discrete time signal in rad/sample.

Let, z = re^{jw} be a point on z-plane, and s = s + jW, be a corresponding point in s-plane.

Consider the transformation,

$$z = e^{sT} \qquad \ldots..(3.72)$$

Put, z = r e^{jw} and s = s + jW in equation (3.72)

$$\therefore\ r\,e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$r\,e^{j\omega} = e^{\sigma T}\,e^{j\Omega T} \qquad .....(3.73)$$

On equating the imaginary part on either side of equation (3.73) we get,

$$\boxed{w = WT} \quad \text{or} \quad \boxed{\Omega = \frac{\omega}{T}} \qquad .....(3.74)$$

When the transformation $e^{sT} = z$ is employed, the equation (3.74) can be used to compute the frequency of discrete time signal for a given frequency of continuous time signal and viceversa.The frequency of discrete time signal w is unique over the range (-p, +p), and so the mapping w = WT implies that the frequency of continuous time signal in the interval -p/T £ W £ +p/T maps into the corresponding values of frequency of discrete time signal in the interval -p £ w £ +p.

The mapping of s-plane to z-plane, using the transformation, $e^{sT} = z$ is not one-to-one. Therefore in general, the interval (2k - 1)p/T £ W £ (2k + 1)p/T, where k is an integer, maps into the corresponding values of -p £ w £ +p. Thus the mapping of the frequency of continuous time signal W to the frequency of discrete time signal w is many-to-one. This reflects the effects of aliasing due to sampling.

Example 3.12

Determine the impulse response h(n) for the system described by the second-order difference equation,

y(n) + 4y(n – 1) + 3y(n – 2) = x(n – 1).

Solution

The difference equation governing the system is,

$$y(n) + 4y(n-1) + 3y(n-2) = x(n-1)$$

Let us take Z-transform of the difference equation governing the system with zero initial conditions.

$$\therefore\ Z\{y(n) + 4y(n-1) + 3y(n-2)\} = Z\{x(n-1)\}$$

$$Z\{y(n)\} + 4\,Z\{y(n-1)\} + 3\,Z\{y(n-2)\} = Z\{x(n-1)\}$$

$$Y(z) + 4z^{-1}\,Y(z) + 3z^{-2}\,Y(z) = z^{-1}\,X(z)$$

$$(1 + 4z^{-1} + 3z^{-2})\,Y(z) = z^{-1}\,X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 + 4z^{-1} + 3z^{-2}}$$

If Z{x(n)} = X(z)
then by shifting property
$Z\{x(n-m\} = z^{-m}\,X(z)$

If Z{y(n)} = Y(z)
then by shifting property
$Z\{y(n-m\} = z^{-m}\,Y(z)$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore\ H(z) = \frac{z^{-1}}{1 + 4z^{-1} + 3z^{-2}} = \frac{z^{-1}}{z^{-2}(z^2 + 4z + 3)} = \frac{z}{(z+1)\,(z+3)}$$

Using partial fraction expansion technique we can write,

$$\frac{H(z)}{z} = \frac{1}{(z+1)\,(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$A = \cancel{(z+1)}\,\frac{1}{\cancel{(z+1)}\,(z+3)}\bigg|_{z=-1} = \frac{1}{-1+3} = \frac{1}{2} = 0.5$$

$$B = \cancel{(z+3)}\,\frac{1}{(z+1)\,\cancel{(z+3)}}\bigg|_{z=-3} = \frac{1}{-3+1} = -\frac{1}{2} = -0.5$$

The roots of quadratic $z^2 + 4z + 3$ are,

$$z = \frac{-4 \pm \sqrt{4^2 - 4\times 3}}{2} = \frac{-4 \pm 2}{2} = -1, -3$$

$$\therefore\ \frac{H(z)}{z} = \frac{0.5}{z+1} - \frac{0.5}{z+3} \quad \Rightarrow \quad H(z) = \frac{0.5z}{z+1} - \frac{0.5z}{z+3}$$

The impulse response h(n) is given by inverse Z-transform of H(z).

$$\text{Impulse response, } h(n) = Z^{-1}\{H(z)\} = Z^{-1}\left\{\frac{0.5z}{z+1} - \frac{0.5z}{z+3}\right\} = 0.5\, Z^{-1}\left\{\frac{z}{z-(-1)}\right\} - 0.5\, Z^{-1}\left\{\frac{z}{z-(-3)}\right\}$$

$$= 0.5(-1)^n\, u(n) - 0.5\,(-3)^n\, u(n) = 0.5\,[(-1)^n - (-3)^n]\, u(n)$$

Example 3.13

Find the transfer function and unit sample response of the second-order difference equation with zero initial condition,

$y(n) = x(n) - 0.25y(n-2)$.

Solution

The difference equation governing the system is,

$$y(n) = x(n) - 0.25\, y(n-2)$$

Let us take Z-transform of the difference equation governing the system with zero initial condition.

$$Z\{y(n)\} = Z\{x(n) - 0.25\, y(n-2)\}$$

$$Z\{y(n)\} = Z\{x(n)\} - 0.25\, Z\{y(n-2)\}$$

$$Y(z) = X(z) - 0.25\, z^{-2}\, Y(z)$$

$$Y(z) + 0.25z^{-2}\, Y(z) = X(z)$$

$$[1 + 0.25z^{-2}]\, Y(z) = X(z)$$

$Z\{x(n)\} = X(z)$
$Z\{y(n)\} = Y(z)$ $Z\{y(n-2)\} = z^{-2}\, Y(z)$ (Using shifting property)

$$\therefore \text{ Transfer function, } \frac{Y(z)}{X(z)} = \frac{1}{1 + 0.25z^{-2}}$$

We know that, $\frac{Y(z)}{X(z)} = H(z)$

$(a+b)\,(a-b) = a^2 - b^2$	$j^2 = -1$

$$\therefore H(z) = \frac{1}{1 + 0.25\, z^{-2}} = \frac{1}{z^{-2}(z^2 + 0.25)} = \frac{z^2}{(z + j\,0.5)\,(z - j\,0.5)}$$

Using partial fraction expansion technique we can write,

$$\frac{H(z)}{z} = \frac{z}{(z + j0.5)\,(z - j0.5)} = \frac{A}{z + j0.5} + \frac{A^*}{z - j0.5} \quad ; \quad \text{where } A^* \text{ is conjugate of } A.$$

$$A = (z + j0.5)\, \frac{H(z)}{z}\bigg|_{z=-j0.5} = \cancel{(z + j0.5)}\, \frac{z}{\cancel{(z + j0.5)}\,(z - j0.5)}\bigg|_{z=-j0.5}$$

$$= \frac{z}{z - j0.5}\bigg|_{z=-j0.5} = \frac{-j0.5}{-j0.5 - j0.5} = \frac{-j0.5}{2(-j0.5)} = \frac{1}{2} = 0.5$$

$$\therefore A^* = 0.5$$

$$\frac{H(z)}{z} = \frac{A}{z + j0.5} + \frac{A^*}{z - j0.5} = \frac{0.5}{z + j0.5} + \frac{0.5}{z - j0.5}$$

$$\therefore H(z) = \frac{0.5z}{z + j0.5} + \frac{0.5z}{z - j0.5} = \frac{0.5z}{z - (-j0.5)} + \frac{0.5z}{z - j0.5}$$

The impulse response is obtained by taking inverse Z-transform of H(z).

$\therefore$ Impulse response, $h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{0.5z}{z-(-j0.5)} + \frac{0.5z}{z-j0.5}\right\}$

$$= 0.5\left[\mathcal{Z}^{-1}\left\{\frac{z}{z-(-j0.5)}\right\} + \mathcal{Z}^{-1}\left\{\frac{z}{z-j0.5}\right\}\right]$$

$$= 0.5\left[(-j0.5)^n\,u(n) + (j0.5)^n\,u(n)\right]$$

$$\boxed{\mathcal{Z}\{a^n\,u(n)\} = \frac{z}{z-a}}$$

Alternatively the impulse response can be expressed as shown below.

$$-j0.5 = 0.5\angle-90^\circ = 0.5\angle-\pi/2 = 0.5\angle-0.5\pi$$

$$+j0.5 = 0.5\angle 90^\circ = 0.5\angle\pi/2 = 0.5\angle 0.5\pi$$

$$\therefore\ h(n) = 0.5\left[(0.5\angle-0.5\pi)^n + (0.5\angle 0.5\pi)^n\right]u(n)$$

$$= 0.5\left[0.5^n\angle-0.5n\pi + 0.5^n\angle 0.5n\pi\right]u(n)$$

$$= 0.5\,(0.5)^n\left[\cos 0.5n\pi - j\sin 0.5n\pi + \cos 0.5n\pi + j\sin 0.5n\pi\right]u(n)$$

$$= 0.5\,(0.5)^n\,[2\cos 0.5n\pi]\,u(n)$$

$$= 0.5^n\cos(0.5n\pi)\,u(n)$$

$180^\circ = \pi$ rad

$\therefore\ 1^\circ = \frac{\pi}{180}$ rad

$\therefore\ 90^\circ = 90\times\frac{\pi}{180} = 0.5\,\pi$ rad

Example 3.14

Determine the impulse response sequence of the discrete time LTI system defined by,

$y(n) - 4y(n-1) + 4y(n-2) = x(n) - 5x(n-3)$.

Solution

The difference equation governing the LTI system is,

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - 5x(n-3)$$

Let us assume that the initial conditions are zero.

On taking $\mathcal{Z}$-transform of the difference equation governing the system we get,

$\mathcal{Z}\{x(n)\} = X(z)$, $\therefore$	$\mathcal{Z}\{ax(n-m)\} = az^{-m}X(z)$
$\mathcal{Z}\{y(n)\} = Y(z)$, $\therefore$	$\mathcal{Z}\{ay(n-m)\} = az^{-m}Y(z)$

$$\mathcal{Z}\{y(n) - 4y(n-1) + 4y(n-2)\} = \mathcal{Z}\{x(n) - 5x(n-3)\}$$

$$\mathcal{Z}\{y(n)\} - 4\,\mathcal{Z}\{y(n-1)\} + 4\mathcal{Z}\{y(n-2)\} = \mathcal{Z}\{x(n)\} - 5\,\mathcal{Z}\{x(n-3)\}$$

$$Y(z) - 4z^{-1}\,Y(z) + 4z^{-2}\,Y(z) = X(z) - 5z^{-3}\,X(z)$$

$$[1 - 4z^{-1} + 4z^{-2}]\,Y(z) = [1 - 5z^{-3}]\,X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{1-5z^{-3}}{1-4z^{-1}+4z^{-2}}$$

$\mathcal{Z}\{a^n u(n)\} = \frac{z}{z-a}$

$\mathcal{Z}\{na^n u(n)\} = \frac{az}{(z-a)^2}$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property $\mathcal{Z}\{x(n\pm m)\} = z^{\pm m}X(z)$

We know that, $\frac{Y(z)}{X(z)} = H(z)$

$$\therefore\ H(z) = \frac{1-5z^{-3}}{1-4z^{-1}+4z^{-2}} = \frac{1-5z^{-3}}{z^{-2}(z^2-4z+4)} = \frac{z^2-5z^{-1}}{(z-2)^2}$$

$$= \frac{z^2}{(z-2)^2} - \frac{5z^{-1}}{(z-2)^2} = \frac{1}{2}z\,\frac{2z}{(z-2)^2} - \frac{5}{2}z^{-2}\,\frac{2z}{(z-2)^2}$$

$$\boxed{(a-b)^2 = a^2 - 2ab + b^2}$$

The impulse response is obtained by taking inverse $\mathcal{Z}$-transform of H(z).

$\therefore$ Impulse response, $h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{2}z\,\frac{2z}{(z-2)^2} - \frac{5}{2}z^{-2}\,\frac{2z}{(z-2)^2}\right\}$

$$= \frac{1}{2}\mathcal{Z}^{-1}\left\{z\,\frac{2z}{(z-2)^2}\right\} - \frac{5}{2}\mathcal{Z}^{-1}\left\{z^{-2}\,\frac{2z}{(z-2)^2}\right\}$$

$$= \frac{1}{2}(n+1)\,(2)^{n+1}u(n+1) - \frac{5}{2}(n-2)\,(2)^{n-2}u(n-2)$$

Example 3.15

Find the impulse response of the system described by the difference equation,
$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$.

Solution

$Z\{y(n)\} = Y(z)$; \ $Z\{y(n-m)\} = z^{-m}\,Y(z)$
$Z\{x(n)\} = X(z)$; \ $Z\{x(n-m)\} = z^{-m}\,X(z)$

The difference equation governing the LTI system is,

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

On taking Z-transform we get,

$$Y(z) - 3z^{-1}Y(z) - 4\,z^{-2}Y(z) = X(z) + 2z^{-1}\,X(z)$$

$$[1 - 3z^{-1} - 4z^{-2}]\,Y(z) = [1 + 2z^{-1}]\,X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}}$$

The roots of the quadratic,
$z^2 - 3z - 4 = 0$ are,
$z = \dfrac{3 \pm \sqrt{3^2 + 4 \times 4}}{2} = 4 \text{ or } -1$

We know that $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore\ H(z) = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z^{-2}(z^2 + 2z)}{z^{-2}(z^2 - 3z - 4)} = \frac{z^2 + 2z}{(z-4)\,(z+1)}$$

By partial fraction expansion technique,

$$\frac{H(z)}{z} = \frac{z+2}{(z-4)(z+1)} = \frac{A}{z-4} + \frac{B}{z+1}$$

$$A = (z-4)\left.\frac{H(z)}{z}\right|_{z=4} = \cancel{(z-4)}\left.\frac{z+2}{\cancel{(z-4)}\,(z+1)}\right|_{z=4} = \left.\frac{z+2}{z+1}\right|_{z=4} = \frac{4+2}{4+1} = \frac{6}{5} = 1.2$$

$$B = (z+1)\left.\frac{H(z)}{z}\right|_{z=-1} = \cancel{(z+1)}\left.\frac{z+2}{(z-4)\,\cancel{(z+1)}}\right|_{z=-1} = \left.\frac{z+2}{z-4}\right|_{z=-1} = \frac{-1+2}{-1-4} = \frac{1}{-5} = -0.2$$

$$\therefore\ \frac{H(z)}{z} = \frac{A}{z-4} + \frac{B}{z+1} = \frac{1.2}{z-4} - \frac{0.2}{z+1}$$

$$\therefore\ H(z) = 1.2\,\frac{z}{z-4} - 0.2\frac{z}{z+1} = 1.2\left(\frac{z}{z-4}\right) - 0.2\left(\frac{z}{z-(-1)}\right)$$

$$\boxed{Z\left\{\frac{z}{z-a}\right\} = a^n}$$

The impulse response is obtained by taking inverse Z-transform of H(z).

\ Impulse response, $h(n) = 1.2(4)^n\,u(n) - 0.2(-1)^n\,u(n)$

Example 3.16

Determine the steady state response for the system with impulse function, $h(n) = (j0.8)^n\,u(n)$ for an input, $x(n) = \cos(pn)\,u(n)$.

Solution

Let y(n) be the steady state response of the system, which is given by convolution of x(n) and h(n).

\ Steady state response, $y(n) = x(n) * h(n)$

On taking Z-transform of the above equation we get,

$$Z\{y(n)\} = Z\{x(n) * h(n)\}$$

\ $Y(z) = X(z)\,H(z)$

Using convolution property.

\ $y(n) = Z^{-1}\{X(z)\,H(z)\}$

Given that, $h(n) = (j0.8)^n\ u(n)$

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \frac{z}{z - j0.8}$$

$$\boxed{\begin{aligned} \mathcal{Z}\{a^n u(n)\} &= \frac{z}{z-a} \\ \mathcal{Z}\{\cos(\omega n)\ u(n)\} &= \frac{z(z-\cos\omega)}{z^2 - 2z\cos\omega + 1} \end{aligned}}$$

Given that, $x(n) = \cos(\pi n)\ u(n)$

$$\boxed{\cos \pi = -1}$$

$$\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \frac{z(z - \cos\pi)}{z^2 - 2z\cos\pi + 1} = \frac{z(z+1)}{z^2 + 2z + 1} = \frac{z(z+1)}{(z+1)^2} = \frac{z}{z+1}$$

$$\therefore\ Y(z) = X(z)\,H(z) = \frac{z}{z+1} \times \frac{z}{z - j0.8} = \frac{z^2}{(z+1)(z - j0.8)}$$

By partial fraction expansion technique we can write,

$$\frac{Y(z)}{z} = \frac{z}{(z+1)(z-j0.8)} = \frac{A}{z+1} + \frac{B}{z - j0.8}$$

$$A = (z+1)\left.\frac{Y(z)}{z}\right|_{z=-1} = \cancel{(z+1)}\left.\frac{z}{\cancel{(z+1)}(z-j0.8)}\right|_{z=-1} = \left.\frac{z}{z-j0.8}\right|_{z=-1} = \frac{-1}{-1-j0.8}$$

$$= \frac{-1}{-1-j0.8} \times \frac{-1+j0.8}{-1+j0.8} = \frac{1-j0.8}{1^2+0.8^2} = \frac{1-j0.8}{1.64} = 0.61 - j0.49$$

$$B = (z-j0.8)\left.\frac{Y(z)}{z}\right|_{z=j0.8} = \cancel{(z-j0.8)}\left.\frac{z}{(z+1)\cancel{(z-j0.8)}}\right|_{z=j0.8} = \left.\frac{z}{z+1}\right|_{z=j0.8} = \frac{j0.8}{j0.8+1}$$

$$= \frac{j0.8}{1+j0.8} \times \frac{1-j0.8}{1-j0.8} = \frac{j0.8 - (j0.8)^2}{1^2+0.8^2} = \frac{0.64 + j0.8}{1.64} = 0.39 + j0.49$$

$$\therefore\ \frac{Y(z)}{z} = \frac{A}{z+1} + \frac{B}{z-j0.8} = \frac{0.61 - j0.49}{z+1} + \frac{0.39 + j0.49}{z - j0.8}$$

$$\therefore\ Y(z) = (0.61 - j0.49)\frac{z}{z+1} + (0.39 + j0.49)\frac{z}{z-j0.8}$$

$$\boxed{\mathcal{Z}\{a^n\ u(n)\} = \frac{z}{z-a}}$$

$$= (0.61 - j0.49)\frac{z}{z-(-1)} + (0.39 + j0.49)\frac{z}{z-j0.8}$$

The steady state response is obtained by taking inverse $\mathcal{Z}$-transform of Y(z).

$\therefore$ Steady state response, $y(n) = (0.61 - j0.49)\,(-1)^n\,u(n) + (0.39 + j0.49)\,(j0.8)^n\,u(n)$

Alternatively the steady state response can be expressed as shown below.

Here,
$$0.61 - j0.49 = 0.78\ \angle -38.2^\circ = 0.78\ \angle -0.21\pi$$
$$0.39 + j0.49 = 0.63\ \angle 51.5^\circ = 0.63\ \angle 0.29\pi$$
$$-1 = 1\ \angle 180^\circ = 1\ \angle \pi$$
$$j0.8 = 0.8\ \angle 90^\circ = 0.8\ \angle 0.5\pi$$

$$\therefore\ y(n) = 0.78\ \angle -0.21\pi\ [1\angle\pi]^n\,u(n) + 0.63\angle 0.29\pi\ [0.8\angle 0.5\pi]^n\ u(n)$$
$$= 0.78\ \angle -0.21\pi\ 1^n \angle n\pi\,u(n) + 0.63\angle 0.29\pi\ 0.8^n \angle 0.5n\pi\ u(n)$$
$$= 0.78\ \angle(n - 0.21)\pi\ u(n) + 0.63\ (0.8)^n \angle(0.5n + 0.29)\pi\ u(n)$$

$$180^\circ = \pi \text{ rad}$$
$$\therefore\ 1^\circ = \frac{\pi}{180} \text{ rad}$$
$$38.2^\circ = \frac{38.2}{180}\pi = 0.21\pi \text{ rad}$$
$$51.5^\circ = \frac{51.5}{180}\pi = 0.29\pi \text{ rad}$$
$$90^\circ = \frac{90}{180}\pi = 0.5\pi \text{ rad}$$

Example 3.17

Obtain and sketch the impulse response of shift invariant system described by,
y(n) = 0.4 x(n) + x(n – 1) + 0.2 x(n – 2) + x(n – 3) + 0.6 x(n – 4).

Solution

The difference equation governing the system is,

$$y(n) = 0.4\,x(n) + x(n-1) + 0.2\,x(n-2) + x(n-3) + 0.6\,x(n-4)$$

On taking Z-transform we get,

$$Y(z) = 0.4X(z) + z^{-1}X(z) + 0.2z^{-2}X(z) + z^{-3}X(z) + 0.6z^{-4}X(z)$$

$$Y(z) = [0.4 + z^{-1} + 0.2z^{-2} + z^{-3} + 0.6z^{-4}]\,X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \left[0.4 + z^{-1} + 0.2z^{-2} + z^{-3} + 0.6z^{-4}\right]$$

If $Z\{x(n)\} = X(z)$ then by shifting property
$Z\{x(n-k)\} = z^{-k}\,X(z)$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore\ H(z) = 0.4 + z^{-1} + 0.2z^{-2} + z^{-3} + 0.6z^{-4} \qquad .....(1)$$

By the definition of one sided Z-transform we get,

$$H(z) = \sum_{n=0}^{+\infty} h(n)z^{-n}$$

$$= h(0)\,z^{0} + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3} + h(4)\,z^{-4} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

h(0) = 0.4	h(3) = 1
h(1) = 1	h(4) = 0.6
h(2) = 0.2	h(n) = 0 ; for n < 0 and n > 4

\ Impulse response, h(n) = {0.4, 1.0, 0.2, 1.0, 0.6}

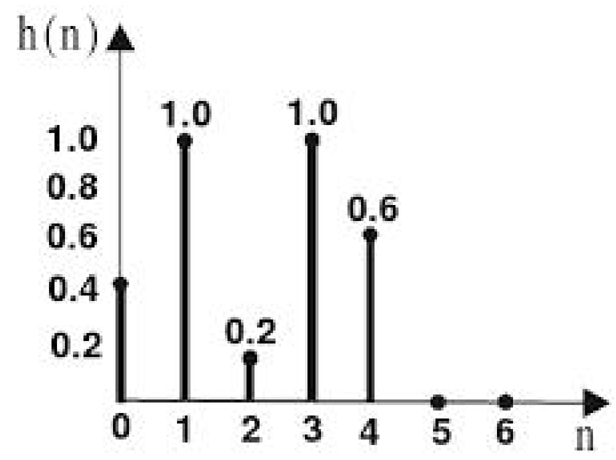


Fig 1: Graphical representation of impulse response h(n).

Example 3.18

Determine the response of discrete time LTI system governed by the difference equation, y(n) = – 0.8 y(n – 1) + x(n), when the input is unit step and initial condition, a) y(–1) = 0 and b) y(–1) = 2/9.

Solution

Given that, x(n) = u(n) ; $\therefore\ X(z) = Z\{x(n)\} = Z\{u(n)\} = \dfrac{z}{z-1}$ .....(1)

Given that, y(n) = – 0.8 y(n – 1) + x(n)

\ y(n) + 0.8 y(n – 1) = x(n)

On taking Z-transform of above equation we get,

If $Z\{y(n)\} = Y(z)$
then $Z\{y(n-1)\} = z^{-1}\,Y(z) - y(-1)$

$$Y(z) + 0.8\left[z^{-1}\,Y(z) + y(-1)\right] = X(z)$$

$$Y(z)\left[1 + 0.8\,z^{-1}\right] + 0.8\,y(-1) = \frac{z}{z-1}$$

Using equation (1).

$$Y(z)\left(1 + \frac{0.8}{z}\right) = \frac{z}{z-1} - 0.8\,y(-1)$$

$$Y(z)\left(\frac{z + 0.8}{z}\right) = \frac{z}{z-1} - 0.8\,y(-1)$$

$$\therefore\ Y(z) = \frac{z^2}{(z-1)\,(z+0.8)} - 0.8\ \frac{z\,y(-1)}{z+0.8}$$

Let, $P(z) = \frac{z^2}{(z-1)(z+0.8)} \quad \Rightarrow \quad \frac{P(z)}{z} = \frac{z}{(z-1)(z+0.8)}$

Let, $\frac{z}{(z-1)(z+0.8)} = \frac{A}{z-1} + \frac{B}{z+0.8}$

$$A = \frac{z}{\cancel{(z-1)}(z+0.8)} \times \cancel{(z-1)}\Big|_{z=1} = \frac{1}{1+0.8} = \frac{1}{1.8} = \frac{10}{18} = \frac{5}{9}$$

$$B = \frac{z}{(z-1)\cancel{(z+0.8)}} \times \cancel{(z+0.8)}\Big|_{z=-0.8} = \frac{-0.8}{-0.8-1} = \frac{-0.8}{-1.8} = \frac{8}{18} = \frac{4}{9}$$

$$\therefore \frac{P(z)}{z} = \frac{5}{9}\,\frac{1}{z-1} + \frac{4}{9}\,\frac{1}{z+0.8} \quad \Rightarrow \quad P(z) = \frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8}$$

$$\therefore Y(z) = \frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8} - 0.8\,\frac{z\,y(-1)}{z+0.8} \qquad .....(2)$$

a) When y(−1) = 0

From equation (2), when $y(-1) = 0$, we get,

$$Y(z) = \frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8}$$

$$\therefore \text{Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8}\right\}$$

$$= \frac{5}{9}\,u(n) + \frac{4}{9}\,(-0.8)^n\,u(n)$$

b) When y(−1) = 2/9

From equation (2), when $y(-1) = 2/9$, we get,

$$Y(z) = \frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8} - 0.8 \times \frac{2}{9}\,\frac{z}{z+0.8} = \frac{5}{9}\,\frac{z}{z-1} + \frac{2.4}{9}\,\frac{z}{z+0.8}$$

$$= \frac{5}{9}\,\frac{z}{z-1} + \frac{24}{90}\,\frac{z}{z+0.8} = \frac{5}{9}\,\frac{z}{z-1} + \frac{12}{45}\,\frac{z}{z+0.8}$$

$$\therefore \text{Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{5}{9}\,\frac{z}{z-1} + \frac{12}{45}\,\frac{z}{z+0.8}\right\}$$

$$= \left[\frac{5}{9} + \frac{12}{45}(-0.8)^n\right] u(n)$$

Note : Compare the result with example 2.8 of Chapter 2.

Example 3.19

Determine the response of LTI discrete time system governed by the difference equation, $y(n) - 0.2\,y(n-1) - 0.03\,y(n-2) = x(n) + 0.4\,x(n-1)$ for the input, $x(n) = 0.2^n\,u(n)$ and with initial condition, $y(-2) = 0$, $y(-1) = 0.5$.

Solution

Given that, $x(n) = 0.2^n\,u(n)$; $\quad \therefore X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{0.2^n\,u(n)\} = \frac{z}{z-0.2}$ .....(1)

Given that, $y(n) - 0.2\,y(n-1) - 0.03\,y(n-2) = x(n) + 0.4\,x(n-1)$

On taking $\mathcal{Z}$-transform of above equation we get,

$$Y(z) - 0.2\left[z^{-1}Y(z) + y(-1)\right] - 0.03\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] = X(z) + 0.4\left[z^{-1}X(z) + x(-1)\right] \qquad .....(2)$$

If $\mathcal{Z}\{y(n)\} = Y(z)$, then $\mathcal{Z}\{y(n-1)\} = z^{-1}Y(z) + y(-1)$

and $\mathcal{Z}\{y(n-2)\} = z^{-2}Y(z) + z^{-1}y(-1) + y(-2)$

Given that, $y(-2)= 0$, $y(-1) = 0.5$

$$x(n) = 0.2^n\ u(n) = 0.2^n\ ; \text{ for } n \geq 0 \qquad \Rightarrow \quad x(-1) = 0$$
$$= 0 \quad ; \text{ for } n < 0$$

On substituting the above initial conditions in equation (2) we get,

$$Y(z) - 0.2\,z^{-1}\,Y(z) - 0.2 \times 0.5 - 0.03\,z^{-2}\,Y(z) - 0.03\,z^{-1} \times 0.5 + 0 = X(z) + 0.4\,z^{-1}\,X(z) + 0$$

$$Y(z) - \frac{0.2}{z}Y(z) - 0.1 - \frac{0.03}{z^2}Y(z) - \frac{0.015}{z} = X(z) + \frac{0.4}{z}X(z)$$

$$\therefore\ Y(z)\left(1 - \frac{0.2}{z} - \frac{0.03}{z^2}\right) - \left(\frac{0.015}{z} + 0.1\right) = X(z)\left(1 + \frac{0.4}{z}\right)$$

$$Y(z)\left(\frac{z^2 - 0.2\,z - 0.03}{z^2}\right) - \left(\frac{0.015 + 0.1z}{z}\right) = \left(\frac{\cancel{z}}{z - 0.2}\right)\left(\frac{z + 0.4}{\cancel{z}}\right)$$

Using equation (1).

$$Y(z)\frac{(z-0.3)\,(z+0.1)}{z^2} = \frac{z+0.4}{z-0.2} + \frac{0.015+0.1z}{z}$$

$$Y(z)\frac{(z-0.3)\,(z+0.1)}{z^2} = \frac{z(z+0.4)\ + (0.015+0.1z)\,(z-0.2)}{z(z-0.2)}$$

$$Y(z)\frac{(z-0.3)\,(z+0.1)}{z^2} = \frac{z^2 + 0.4\,z + 0.015\,z - 0.003 + 0.1z^2 - 0.02\,z}{z(z-0.2)}$$

The roots of quadratic $z^2 - 0.2z - 0.03 = 0$ are,

$$z = \frac{0.2 \pm \sqrt{0.2^2 - 4 \times (-0.03)}}{2} = \frac{0.2 \pm 0.4}{2} = 0.3,\ -0.1$$

$$Y(z)\frac{(z-0.3)\,(z+0.1)}{z^2} = \frac{1.1z^2 + 0.395z - 0.003}{z(z-0.2)}$$

$$Y(z) = \frac{1.1z^2 + 0.395z - 0.003}{z(z-0.2)} \times \frac{z^2}{(z-0.3)\,(z+0.1)}$$

$$= \frac{z(1.1z^2 + 0.395z - 0.003)}{(z-0.2)\,(z-0.3)\,(z+0.1)}$$

Let, $$\frac{Y(z)}{z} = \frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,(z-0.3)\,(z+0.1)} = \frac{A}{z-0.2} + \frac{B}{z-0.3} + \frac{C}{z+0.1}$$

$$A = \frac{1.1z^2 + 0.395z - 0.003}{\cancel{(z-0.2)}\,(z-0.3)\,(z+0.1)} \times \cancel{(z-0.2)}\Bigg|_{z=0.2} = \frac{1.1z^2 + 0.395z - 0.003}{(z-0.3)\,(z+0.1)}\Bigg|_{z=0.2}$$

$$= \frac{1.1 \times 0.2^2 + 0.395 \times 0.2 - 0.003}{(0.2-0.3)\,(0.2+0.1)} = -4$$

$$B = \frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,\cancel{(z-0.3)}\,(z+0.1)} \times \cancel{(z-0.3)}\Bigg|_{z=0.3} = \frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,(z+0.1)}\Bigg|_{z=0.3}$$

$$= \frac{1.1 \times 0.3^2 + 0.395 \times 0.3 - 0.003}{(0.3-0.2)\,(0.3+0.1)} = 5.3625$$

$$C = \frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,(z-0.3)\,\cancel{(z+0.1)}} \times \cancel{(z+0.1)}\Bigg|_{z=-0.1} = \frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,(z-0.3)}\Bigg|_{z=-0.1}$$

$$= \frac{1.1 \times (-0.1)^2 + 0.395 \times (-0.1) - 0.003}{(-0.1-0.2)\,(-0.1-0.3)} = -0.2625$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{-4}{z-0.2} + \frac{5.3625}{z-0.3} - \frac{0.2625}{z+0.1} \quad \Rightarrow \quad Y(z) = -4\frac{z}{z-0.2} + 5.3625\frac{z}{z-0.3} - 0.2625\frac{z}{z-(-0.1)}$$

$$\therefore \text{ Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{-4\frac{z}{z-0.2} + 5.3625\frac{z}{z-0.3} - 0.2625\frac{z}{z-(-0.1)}\right\}$$

$$= -4(0.2)^n u(n) + 5.3625(0.3)^n u(n) - 0.2625(-0.1)^n u(n)$$

$$= \left[-4(0.2)^n + 5.3625(0.3)^n - 0.2625(-0.1)^n\right]u(n)$$

Note : Compare the result with example 2.9 of Chapter 2.

Example 3.20

Find the response of the time invariant system with impulse response, $h(n) = \{1, 2, -1, -2\}$ to an input signal, $x(n) = \{1, 2, 3, 4\}$.

Solution

Let, $y(n)$ = Response or Output of an LTI system.

The response of an LTI system is given by the convolution of input signal and impulse response.

\ $y(n) = x(n) * h(n)$

On taking $\mathcal{Z}$-transform we get,

By convolution property
$\mathcal{Z}\{x(n) * h(n)\} = X(z)\,H(z)$

$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) * h(n)\}$

\ $Y(z) = X(z)\,H(z)$

Given that, $x(n) = \{1, 2, 3, 4\}$

By definition of one-sided $\mathcal{Z}$-transform,

$$X(z) = \sum_{n=0}^{\infty} x(n)\,z^{-n} = \sum_{n=0}^{3} x(n)\,z^{-n} = x(0)\,z^0 + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

Given that, $h(n) = \{1, 2, -1, -2\}$

By definition of one-sided $\mathcal{Z}$-transform,

$$H(z) = \sum_{n=0}^{\infty} h(n)\,z^{-n} = \sum_{n=0}^{3} h(n)\,z^{-n} = h(0)\,z^0 + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3}$$

$$= 1 + 2z^{-1} - z^{-2} - 2z^{-3}$$

\ $Y(z) = X(z)\,H(z)$

$$= [1 + 2z^{-1} + 3z^{-2} + 4z^{-3}]\,[1 + 2z^{-1} - z^{-2} - 2z^{-3}]$$

$$= 1 + 2z^{-1} - z^{-2} - 2z^{-3}$$
$$+ 2z^{-1} + 4z^{-2} - 2z^{-3} - 4z^{-4}$$
$$+ 3z^{-2} + 6z^{-3} - 3z^{-4} - 6z^{-5}$$
$$+ 4z^{-3} + 8z^{-4} - 4z^{-5} - 8z^{-6}$$

$$= 1 + 4z^{-1} + 6z^{-2} + 6z^{-3} + z^{-4} - 10z^{-5} - 8z^{-6} \qquad .....(1)$$

By definition of one-sided $\mathcal{Z}$-transform we get,

$$Y(z) = \sum_{n=0}^{\infty} y(n)\,z^{-n}$$

$$= y(0)\,z^0 + y(1)\,z^{-1} + y(2)\,z^{-2} + y(3)\,z^{-3} + y(4)\,z^{-4} + y(5)\,z^{-5} + y(6)\,z^{-6} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$y(0) = 1$	$y(2) = 6$	$y(4) = 1$	$y(6) = -8$
$y(1) = 4$	$y(3) = 6$	$y(5) = -10$	

\ The response of the system, $y(n) = \{1, 4, 6, 6, 1, -10, -8\}$

Example 3.21

Using Z-transform, perform deconvolution of the response, $y(n) = \{1, \ 4, \ 6, \ 6, \ 1, \ -10, \ -8\}$ and impulse response $h(n) = \{1, \ 2, \ -1, \ -2\}$ to extract the input x(n).

Solution

Given that, $y(n) = \{1, \ 4, \ 6, \ 6, \ 1, \ -10, \ -8\}$

$$\therefore Y(z) = Z\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\ z^{-n} = \sum_{n=0}^{6} y(n)\ z^{-n}$$

$$= y(0) + y(1)\ z^{-1} + y(2)\ z^{-2} + y(3)\ z^{-3} + y(4)\ z^{-4} + y(5)\ z^{-5} + y(6)\ z^{-6}$$

$$= 1 + 4\,z^{-1} + 6\,z^{-2} + 6\,z^{-3} + z^{-4} - 10\,z^{-5} - 8z^{-6}$$

Given that, $h(n) = \{1, \ 2, \ -1, \ -2\}$

$$\therefore H(z) = Z\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\ z^{-n} = \sum_{n=0}^{3} h(n)\ z^{-n} = h(0) + h(1)\ z^{-1} + h(2)\ z^{-2} + h(3)\ z^{-3}$$

$$= 1 + 2\,z^{-1} - z^{-2} - 2z^{-3}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore X(z) = \frac{Y(z)}{H(z)} = \frac{1 + 4\,z^{-1} + 6\,z^{-2} + 6\,z^{-3} + z^{-4} - 10\,z^{-5} - 8z^{-6}}{1 + 2\,z^{-1} - z^{-2} - 2z^{-3}}$$

$$= 1 + 2\,z^{-1} + 3\,z^{-2} + 4z^{-3} \qquad \text{.....(1)}$$

$$\begin{array}{r|l}
 & 1 + 2\,z^{-1} + 3\,z^{-2} + 4z^{-3} \\
\hline
1 + 2\,z^{-1} - z^{-2} - 2z^{-3} & 1 + 4\,z^{-1} + 6\,z^{-2} + 6\,z^{-3} + z^{-4} - 10\,z^{-5} - 8z^{-6} \\
 & \underset{(-)}{1} \underset{(-)}{+ 2\,z^{-1}} \underset{(+)}{- z^{-2}} \underset{(+)}{- 2z^{-3}} \\
\hline
 & \quad 2\,z^{-1} + 7\,z^{-2} + 8\,z^{-3} + z^{-4} \\
 & \quad \underset{(-)}{2\,z^{-1}} \underset{(-)}{+ 4\,z^{-2}} \underset{(+)}{- 2\,z^{-3}} \underset{(+)}{- 4z^{-4}} \\
\hline
 & \qquad 3\,z^{-2} + 10\,z^{-3} + 5\,z^{-4} - 10\,z^{-5} \\
 & \qquad \underset{(-)}{3\,z^{-2}} \underset{(-)}{+ 6\,z^{-3}} \underset{(+)}{- 3\,z^{-4}} \underset{(+)}{- 6\,z^{-5}} \\
\hline
 & \qquad\quad 4z^{-3} + 8\,z^{-4} - 4z^{-5} - 8z^{-6} \\
 & \qquad\quad \underset{(-)}{4z^{-3}} \underset{(-)}{+ 8\,z^{-4}} \underset{(+)}{- 4z^{-5}} \underset{(+)}{- 8z^{-6}} \\
\hline
 & \qquad\qquad 0
\end{array}$$

By the definition of Z-transform,

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\ z^{-n}$$

On expanding the above summation we get,

$$X(z) = \ \ldots\ldots\ + x(0) + x(1)\ z^{-1} + x(2)\ z^{-2} + x(3)\ z^{-3} + \ldots\ldots \qquad \text{.....(2)}$$

On comparing equations (1) and (2) we get,

$x(0) = 1 \quad ; \quad x(1) = 2 \quad ; \quad x(2) = 3 \quad ; \quad x(3) = 4$

$\therefore$ Input, $x(n) = \{\underset{\uparrow}{1}, \ 2, \ 3, \ 4\}$

Example 3.22

An LTI system is described by the equation, $y(n) = x(n) + 0.8\,x(n-1) + 0.8\,x(n-2) - 0.49\,y(n-2)$. Determine the transfer function of the system. Sketch the poles and zeros on the z-plane.

Solution

Given that, $y(n) = x(n) + 0.8\,x(n-1) + 0.8\,x(n-2) - 0.49\,y(n-2)$

On taking Z-transform we get,

$Z\{y(n)\} = Y(z)$; \ $Z\{y(n-m)\} = z^{-m}Y(z)$
$Z\{x(n)\} = X(z)$; \ $Z\{x(n-m)\} = z^{-m}X(z)$

$$Y(z) = X(z) + 0.8z^{-1}X(z) + 0.8z^{-2}X(z) - 0.49z^{-2}Y(z)$$

$$Y(z) + 0.49z^{-2}Y(z) = X(z) + 0.8z^{-1}X(z) + 0.8z^{-2}X(z)$$

$$[1 + 0.49z^{-2}]\,Y(z) = [1 + 0.8z^{-1} + 0.8z^{-2}]\,X(z)$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1} + 0.8z^{-2}}{1 + 0.49z^{-2}} \qquad .....(1)$$

The equation(1) is the transfer function of the LTI system.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1} + 0.8z^{-2}}{1 + 0.49z^{-2}} = \frac{z^{-2}(z^2 + 0.8z + 0.8)}{z^{-2}(z^2 + 0.49)}$$

$$= \frac{z^2 + 0.8z + 0.8}{z^2 + 0.49}$$

The poles are the roots of the denominator polynomial,

$$z^2 + 0.49 = 0$$

$$\backslash \; z^2 = -0.49$$

$$z = \pm\sqrt{-0.49} = \pm\, j0.7$$

\ The poles are, $p_1 = j0.7$, $p_2 = -j0.7$

The zeros are the roots of the numerator polynomial,

$$z^2 + 0.8z + 0.8 = 0$$

$$z = \frac{-0.8 \pm \sqrt{0.8^2 - 4 \times 0.8}}{2} = \frac{-0.8 \pm \sqrt{-2.56}}{2}$$

$$= \frac{-0.8 \pm j0.16}{2} = -0.4 \pm j0.8$$

\ The zeros are, $z_1 = -0.4 + j0.8$ and $z_2 = -0.4 - j0.8$

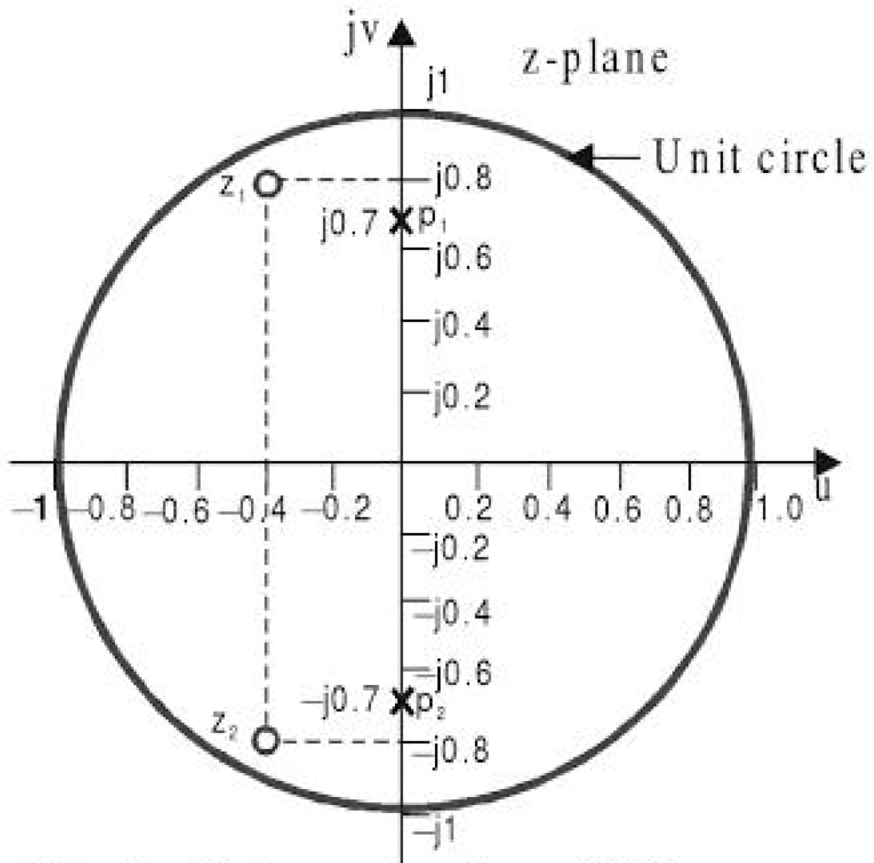


Fig 1 : Pole-zero plot of LTI system.

$$\therefore \quad H(z) = \frac{z^2 + 0.8z + 0.8}{z^2 + 0.49} = \frac{(z + 0.4 - j0.8)\,(z + 0.4 + j0.8)}{(z - j0.7)\,(z + j0.7)}$$

The fig1 Shows the location of poles and zeros on the z-plane. The poles are marked as "X" and Zeros as "O".

Example 3.23

Determine the step response of an LTI system whose impulse response h(n) is given by,

$h(n) = a^{-n}\,u(-n)$; $0 < a < 1$.

Solution

$\because u(-n) = 1 \;;\; n \le 0$
$= 0 \;;\; n > 0$

On taking Z-transform of impulse response h(n) we get,

$$H(z) = Z\{h(n)\} = \sum_{n=-\infty}^{\infty} h(n)\,z^{-n} = \sum_{n=-\infty}^{0} a^{-n}\,z^{-n} = \sum_{n=0}^{\infty} a^n z^n = \sum_{n=0}^{\infty} (az)^n \qquad ....(1)$$

If $|az| < 1$, then using infinite geometric series sum formula,

$$H(z) = \sum_{n=0}^{\infty} (az)^n = \frac{1}{1 - az} = \frac{-\frac{1}{a}}{z - \frac{1}{a}} \;;\; \text{ROC } |z| < \left|\frac{1}{a}\right|$$

Here, $|a\,z| < 1 \quad \Rightarrow \quad |z| < \left|\frac{1}{a}\right|$

$\therefore$ ROC is $|z| < \left|\frac{1}{a}\right|$. Since $|a| < 1$, $\left|\frac{1}{a}\right| > 1$, and so ROC includes unit circle.

The step input, $u(n) = 1 \;;\; n \geq 0$

$= 0 \;;\; n < 0$

On taking Z-transform of unit step signal we get,

Refer table 3.4

$$U(z) = Z\{u(n)\} = \frac{z}{z-1} \;;\; \text{ROC } |z| > 1 \qquad .....(2)$$

Let y(n) be step response. Now the step response is given by convolution of step input, u(n) and impulse response, h(n).

$\therefore\; y(n) = u(n) * h(n)$

By convolution property,
$Z\{u(n) * h(n)\} = U(z)\,H(z)$

On taking Z-transform we get,

$Z\{y(n)\} = Z\{u(n) * h(n)\}$

$\therefore\; Y(z) = U(z)\,H(z)$

On substituting for U(z) and H(z) from equations (1) and (2) respectively we get,

$$Y(z) = U(z)\;H(z) = \left(\frac{z}{z-1}\right)\left(\frac{-1/a}{z-1/a}\right)$$

By partial fraction expansion we can write,

$$\frac{Y(z)}{z} = \frac{-1/a}{(z-1)\,(z-1/a)} = \frac{A}{z-1} + \frac{B}{z-1/a}$$

$$A = (z-1)\left.\frac{Y(z)}{z}\right|_{z=1} = \left.\frac{-1/a}{z-1/a}\right|_{z=1} = \frac{-1/a}{1-1/a} = \frac{-1/a}{\frac{a-1}{a}} = \frac{-1}{a-1} = \frac{1}{1-a}$$

$$B = (z-1/a)\left.\frac{Y(z)}{z}\right|_{z=1/a} = \left.\frac{-1/a}{z-1}\right|_{z=1/a} = \frac{-1/a}{1/a-1} = \frac{-1/a}{\frac{1-a}{a}} = \frac{-1}{1-a}$$

$$\therefore\; \frac{Y(z)}{z} = \frac{1}{(1-a)}\,\frac{1}{(z-1)} - \frac{1}{(1-a)}\,\frac{1}{(z-1/a)}$$

$$\therefore\; Y(z) = \frac{1}{(1-a)}\,\frac{z}{(z-1)} - \frac{1}{(1-a)}\,\frac{z}{(z-1/a)}$$

$Z\{-u(-n-1)\} = \frac{z}{z-1}$

$Z\{-b^n\,u(-n-1)\} = \frac{z}{z-b}$

Note : Since impulse response is anticausal, the step response is also anticausal.

On taking inverse Z-transform of Y(z) we get step response.

$$\therefore \text{ Step response, } y(n) = -\frac{1}{1-a}\;u(-n-1) + \frac{1}{1-a}\left(\frac{1}{a}\right)^n u(-n-1) = \left[\left(\frac{1}{a}\right)^n - 1\right]\left(\frac{1}{1-a}\right)u(-n-1)$$

Example 3.24

Test the stability of the first-order system governed by the equation, $y(n) = x(n) + b\,y(n-1)$, where $|b| < 1$.

Solution

$Z\{y(n)\} = Y(z) \;;\quad Z\{y(n-1)\} = z^{-1}Y(z) \;;\quad Z\{x(n)\} = X(z)$

Given that, $y(n) = x(n) + b\,y(n-1)$

On taking Z-transform we get,

$$Y(z) = X(z) + b\,z^{-1}\,Y(z) \;\Rightarrow\; Y(z) - b\,z^{-1}\,Y(z) = X(z) \;\Rightarrow\; (1 - b\,z^{-1})\,Y(z) = X(z)$$

$$\therefore\; \frac{Y(z)}{X(z)} = \frac{1}{1 - b\,z^{-1}}$$

We know that, Y(z)/X(z) is equal to H(z).

$$\therefore\ H(z) = \frac{1}{1 - b\ z^{-1}} = \frac{1}{z^{-1}(z - b)} = \frac{z}{z - b}$$

On taking inverse Z-transform of H(z) we get the impulse response h(n).

$$\boxed{\mathcal{Z}\left\{a^n u(n)\right\} = \frac{z}{z-a}}$$

\ Impulse response, $h(n) = b^n\ u(n)$

The condition to be satisfied for the stability of the system is, $\sum\limits_{n=-\infty}^{\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |b^n| = \sum_{n=0}^{\infty} |b|^n$$

Since |b| < 1, using the infinite geometric series sum formula we can write,

$$\sum_{n=0}^{\infty} |b|^n = \frac{1}{1-|b|}$$

$$\therefore\ \sum_{n=-\infty}^{\infty} |h(n)| = \frac{1}{1-|b|} = \text{constant}$$

Infinite geometric series sum formula
$\sum\limits_{n=0}^{\infty} C^n = \frac{1}{1-C}\ ;\ \text{if, } 0 < \lvert C \rvert < 1$

The term 1/(1–|b|) is less than infinity and so the system is stable.

Example 3.25

Using Z-transform, find the autocorrelation of the causal sequence, $x(n) = a^n\ u(n),\ -1 < a < 1$.

Solution

Given that, $x(n) = a^n\ u(n)$

$$\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\left\{a^n u(n)\right\} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

$$\therefore\ X(z^{-1}) = X(z)\Big|_{z=z^{-1}} = \frac{1}{1 - a z} = -\frac{1}{a}\ \frac{1}{z - 1/a}$$

Let, $r_{xx}(m)$ be autocorrelation sequence.

By correlation property of Z - transform,

$$\mathcal{Z}\{r_{xx}(m)\} = X(z)\ X(z^{-1}) = \frac{z}{z-a} \times \frac{1}{-a}\ \frac{1}{z - 1/a} = -\frac{1}{a}\ \frac{z}{(z-a)\ (z - 1/a)}$$

Let, $\frac{z}{(z-a)\ (z-1/a)} = \frac{A}{z-a} + \frac{B}{z - 1/a}$

$$A = \frac{z}{\cancel{(z-a)}\ (z-1/a)} \times \cancel{(z-a)}\Big|_{z=a} = \frac{z}{z-1/a}\Bigg|_{z=a} = \frac{a}{a - 1/a} = \frac{a}{\frac{a^2-1}{a}} = \frac{a^2}{a^2-1}$$

$$B = \frac{z}{(z-a)\ \cancel{(z-1/a)}} \times \cancel{(z-1/a)}\Big|_{z=1/a} = \frac{z}{z-a}\Bigg|_{z=1/a} = \frac{1/a}{1/a - a} = \frac{1/a}{\frac{1-a^2}{a}} = \frac{1}{1-a^2} = \frac{-1}{a^2-1}$$

$$\therefore\ \mathcal{Z}\{r_{xx}(m)\} = -\frac{1}{a}\left(\frac{a^2}{a^2-1}\ \frac{1}{z-a} - \frac{1}{a^2-1}\ \frac{1}{z-1/a}\right) = -\frac{a}{a^2-1}\ \frac{1}{z-a} + \frac{1}{a(a^2-1)}\ \frac{1}{z-1/a}$$

$$\therefore\ r_{xx}(m) = \mathcal{Z}^{-1}\left\{-\frac{a}{a^2-1}\ \frac{1}{z-a} + \frac{1}{a(a^2-1)}\ \frac{1}{z-1/a}\right\}$$

$$= -\frac{a}{a^2-1}\ \mathcal{Z}^{-1}\left\{z^{-1}\frac{z}{z-a}\right\} + \frac{1}{a(a^2-1)}\ \mathcal{Z}^{-1}\left\{z^{-1}\frac{z}{z-1/a}\right\}$$

$$= -\frac{a}{a^2-1}\ a^{(n-1)}\ u(n-1) + \frac{1}{a(a^2-1)}\left(\frac{1}{a}\right)^{n-1} u(n-1)$$

$$= \frac{1}{a^2-1}\left[\frac{1}{a}\left(\frac{1}{a}\right)^{n-1} - a\ (a)^{n-1}\right] u(n-1) = \frac{1}{a^2-1}\left[\left(\frac{1}{a}\right)^{n} - a^{n}\right] u(n-1)$$

$$\boxed{z^{-1}\, z = z^0 = 1}$$

$$\boxed{\begin{array}{l}\text{If } \mathcal{Z}\{x(n)\} = X(z)\\ \text{then by shifting property}\\ \mathcal{Z}\{x(n-m)\} = z^{-m}\, X(z)\end{array}}$$

$$\boxed{\begin{array}{l}\mathcal{Z}\left\{a^n u(n)\right\} = \dfrac{z}{z-a}\\ \mathcal{Z}\left\{a^{n-1}u(n-1)\right\} = z^{-1}\dfrac{z}{z-a}\end{array}}$$

3.8 Structures for Realization of LTI Discrete Time Systems in z-Domain

A discrete time system is a system that accepts a discrete time signal as input and processes it, and delivers the processed discrete time signal as output. Mathematically, a discrete time system is represented by a difference equation. Physically, a discrete time system is realized or implemented either as a digital hardware (like special purpose Microprocessor / Microcontroller) or as a software running on a digital hardware (like PC-Personal Computer).

The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication and delay. Also the calculations are performed either by using fixed point arithmetic or floating point arithmetic. The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithmetic used for computation. These issues are addressed in structures for realization of discrete time systems.

From the implementation point of view, the discrete time systems are basically classified as IIR and FIR systems. The various structures proposed for IIR and FIR systems, attempt to reduce the computational complexity, errors in computation, memory requirement and finite word length effects in computations.

Discrete Time IIR System

Let, H(z) = Transfer function of discrete time IIR system.

The general form of transfer function of IIR system is,

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}}$$

Let, X(z) = Input of the discrete time system in z-domain.

Y(z) = Output of the discrete time system in z-domain.

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}} \qquad \ldots..(3.75)$$

On cross multiplying the equation (3.75) we get,

$$\left[1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}\right]Y(z) = \left[b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}\right]X(z)$$

$$Y(z) + a_1 z^{-1}\, Y(z) + a_2 z^{-2}\, Y(z) + \ldots\ldots + a_N z^{-N} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots\ldots + b_M z^{-M} X(z)$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

If $\mathcal{Z}\{x(n)\} = X(z)$ then,
$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z)$

$$y(n) + a_1\, y(n-1) + a_2\, y(n-2) + \ldots.. + a_N\, y(n-N)$$
$$= b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots.. + b_M\, x(n-M)$$

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - \ldots.. - a_N\, y(n-N)$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots.. + b_M\, x(n-M)$$

$$\therefore y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \qquad .....(3.76)$$

The equation (3.75) is the transfer function of discrete time IIR system and the equation (3.76) is the time domain difference equation governing discrete time IIR system. From equation (3.76), it is observed that the output at any time n depends on past outputs and so the IIR systems are recursive systems.

Discrete Time FIR system

Let, H(z) = Transfer function of discrete time FIR system.

The general form of transfer function of FIR system is,

$$H(z) = b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \ldots.. + b_{N-1}\, z^{-(N-1)}$$

Let, X(z) = Input of the discrete time system in z-domain.

Y(z) = Output of the discrete time system in z-domain.

$$\therefore H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \ldots.. + b_{N-1}\, z^{-(N-1)} \qquad .....(3.77)$$

On cross multiplying the equation (3.77) we get,

$$Y(z) = \left[b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \ldots.. + b_{N-1}\, z^{-(N-1)}\right]X(z)$$
$$= b_0X(z) + b_1\, z^{-1}\, X(z) + b_2\, z^{-2}\, X(z) + \ldots.. + b_{N-1}\, z^{-(N-1)}\, X(z)$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

$$y(n) = b_0x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots.. + b_{N-1}\, x(n-(N-1))$$

$$\therefore y(n) = \sum_{m=0}^{N-1} b_m x(n-m) \qquad .....(3.78)$$

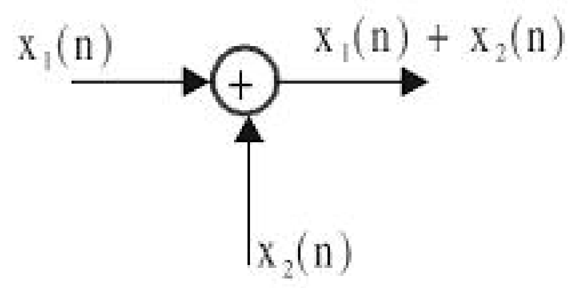
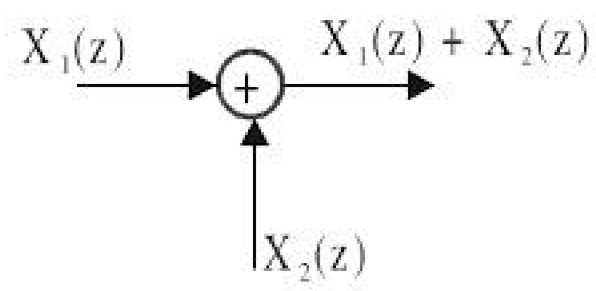
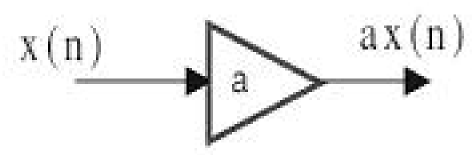
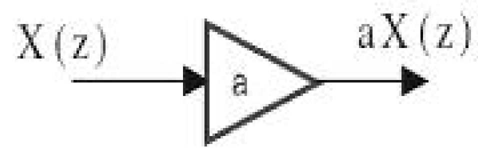
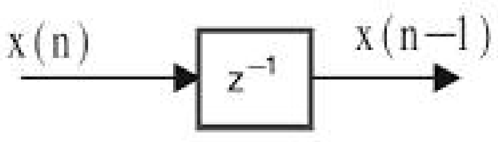
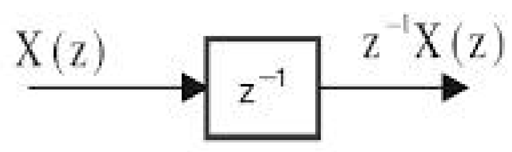
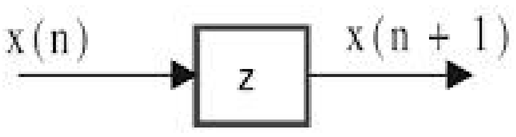
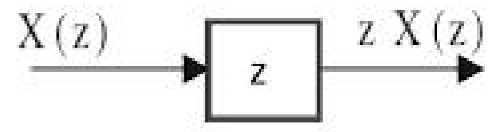
The equation (3.77) is the transfer function of discrete time FIR system and the equation (3.78) is the time domain difference equation governing discrete time FIR system. From equation (3.78), it is observed that the output at any time n does not depend on past outputs and so the FIR systems are nonrecursive systems.

Basic Elements of Block Diagram

The difference equations of IIR and FIR systems can be viewed as a computational procedure (or algorithm) to determine the output signal y(n) from the input signal x(n). The computations in the above difference equation of a system can be arranged into various equivalent sets of difference equations.

For each set of equations, we can construct a block diagram consisting of adder, constant multiplier, unit delay element and Unit advance element. Such block diagrams are referred to as realization of system or equivalently as structure for realizing system.The basic elements used to construct block diagrams are listed in table 3.6.

Table 3.6 : Basic elements of block diagram in time domain and z-domain

Elements of block diagram	Time domain representation	z-domain representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

3.9 Structures for Realization of IIR Systems

In general, the time domain representation of an N^{th} order IIR system is,

$$y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m)$$

and the z-domain representation of an N^{th} order IIR system is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

The above two representations of IIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence y(n) from the input sequence x(n). Also, in the above representations the value of M gives the number of zeros and the value of N gives the number of poles of the IIR system.

The computations in the above equation can be arranged into various equivalent sets of difference equations, which leads to different types of structures for realizing IIR systems.

Some of the structures of the system gives a direct relation between the time domain equation and the z-domain equation.

The different types of structures for realizing the IIR systems are,

1. Direct form-I structure
2. Direct form-II structure
3. Cascade form structure
4. Parallel form structure

3.9.1 Direct Form-I Structure of IIR System

Consider the difference equation governing an IIR system.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - \ldots\ldots - a_N\, y(n-N)$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots\ldots + b_M\, x(n-M)$$

On taking $\mathbb{Z}$-transform of the above equation we get,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \ldots\ldots - a_N z^{-N} Y(z)$$
$$+ b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots\ldots + b_M z^{-M} X(z) \qquad \ldots\ldots(3.79)$$

The equation of Y(z) [equation (3.79)] can be directly represented by a block diagram as shown in fig 3.15 and this structure is called direct form-I structure. The direct form-I structure provides a direct relation between time domain and z-domain equations. The direct form-I structure requires separate delays (z^{-1}) for input and output samples. Hence for realizing direct form-I structure more memory is required.

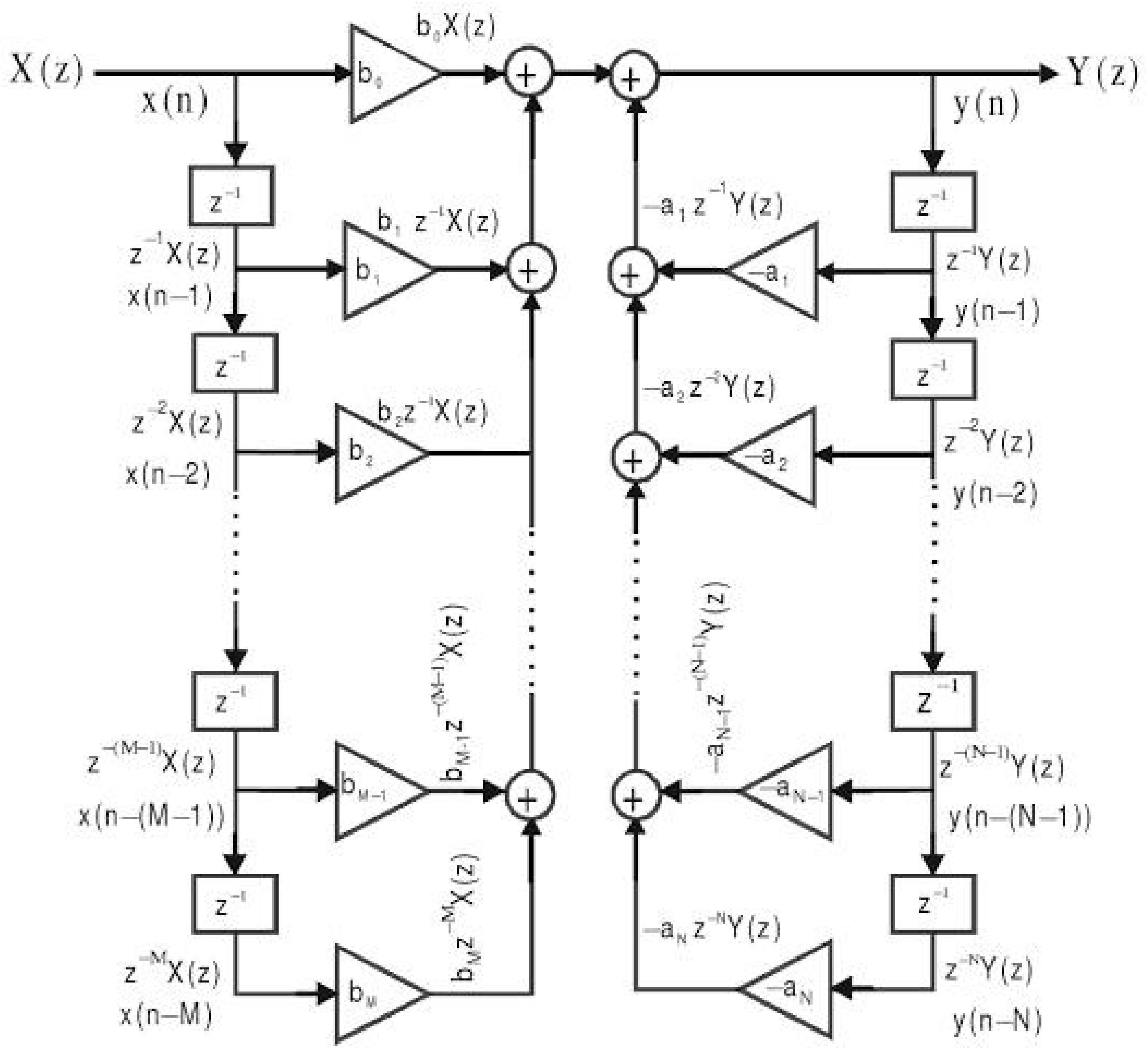


Fig 3.15 : Direct form-I structure of IIR system.

From the direct form-I structure it is observed that the realization of an N^{th} order discrete time system with M number of zeros and N number of poles, involves M+N+1 number of multiplications and M+N number of additions. Also this structure involves M+N delays and so M+N memory locations are required to store the delayed signals.

When the number of delays in a structure is equal to the order of the system, the structure is called ***canonic structure***. In direct form-I structure the number of delays is not equal to order of the system and so direct form-I structure is ***noncanonic structure***.

3.9.2 Direct Form-II Structure of IIR System

An alternative structure called direct form-II structure can be realized which uses less number of delay elements than the direct form-I structure.

Consider the general difference equation governing an IIR system.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - .....- a_N\, y(n-N)$$
$$+\, b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + ... + b_M\, x(n-M)$$

On taking Z-transform of the above equation we get,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - ..... - a_N z^{-N} Y(z)$$
$$+\, b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_M z^{-M} X(z)$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + ... + a_N z^{-N} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_M z^{-M} X(z)$$

$$Y(z)\left[1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}\right]$$
$$= X(z)\left[b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)} \times \dfrac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}} \qquad .....(3.80)$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M} \qquad .....(3.81)$$

On cross multiplying equation (3.80) we get,

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + ... + a_N z^{-N} W(z) = X(z)$$

$$\therefore\ W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - ..... - a_N z^{-N} W(z) \qquad ..... (3.82)$$

On cross multiplying equation (3.81) we get,

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + ... + b_M z^{-M} W(z) \qquad ..... (3.83)$$

The equations (3.82) and (3.83) represent the IIR system in z-domain and can be realized by a direct structure called direct form-II structure as shown in fig 3.16. In direct form-II structure the number of delays is equal to order of the system and so the direct form-II structure is canonic structure.

From the direct form-II structure it is observed that the realization of an N^{th} order discrete time system with M number of zeros and N number of poles, involves M+N+1 number of multiplications and M+N number of additions. In a realizable system, $N \geq M$, and so the number of delays in direct form-II structure will be equal to N. Hence, when a system is realized using direct form-II structure, N memory locations are required to store the delayed signals.

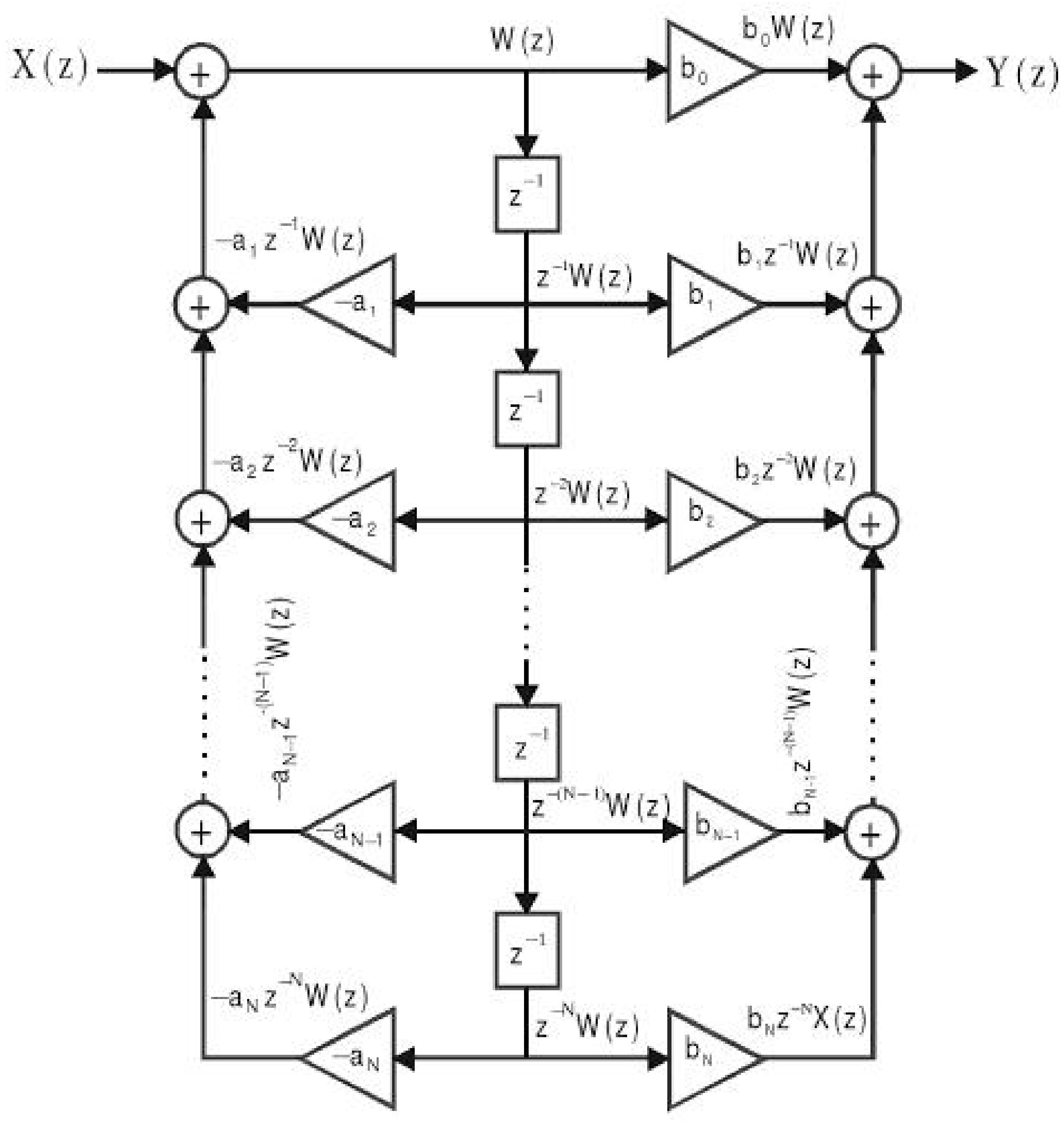


Fig 3.16 : Direct form-II structure of IIR system for N = M.

Conversion of Direct Form-I Structure to Direct Form-II Structure

The direct form-I structure can be converted to direct form-II structure by considering the direct form-I structure as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig 3.17. By linearity property the order of cascading can be interchanged as shown in fig 3.18 and fig 3.19.

In fig 3.19 we can observe that the input to the delay elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of delay elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same. Therefore instead of having separate delays for $\mathcal{H}_1$ and $\mathcal{H}_2$, a single set of delays can be used. Hence the delays can be merged to combine the cascaded systems to a single system and the resultant structure will be direct form-II structure as that of fig 3.16.

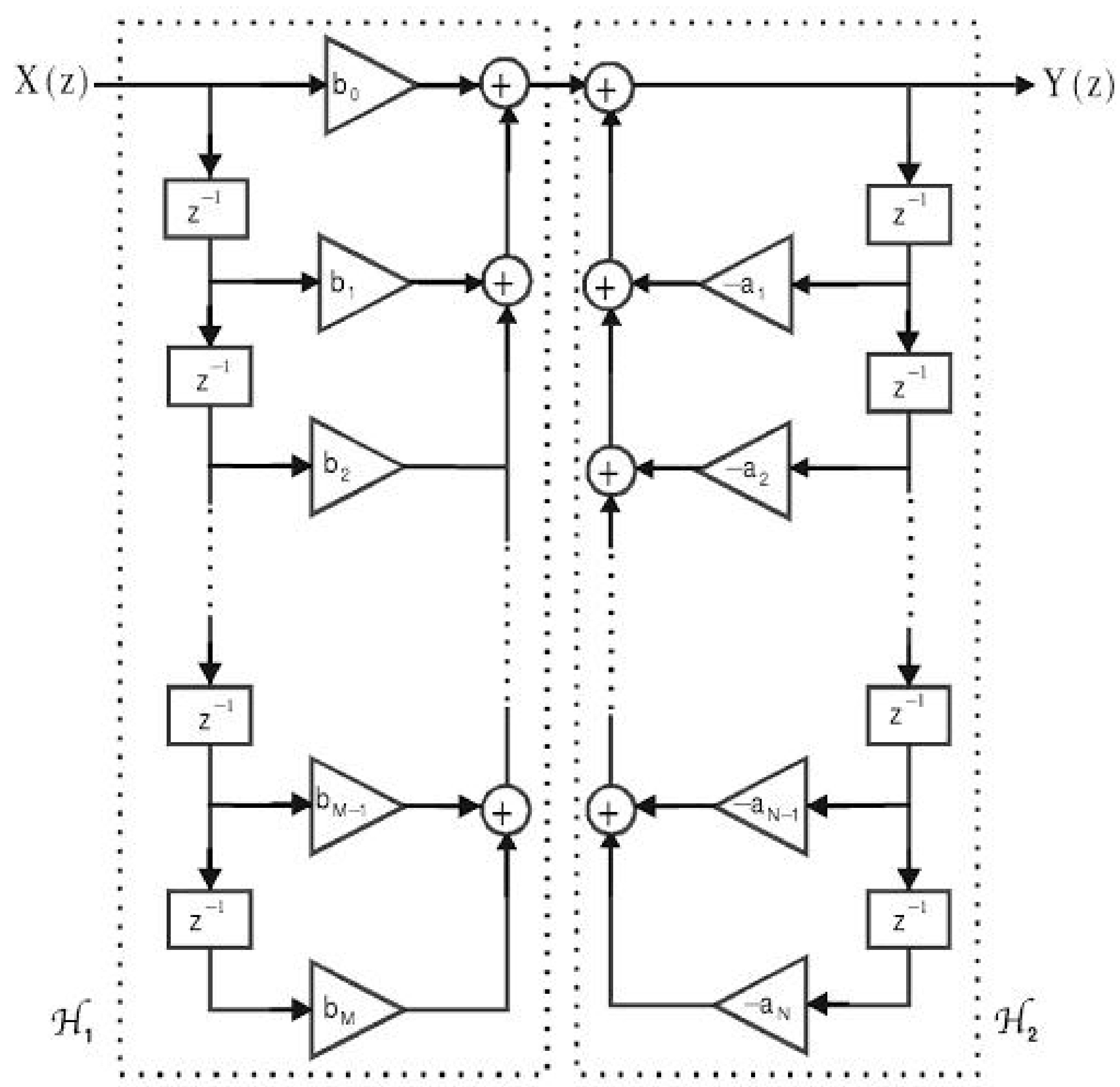


Fig 3.17 : Direct form-I structure as cascade of two systems.

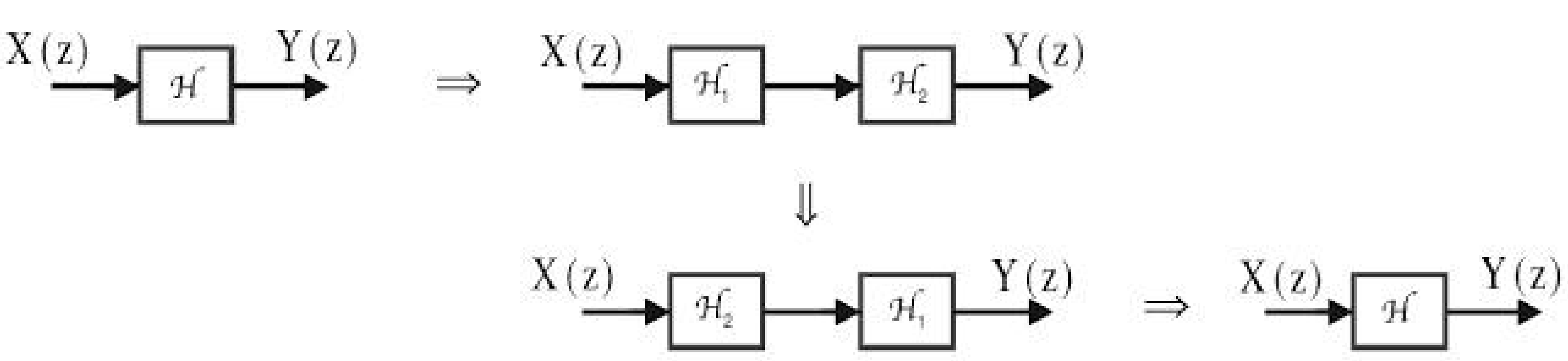


Fig 3.18 : Conversion of Direct form-I structure to Direct form-II structure.

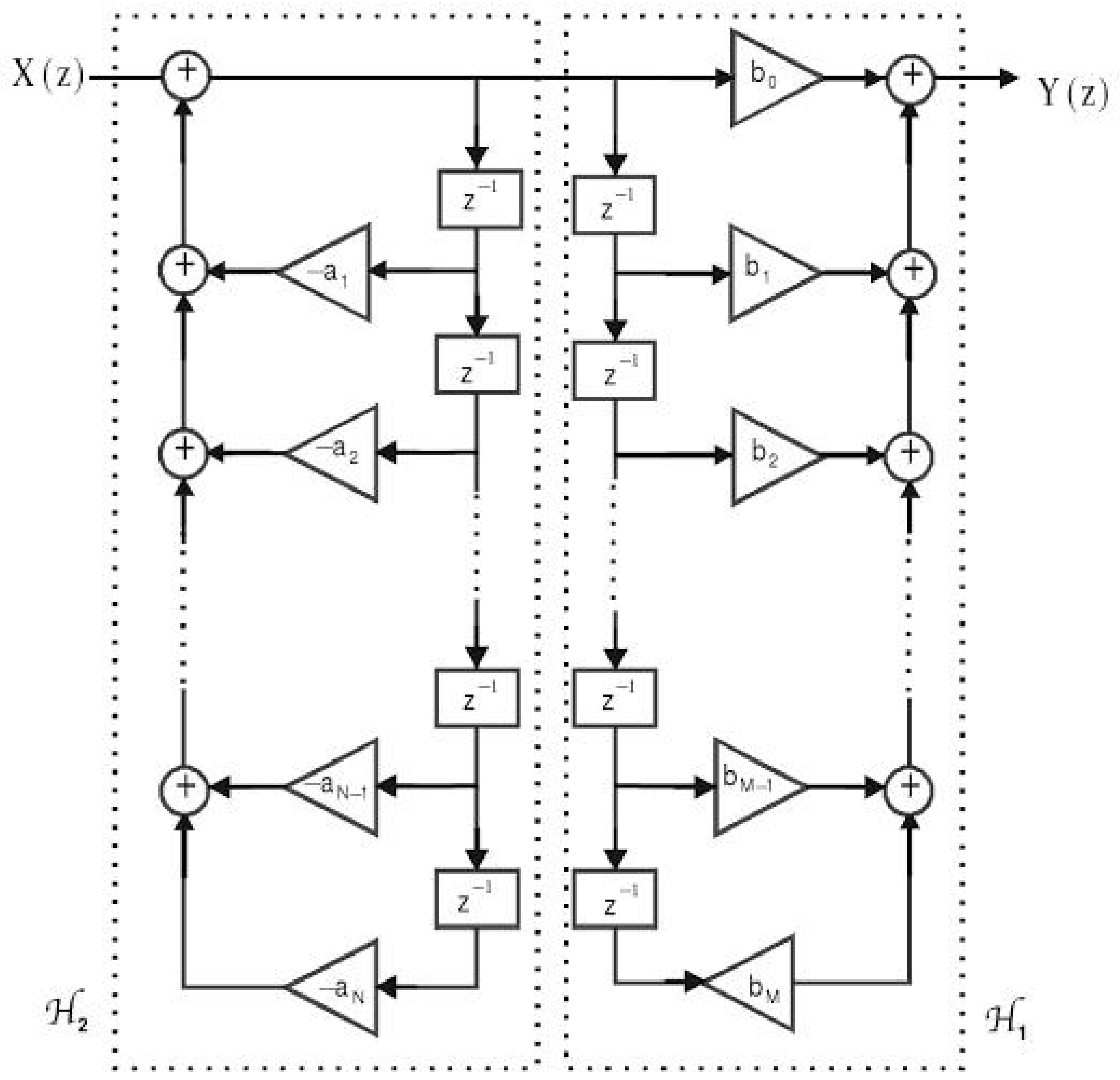


Fig 3.19 : *Direct form-I structure after interchanging the order of cascading.*

3.9.3 Cascade Form Realization of IIR System

The transfer function H(z) can be expressed as a product of a number of second-order or first-order sections, as shown in equation (3.84).

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) \,.....\, H_m(z) = \prod_{i=1}^{m} H_i(z) \qquad .....(3.84)$$

$$\text{where, } H_i(z) = \frac{c_{0i} + c_{1i}\, z^{-1} + c_{2i}\, z^{-2}}{d_{0i} + d_{1i}\, z^{-1} + d_{2i}\, z^{-2}} \qquad \boxed{\text{Second-order section}}$$

$$\text{or, } H_i(z) = \frac{c_{0i} + c_{1i}\, z^{-1}}{d_{0i} + d_{1i}\, z^{-1}} \qquad \boxed{\text{First-order section}}$$

The individual second-order or first-order sections can be realized either in direct form-I or direct form-II structures. The overall system is obtained by cascading the individual sections as shown in fig 3.20. The number of calculations and the memory requirement depends on the realization of individual sections.

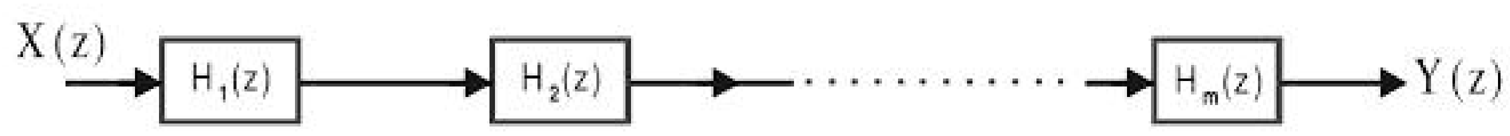


Fig 3.20 : *Cascade form realization of IIR system.*

The difficulty in cascade structure are,

1. Decision of pairing poles and zeros.
2. Deciding the order of cascading the first and second-order sections.
3. Scaling multipliers should be provided between individual sections to prevent the system variables from becoming too large or too small.

Direct Form-II

Consider equation (2).

$$Y(z) = -\frac{3}{8}z^{-1}\,Y(z) + \frac{3}{32}z^{-2}\,Y(z) + \frac{1}{64}z^{-3}\,Y(z) + X(z) + 3z^{-1}\,X(z) + 2z^{-2}\,X(z)$$

$$Y(z) + \frac{3}{8}z^{-1}\,Y(z) - \frac{3}{32}z^{-2}\,Y(z) - \frac{1}{64}z^{-3}\,Y(z) = X(z) + 3z^{-1}\,X(z) + 2z^{-2}\,X(z)$$

$$Y(z)\left[1+\frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right] = X(z)\left[1+3z^{-1}+2z^{-2}\right]$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{1+3z^{-1}+2z^{-2}}{1+\frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(3)$$

Let, $\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)}\ \frac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1+\frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(4)$$

$$\frac{Y(z)}{W(z)} = 1+3z^{-1}+2z^{-2} \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$W(z) + \frac{3}{8}z^{-1}\,W(z) - \frac{3}{32}z^{-2}\,W(z) - \frac{1}{64}z^{-3}\,W(z) = X(z)$$

$$\text{or}\quad W(z) = X(z) - \frac{3}{8}z^{-1}\,W(z) + \frac{3}{32}z^{-2}\,W(z) + \frac{1}{64}z^{-3}\,W(z) \qquad .....(6)$$

On cross multiplying equation (5) we get,

$$Y(z) = W(z) + 3z^{-1}\,W(z) + 2z^{-2}\,W(z) \qquad .....(7)$$

The equations (6) and (7) can be realized by a direct form-II structure as shown in fig 2.

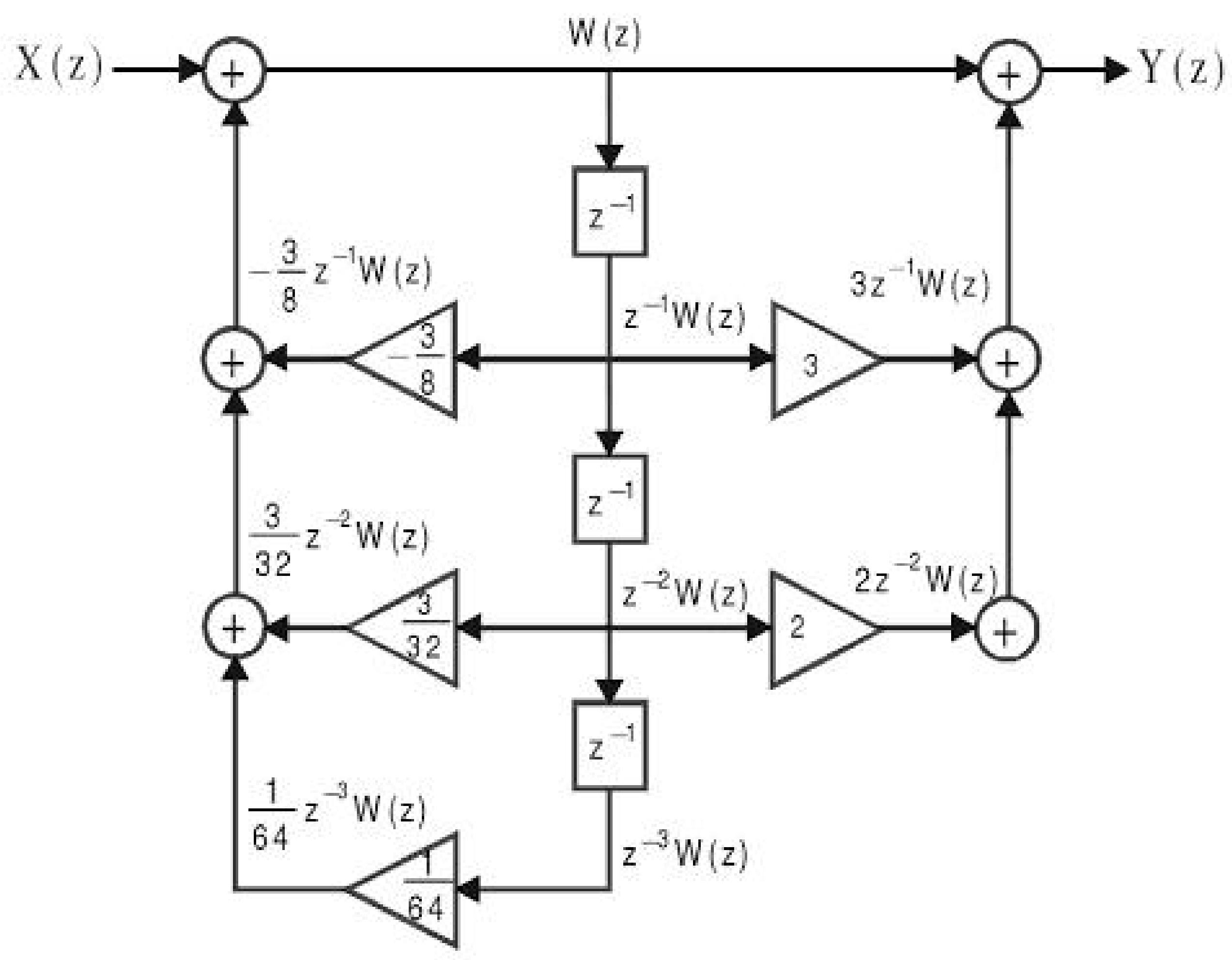


***Fig 2 :** Direct form-II realization structure.*

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$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{2z^3 - 4z^2 + 11z - 8}{(z-8)\left(z^2 - z + 3\right)} \quad = \quad \frac{2z^3 - 4z^2 + 11z - 8}{z^3 - z^2 + 3z - 8z^2 + 8z - 24}$$

$$= \frac{2z^3 - 4z^2 + 11z - 8}{z^3 - 9z^2 + 11z - 24} \quad = \quad \frac{z^3\left(2 - 4z^{-1} + 11z^{-2} - 8z^{-3}\right)}{z^3\left(1 - 9z^{-1} + 11z^{-2} - 24z^{-3}\right)}$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{2 - 4z^{-1} + 11z^{-2} - 8z^{-3}}{1 - 9z^{-1} + 11z^{-2} - 24z^{-3}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y(z) - 9z^{-1}\,Y(z) + 11z^{-2}\,Y(z) - 24z^{-3}\,Y(z) = 2X(z) - 4z^{-1}\,X(z) + 11z^{-2}\,X(z) - 8z^{-3}\,X(z)$$

$$\therefore \quad Y(z) = 2X(z) - 4z^{-1}\,X(z) + 11z^{-2}\,X(z) - 8z^{-3}\,X(z)$$
$$+ 9z^{-1}\,Y(z) - 11z^{-2}\,Y(z) + 24z^{-3}\,Y(z) \qquad .....(2)$$

The direct form-I structure can be obtained from equation (2) as shown in fig 1.

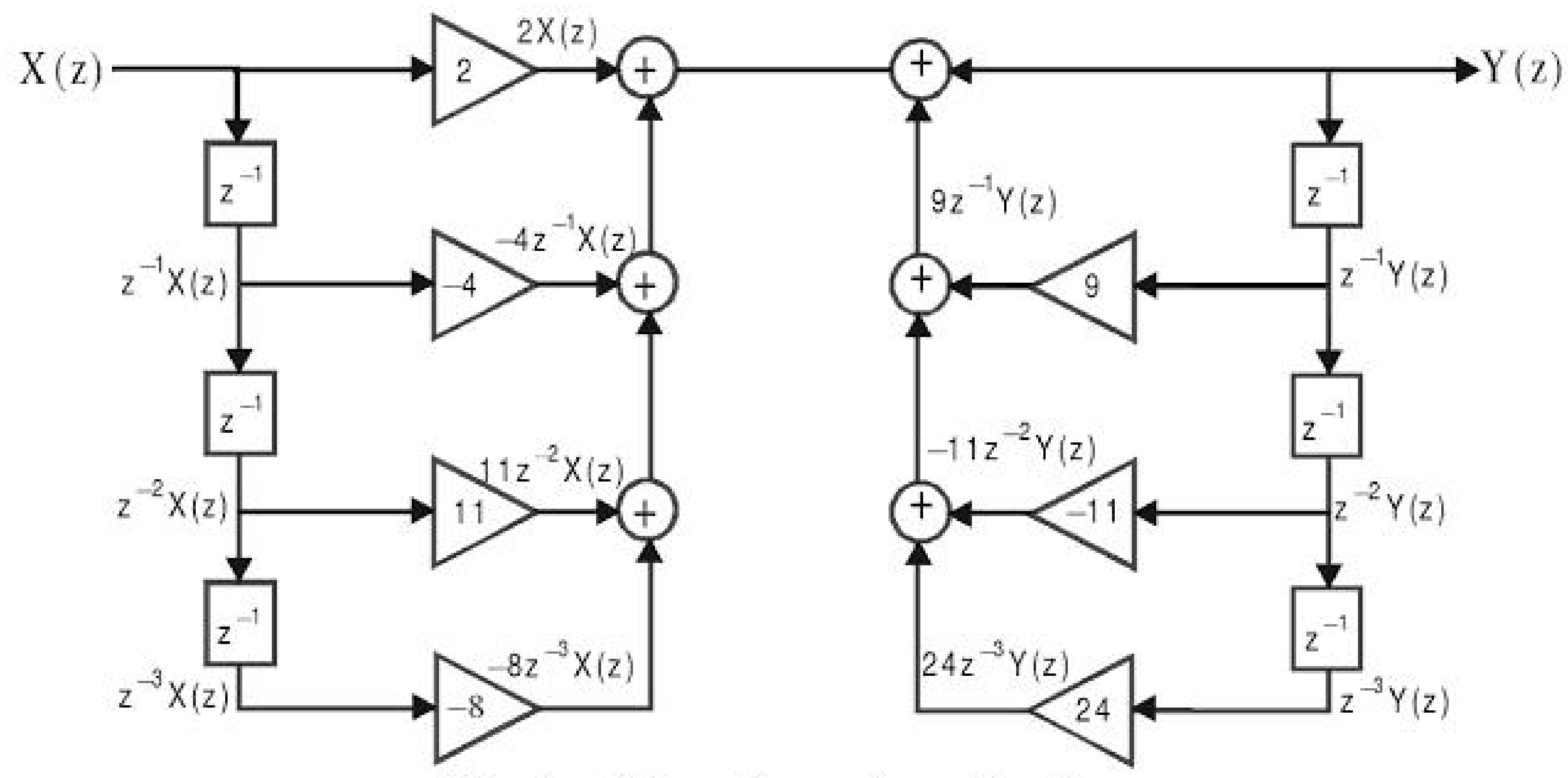


Fig 1 : Direct form-I realization.

Direct Form-II

From equation (1) we get,

$$\frac{Y(z)}{X(z)} = \frac{2 - 4z^{-1} + 11z^{-2} - 8z^{-3}}{1 - 9z^{-1} + 11z^{-2} - 24z^{-3}}$$

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)}\,\dfrac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 - 9z^{-1} + 11z^{-2} - 24z^{-3}} \qquad .....(3)$$

$$\frac{Y(z)}{W(z)} = 2 - 4z^{-1} + 11z^{-2} - 8z^{-3} \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$W(z) - 9z^{-1}\,W(z) + 11z^{-2}\,W(z) - 24z^{-3}\,W(z) = X(z)$$

$$\therefore \quad W(z) = X(z) + 9z^{-1}\,W(z) - 11z^{-2}\,W(z) + 24z^{-3}\,W(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = 2W(z) - 4z^{-1}\,W(z) + 11z^{-2}\,W(z) - 8z^{-3}\,W(z) \qquad .....(6)$$

The equations (5) and (6) can be realized by a direct form-II Structure as shown in fig 2.

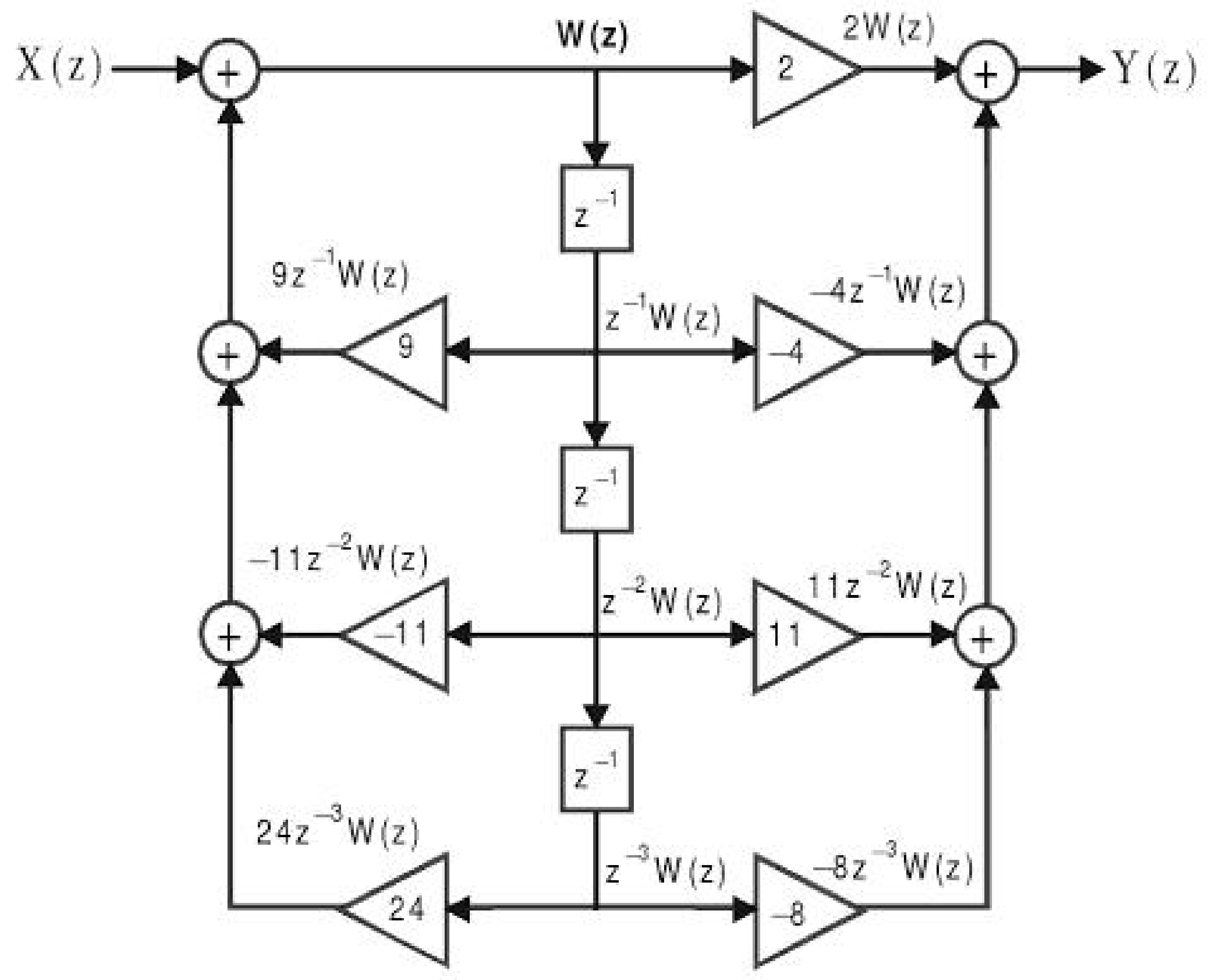


***Fig 2 :** Direct form-II realization.*

Example 3.28

Find the digital network in direct form-I and II for the system described by the difference equation,

$$y(n) = x(n) + 0.3\,x(n-1) - 0.4\,x(n-2) - 0.8\,y(n-1) + 0.7\,y(n-2).$$

Solution

Given that, $y(n) = x(n) + 0.3\,x(n-1) - 0.4\,x(n-2) - 0.8\,y(n-1) + 0.7\,y(n-2)$

On taking Z-transform we get,

$$Y(z) = X(z) + 0.3z^{-1}X(z) - 0.4z^{-2}X(z) - 0.8z^{-1}Y(z) + 0.7z^{-2}\ Y(z) \qquad .....(1)$$

The direct form-I digital network can be realized using equation (1) as shown in fig 1.

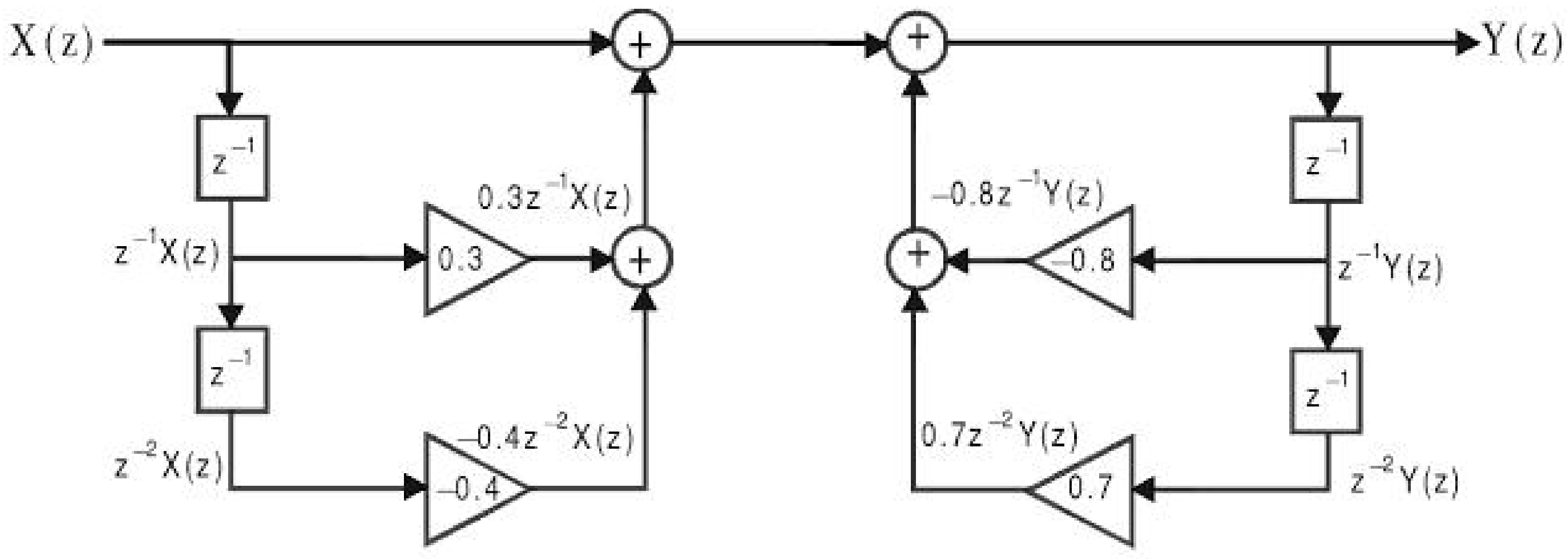


***Fig 1 :** Direct form-I digital network.*

On rearranging equation (1) we get,

$$Y(z) + 0.8z^{-1}Y(z) - 0.7z^{-2}Y(z) = X(z) + 0.3z^{-1}X(z) - 0.4z^{-2}X(z)$$

$$[1+0.8z^{-1} - 0.7z^{-2}]\,Y(z) = [1 + 0.3z^{-1} - 0.4z^{-2}]\,X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1+0.3z^{-1}-0.4z^{-2}}{1+0.8z^{-1}-0.7z^{-2}} \qquad .....(2)$$

The equation (2) is the transfer function of the system.

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)}\ \dfrac{Y(z)}{W(z)}$

where, $\dfrac{W(z)}{X(z)} = \dfrac{1}{1 + 0.8z^{-1} - 0.7z^{-2}}$ ..... (3)

$\dfrac{Y(z)}{W(z)} = 1 + 0.3z^{-1} - 0.4z^{-2}$ ..... (4)

On cross multiplying equation (3) we get,

$W(z) + 0.8z^{-1}W(z) + 0.7z^{-2}W(z) = X(z)$

\ $W(z) = X(z) - 0.8z^{-1}W(z) - 0.7z^{-2}W(z)$ ..... (5)

On cross multiplying equation (4) we get,

$Y(z) = W(z) + 0.3z^{-1}W(z) - 0.4z^{-2}W(z)$ ..... (6)

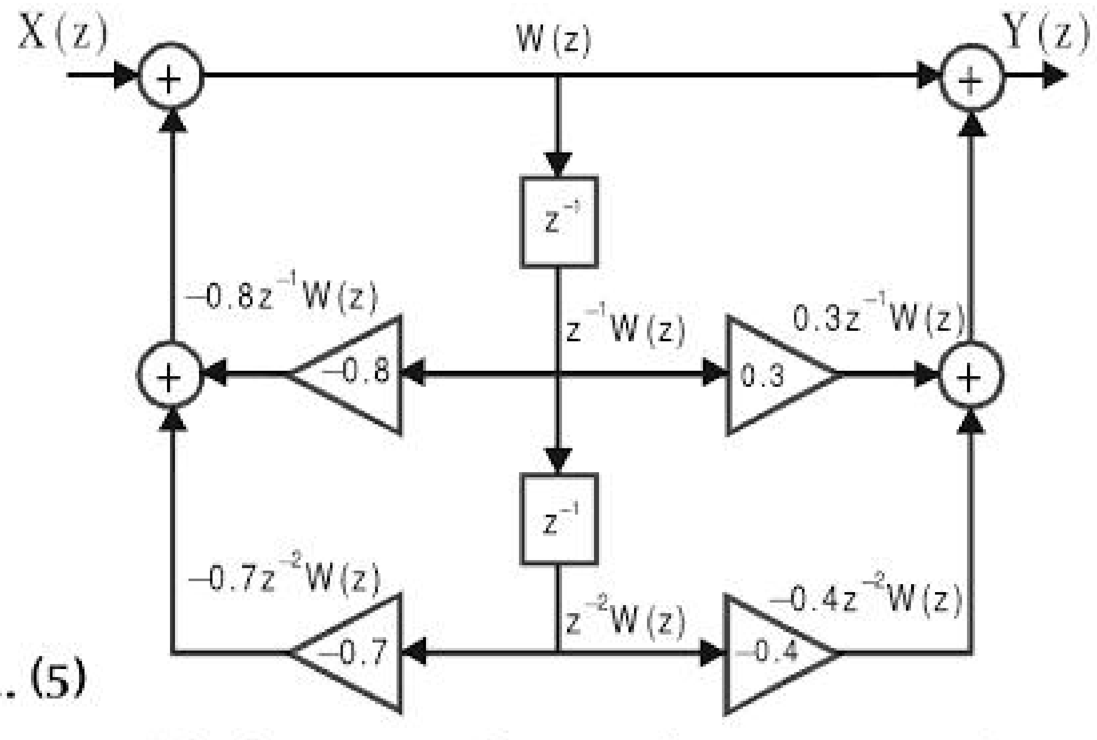


Fig 2 : Direct form-II digital network.

The direct form-II digital network is realized using equations (5) and (6) as shown in fig 2.

Example 3.29

Realize the digital network described by H(z) in two ways. $H(z) = \dfrac{1 - a\ \cos\omega_0\ z^{-1}}{1 - 2a\ \cos\omega_0\ z^{-1} + a^2\ z^{-2}}$

Solution

Let, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 - a\ \cos\omega_0\ z^{-1}}{1 - 2a\ \cos\omega_0\ z^{-1} + a^2\ z^{-2}}$

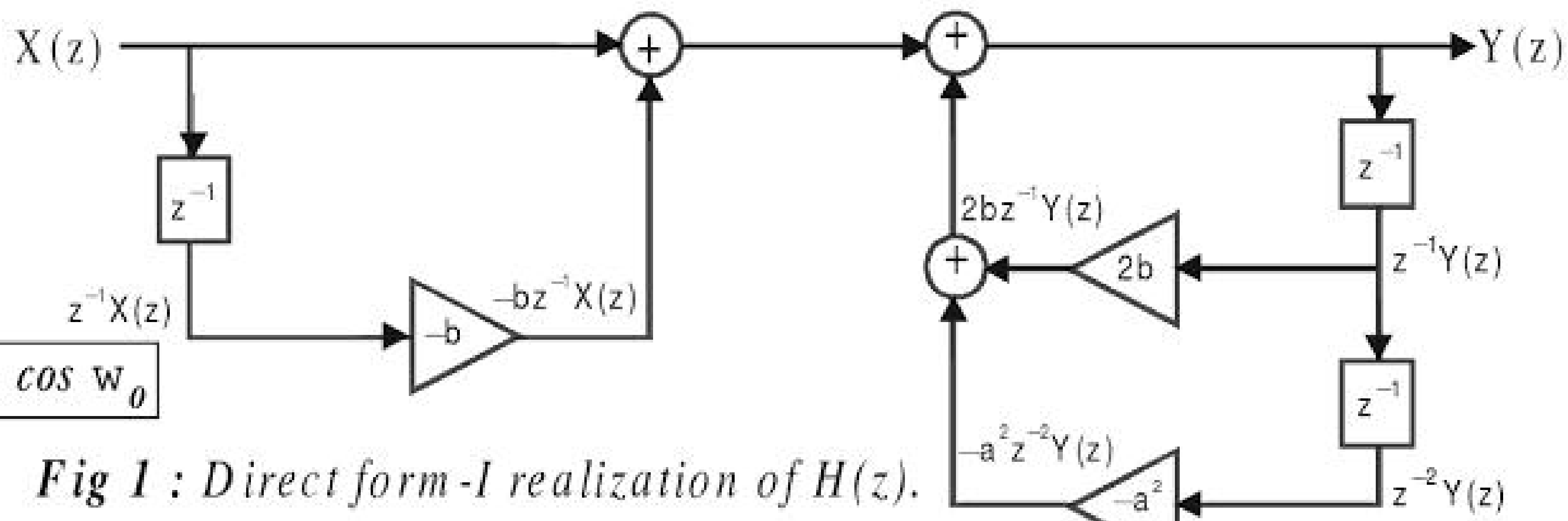
On cross multiplying we get,

$Y(z) - 2a\cos w_0 z^{-1}\,Y(z) + a^2 z^{-2}\,Y(z) = X(z) - a\cos w_0 z^{-1}\,X(z)$

\ $Y(z) = X(z) - a\cos w_0 z^{-1}\,X(z) + 2a\cos w_0 z^{-1}\,Y(z) - a^2 z^{-2}\,Y(z)$

Let, $a\cos w_0 = b.$ \ $Y(z) = X(z) - bz^{-1}\,X(z) + 2bz^{-1}\,Y(z) - a^2 z^{-2}\,Y(z)$ ..... (1)

The equation (1) can be used to construct direct form-I structure of H(z) as shown in fig 1.



Note : $b = a\cos w_0$

Fig 1 : Direct form-I realization of H(z).

Consider the direct form-I structure as cascade of two systems $H_1(z)$ and $H_2(z)$ as shown in fig 2.

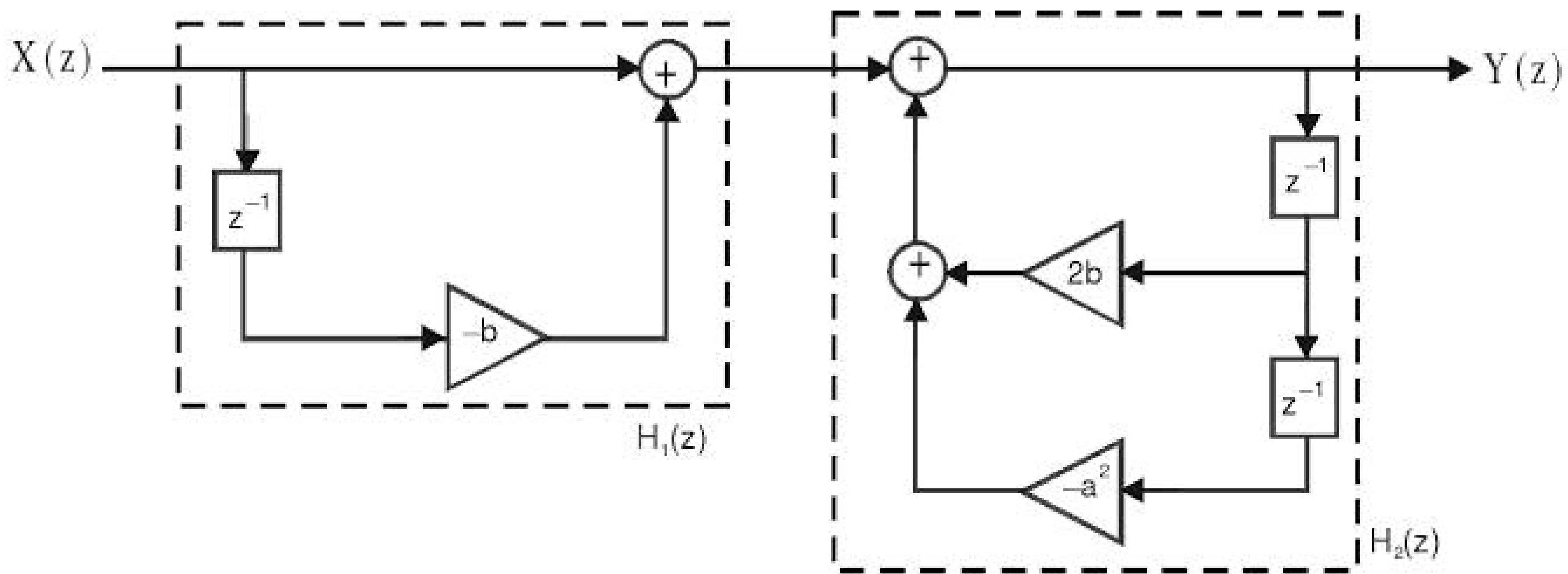


Fig 2 : Direct form-I structure as cascade of two systems.

In an LT1 system, by linearity property, the order of cascading can be changed. Hence the systems $H_1(z)$ and $H_2(z)$ are interchanged and the fig 2 is redrawn as shown in fig 3.

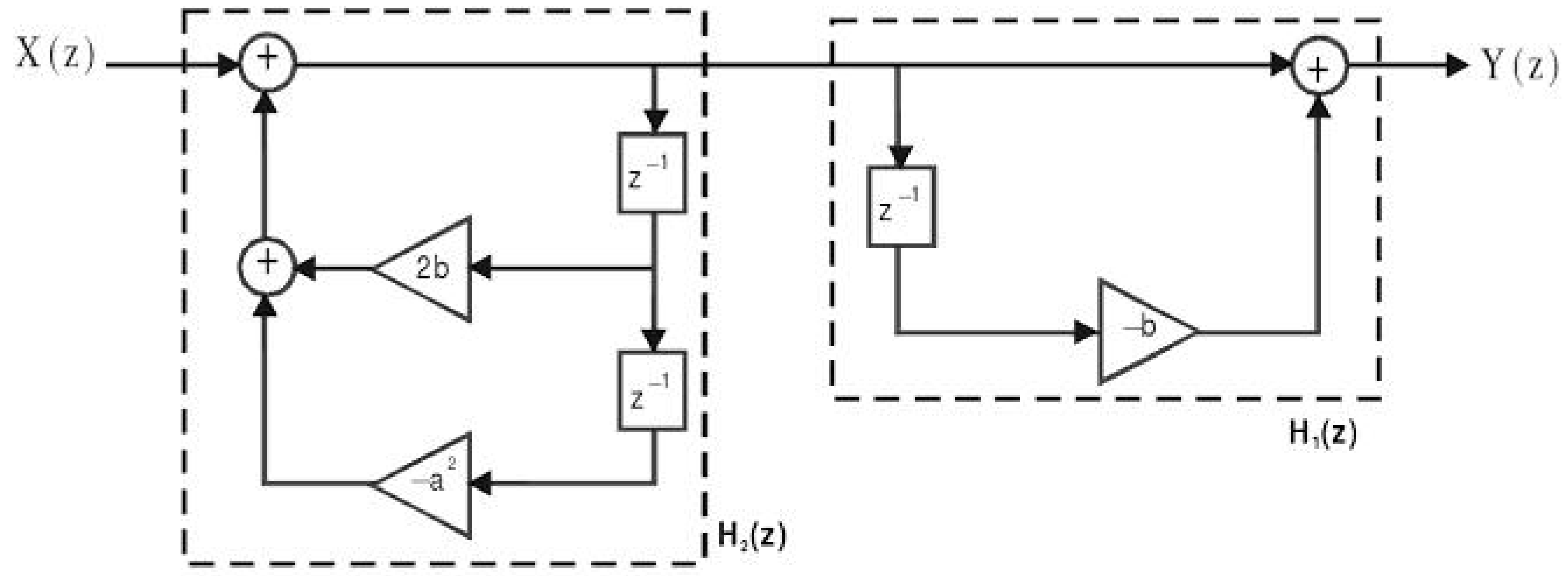


Fig 3 : Direct form-I structure with $H_1(z)$ and $H_2(z)$ interchanged.

Since the input to delay elements in both the systems $H_1(z)$ and $H_2(z)$ of fig 3 are same, the outputs will also be same. Hence the delays can be combined and the resultant structure is direct form-II structure, which is shown in fig 4.

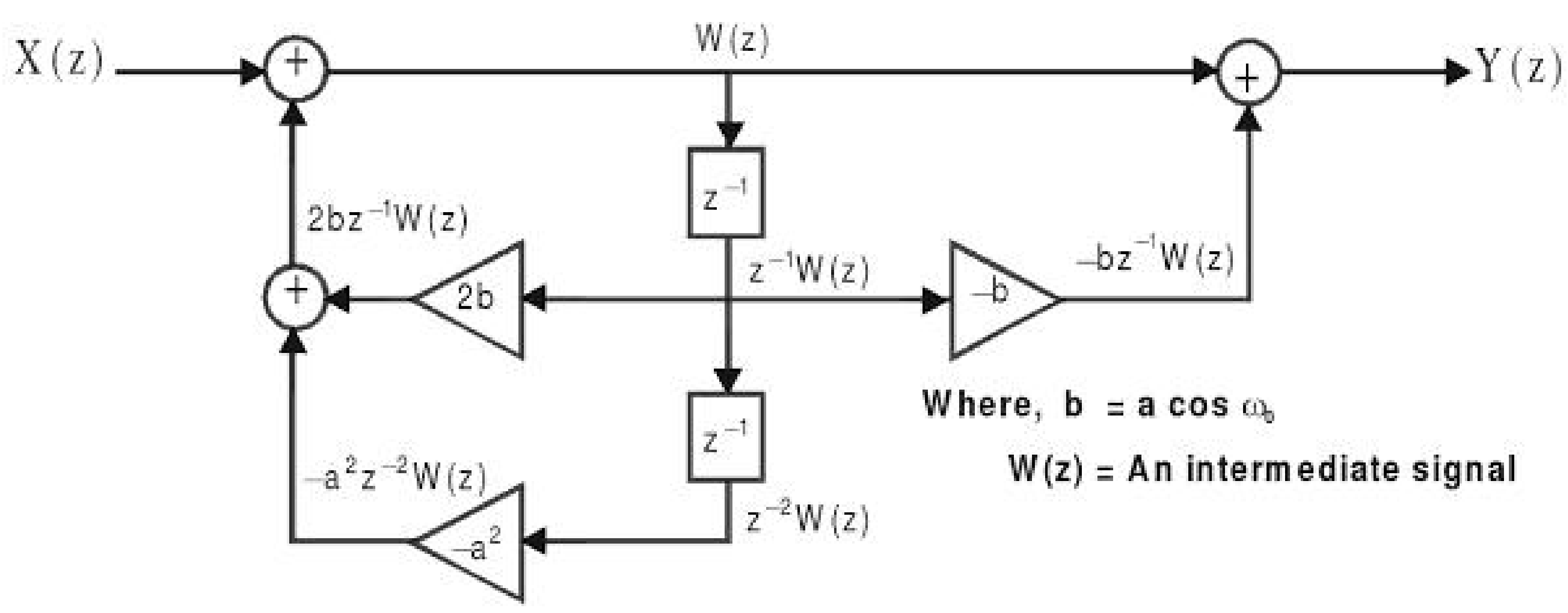


Fig 4 : Direct form-II structure of H(z).

Example 3.30

Realize the given system in cascade and parallel forms.

$$H(z) = \frac{1 + 0.25z^{-1}}{\left(1 - 2z^{-1} + 0.25z^{-2}\right)\left(1 - 3z^{-1} + 0.2z^{-2}\right)}$$

Solution

Cascade Form

Let us realize the system as cascade of two second-order systems.

$$H(z) = \frac{1 + 0.25z^{-1}}{\left(1 - 2z^{-1} + 0.25z^{-2}\right)\left(1 - 3z^{-1} + 0.2z^{-2}\right)} = \frac{1}{1 - 2z^{-1} + 0.25z^{-2}} \times \frac{1 + 0.25z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

where, $H_1(z) = \dfrac{1}{1 - 2z^{-1} + 0.25z^{-2}}$; $H_2(z) = \dfrac{1 + 0.25z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}}$

Let, $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1 - 2z^{-1} + 0.25z^{-2}} \quad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - 2z^{-1}\,Y_1(z) + 0.25z^{-2}\,Y_1(z) = X(z)$$

$$\therefore\ Y_1(z) = X(z) + 2z^{-1}\,Y_1(z) - 0.25z^{-2}\,Y_1(z) \quad .....(2)$$

Using equation (2) the direct form-II structure of $H_1(z)$ is realized as shown in fig 1.

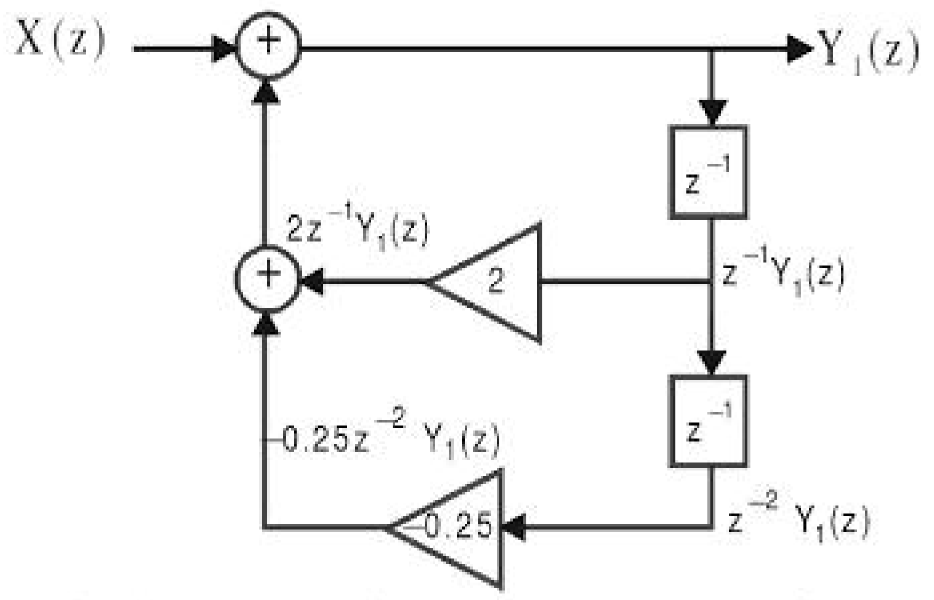


Fig 1 : Direct form-II structure of $H_1(z)$.

Let, $\quad H_2(z) = \dfrac{Y(z)}{Y_1(z)} = \dfrac{1 + 0.25z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}}$

Let, $\quad \dfrac{Y(z)}{Y_1(z)} = \dfrac{W_2(z)}{Y_1(z)}\ \dfrac{Y(z)}{W_2(z)}$

where, $\quad \dfrac{W_2(z)}{Y_1(z)} = \dfrac{1}{1 - 3z^{-1} + 0.2z^{-2}} \quad .....(3)$

$$\frac{Y(z)}{W_2(z)} = 1 + 0.25z^{-1} \quad .....(4)$$

On cross multiplying equation (3) we get,

$$W_2(z) - 3z^{-1}\,W_2(z) + 0.2z^{-2}\,W_2(z) = Y_1(z)$$

$$\therefore\ W_2(z) = Y_1(z) + 3z^{-1}\,W_2(z) - 0.2z^{-2}\,W_2(z) \quad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = W_2(z) + 0.25z^{-1}\,W_2(z) \quad .....(6)$$

Using equations (5) and (6) the direct form-II structure of $H_2(z)$ is realized as shown in fig 2.

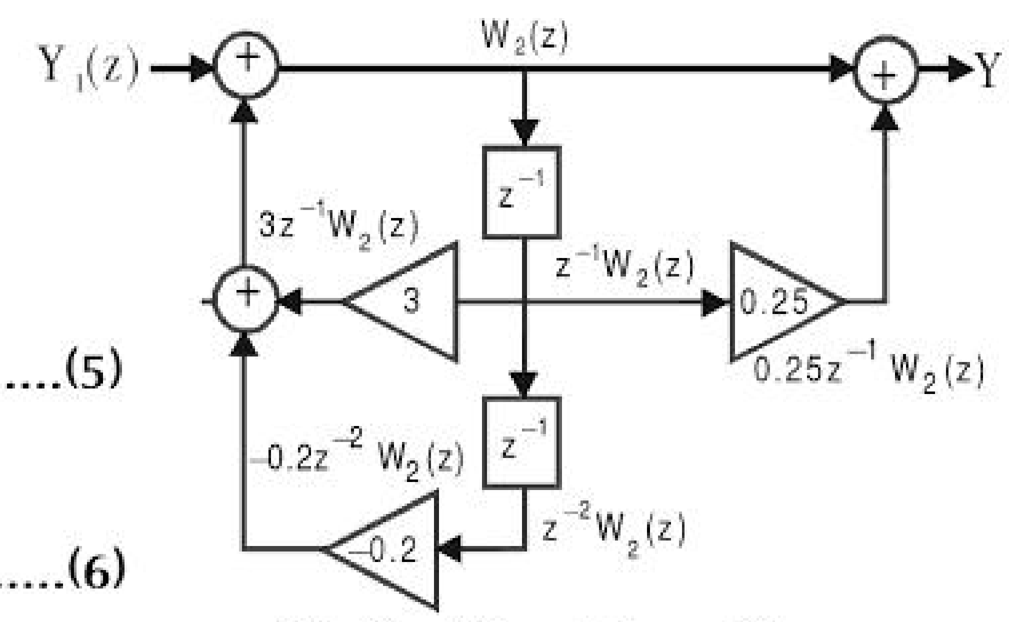


Fig 2 : Direct form-II structure of $H_2(z)$.

The cascade structure of H(z) is obtained by connecting the structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

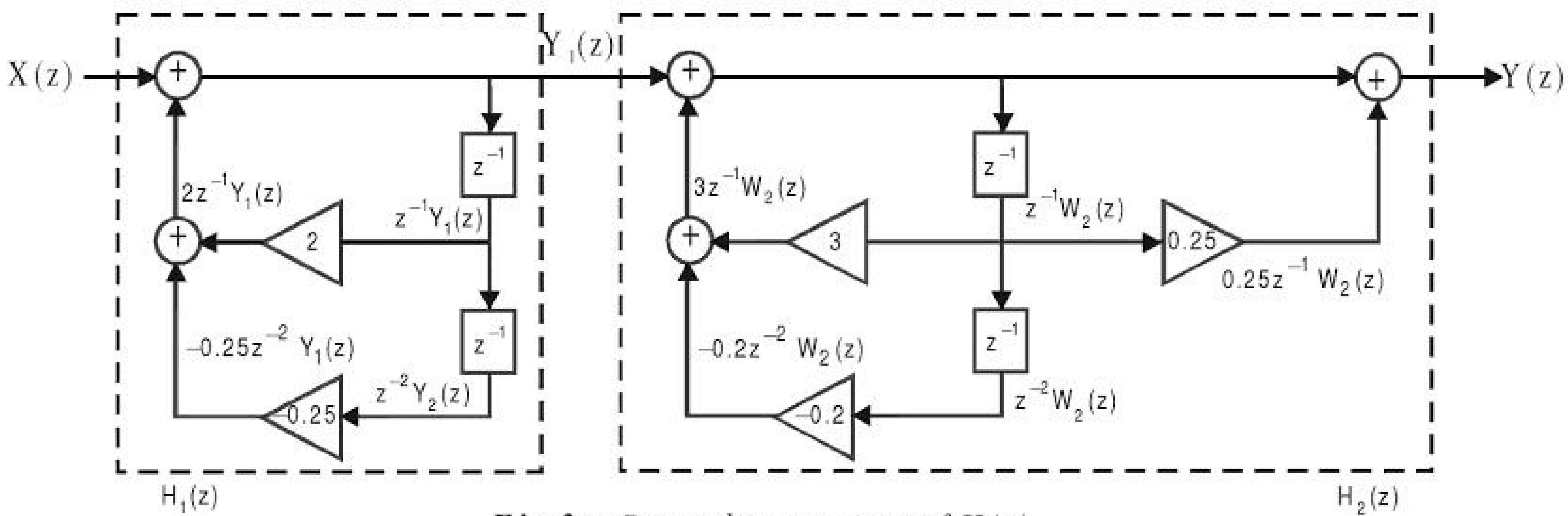


Fig 3 : Cascade structure of H(z).

Parallel Realization

Given that, $H(z) = \dfrac{1 + 0.25z^{-1}}{\left(1 - 2z^{-1} + 0.25z^{-2}\right)\left(1 - 3z^{-1} + 0.2z^{-2}\right)}$

By partial fraction expansion we can write,

$$H(z) = \frac{1 + 0.25z^{-1}}{\left(1 - 2z^{-1} + 0.25z^{-2}\right)\left(1 - 3z^{-1} + 0.2z^{-2}\right)} = \frac{A + Bz^{-1}}{1 - 2z^{-1} + 0.25z^{-2}} + \frac{C + Dz^{-1}}{1 - 3z^{-1} + 0.2z^{-2}} \quad .....(7)$$

On cross multiplying equation (7) we get,

$$1 + 0.25z^{-1} = (A + Bz^{-1})\left(1 - 3z^{-1} + 0.2\,z^{-2}\right) + (C + Dz^{-1})\left(1 - 2z^{-1} + 0.25\,z^{-2}\right)$$

$$1 + 0.25z^{-1} = A - 3Az^{-1} + 0.2Az^{-2} + Bz^{-1} - 3Bz^{-2} + 0.2Bz^{-3} + C - 2Cz^{-1} + 0.25Cz^{-2} + Dz^{-1} - 2Dz^{-2} + 0.25Dz^{-3}$$

$$1 + 0.25z^{-1} = (A + C) + (-3A + B - 2C + D)\,z^{-1} + (0.2A - 3B + 0.25C - 2D)z^{-2} + (0.2B + 0.25D)z^{-3} \qquad .....(8)$$

On equating the constants in equation (8) we get,

$$A + C = 1 \quad \Rightarrow \quad C = 1 - A$$

On equating the coefficients of z^{-3} in equation (8) we get,

$$0.2B + 0.25D = 0 \qquad \therefore\ 0.25D = -0.2B \quad \Rightarrow \quad D = -\frac{0.2}{0.25}B = -0.8B$$

On equating the coefficients of z^{-1} in equation (8) we get,

$$-3A + B - 2C + D = 0.25$$

On substituting $C = 1 - A$ and $D = -0.8B$ in the above equation we get,

$$-3A + B - 2(1 - A) + (-0.8B) = 0.25 \quad \Rightarrow \quad -A + 0.2B = 2.25 \quad \Rightarrow \quad A = 0.2B - 2.25$$

On equating the coefficients of z^{-2} in equation (8) we get,

$$0.2A - 3B + 0.25C - 2D = 0$$

On substituting $C = 1 - A$, and $D = -0.8B$ in the above equation we get,

$$0.2A - 3B + 0.25(1 - A) - 2\,(-0.8B) = 0 \quad \Rightarrow \quad -0.05A - 1.4B = -0.25$$

On substituting $A = 0.2B - 2.25$ in the above equation we get,

$$-0.05\,(0.2B - 2.25) - 1.4B = -0.25 \quad \Rightarrow \quad -1.41B = -0.3625 \quad \Rightarrow \quad B = \frac{0.3625}{1.41} = 0.26$$

$$\therefore\ A = 0.2B - 2.25 = 0.2 \times 0.26 - 2.25 = -2.2$$

$$\therefore\ C = 1 - A = 1 + 2.2 = 3.2$$

$$\therefore\ D = -0.8B = -0.8 \times 0.26 = -0.21$$

$$\therefore\ H(z) = \frac{A + Bz^{-1}}{1 - 2z^{-1} + 0.25z^{-2}} + \frac{C + Dz^{-1}}{1 - 3z^{-1} + 0.2z^{-2}} = \frac{-2.2 + 0.26z^{-1}}{1 - 2z^{-1} + 0.25z^{-2}} + \frac{3.2 - 0.21z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}}$$

Let, $H(z) = \dfrac{-2.2 + 0.26z^{-1}}{1 - 2z^{-1} + 0.25z^{-2}} + \dfrac{3.2 - 0.21z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}} = H_1(z) + H_2(z)$

where, $H_1(z) = \dfrac{-2.2 + 0.26z^{-1}}{1 - 2z^{-1} + 0.25z^{-2}}$

$$H_2(z) = \frac{3.2 - 0.21z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}}$$

Let, $H(z) = \dfrac{Y(z)}{X(z)}$; $H_1(z) = \dfrac{Y_1(z)}{X(z)}$; $H_2(z) = \dfrac{Y_2(z)}{X(z)}$

$$\therefore\ H(z) = H_1(z) + H_2(z) \quad \Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)} \quad \Rightarrow \quad Y(z) = Y_1(z) + Y_2(z)$$

Realization of $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{-2.2 + 0.26z^{-1}}{1 - 2z^{-1} + 0.25z^{-2}}$$

Let, $\frac{Y_1(z)}{X(z)} = \frac{W_1(z)}{X(z)} \frac{Y_1(z)}{W_1(z)}$

where, $$\frac{W_1(z)}{X(z)} = \frac{1}{1 - 2z^{-1} + 0.25z^{-2}} \quad .....(9)$$

$$\frac{Y_1(z)}{W_1(z)} = -2.2 + 0.26z^{-1} \quad .....(10)$$

On cross multiplying equation (9) we get,

$$W_1(z) - 2z^{-1} W_1(z) + 0.25z^{-2} W_1(z) = X(z)$$

$$\therefore \; W_1(z) = X(z) + 2z^{-1} W_1(z) - 0.25z^{-2} W_1(z) \quad .....(11)$$

On cross multiplying equation (10) we get,

$$Y_1(z) = -2.2W_1(z) + 0.26z^{-1} W_1(z) \quad .....(12)$$

The direct form-II structure of system $H_1(z)$ can be realized using equations (11) and (12) as shown in fig 4.

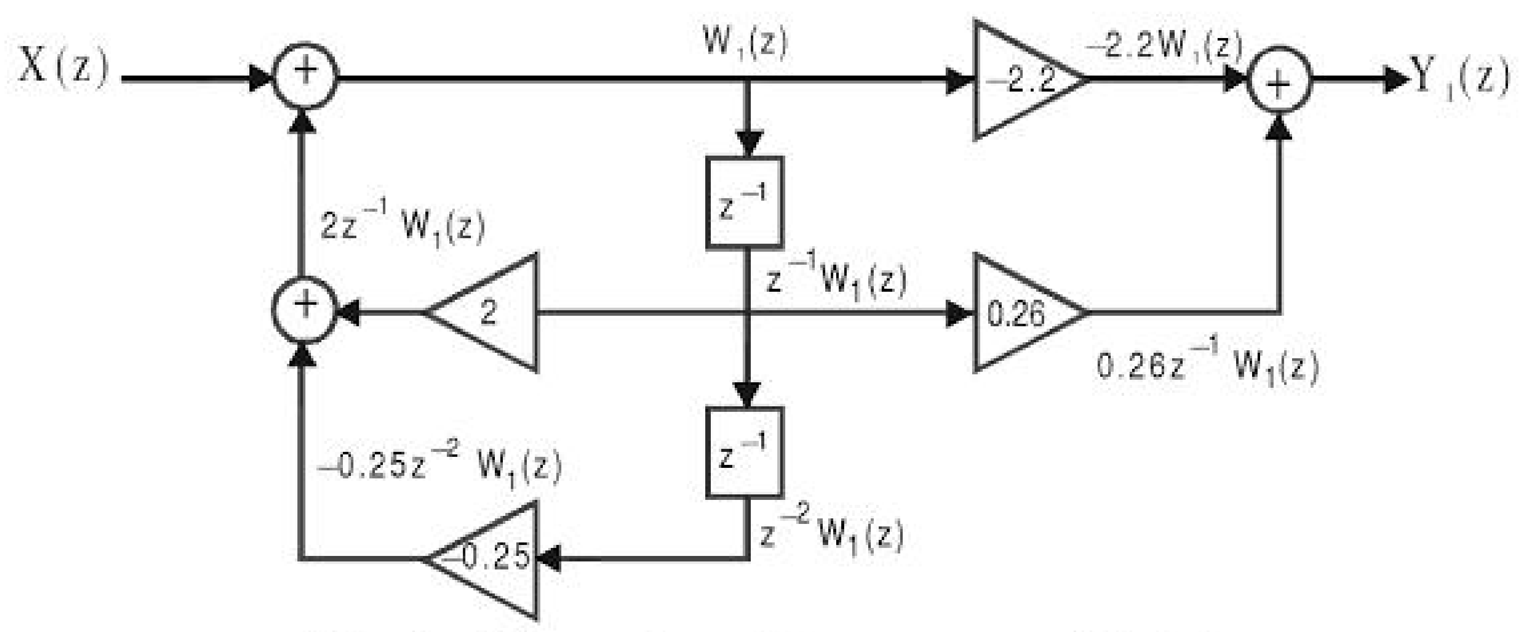


Fig 4 : *Direct form-II structure of $H_1(z)$.*

Realization of $H_2(z)$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{3.2 - 0.21z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}}$$

Let, $\frac{Y_2(z)}{X(z)} = \frac{W_2(z)}{X(z)} \frac{Y_2(z)}{W_2(z)}$

where, $$\frac{W_2(z)}{X(z)} = \frac{1}{1 - 3z^{-1} + 0.2z^{-2}} \quad .....(13)$$

$$\frac{Y_2(z)}{W_2(z)} = 3.2 - 0.21z^{-1} \quad .....(14)$$

On cross multiplying equation (13) we get,

$$W_2(z) - 3z^{-1} W_2(z) + 0.2z^{-2} W_2(z) = X(z)$$

$$\therefore \; W_2(z) = X(z) + 3z^{-1} W_2(z) - 0.2z^{-2} W_2(z) \quad .....(15)$$

On cross multiplying equation (14) we get,

$$Y_2(z) = 3.2\, W_2(z) - 0.21z^{-1} W_2(z) \quad ..... (16)$$

The direct form-II structure of system $H_2(z)$ can be realized using equations (15) and (16) as shown in fig 5.

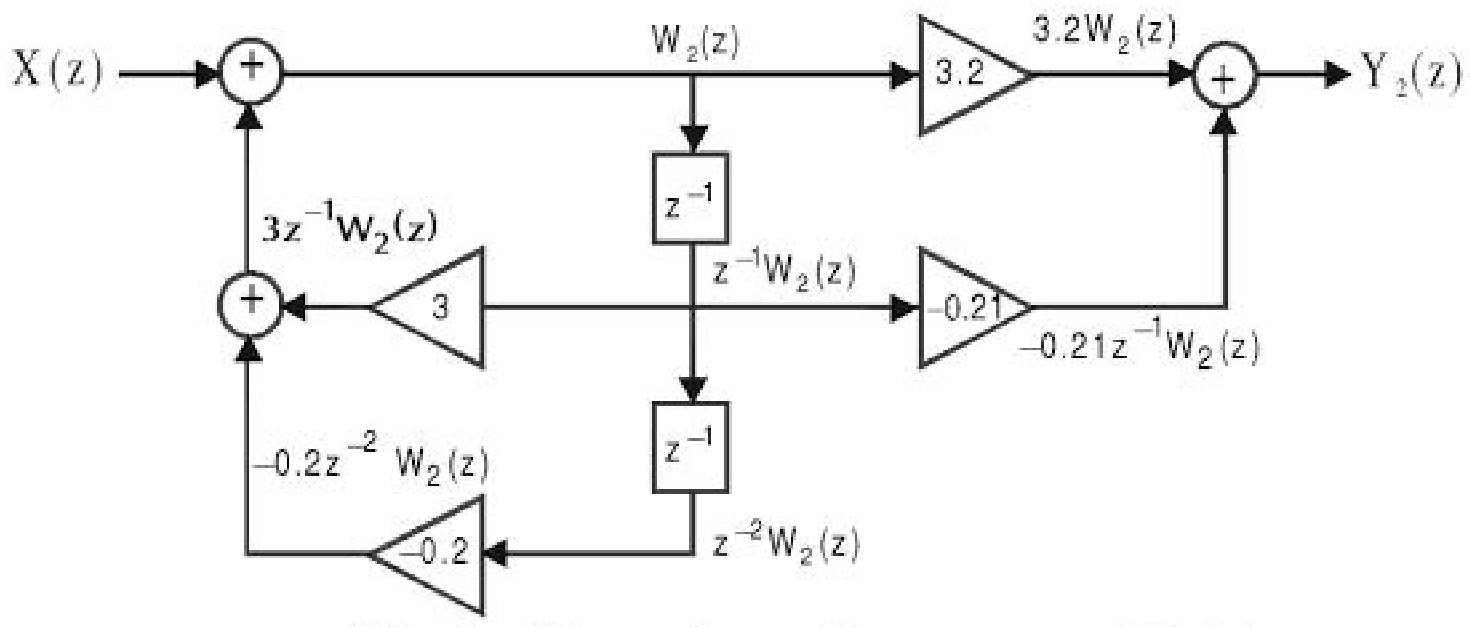


Fig 5 : Direct form-II structure of $H_2(z)$.

The parallel form structure of H(z) is obtained by connecting the direct form-II structure of $H_1(z)$ and $H_2(z)$ in parallel as shown in fig 6.

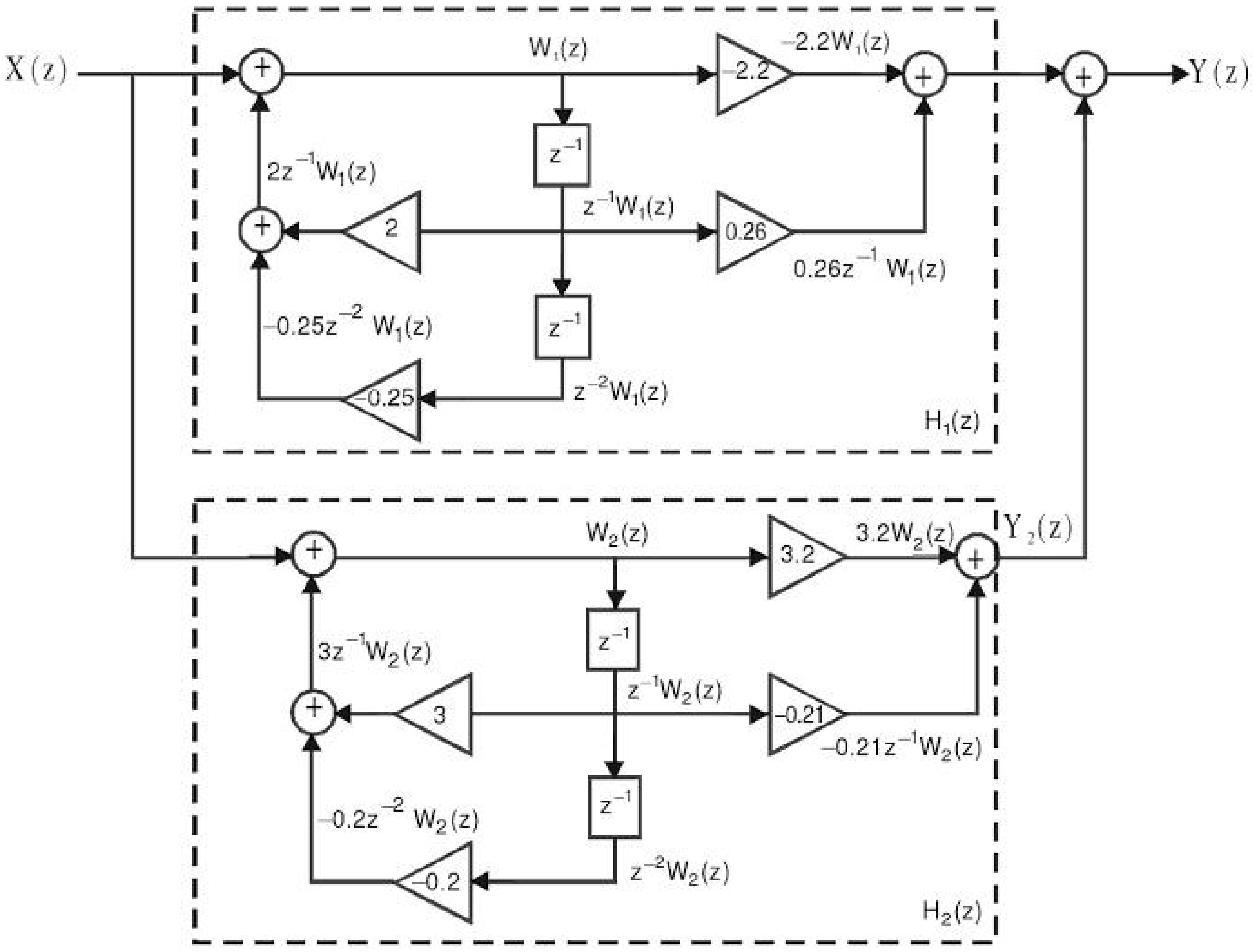


Fig 6 : Parallel form realization of system $H(z)$.

Example 3.31

Obtain the cascade realization of the system, $H(z) = \dfrac{2+3z^{-1}+4z^{-2}}{\left(1+\frac{1}{7}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{9}z^{-1}\right)}$

Solution

Given that, $H(z) = \dfrac{2+3z^{-1}+4z^{-2}}{\left(1+\frac{1}{7}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{9}z^{-1}\right)}$

On examining the roots of numerator polynomial it is found that the roots are complex conjugate. Hence H(z) can be realized as cascade of one first-order and one second-order system.

$2+3z^{-1}+4z^{-2} = 2z^{-2}(z^2+1.5z+2)$

The roots of quadratic,

$z^2+1.5z+2=0$ are,

$$z = \frac{-1.5 \pm \sqrt{1.5^2 - 4\times 2}}{2} = \frac{-1.5 \pm j2.4}{2}$$

$$\therefore\ H(z) = \frac{1}{1-\frac{1}{4}z^{-1}} \times \frac{2+3z^{-1}+4z^{-2}}{\left(1+\frac{1}{7}z^{-1}\right)\left(1+\frac{1}{9}z^{-1}\right)} = \frac{1}{1-\frac{1}{4}z^{-1}} \times \frac{2+3z^{-1}+4z^{-2}}{1+\frac{16}{63}z^{-1}+\frac{1}{63}z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

$$\text{where, } H_1(z) = \frac{1}{1-\frac{1}{4}z^{-1}} \quad \text{and} \quad H_2(z) = \frac{2+3z^{-1}+4z^{-2}}{1+\frac{16}{63}z^{-1}+\frac{1}{63}z^{-2}}$$

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-\frac{1}{4}z^{-1}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - \frac{1}{4}z^{-1}Y_1(z) = X(z) \quad \Rightarrow \quad Y_1(z) = X(z) + \frac{1}{4}z^{-1}Y_1(z) \qquad .....(2)$$

The direct form-II structure of $H_1(z)$ can be obtained from equation (2) as shown in fig 1.

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{2+3z^{-1}+4z^{-2}}{1+\frac{16}{63}z^{-1}+\frac{1}{63}z^{-2}}$$

$$\text{Let, } \frac{Y(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\ \frac{Y(z)}{W_2(z)}$$

$$\text{where, } \frac{W_2(z)}{Y_1(z)} = \frac{1}{1+\frac{16}{63}z^{-1}+\frac{1}{63}z^{-2}} \qquad .....(3)$$

$$\frac{Y(z)}{W_2(z)} = 2+3z^{-1}+4z^{-2} \qquad .....(4)$$

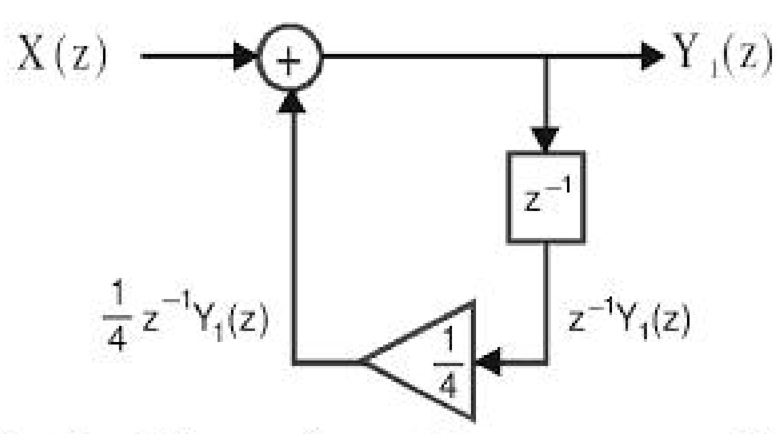


Fig 1 : Direct form-II structure of $H_1(z)$.

On cross multiplying equation (3) we get,

$$W_2(z) + \frac{16}{63}z^{-1}W_2(z) + \frac{1}{63}z^{-2}W_2(z) = Y_1(z)$$

$$\therefore\ W_2(z) = Y_1(z) - \frac{16}{63}z^{-1}W_2(z) - \frac{1}{63}z^{-2}W_2(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = 2W_2(z) + 3z^{-1}W_2(z) + 4z^{-2}W_2(z) \qquad .....(6)$$

The direct form-II structure of $H_2(z)$ can be obtained using equations (5) and (6) as shown in fig 2.

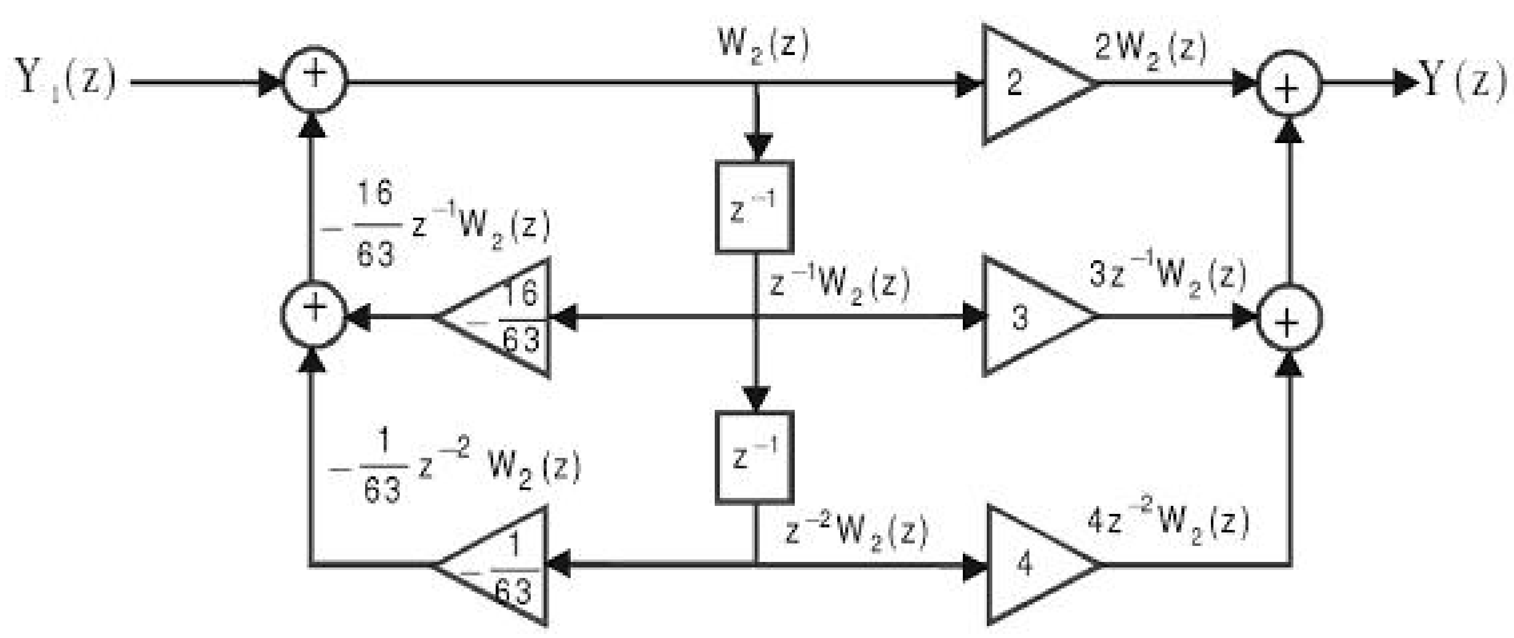


Fig 2 : Direct form-II structure of $H_2(z)$.

The cascade realization of H(z) is obtained by connecting the direct form-II structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

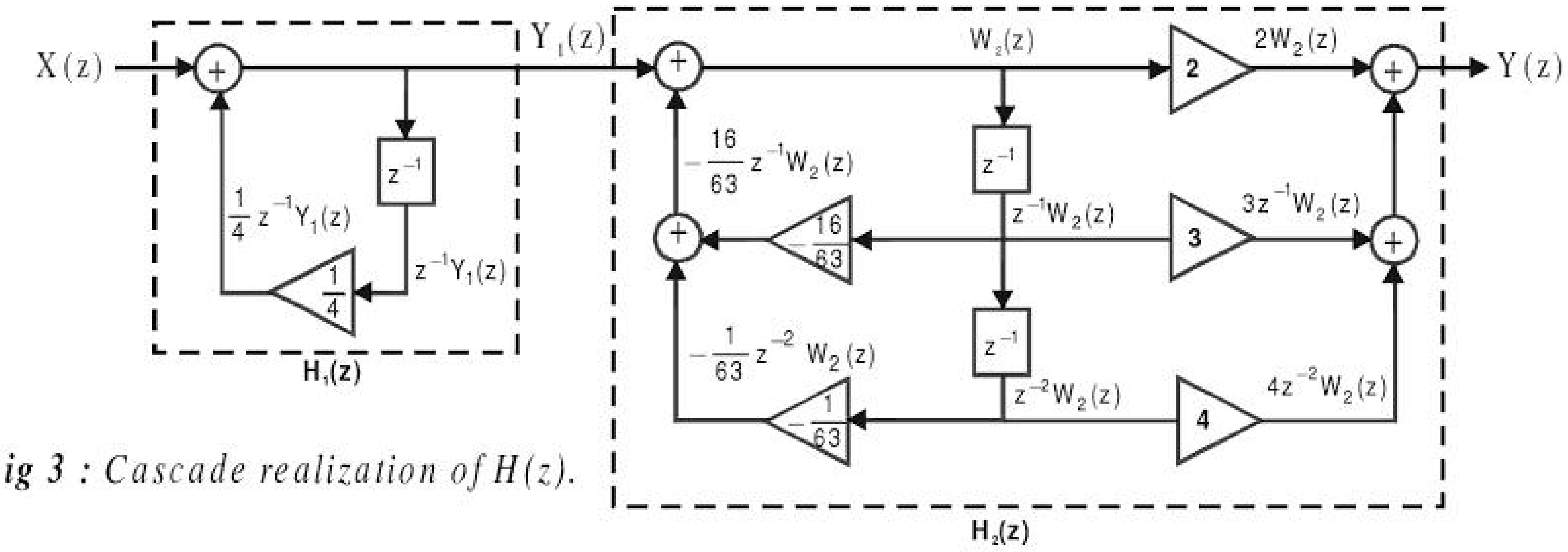


Fig 3 : Cascade realization of H(z).

Example 3.32

The transfer function of a system is given by, $H(z) = \dfrac{\left(2 - z^{-1}\right)\left(1 - z^{-1}\right)^2}{\left(1 - 2z^{-1}\right)\left(5 - 3z^{-1} + 2z^{-2}\right)}$

Realize the system in cascade and parallel structures.

Solution

Cascade Realization

Given that, $H(z) = \dfrac{\left(2 - z^{-1}\right)\left(1 - z^{-1}\right)^2}{\left(1 - 2z^{-1}\right)\left(5 - 3z^{-1} + 2z^{-2}\right)}$

> $5 - 3z^{-1} + 2z^{-2} = z^{-2}(5z^2 - 3z + 2)$
>
> The roots of quadratic,
>
> $5z^2 - 3z + 2 = 0$ are,
>
> $z = \dfrac{3 \pm \sqrt{3^2 - 4 \times 5 \times 2}}{2} = -\dfrac{3 \pm j5.6}{2}$

On examining the roots of the quadratic factor in the denominator it is observed that the roots are complex conjugate. Hence the system has to be realized as cascade of one first-order section and one second-order section.

$$\therefore\ H(z) = \frac{2 - z^{-1}}{1 - 2z^{-1}} \times \frac{\left(1 - z^{-1}\right)^2}{5 - 3z^{-1} + 2z^{-2}} = \frac{2 - z^{-1}}{1 - 2z^{-1}} \times \frac{1 - 2z^{-1} + z^{-2}}{5 - 3z^{-1} + 2z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

$$\text{where, } H_1(z) = \frac{2 - z^{-1}}{1 - 2z^{-1}} \quad \text{and} \quad H_2(z) = \frac{1 - 2z^{-1} + z^{-2}}{5 - 3z^{-1} + 2z^{-2}}$$

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{2 - z^{-1}}{1 - 2z^{-1}} \qquad \text{..... (1)}$$

On cross multiplying equation (1) we get,

$$Y_1(z) - 2z^{-1}\,Y_1(z) = 2X(z) - z^{-1}\,X(z)$$

$$\therefore\ Y_1(z) = 2X(z) - z^{-1}\,X(z) + 2z^{-1}\,Y_1(z) \qquad \text{..... (2)}$$

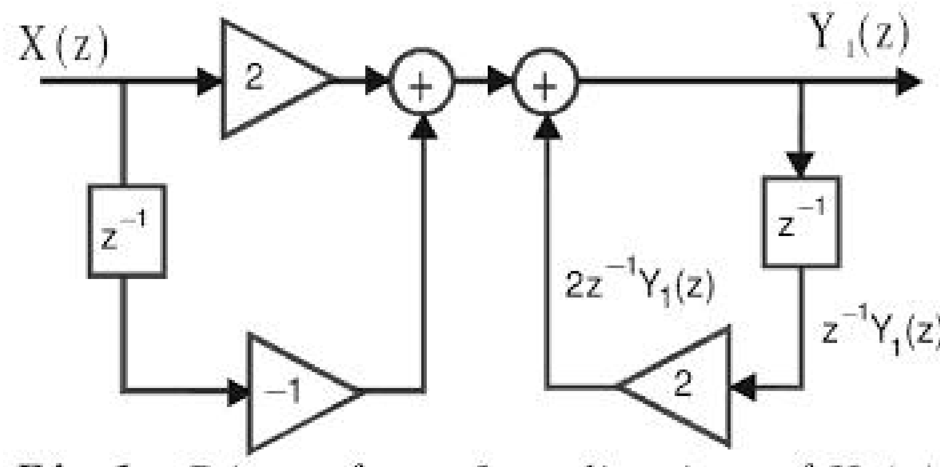


Fig 1 : Direct form-I realization of $H_1(z)$.

The direct form-I structure of $H_1(z)$ can be drawn using equation (2) as shown in fig 1.

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1 - 2z^{-1} + z^{-2}}{5 - 3z^{-1} + 2z^{-2}} \qquad \text{.....(3)}$$

On cross multiplying equation (3) we get

$$5Y(z) - 3z^{-1}Y(z) + 2z^{-2}Y(z) = Y_1(z) - 2z^{-1}Y_1(z) + z^{-2}Y_1(z)$$

$$\therefore\ Y(z) = \frac{1}{5}Y_1(z) - \frac{2}{5}z^{-1}Y_1(z) + \frac{1}{5}z^{-2}Y_1(z) + \frac{3}{5}z^{-1}Y(z) - \frac{2}{5}z^{-2}Y(z) \qquad .....(4)$$

The direct form-I structure of $H_2(z)$ can be drawn using equation (4) as shown in fig 2.

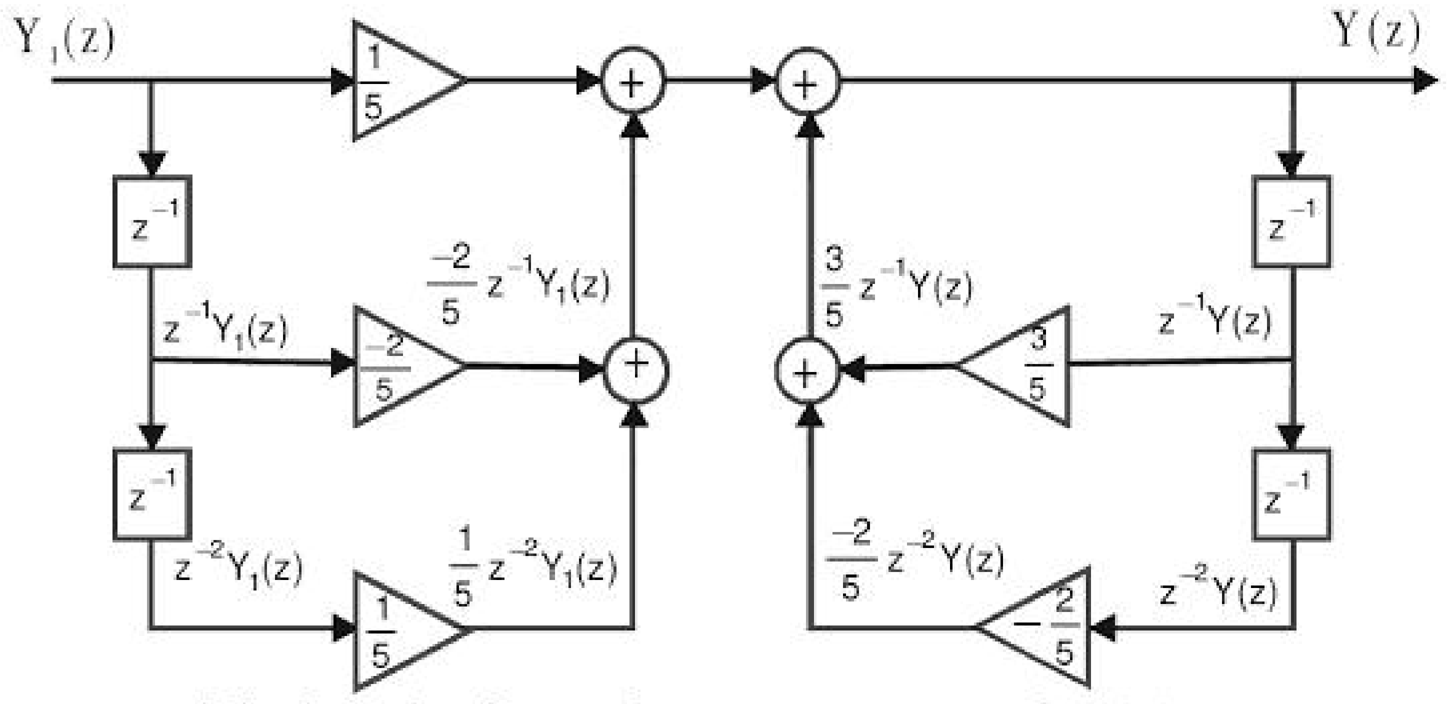


Fig 2 : The direct form-I structure of $H_2(z)$.

The cascade realization of H(z) is obtained by connecting the direct form-I structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

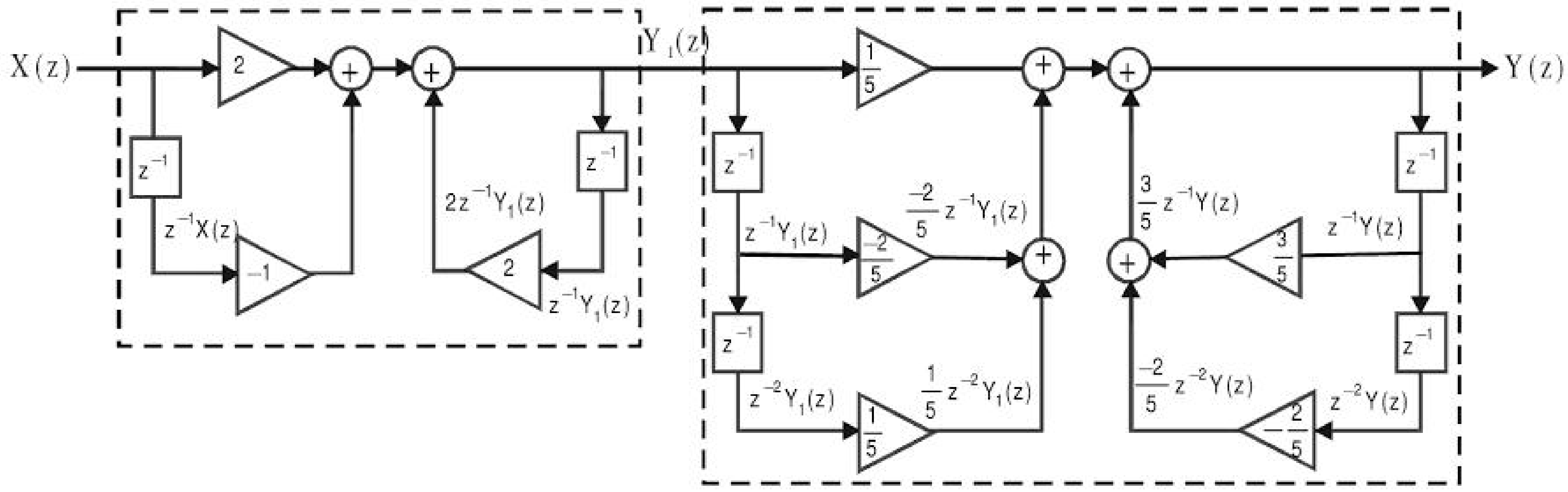


Fig 3 : Cascade realization of H(z).

Parallel Realization

$$\text{Given that, } H(z) = \frac{(2 - z^{-1})(1 - z^{-1})^2}{(1-2z^{-1})(5-3z^{-1}+2z^{-2})} \qquad .....(5)$$

$$= \frac{(2 - z^{-1})(1 - 2z^{-1} + z^{-2})}{(1-2z^{-1})(5-3z^{-1}+2z^{-2})} = \frac{2-4z^{-1}+2z^{-2}-z^{-1}+2z^{-2}-z^{-3}}{5-3z^{-1}+2z^{-2}-10z^{-1}+6z^{-2}-4z^{-3}}$$

$$\therefore H(z) = \frac{2-5z^{-1}+4z^{-2}-z^{-3}}{5-13z^{-1}+8z^{-2}-4z^{-3}}$$

$$= 0.4 + \frac{0.2z^{-1}+0.8z^{-2}+0.6z^{-3}}{5-13z^{-1}+8z^{-2}-4z^{-3}}$$

$$\begin{array}{r|l} & 0.4 \\ \hline 5-13z^{-1}+8z^{-2}-4z^{-3} & 2-5z^{-1}+4z^{-2}-z^{-3} \\ & 2-5.2z^{-1}+3.2z^{-2}-1.6z^{-3} \\ & (-)\quad(+)\quad(-)\quad(+) \\ \hline & 0.2z^{-1}+0.8z^{-2}+0.6z^{-3} \end{array}$$

$$= 0.4 + z^{-1}\left[\frac{0.2+0.8z^{-1}+0.6z^{-2}}{(1-2z^{-1})(5-3z^{-1}+2z^{-2})}\right] \qquad .....(6)$$

Using equation (5).

By partial fraction expansion we can write,

$$\frac{0.2+0.8z^{-1}-0.6z^{-2}}{\left(1-2z^{-1}\right)\left(5-3z^{-1}+2z^{-2}\right)}=\frac{A}{1-2z^{-1}}+\frac{B\ +\ Cz^{-1}}{5-3z^{-1}+2z^{-2}} \qquad .....(7)$$

On cross multiplying equation (7) we get,

$$0.2+0.8z^{-1}+0.6z^{-2} = A\left(5-3z^{-1}+2z^{-2}\right)+(B\ +\ Cz^{-1})\left(1-2z^{-1}\right)$$

$$0.2+0.8z^{-1}+0.6z^{-2} = 5A-3Az^{-1}+2Az^{-2}+B-2Bz^{-1}+Cz^{-1}-2Cz^{-2} \qquad .....(8)$$

$$0.2+0.8z^{-1}+0.6z^{-2} = (5A+B)+(-3A-2B+C)z^{-1}+(2A-2C)z^{-2}$$

On equating constants of equation (8),

$$5A+B=0.2$$

$$\backslash \quad B=0.2-5A$$

On equating coefficients of z^{-1} of equation (8),

$$-3A-2B+C=0.8$$

Put, $B=0.2-5A$

$$\backslash \quad -3A-2(0.2-5A)+C=0.8$$

$$-3A-0.4+10A+C=0.8$$

$$\backslash \quad C=1.2-7A$$

On equating coefficients of z^{-2} of equation (8),

$$2A-2C=0.6$$

$$2A-2(1.2-7A)=0.6$$

$$2A-2.4+14A=0.6$$

$$\therefore\ 16A=3 \Rightarrow A=\frac{3}{16}$$

Here, $A=\dfrac{3}{16}$

$$\therefore\ B=0.2-5A=0.2-5\times\frac{3}{16}=\frac{2}{10}-\frac{15}{16}=\frac{32-150}{160}=-\frac{118}{160}=-\frac{59}{80}$$

$$\therefore\ C=1.2-7A=1.2-7\times\frac{3}{16}=\frac{12}{10}-\frac{21}{16}=\frac{192-210}{160}=-\frac{18}{160}=-\frac{9}{80}$$

From equations (6) and (7) we can write,

$$H(z)=0.4+z^{-1}\left[\frac{A}{1-2z^{-1}}+\frac{B+Cz^{-1}}{5-3z^{-1}+2z^{-2}}\right]=0.4+\frac{\frac{3}{16}z^{-1}}{1-2z^{-1}}+\frac{-\frac{59}{80}z^{-1}-\frac{9}{80}z^{-2}}{5-3z^{-1}+2z^{-2}}$$

Let, $$H(z)=0.4+\frac{\frac{3}{16}z^{-1}}{1-2z^{-1}}+\frac{-\frac{59}{80}z^{-1}-\frac{9}{80}z^{-2}}{5-3z^{-1}+2z^{-2}}=H_1(z)+H_2(z)+H_3(z)$$

where, $H_1(z)=0.4$; $H_2(z)=\dfrac{\frac{3}{16}z^{-1}}{1-2z^{-1}}$; $H_3(z)=\dfrac{-\frac{59}{80}z^{-1}-\frac{9}{80}z^{-1}}{5-3z^{-1}+2z^{-2}}$

Let, $H(z)=\dfrac{Y(z)}{X(z)}$; $H_1(z)=\dfrac{Y_1(z)}{X(z)}$; $H_2(z)=\dfrac{Y_2(z)}{X(z)}$; $H_3(z)=\dfrac{Y_3(z)}{X(z)}$

$$\therefore\ H(z)=H_1(z)+H_2(z)+H_3(z) \Rightarrow \frac{Y(z)}{X(z)}=\frac{Y_1(z)}{X(z)}+\frac{Y_2(z)}{X(z)}+\frac{Y_3(z)}{X(z)}$$

$$\therefore\ Y(z)=Y_1(z)+Y_2(z)+Y_3(z)$$

Realization of $H_1(z)$

$$H_1(z)=\frac{Y_1(z)}{X(z)}=0.4 \quad\Rightarrow\quad Y_1(z)=0.4\,X(z)$$

Using the above equation, the direct form-I structure of $H_1(z)$ is drawn as shown in fig 4.

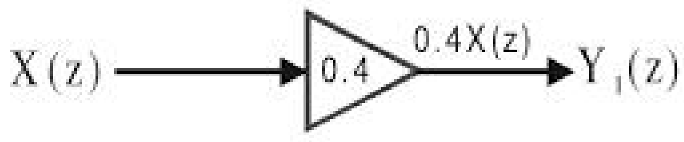


Fig 4 : Direct form-I structure of $H_1(z)$.

Realization of $H_2(z)$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{\frac{3}{16} z^{-1}}{1 - 2z^{-1}}$$

On cross multiplying the above equation we get,

$$Y_2(z) - 2z^{-1} Y_2(z) = \frac{3}{16} z^{-1} X(z)$$

$$\therefore \quad Y_2(z) = \frac{3}{16} z^{-1} X(z) + 2z^{-1} Y_2(z) \qquad \text{..... (9)}$$

Using equation (9) the direct form-I structure of $H_2(z)$ is drawn as shown in fig 5.

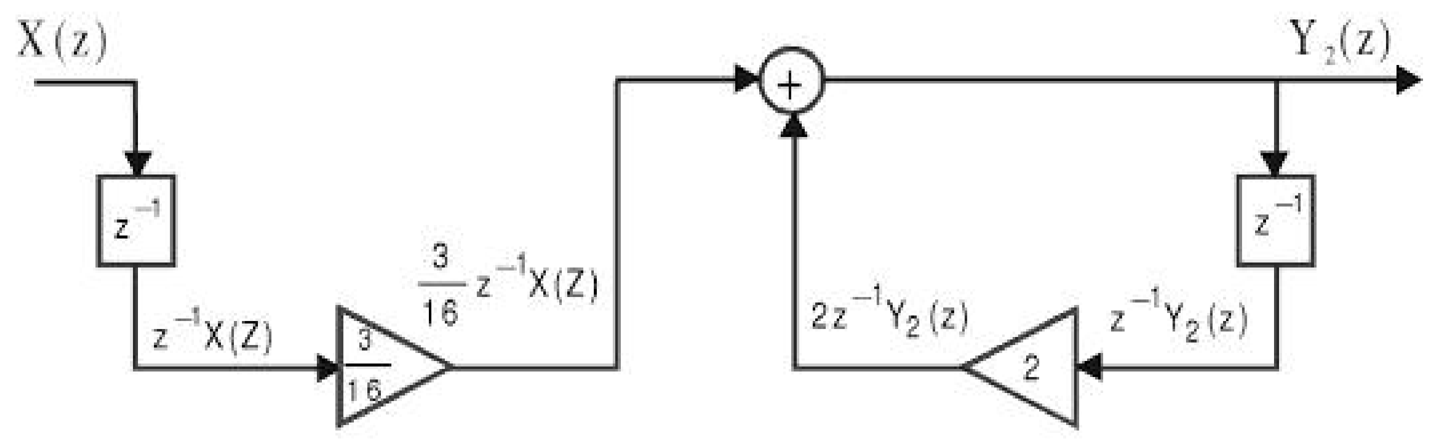


Fig 5 : Direct form-I structure of $H_2(z)$.

Realization of $H_3(z)$

$$H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{-\frac{59}{80} z^{-1} - \frac{9}{80} z^{-2}}{5 - 3z^{-1} + 2z^{-2}}$$

On cross multiplying the above equation we get,

$$5Y_3(z) - 3z^{-1} Y_3(z) + 2z^{-2} Y_3(z) = -\frac{59}{80} z^{-1} X(z) - \frac{9}{80} z^{-2} X(z)$$

$$\therefore \; Y_3(z) = -\frac{59}{400} z^{-1} X(z) - \frac{9}{400} z^{-2} X(z) + \frac{3}{5} z^{-1} Y_3(z) - \frac{2}{5} z^{-2} Y_3(z) \qquad \text{.....(10)}$$

Using equation (10) the direct form-I structure of $H_2(z)$ is drawn as shown in fig 6.

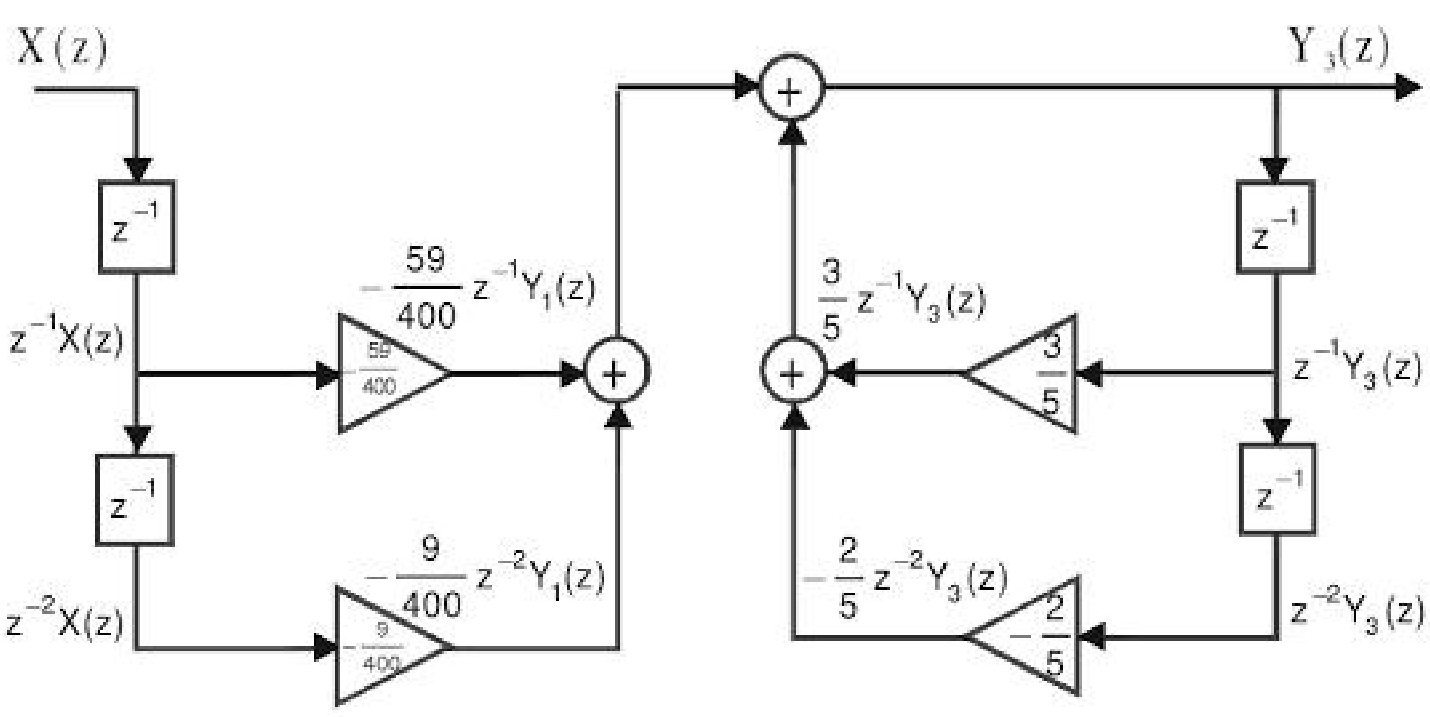


Fig 6 : Direct form-I structure of $H_3(z)$.

Parallel Structure

The parallel structure of H(z) is obtained by connecting the direct form-I structure of $H_1(z)$, $H_2(z)$ and $H_3(z)$ as shown in fig 7.

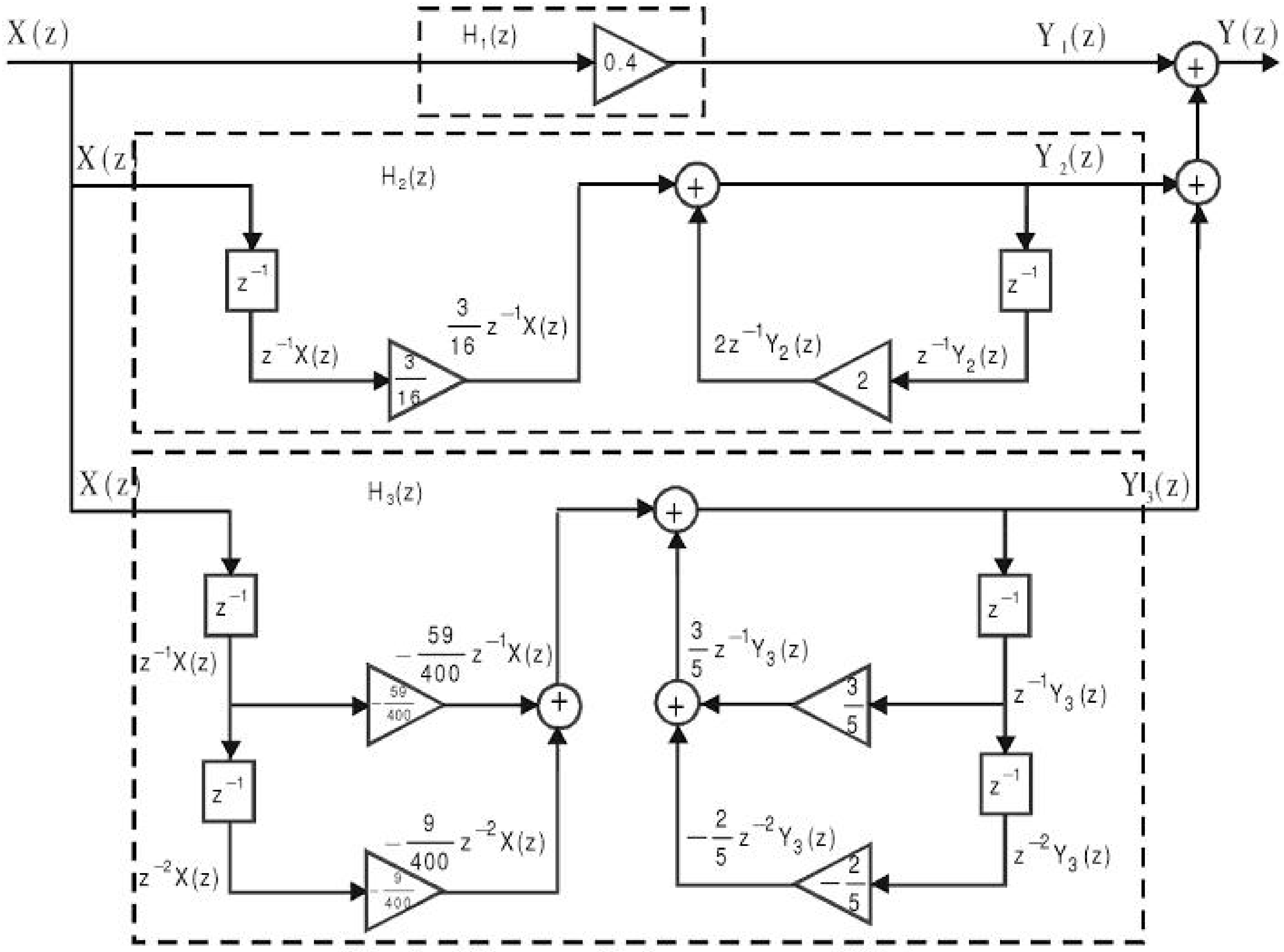


Fig 7 : The parallel structure of H(z).

Example 3.33

An LTI System is described by the equation, $y(n) - \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) = x(n)$.
Determine the cascade realization structure of the system.

Solution

Given that, $y(n) - \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) = x(n)$

On taking Z-transform we get,

$$Y(z) - \frac{1}{2}z^{-1}Y(z) - \frac{1}{4}z^{-2}Y(z) = X(z)$$

$$\left(1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2}\right)Y(z) = X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2}}$$

The roots of quadratic
$z^2 - 0.5z - 0.25 = 0$ are,
$z = \frac{0.5 \pm \sqrt{0.5^2 + 4 \times 0.25}}{2}$
$= \frac{0.5 \pm 1.118}{2} = 0.809, \; -0.309$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2}} = \frac{1}{z^{-2}\left(z^2 - 0.5z - 0.25\right)}$$

$$= \frac{1}{z^{-2}(z - 0.809)(z + 0.309)} = \frac{1}{(1 - 0.809z^{-1})(1 + 0.309z^{-1})}$$

Let, $H(z) = H_1(z)\,H_2(z)$

$$\text{where, } H_1(z) = \frac{1}{1 - 0.809z^{-1}}\;; \qquad H_2(z) = \frac{1}{1 + 0.309z^{-1}}$$

Let, $H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-0.809z^{-1}}$ .....(1)

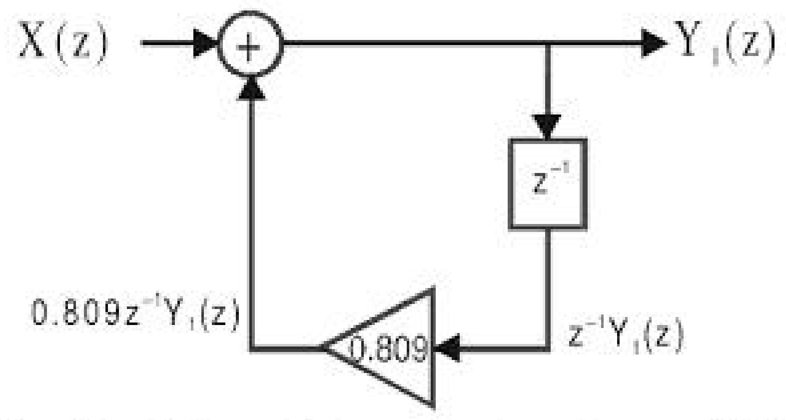


Fig 1 : Direct form-I structure of $H_1(z)$.

On cross multiplying equation (1) we get,

$$Y_1(z) - 0.809z^{-1}\,Y_1(z) = X(z)$$

$$\therefore\ Y_1(z) = X(z) + 0.809z^{-1}\,Y_1(z) \quad ..... (2)$$

The direct form-I structure of $H_1(z)$ is obtained using equation (2) as shown in fig 1.

Let, $H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1 + 0.309z^{-1}}$ ..... (3)

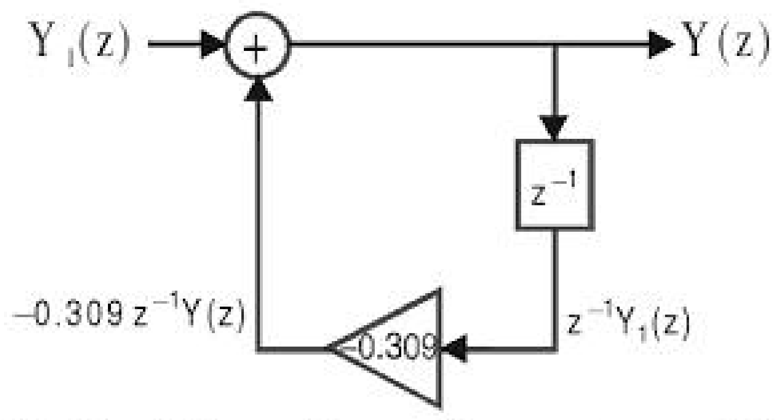


Fig 2 : Direct form-I structure of $H_2(z)$.

On cross multiplying equation (3) we get,

$$Y(z) + 0.309\ z^{-1}\ Y(z) = Y_1(z)$$

$$Y(z) = Y_1(z) - 0.309\ z^{-1}\ Y(z) \quad ..... (4)$$

The direct form-I structure of $H_2(z)$ is obtained using equation (4) as shown in fig 2.The cascade structure is obtained by connecting the direct form structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

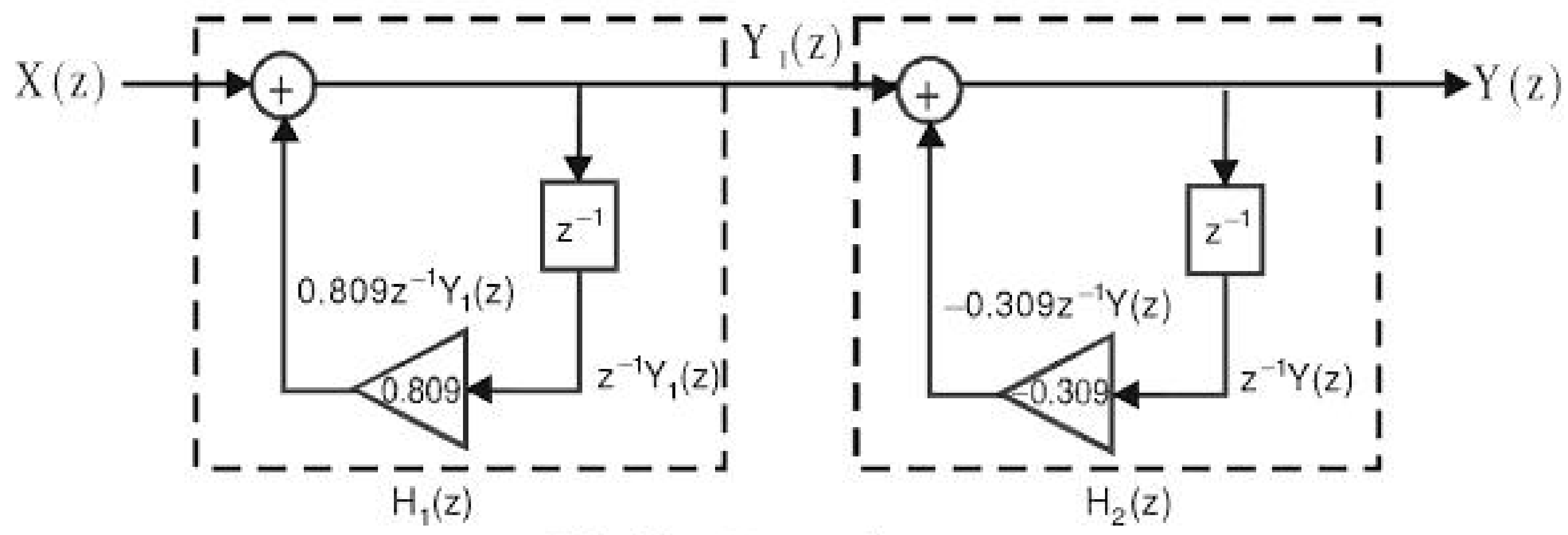


Fig 3 : Cascade structure.

3.10 Structures for Realization of FIR Systems

In general, the time domain representation of an N^{th} order FIR system is,

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m) = b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + ... + b_{N-1}\, x(n-(N-1))$$

and the z-domain representation of a FIR system is,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}$$

The above two representations of FIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence y(n) from the input sequence x(n). Also in the above representations the value of N gives the number of zeros of the FIR system. The computations in the above equation can be arranged into various equivalent sets of difference equations, which leads to different types of structures for realizing FIR systems. Some of the structures of the system gives a direct relation between time domain equation and z-domain equation.

The different types of structures for realizing FIR systems are,

1. Direct form realization
2. Cascade realization
3. Linear phase realization

3.10.1 Direct Form Realization of FIR System

Consider the difference equation governing a FIR system,

If $\mathcal{Z}\{x(n)\} = X(z)$ then, $\mathcal{Z}\{x(n-k)\} = z^{-k}X(z)$

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m)$$

$$= b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \;.....\; + b_{N-1}\, x(n-(N-1))$$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$Y(z) = b_0\, X(z) + b_1\, z^{-1}\, X(z) + b_2\, z^{-2}\, X(z) + b_3\, z^{-3}\, X(z) + \;.....\; + b_{N-2}\, z^{-(N-2)}\, X(z) + b_{N-1}\, z^{-(N-1)}\, X(z) \qquad .....(3.86)$$

The equation of Y(z) [equation (3.86)] can be directly represented by a block diagram as shown in fig 3.22 and this structure is called direct form structure. The direct form structure provides a direct relation between time domain and z-domain equations.

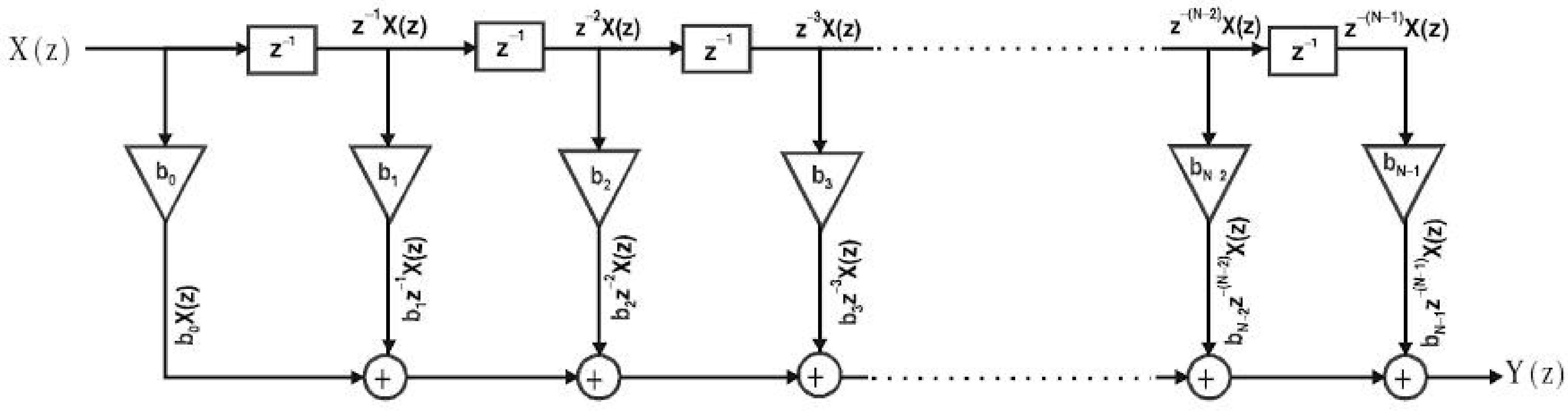


Fig 3.22 : Direct form structure of FIR system.

From the direct form structure it is observed that the realization of an N^{th} order FIR discrete time system involves N number of multiplications and N-1 number of additions. Also the structure involves N-1 delays and so N-1 memory locations are required to store the delayed signals.

3.10.2 Cascade Form Realization of FIR System

Consider the transfer function of a FIR system,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \;.....\; + b_{N-1}\, z^{-(N-1)}$$

The transfer function of FIR system is $(N-1)^{th}$ order polynomial in z. This polynomial can be factorized into first and second-order factors and the transfer function can be expressed as a product of first and second-order factors or sections as shown in equation (3.87).

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) \;.....\; H_m(z) = \prod_{i=1}^{m} H_i(z) \qquad ...(3.87)$$

where, $H_i(z) = c_{0i} + c_{1i}\, z^{-1} + c_{2i}\, z^{-2}$ Second-order section

or, $H_i(z) = c_{0i} + c_{1i}\, z^{-1}$ First-order section

The individual second-order or first-order sections can be realized either in direct form structure or linear phase structure. The overall system is obtained by cascading the individual sections as shown in fig 3.23. The number of calculations and the memory requirement depends on the realization of individual sections.

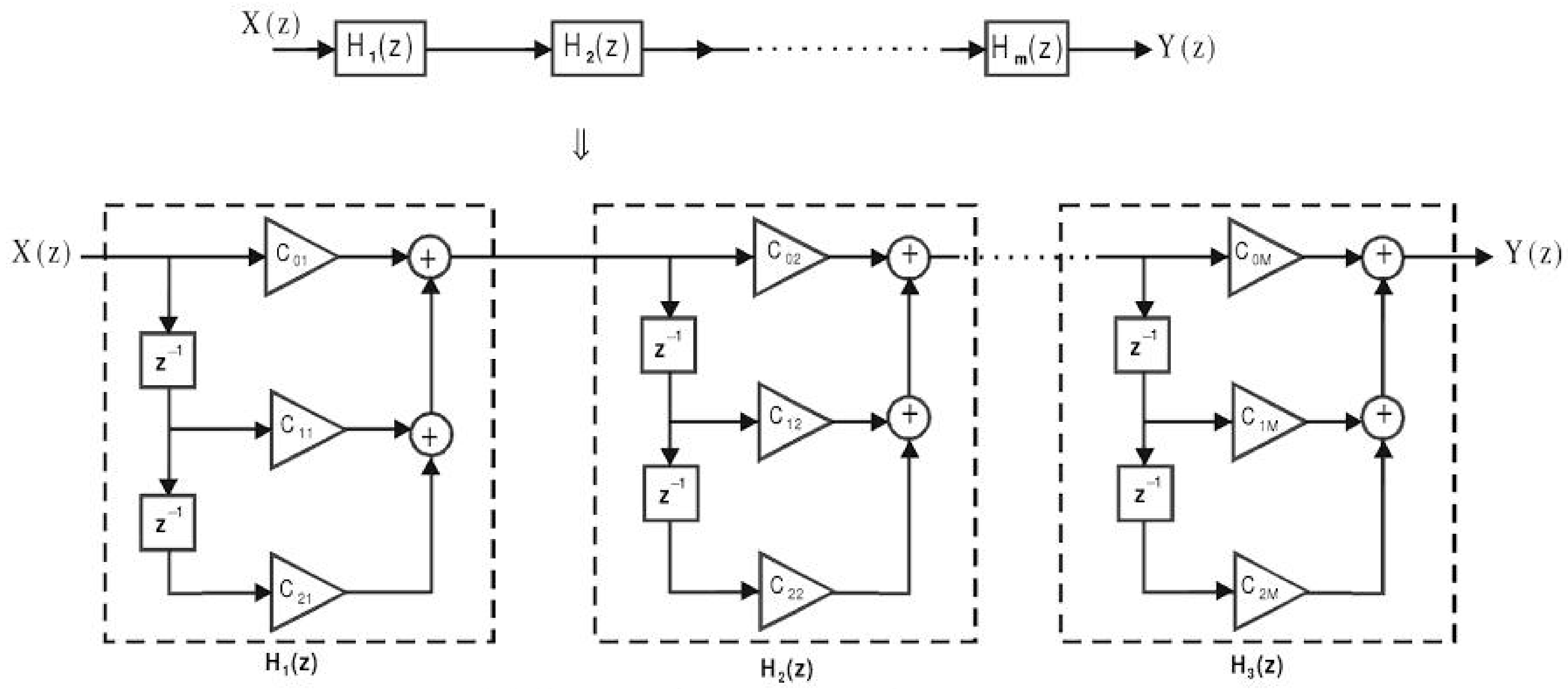


Fig 3.23 : Cascade structure of FIR system.

3.10.3 Linear Phase Realization of FIR System

Consider the impulse response h(n) of FIR system,

$$h(n) = \{b_0, b_1, b_2, \ldots\ldots\ldots\ldots\ldots\ldots b_{N-1}\}$$

In FIR system, for linear phase response the impulse response should be symmetrical.

The condition for symmetry is,

$$h(n) = h(N-1-n)$$

Proof :

Let, N =7, \ h(n) = h(6–n)	**Let, N =8, \ h(n) = h(7–n)**
n = 0, 1, 2, 3, 4, 5, 6	n = 0, 1, 2, 3, 4, 5, 6, 7
When n = 0; h(0) = h(6)	When n = 0; h(0) = h(7)
When n = 1; h(1) = h(5)	When n = 1; h(1) = h(6)
When n = 2; h(2) = h(4)	When n = 2; h(2) = h(5)
When n = 3; h(3) = h(3)	When n = 3; h(3) = h(4)

When the impulse response is symmetric, the samples of impulse response will satisfy the condition,

$$b_n = b_{N-1-n}$$

By using the above symmetry condition it is possible to reduce the number of multipliers required for the realization of FIR system. Hence, the linear phase realization is also called ***realization with minimum number of multipliers.***

Consider the transfer function of FIR system,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_{N-1} z^{-(N-1)}$$

The linear phase realization of the FIR system using the above equation for even and odd values of N are discussed below.

Case i : When N is even

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}$$

$$= \sum_{m=0}^{N-1} b_m z^{-m} = \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=\frac{N}{2}}^{N-1} b_m z^{-m}$$

Dividing the summation of N terms into two summations with N/2 terms.

Let, p = N–1–m, $\backslash$ m = N–1–p

When, $m = \frac{N}{2}$; $p = N-1-\frac{N}{2} = \frac{N}{2}-1$

When, m = N–1; p = N–1–(N–1) = 0

$$\therefore \frac{Y(z)}{X(z)} = \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{p=0}^{\frac{N}{2}-1} b_{N-1-p} z^{-(N-1-p)}$$

Let, m = p in the second summation

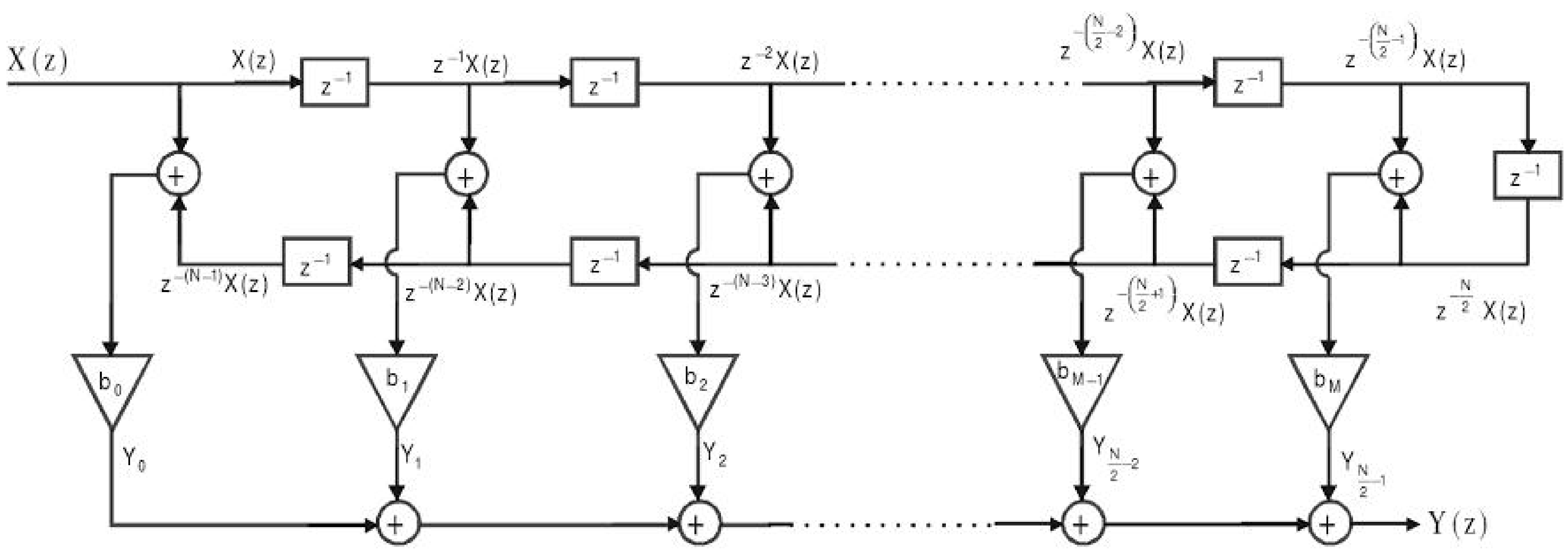
$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=0}^{\frac{N}{2}-1} b_{N-1-m} z^{-(N-1-m)}$$

Let, p = m

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-(N-1-m)}$$

When impulse response is symmetric, $b_m = b_{N-1-m}$

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m \left[z^{-m} + z^{-(N-1-m)}\right]$$



where, $M = \frac{N}{2}-1$; $Y_0 = b_0\left[X(z) + z^{-(N-1)}X(z)\right]$; $Y_{\frac{N}{2}-2} = b_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)}X(z) + z^{-\left(\frac{N}{2}+1\right)}X(z)\right]$

$Y_1 = b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right]$; $Y_{\frac{N}{2}-1} = b_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\frac{N}{2}}X(z)\right]$

Fig 10.24 : *Direct form realization of a linear phase FIR system when N is even.*

$$\therefore\ Y(z) = b_0\left[X(z) + z^{-(N-1)}X(z)\right] + b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right] + ...$$

$$+\ b_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)}X(z) + z^{-\left(\frac{N}{2}+1\right)}X(z)\right] + b_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\frac{N}{2}}X(z)\right]$$

When N is even, the above equation can be used to construct the direct form structure of linear phase FIR system with minimum number of multipliers, as shown in fig 3.24. From the direct form linear phase structure it is observed that the realization of an N^{th} order FIR discrete time system for even values of N involves N/2 number of multiplications and N - 1 number of additions. Also the structure involves N - 1 delays and so N - 1 memory locations are required to store the delayed signals.

Case ii : When N is odd

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)} = \sum_{m=0}^{N-1} b_m z^{-m}$$

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=\frac{N+1}{2}}^{N-1} b_m z^{-m}$$

Dividing the summation of N terms into two summations with $\frac{N-1}{2}$ terms.

Let, p = N–1–m, $\quad$ \ m = N–1–p

When, $m = \frac{N+1}{2}$; $\quad p = N-1-\frac{N+1}{2} = \frac{N-3}{2}$

When, m = N–1 ; $\quad$ p = N–1 – (N–1) = 0

$$\therefore\ \frac{Y(z)}{X(z)} = \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{p=0}^{\frac{N-3}{2}} b_{N-1-p}\, z^{-(N-1-p)}$$

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_{N-1-m}\, z^{-(N-1-m)}$$

Let, p = m

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_m\, z^{-(N-1-m)}$$

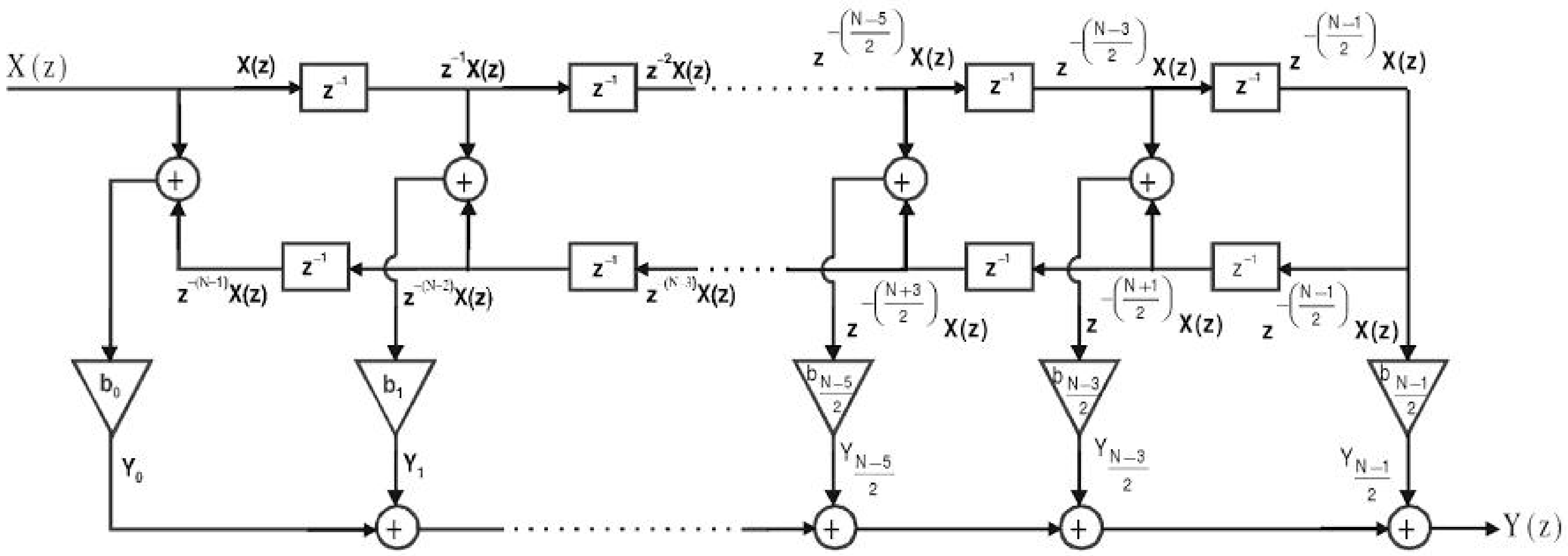
When impulse response is symmetric, $b_m = b_{N-1-m}$

$$= b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_m \left[z^{-m} + z^{-(N-1-m)}\right]$$

$$\therefore\ Y(z) = b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} X(z) + b_0\left[X(z) + z^{-(N-1))}X(z)\right] + b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right] +$$

$$.....\ +\ b_{\frac{N-5}{2}}\left[z^{-\left(\frac{N-5}{2}\right)}X(z) + z^{-\left(\frac{N+3}{2}\right)}X(z)\right] + b_{\frac{N-3}{2}}\left[z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-\left(\frac{N+1}{2}\right)}X(z)\right]$$

When N is odd, the above equation can be used to construct the direct form structure of linear phase FIR system with minimum number of multipliers, as shown in fig 3.25.



$$Y_0 = b_0\left[X(z) + z^{-(N-1)}X(z)\right]; \quad Y_{\frac{N-5}{2}} = b_{\frac{N-5}{2}}\left[z^{-\left(\frac{N-5}{2}\right)}X(z) + z^{-\left(\frac{N-3}{2}\right)}X(z)\right]; \quad Y_{\frac{N-1}{2}} = b_{\frac{N-1}{2}}\, z^{-\left(\frac{N-1}{2}\right)}X(z)$$

$$Y_1 = b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right]; \quad Y_{\frac{N-3}{2}} = b_{\frac{N-3}{2}}\left[z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-\left(\frac{N-1}{2}\right)}X(z)\right]$$

***Fig 3.25 :** Direct form realization of a linear phase FIR system when N is odd.*

From the direct form linear phase structure it is observed that the realization of an N^{th} order FIR discrete time system for odd values of N involves (N+1)/2 number of multiplications and N-1 number of additions. Also the structure involves N-1 delays and so N-1 memory locations are required to store the delayed signals.

Example 3.34

Draw the direct form structure of the FIR system described by the transfer function,

$$H(z) = 1+\frac{1}{2}z^{-1}+\frac{3}{8}z^{-2}+\frac{5}{4}z^{-3}+\frac{1}{2}z^{-4}+\frac{7}{8}z^{-5}$$

Solution

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 1+\frac{1}{2}z^{-1}+\frac{3}{8}z^{-2}+\frac{5}{4}z^{-3}+\frac{1}{2}z^{-4}+\frac{7}{8}z^{-5}$$

$$\therefore\ Y(z) = X(z)+\frac{1}{2}z^{-1}X(z)+\frac{3}{8}z^{-2}X(z)+\frac{5}{4}z^{-3}X(z)+\frac{1}{2}z^{-4}X(z)+\frac{7}{8}z^{-5}X(z) \quad .....(1)$$

The direct form structure of FIR system can be obtained directly from equation (1).

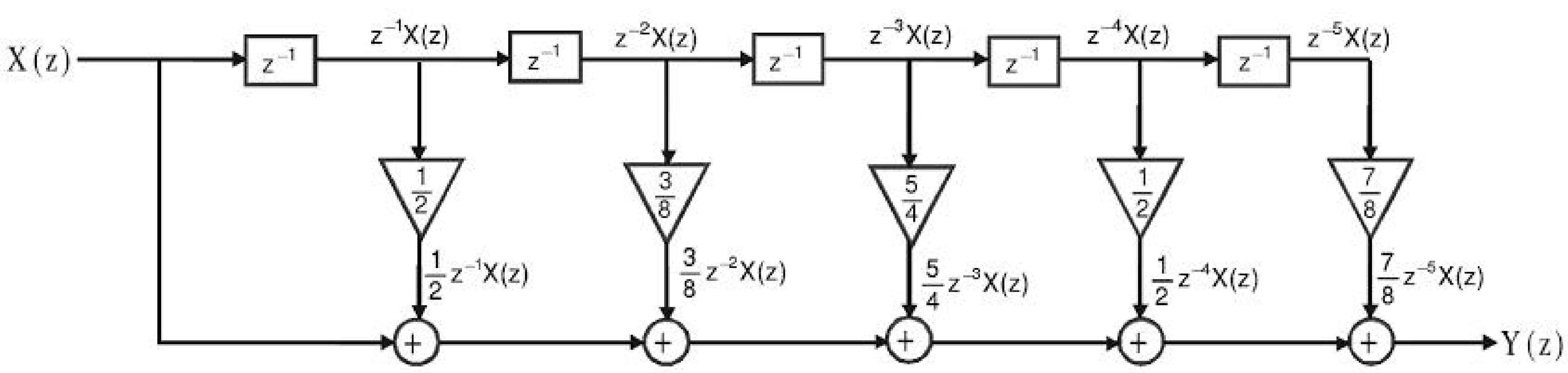


***Fig 1 :** Direct form structure of H(z).*

Example 3.35

Realize the following system with minimum number of multipliers.

a) $H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$

b) $H(z) = \frac{1}{3} + \frac{1}{4}z^{-1} + \frac{3}{2}z^{-2} + \frac{3}{2}z^{-3} + \frac{1}{4}z^{-4} + \frac{1}{3}z^{-5}$

c) $H(z) = \left(\frac{1}{5} + \frac{1}{2}z^{-1} + \frac{1}{5}z^{-2}\right)\left(\frac{1}{7} + \frac{1}{4}z^{-1} + \frac{1}{7}z^{-2}\right)$

Solution

a) Given that, $\mathbf{H(z)} = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$ ..... (1)

By the definition of Z-transform we get,

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + \ldots \qquad \text{..... (2)}$$

On comparing equations (1) and (2) we get,

Impulse response, $h(n) = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right\}$

Here h(n) satisfies the condition $h(n) = h(N - 1 - n)$ and so impulse response is symmetrical. Hence the system has linear phase and can be realized with minimum number of multipliers.

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$

$$\therefore\ Y(z) = \frac{1}{4}X(z) + \frac{1}{2}z^{-1}X(z) + \frac{3}{4}z^{-2}X(z) + \frac{1}{2}z^{-3}X(z) + \frac{1}{4}z^{-4}X(z)$$

$$= \frac{1}{4}\left[X(z) + z^{-4}X(z)\right] + \frac{1}{2}\left[z^{-1}X(z) + z^{-3}X(z)\right] + \frac{3}{4}z^{-2}X(z) \qquad \text{.....(3)}$$

The direct form structure of linear phase FIR system is constructed using equation (3) as shown in fig 1.

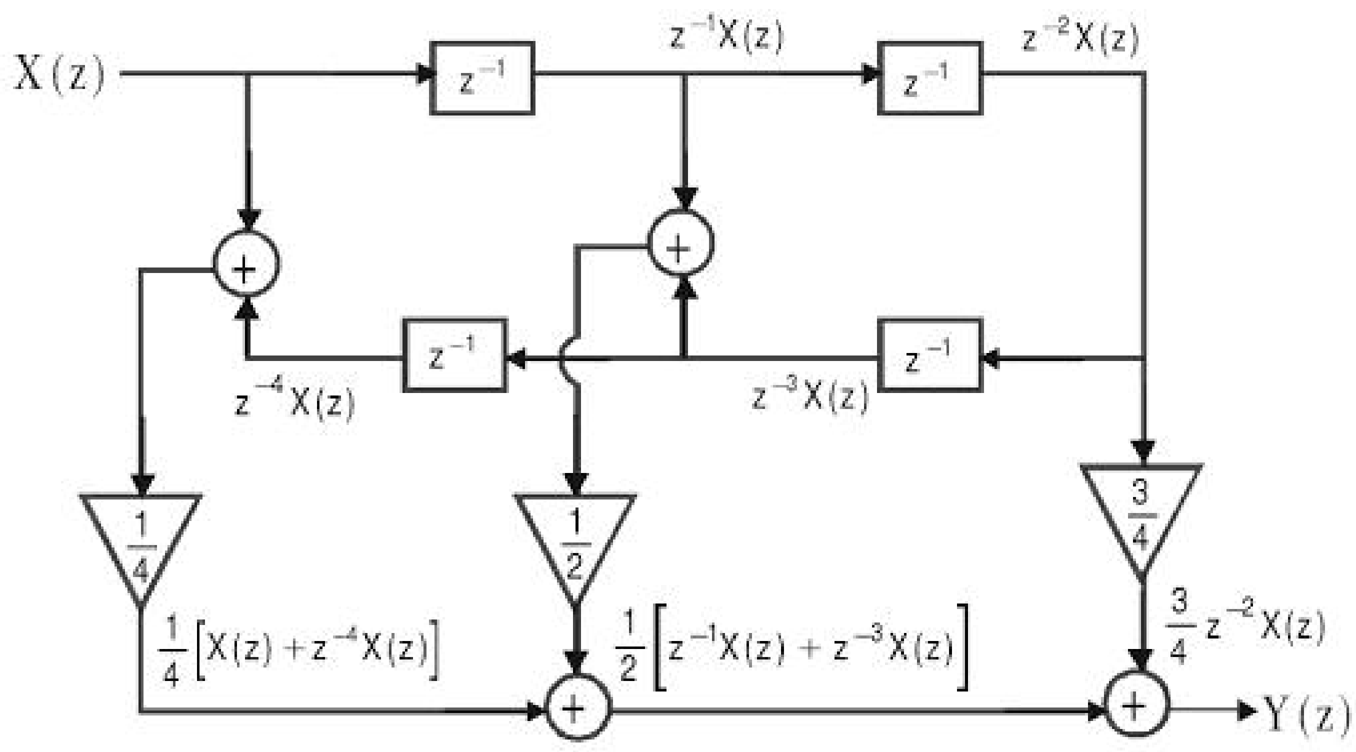


Fig 1 : Linear phase realization of H(z).

b) Given that, $\mathbf{H(z)} = \frac{1}{3} + \frac{1}{4}z^{-1} + \frac{3}{2}z^{-2} + \frac{3}{2}z^{-3} + \frac{1}{4}z^{-4} + \frac{1}{3}z^{-5}$

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{3} + \frac{1}{4}z^{-1} + \frac{3}{2}z^{-2} + \frac{3}{2}z^{-3} + \frac{1}{4}z^{-4} + \frac{1}{3}z^{-5}$

$$\therefore\ Y(z) = \frac{1}{3}X(z) + \frac{1}{4}z^{-1}X(z) + \frac{3}{2}z^{-2}X(z) + \frac{3}{2}z^{-3}X(z) + \frac{1}{4}z^{-4}X(z) + \frac{1}{3}z^{-5}X(z)$$

$$= \frac{1}{3}\left[X(z) + z^{-5}X(z)\right] + \frac{1}{4}\left[z^{-1}X(z) + z^{-4}X(z)\right] + \frac{3}{2}\left[z^{-2}X(z) + z^{-3}X(z)\right] \qquad \text{.....(4)}$$

The direct form realization of H(z) with minimum number of multipliers (i.e., linear phase realization) is obtained using equation (4) as shown in fig 2.

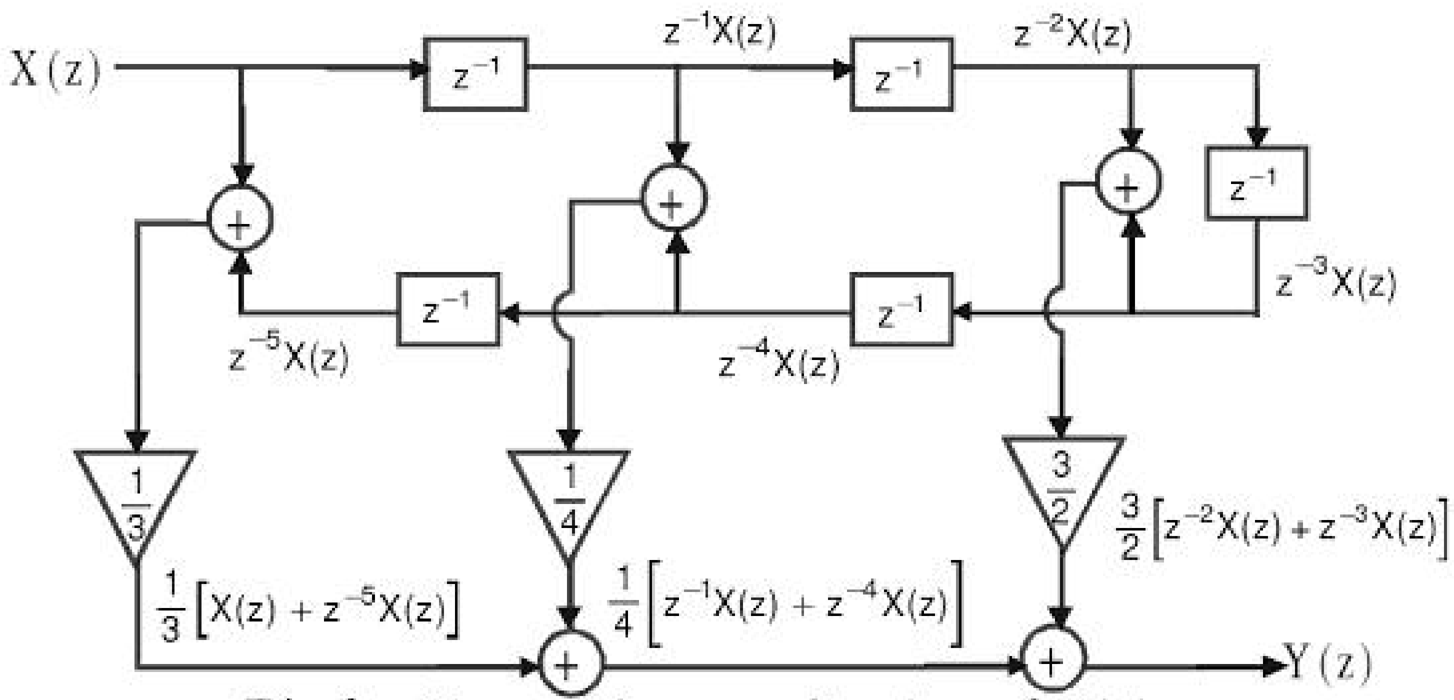


Fig 2 : Linear phase realization of H(z).

c) Given that, $\mathbf{H(z) = \left(\frac{1}{5} + \frac{1}{2}z^{-1} + \frac{1}{5}z^{-2}\right)\left(\frac{1}{7} + \frac{1}{4}z^{-1} + \frac{1}{7}z^{-2}\right)}$

The given system can be realized as cascade of two second-order systems. Each system can be realized with minimum number of multipliers.

Let, $H(z) = H_1(z)\, H_2(z)$

where, $H_1(z) = \frac{1}{5} + \frac{1}{2}z^{-1} + \frac{1}{5}z^{-2}$; $H_2(z) = \frac{1}{7} + \frac{1}{4}z^{-1} + \frac{1}{7}z^{-2}$

Let, $H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{5} + \frac{1}{2}z^{-1} + \frac{1}{5}z^{-2}$

$$\therefore\ Y_1(z) = \frac{1}{5}X(z) + \frac{1}{2}z^{-1}X(z) + \frac{1}{5}z^{-2}X(z)$$

$$= \frac{1}{5}\left[X(z) + z^{-2}X(z)\right] + \frac{1}{2}z^{-1}X(z) \qquad .....(5)$$

The linear phase realization structure of $H_1(z)$ is obtained using equation (5) as shown in fig 3.

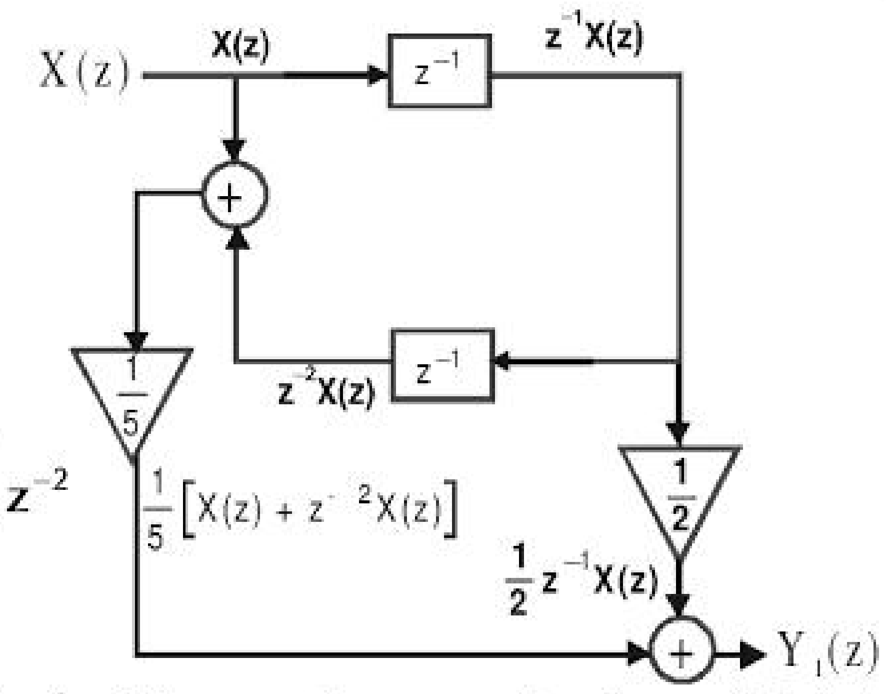


Fig 3 : Linear phase realization of $H_1(z)$.

Let, $H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{7} + \frac{1}{4}z^{-1} + \frac{1}{7}z^{-2}$

$$\therefore\ Y(z) = \frac{1}{7}Y_1(z) + \frac{1}{4}z^{-1}Y_1(z) + \frac{1}{7}z^{-2}Y_1(z)$$

$$= \frac{1}{7}\left[Y_1(z) + z^{-2}Y_1(z)\right] + \frac{1}{4}z^{-1}Y_1(z) \qquad .....(6)$$

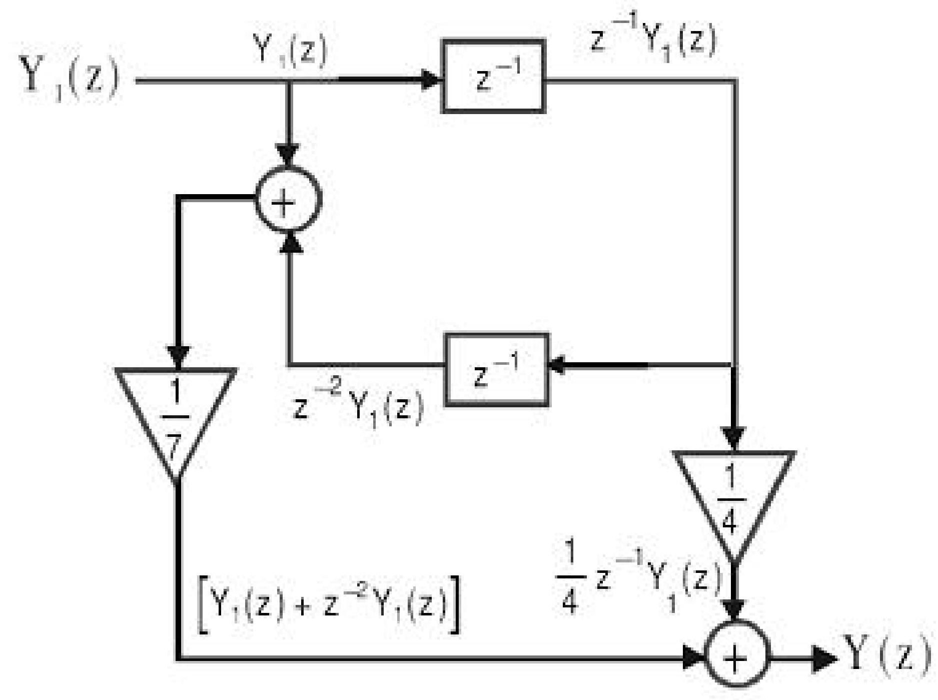


Fig 4 : Linear phase realization of $H_2(z)$.

The linear phase realization structure of $H_2(z)$ is obtained using equation (6) as shown in fig 4.

The linear phase structure of H(z) is obtained by connecting the linear phase realization structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 5.

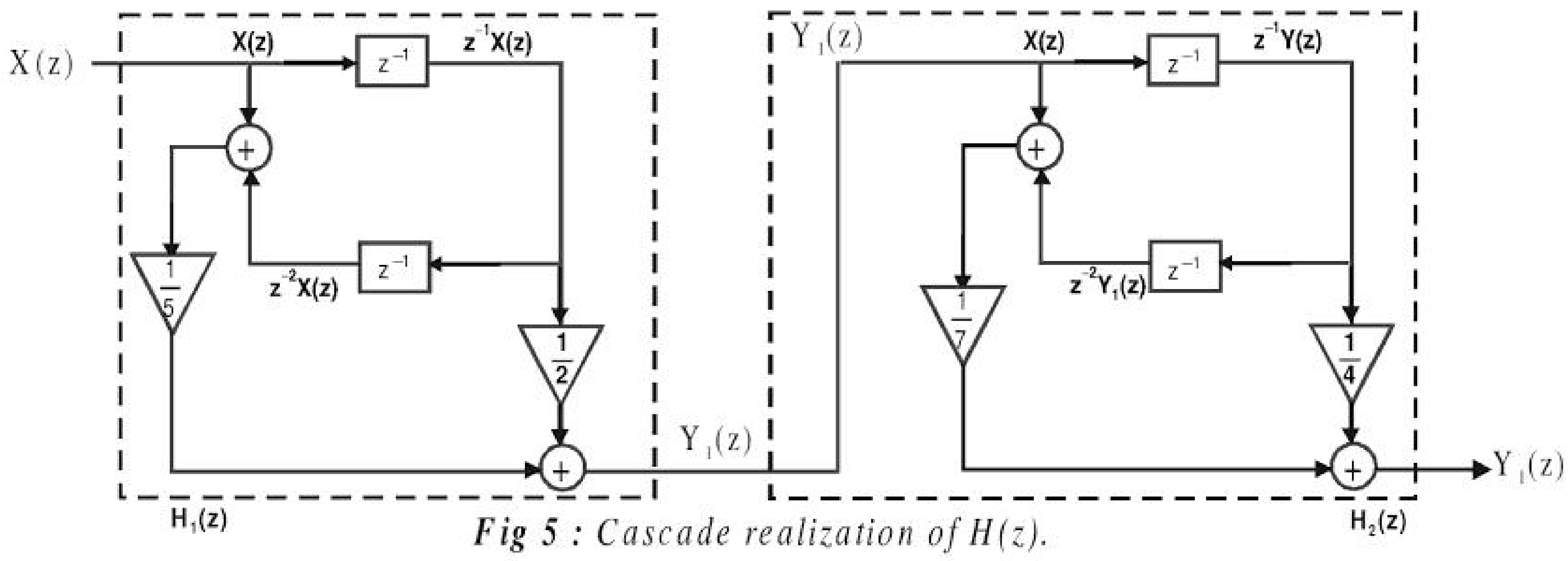


Fig 5 : Cascade realization of H(z).

3.11 Summary of Important Concepts

1. The $\mathcal{Z}$-transform provides a method for analysis of discrete time signals and systems in frequency domain.
2. The ROC of X(z) is a set of all values of z, for which X(z) attains a finite value.
3. Since ROC is a set of values of z, it will be a ring or disk in z-plane, with centre at origin.
4. The zeros are defined as values of z at which the function X(z) becomes zero.
5. The poles are defined as values of z at which the function X(z) becomes infinite.
6. In a realizable system, the number of zeros will be less than or equal to number of poles.
7. If x(n) is finite duration right-sided (causal) signal, then the ROC is entire z-plane except z = 0.
8. If x(n) is finite duration left-sided (anticausal) signal, then the ROC is entire z-plane except z = ¥.
9. If x(n) is finite duration two-sided (noncausal) signal, then the ROC is entire z-plane except z = 0 and z = ¥.
10. If x(n) is infinite duration right-sided (causal) signal, then the ROC is exterior of a circle of radius r_1.
11. If x(n) is infinite duration left-sided (anticausal) signal, then the ROC is interior of a circle of radius r_2.
12. If x(n) is infinite duration two-sided (noncausal) signal, then the ROC is the region in between two circles of radius r_1 and r_2.
13. If X(z) is rational, [where X(z) is $\mathcal{Z}$-transform of x(n)], then the ROC does not include any poles of X(z).
14. If X(z) is rational, [where X(z) is $\mathcal{Z}$-transform of x(n)], and if x(n) is right-sided, then ROC is exterior of a circle whose radius corresponds to pole with largest magnitude.
15. If X(z) is rational, [where X(z) is $\mathcal{Z}$-transform of x(n)], and if x(n) is left-sided, then ROC is interior of a circle whose radius corresponds to pole with smallest magnitude.
16. If X(z) is rational, [where X(z) is $\mathcal{Z}$-transform of x(n)], and if x(n) is two-sided, then ROC is region in between two circles whose radii corresponds to pole of causal part with largest magnitude and pole of anticausal part with smallest magnitude.
17. The inverse $\mathcal{Z}$-transform is the process of recovering the discrete time signal x(n) from its $\mathcal{Z}$-transform X(z).
18. The transfer function of an LTI discrete time system is defined as the ratio of $\mathcal{Z}$-transform of output and $\mathcal{Z}$-transform of input.
19. The transfer function of an LTI discrete time system is also given by $\mathcal{Z}$-transform of the impulse response.
20. The inverse $\mathcal{Z}$-transform of transfer function is the impulse response of the system.
21. The zero-input response $y_{zi}(n)$ is mainly due to initial output (or initial stored energy) in the system.
22. The zero-state response $y_{zs}(n)$ is the response of the system due to input signal and with zero initial output.
23. The total response y(n) is the response of the system due to input signal and initial output (or initial stored energy).
24. The convolution operation is performed to find the response y(n) of an LTI discrete time system from the input x(n) and impulse response h(n).
25. The deconvolution operation is performed to extract the input x(n) of an LTI system from the response y(n) and impulse response h(n) of the system.
26. A point-s_1 on left half of s-plane (LHP), will map as a point-z_1 inside the unit circle in z-plane.
27. A point-s_1 on imaginary axis of s-plane, will map as a point-z_1 on the unit circle in z-plane.
28. A point-s_1 on the right half of s-plane (RHP), will map as a point-z_1 outside the unit circle in z-plane.
29. The mapping of s-plane to z-plane, using the transformation, $e^{sT} = z$ is not one-to-one.
30. The mapping of frequency of continuous time signal W to the frequency of discrete time signal w is many-to-one.
31. Mathematically, a discrete time system is represented by a difference equation.

32. Physically, a discrete time system is realized or implemented either as a digital hardware or as a software running on a digital hardware.
33. The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication, and delay.
34. The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithmetic used for computation.
35. The various structures proposed for IIR and FIR systems, attempt to reduce the computational complexity, errors in computation and the memory requirement of the system.
36. When a discrete time system is designed by considering all the infinite samples of the impulse response, then the system is called IIR (Infinite Impulse Response) system.
37. When a discrete time system is designed by choosing only finite samples (usually N-samples) of the impulse response, then the system is called FIR (Finite Impulse Response) system.
38. The IIR systems are recursive systems, whereas the FIR systems are nonrecursive systems.
39. The direct form-I structure of IIR system offers a direct relation between time domain and z-domain equations.
40. Since separate delays are employed for input and output samples, realizing IIR system using direct form-I structure require more memory.
41. The direct form-I and II structure realization of an N^{th} order IIR discrete time system involves M+N+1 number of multiplications and M+N number of additions.
42. The direct form-I structure realization of an N^{th} order IIR discrete time system involves M+N delays and so M+N memory locations are required to store the delayed signals.
43. In a realizable N^{th} order IIR discrete time system, the direct form-II structure realization involves N delays and so N memory locations are required to store the delayed signals.
44. In canonic structure, the number of delays will be equal to the order of the system.
45. The direct form-II structure of IIR system is canonic whereas the direct form-I structure is noncanonic.
46. In cascade realization of IIR system, the N^{th} order transfer function is divided into first and second-order sections and they are realized in direct form-I or II structure and then connected in cascade.
47. In parallel realization of IIR system, the N^{th} order transfer function is divided into first and second-order sections and they are realized in direct form-I or II structure and then connected in parallel.
48. In cascade and parallel realization of IIR systems, the number of calculations and the memory requirement depends on the realization of individual sections.
49. Direct form structure of FIR system provides a direct relation between time domain and z-domain equations.
50. The realization of an N^{th} order FIR discrete time system using direct form structure and linear phase structure involves N number of multiplications and N-1 number of additions.
51. The realization of an N^{th} order FIR discrete time system using direct form structure involves N-1 delays and so N-1 memory locations are required to store the delayed signals.
52. The condition for symmetry of impulse response of FIR system is, $h(n) = h(N-1-n)$.
53. The linear phase realization is also called realization with minimum number of multipliers.
54. In cascade realization of FIR system, the N^{th} order transfer function is divided into first and second-order sections and they are realized in direct form or linear phase structure and then connected in cascade.
55. The direct form linear phase realization structure of an N^{th} order FIR discrete time system for even values of N involves N/2 number of multiplications, and N-1 number of additions.
56. The direct form linear phase realization structure of an N^{th} order FIR discrete time system for odd values of N involves (N+1)/2 number of multiplications, and N-1 number of additions.

3.12 Short Questions and Answers

Infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C} \quad ; \text{ if, } 0 < |C| < 1$$

Q3.1 ***Find the Z-transform of $a^n\, u(n)$.***

By the definition of Z-transform,

$$Z\{a^n\, u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a\, z^{-1})^n = \frac{1}{1-az^{-1}} = \frac{1}{1-a/z} = \frac{z}{z-a}$$

Q3.2 ***Find the Z-transform of $e^{-anT}\, u(n)$.***

By the definition of Z-transform,

$$Z\{e^{-anT}\, u(n)\} = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty} (e^{-aT} z^{-1})^n = \frac{1}{1-e^{-aT}\, z^{-1}} = \frac{1}{1-e^{-aT}\,/\,z} = \frac{z}{z-e^{-aT}}$$

Q3.3 ***Find the Z-transform of x(n) defined as,***

$x(n) = b^n \quad ; \quad 0 \le n \le N-1$

$\quad = 0 \quad ; \quad$ *otherwise*

Finite geometric series sum formula,

$$\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}$$

Solution

By the definition of Z-transform,

$$Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=0}^{N-1} b^n\, z^{-n}$$

$$= \sum_{n=0}^{N-1} (b\, z^{-1})^n = \frac{1-(b\, z^{-1})^N}{1-b\, z^{-1}} = \frac{1-b^N z^{-N}}{1-b\, z^{-1}} = \frac{z^{-N}\left(z^N - b^N\right)}{z^{-1}(z-b)} = \frac{z^{-N+1}\left(z^N - b^N\right)}{z-b}$$

Q3.4 ***Find the Z-transform of $x(n) = a^{n+1}\, u(n+1)$.***

Solution

Given that, $x(n) = a^{n+1}\, u(n+1) = a^{n+1} \quad ; \quad n \ge -1$

By the definition of Z-transform,

$$Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=-1}^{+\infty} a^{n+1}\, z^{-n} = a^{n+1}\, z^{-n}\Big|_{n=-1} + \sum_{n=0}^{+\infty} a^{n+1}\, z^{-n} = a^0\, z + \sum_{n=0}^{+\infty} a^n\, a\, z^{-n}$$

$$= z + a \sum_{n=0}^{+\infty} \left(a\, z^{-1}\right)^n = z + a\, \frac{1}{1-a\, z^{-1}} = z + \frac{az}{z-a} = \frac{z(z-a)+az}{z-a} = \frac{z^2}{z-a}$$

Q3.5 ***Determine the inverse Z-transform of $X(z) = \log\,(1 + az^{-1}) \quad ; \quad |z| > |a|$***

Solution

Given that, $X(z) = \log\,(1 + az^{-1}) \quad ; \quad |z| > |a|$

Let, $x(n) = Z^{-1}\{X(z)\}$

Since ROC is exterior of a circle of radius "a", the x(n) should be a causal signal.

By differentiation property of Z-transform we get,

$$Z\{n\, x(n)\} = -z\, \frac{d}{dz}\, X(z)$$

$$= -z\, \frac{d}{dz}\left[\log\,(1 + a\, z^{-1})\right] = -z\, \frac{1}{1+a\, z^{-1}}(-a\, z^{-2}) = \frac{a\, z^{-1}}{1+a\, z^{-1}}$$

$$= \frac{a\, z^{-1}}{z^{-1}(z+a)} = \frac{a}{z+a} = a\, z^{-1}\, \frac{z}{z-(-a)}$$

If $Z\{x(n)\} = X(z)$ then by shifting property $Z\{x(n-m)\} = z^{-m}\, X(z)$

$$\therefore \quad n\, x(n) = Z^{-1}\left\{a\, z^{-1}\, \frac{z}{z-(-a)}\right\} = a(-a)^{n-1}\, u(n-1)$$

$$\therefore \quad x(n) = \frac{a}{n}\, (-a)^{n-1}\, u(n-1)$$

Q3.6 ***Determine x(0) if the Z-transform of x(n) is*** $X(z) = \dfrac{2z^2}{(z+3)(z-4)}$.

Solution

By initial value theorem of Z-transform,

$$x(0) = \underset{z\to\infty}{\mathrm{Lt}}\ X(z) = \underset{z\to\infty}{\mathrm{Lt}}\ \frac{2z^2}{(z+3)(z-4)}$$

$$= \underset{z\to\infty}{\mathrm{Lt}}\ \frac{2z^2}{z^2\left(1+\dfrac{3}{z}\right)\left(1-\dfrac{4}{z}\right)} = \underset{z\to\infty}{\mathrm{Lt}}\ \frac{2}{\left(1+\dfrac{3}{z}\right)\left(1-\dfrac{4}{z}\right)} = \frac{2}{\left(1+\dfrac{3}{\infty}\right)\left(1-\dfrac{4}{\infty}\right)} = \frac{2}{(1+0)(1-0)} = 2$$

Q3.7 ***Determine the Z-transform of x(n) = (n − 3) u(n).***

Solution

$$\boxed{\mathcal{Z}\{u(n)\} = \frac{z}{z-1}}$$

$$\boxed{\mathcal{Z}\{n\,x(n)\} = -z\frac{d}{dz}X(z)}$$

$$\boxed{d\frac{u}{v} = v\,du - u\,dv}$$

$$\mathcal{Z}\{x(n)\} = \mathcal{Z}\{(n-3)\,u(n)\} = \mathcal{Z}\{n\,u(n) - 3\,u(n)\}$$

$$= \mathcal{Z}\{n\,u(n)\} - 3\,\mathcal{Z}\{u(n)\}$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) - 3\,\frac{z}{z-1} = -z\,\frac{z-1-z}{(z-1)^2} - \frac{3z}{z-1}$$

$$= \frac{z}{(z-1)^2} - \frac{3z}{z-1} = \frac{z-3z(z-1)}{(z-1)^2} = \frac{z-3z^2+3z}{(z-1)^2} = \frac{-3z^2+4z}{(z-1)^2} = \frac{z(4-3z)}{(z-1)^2}$$

Q3.8 ***Determine the transfer function of the LTI system defined by the equation,***

$y(n) - 0.5\,y(n-1) = x(n) + 0.4\,x(n-1)$

Solution

Given that, $y(n) - 0.5\,y(n-1) = x(n) + 0.4\,x(n-1)$

On taking Z-transform we get,

$$Y(z) - 0.5\,z^{-1}Y(z) = X(z) + 0.4\,z^{-1}X(z) \quad\Rightarrow\quad Y(z)\left[1-0.5\,z^{-1}\right] = X(z)\left[1+0.4\,z^{-1}\right]$$

$\therefore$ Transfer function, $\dfrac{Y(z)}{X(z)} = \dfrac{1+0.4\,z^{-1}}{1-0.5\,z^{-1}}$

Q3.9 ***The transfer function of a system is given by, $H(z) = 1 - z^{-1}$. Find the response of the system for any input, x(n).***

Solution

Given that, $H(z) = 1 - z^{-1}$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$\therefore$ Response in z - domain, $\quad Y(z) = H(z)\,X(z) = (1-z^{-1})\,X(z) = X(z) - z^{-1}\,X(z)$

$\therefore$ Response in time domain, $\;y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\{X(z) - z^{-1}\,X(z)\} = x(n) - x(n-1)$

Q3.10 ***An LTI system is governed by equation, y(n) = −2 y(n − 2) − 0.5 y(n − 1) + 3 x(n − 1) + 5 x(n). Determine the transfer function of the system.***

Solution

Given that, $y(n) = -2\,y(n-2) - 0.5\,y(n-1) + 3\,x(n-1) + 5\,x(n)$

On taking Z-transform of above equation we get,

$$Y(z) = -2\,z^{-2}\,Y(z) - 0.5\,z^{-1}\,Y(z) + 3\,z^{-1}\,X(z) + 5\,X(z)$$

$$Y(z) + 2\,z^{-2}\,Y(z) + 0.5\,z^{-1}\,Y(z) = 3\,z^{-1}\,X(z) + 5\,X(z)$$

$$Y(z)\left[1 + 2\,z^{-2} + 0.5\,z^{-1}\right] = \left[3\,z^{-1} + 5\right] X(z)$$

$$\therefore\ \text{Transfer function, } H(z) = \frac{Y(z)}{X(z)} = \frac{3z^{-1} + 5}{1 + 0.5\,z^{-1} + 2\,z^{-2}} = \frac{5\,z^2 + 3\,z}{z^2 + 0.5\,z + 2}$$

Q3.11 ***The transfer function of an LTI system is*** $H(z) = \dfrac{z-1}{(z-2)\,(z+3)}$***. Determine the impulse response.***

Solution

$$H(z) = \frac{z-1}{(z-2)\,(z+3)} = \frac{A}{z-2} + \frac{B}{z+3}$$

$$A = \frac{z-1}{\cancel{(z-2)}\,(z+3)} \times \cancel{(z-2)}\Big|_{z=2} = \frac{z-1}{z+3}\Big|_{z=2} = \frac{2-1}{2+3} = \frac{1}{5}$$

$$B = \frac{z-1}{(z-2)\,\cancel{(z+3)}} \times \cancel{(z+3)}\Big|_{z=-3} = \frac{z-1}{z-2}\Big|_{z=-3} = \frac{-3-1}{-3-2} = \frac{-4}{-5} = \frac{4}{5}$$

$$\therefore\ H(z) = \frac{1}{5}\,\frac{1}{z-2} + \frac{4}{5}\,\frac{1}{z+3}$$

$$\boxed{\mathcal{Z}\left\{a^n\,u(n)\right\} = \frac{z}{z-a}}$$

$$\boxed{\mathcal{Z}\left\{a^{(n-1)}\,u(n-1)\right\} = z^{-1}\,\frac{z}{z-a}}$$

$$\text{Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{5}\,\frac{1}{z-2} + \frac{4}{5}\,\frac{1}{z+3}\right\}$$

$$= \mathcal{Z}^{-1}\left\{\frac{1}{5}\,z^{-1}\,\frac{z}{z-2} + \frac{4}{5}\,z^{-1}\,\frac{z}{z-(-3)}\right\}$$

$$= \frac{1}{5}\,2^{(n-1)}\,u(n-1) + \frac{4}{5}\,(-3)^{(n-1)}\,u(n-1) = \frac{1}{5}\left[2^{(n-1)} + 4\,(-3)^{(n-1)}\right] u(n-1)$$

Q3.12 ***Determine the response of LTI system governed by the equation, $y(n) - 0.5\,y(n-1) = x(n)$, for input $x(n) = 5^n\,u(n)$, and initial condition $y(-1) = 2$.***

Solution

Given that, $x(n) = 5^n u(n)$; $\therefore X(z) = \mathcal{Z}\{u(n)\} = \dfrac{z}{z-5}$

Given that, $y(n) - 0.5\,y(n-1) = x(n)$,

On taking $\mathcal{Z}$-transform of above equation we get,

$$Y(z) - 0.5\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

$$Y(z) - 0.5\left[z^{-1}Y(z) + 2\right] = \frac{z}{z-5}$$

$$Y(z) - 0.5\,z^{-1}Y(z) - 1 = \frac{z}{z-5} \quad\Rightarrow\quad Y(z)\left[1 - \frac{0.5}{z}\right] = \frac{z}{z-5} + 1 \quad\Rightarrow\quad Y(z)\left[\frac{z-0.5}{z}\right] = \frac{z+z-5}{z-5}$$

$$\therefore\quad Y(z) = \frac{z(2\,z-5)}{(z-0.5)\,(z-5)} \quad\Rightarrow\quad \frac{Y(z)}{z} = \frac{2\,z-5}{(z-0.5)\,(z-5)}$$

$$\text{Let, } \frac{Y(z)}{z} = \frac{2\,z-5}{(z-0.5)\,(z-5)} = \frac{A}{z-0.5} + \frac{B}{z-5}$$

$$A = \frac{2z-5}{\cancel{(z-0.5)}\,(z-5)} \times \cancel{(z-0.5)}\Big|_{z=0.5} = \frac{2\times 0.5-5}{0.5-5} = \frac{-4}{-4.5} = \frac{40}{45} = \frac{8}{9}$$

$$B = \frac{2z-5}{(z-0.5)\,\cancel{(z-5)}} \times \cancel{(z-5)}\Big|_{z=5} = \frac{2\times 5-5}{5-0.5} = \frac{5}{4.5} = \frac{50}{45} = \frac{10}{9}$$

$$\therefore \frac{Y(z)}{z} = \frac{8}{9}\,\frac{1}{z-0.5} + \frac{10}{9}\,\frac{1}{z-5} \qquad \Rightarrow \qquad Y(z) = \frac{8}{9}\,\frac{z}{z-0.5} + \frac{10}{9}\,\frac{z}{z-5}$$

$$\therefore \text{ Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{8}{9}\,\frac{z}{z-0.5} + \frac{10}{9}\,\frac{z}{z-5}\right\}$$

$$= \frac{8}{9}\,0.5^n\,u(n) + \frac{10}{9}\,5^n\,u(n) = \left[\frac{8}{9}\,0.5^n + \frac{10}{9}\,5^n\right]u(n)$$

$$\boxed{\mathcal{Z}\left\{a^n\,u(n)\right\} = \frac{z}{z-a}}$$

Q3.13 ***A signal $x(t) = a^t$ is sampled at a frequency of $1/T$ Hz in the range $-$¥ $< t < 0$. Find the Z-transform of the sampled version of the signal.***

Solution

Given that, $x(t) = a^t$; $-$¥ $< t < 0$

The sampled version of the signal $x(nT)$ is given by, $x(nT) = a^{nT}$; $-$¥ $< nT < 0$

Now the Z - transform of $x(nT)$ is,

$$\mathcal{Z}\{x(nT)\} = \sum_{n=-\infty}^{+\infty} x(nT)\,z^{-n} = \sum_{n=-\infty}^{0} a^{nT}\,z^{-n} = \sum_{n=0}^{\infty} a^{-nT}\,z^{n}$$

$$= \sum_{n=0}^{\infty}\left(a^{-T}z\right)^n = \frac{1}{1-a^{-T}z} = \frac{1}{1-z/a^{T}} = \frac{a^T}{a^T - z}$$

Q3.14 ***The transfer function of a system is given by, $H(z) = \dfrac{1}{1-0.5z^{-1}} + \dfrac{1}{1-2z^{-1}}$. Determine the stability and causality of the system for a) ROC : $|z| > 2$; b) ROC : $|z| < 0.5$.***

Solution

a) ROC is $|z| > 2$

When ROC is $|z| > 2$, the impulse response $h(n)$ should be right-sided signal.

$$\therefore \text{ Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{1-0.5z^{-1}} + \frac{1}{1-2z^{-1}}\right\} = \left(0.5^n + 2^n\right)u(n)$$

1. The ROC does not include unit circle. Hence the system is unstable.
2. The impulse response is right-sided signal. Hence the system is causal.

b) ROC is $|z| < 0.5$

When ROC is $|z| < 0.5$, the impulse response $h(n)$ should be left-sided signal.

$$\therefore \text{ Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{1-0.5z^{-1}} + \frac{1}{1-2z^{-1}}\right\} = \left(-0.5^n - 2^n\right)u(-n-1)$$

1. The ROC does not include unit circle. Hence the system is unstable.
2. The impulse response is left-sided sequence. Hence the system is anticausal.

Q3.15 ***Determine the stability and causality of the system described by the transfer function,***

$$H(z) = \frac{1}{1-0.25\,z^{-1}} + \frac{1}{1-2\,z^{-1}} \quad \textit{for ROC : } 0.25 < |z| < 2.$$

Solution

Given that, ROC is $0.25 < |z| < 2$

When ROC is $0.25 < |z| < 2$, the impulse response h(n) is two-sided signal. Since $|z| > 0.25$, the term with pole $z = 0.25$ corresponds to right-sided signal. Since $|z| < 2$, the term with pole $z = 2$ corresponds to left-sided signal.

$$\therefore \text{ Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{1-0.25\,z^{-1}} + \frac{1}{1-2\,z^{-1}}\right\} = 0.25^n\,u(n) - 2^n\,u(-n-1)$$

1. The ROC includes the unit circle. Hence the system is stable.
2. The impulse response is two-sided noncausal signal. Hence the system is noncausal.

Q3.16 ***Using $\mathcal{Z}$-transform, determine the response of the LTI system with impulse response, $h(n) = \{1,\ \ -1,\ \ 1\}$, for an input $x(n) = \{-2,\ \ 3,\ \ 1\}$.***

Solution

Given that, $x(n) = \{-2,\ \ 3,\ \ 1\}$

$$\therefore X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\,z^{-n} = \sum_{n=0}^{2} x(n)\,z^{-n} = x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} = -2 + 3\,z^{-1} + z^{-2}$$

Given that, $h(n) = \{1,\ \ -1,\ \ 1\}$

$$\therefore H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\,z^{-n} = \sum_{n=0}^{2} h(n)\,z^{-n} = h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} = 1 - z^{-1} + z^{-2}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore Y(z) = X(z)\,H(z) = \left(-2 + 3\,z^{-1} + z^{-2}\right) \times \left(1 - z^{-1} + z^{-2}\right)$$

$$= -2 + 2\,z^{-1} - 2\,z^{-2} + 3\,z^{-1} - 3\,z^{-2} + 3\,z^{-3} + z^{-2} - z^{-3} + z^{-4}$$

$$= -2 + 5\,z^{-1} - 4\,z^{-2} + 2\,z^{-3} + z^{-4} \qquad .....(1)$$

By definition of $\mathcal{Z}$ - transform,

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\,z^{-n}$$

On expanding the above summation we get,

$$Y(z) = ...+ y(0) + y(1)\,z^{-1} + y(2)\,z^{-2} + y(3)\,z^{-3} + y(4)\,z^{-4} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$$y(0) = -2 \ ; \quad y(1) = 5 \ ; \quad y(2) = -4 \ ; \quad y(3) = 2 \ ; \quad y(4) = 1$$

$\therefore$ Response, $y(n) = \{-2,\ \ 5,\ \ -4,\ \ 2,\ \ 1\}$

Q3.17 ***Using $\mathcal{Z}$-transform, perform deconvolution of response $y(n) = \{-2,\ \ 5,\ \ -4,\ \ 2,\ \ 1\}$ and impulse response $h(n) = \{1,\ \ -1,\ \ 1\}$, to extract the input x(n).***

Solution

Given that, $y(n) = \{-2,\ \ 5,\ \ -4,\ \ 2,\ \ 1\}$

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\,z^{-n} = \sum_{n=0}^{4} y(n)\,z^{-n}$$

$$= y(0) + y(1)\,z^{-1} + y(2)\,z^{-2} + y(3)\,z^{-3} + y(4)\,z^{-4} = -2 + 5\,z^{-1} - 4z^{-2} + 2\,z^{-3} + z^{-4}$$

Given that, $h(n) = \{1, \ -1, \ 1\}$

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n} = \sum_{n=0}^{2} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} = 1 - z^{-1} + z^{-2}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore\ X(z) = \frac{Y(z)}{H(z)} = \frac{-2 + 5z^{-1} - 4z^{-2} + 2z^{-3} + z^{-4}}{1 - z^{-1} + z^{-2}}$$

$$= -2 + 3z^{-1} + z^{-2} \qquad .....(1)$$

$$\begin{array}{r|l}
 & -2 + 3z^{-1} + z^{-2} \\
\hline
1 - z^{-1} + z^{-2} & -2 + 5z^{-1} - 4z^{-2} + 2z^{-3} + z^{-4} \\
 & \underset{(+)}{-}2 \underset{(-)}{+} 2z^{-1} \underset{(+)}{-} 2z^{-2} \\
\hline
 & 3z^{-1} - 2z^{-2} + 2z^{-3} \\
 & \underset{(-)}{}3z^{-1} \underset{(+)}{-} 3z^{-2} \underset{(-)}{+} 3z^{-3} \\
\hline
 & z^{-2} - z^{-3} + z^{-4} \\
 & \underset{(-)}{}z^{-2} \underset{(+)}{-} z^{-3} \underset{(-)}{+} z^{-4} \\
\hline
 & 0
\end{array}$$

By definition of $\mathcal{Z}$ - transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

On expanding the above summation we get,

$$X(z) = ...+ x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + ...$$

$$.....(2)$$

On comparing equations (1) and (2) we get,

$$x(0) = -2 \quad ; \quad x(1) = 3 \quad ; \quad x(2) = 1$$

$\therefore$ Input, $x(n) = \{-2, \ 3, \ 1\}$

Q3.18 ***In an LTI system the impulse response $h(n) = C^n$ for $n \le 0$. Determine the range of values of C, for which the system is stable.***

Solution

Given that, $h(n) = C^n$ for $n \le 0$.

$$\therefore\ \sum_{n=-\infty}^{+\infty} h(n) = \sum_{n=-\infty}^{0} C^n = \sum_{n=0}^{\infty} C^{-n} = \sum_{n=0}^{+\infty} (C^{-1})^n$$

If, $0 < \left|C^{-1}\right| < 1$, then $\displaystyle\sum_{n=0}^{+\infty} (C^{-1})^n = \frac{1}{1 - C^{-1}}$

If, $\left|C^{-1}\right| > 1$, then $\displaystyle\sum_{n=0}^{+\infty} (C^{-1})^n = \infty$

$\therefore$ For stability, $\left|C^{-1}\right| < 1 \quad \Rightarrow \quad \dfrac{1}{C} < 1 \quad \Rightarrow \quad C > 1$

Q3.19 ***Using Z-transform, determine the response of the LTI system with impulse response $h(n) = 0.4^n\, u(n)$, for an input $x(n) = 0.2^n\, u(n)$.***

Solution

Given that, $x(n) = 0.2^n\, u(n)$.

$$\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{0.2^n\, u(n)\} = \frac{z}{z - 0.2}$$

Given that, $h(n) = 0.4^n\, u(n)$

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\{0.4^n\, u(n)\} = \frac{z}{z - 0.4}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore Y(z) = X(z)\,H(z) = \frac{z}{z-0.2} \times \frac{z}{z-0.4} = \frac{z^2}{(z-0.2)\,(z-0.4)}$$

Let, $\dfrac{Y(z)}{z} = \dfrac{z}{(z-0.2)\,(z-0.4)} = \dfrac{A}{z-0.2} + \dfrac{B}{z-0.4}$

$$A = \frac{z}{\cancel{(z-0.2)}\,(z-0.4)} \times \cancel{(z-0.2)}\Big|_{z=0.2} = \frac{0.2}{0.2-0.4} = \frac{0.2}{-0.2} = -1$$

$$B = \frac{z}{(z-0.2)\,\cancel{(z-0.4)}} \times \cancel{(z-0.4)}\Big|_{z=0.4} = \frac{0.4}{0.4-0.2} = \frac{0.4}{0.2} = 2$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{-1}{z-0.2} + \frac{2}{z-0.4} \qquad \Rightarrow \qquad Y(z) = -\frac{z}{z-0.2} + 2\,\frac{z}{z-0.4}$$

Response, $y(n) = Z^{-1}\{Y(z)\} = Z^{-1}\left\{-\dfrac{z}{z-0.2} + 2\,\dfrac{z}{z-0.4}\right\}$

$$= -(0.2)^n\ u(n) + 2\ (0.4)^n\ u(n) = \left[2\ (0.4)^n - (0.2)^n\right] u(n)$$

Q3.20 ***Using Z-transform perform deconvolution of response, $y(n) = 2\ (0.4)^n\ u(n) - (0.2)^n\ u(n)$ and impulse response, $h(n) = 0.4^n\ u(n)$, to extract the input $x(n)$.***

Solution

Given that, $y(n) = 2\ (0.4)^n\ u(n) - (0.2)^n\ u(n)$

$$\therefore \quad Y(z) = Z\{y(n)\} = Z\left\{2\ (0.4)^n u(n) - (0.2)^n\ u(n)\right\}$$

$$= \frac{2z}{z-0.4} - \frac{z}{z-0.2} = \frac{2z(z-0.2) - z(z-0.4)}{(z-0.4)\,(z-0.2)} = \frac{2z^2 - 0.4z - z^2 + 0.4z}{(z-0.4)\,(z-0.2)} = \frac{z^2}{(z-0.4)\,(z-0.2)}$$

Given that, $h(n) = 0.4^n\ u(n)$

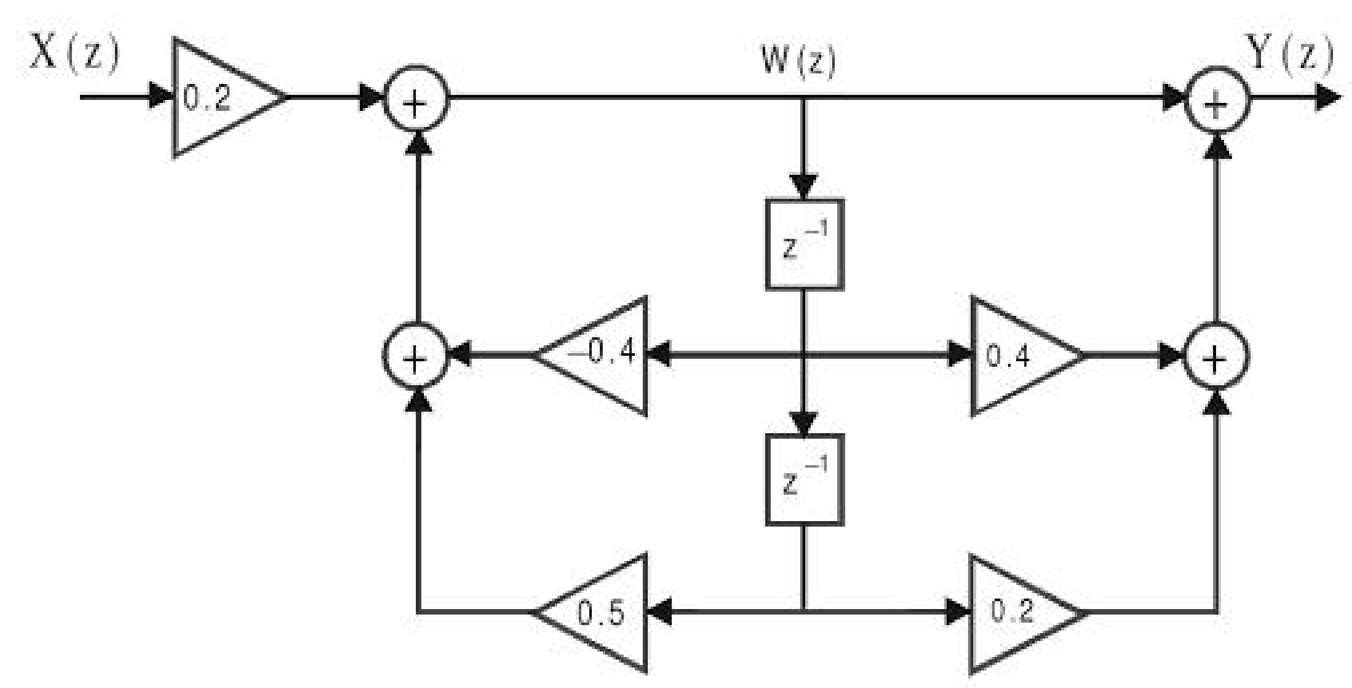
$$\therefore \quad H(z) = Z\{h(n)\} = Z\left\{0.4^n u(n)\right\} = \frac{z}{z-0.4}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore \quad X(z) = \frac{Y(z)}{H(z)} = Y(z) \times \frac{1}{H(z)} = \frac{z^2}{\cancel{(z-0.4)}\,(z-0.2)} \times \frac{\cancel{z-0.4}}{z} = \frac{z}{z-0.2}$$

$$\therefore \quad \text{Input, } x(n) = Z^{-1}\{X(z)\} = Z^{-1}\left\{\frac{z}{z-0.2}\right\} = 0.2^n\ u(n)$$

Q3.21 ***Obtain the transfer function for the following structure.***



Solution

The following z-domain equations can be obtained from the given direct form-II structure.

$$W(z) = -0.4\,z^{-1}W(z) + 0.5\,z^{-2}W(z) + 0.2\,X(z)$$

$$\therefore W(z) + 0.4\,z^{-1}W(z) - 0.5\,z^{-2}W(z) = 0.2\,X(z) \quad \Rightarrow \quad \frac{W(z)}{X(z)} = \frac{0.2}{1 + 0.4\,z^{-1} - 0.5\,z^{-2}}$$

$$Y(z) = W(z) + 0.4\,z^{-1}W(z) + 0.2\,z^{-2}W(z) \quad \Rightarrow \quad \frac{Y(z)}{W(z)} = 1 + 0.4\,z^{-1} + 0.2\,z^{-2}$$

The given direct form-II digital network can be realized by the transfer function,

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.2\,(1 + 0.4\,z^{-1} + 0.2\,z^{-2})}{1 + 0.4\,z^{-1} - 0.5\,z^{-2}}$$

Q3.22 ***Realize the following FIR system with minimum number of multipliers.***

$h(n) = \{ -0.5,\ 0.8,\ -0.5 \}$

Solution

Given that, $h(n) = \{-0.5,\ 0.8,\ -0.5\}$

On taking Z- transform,

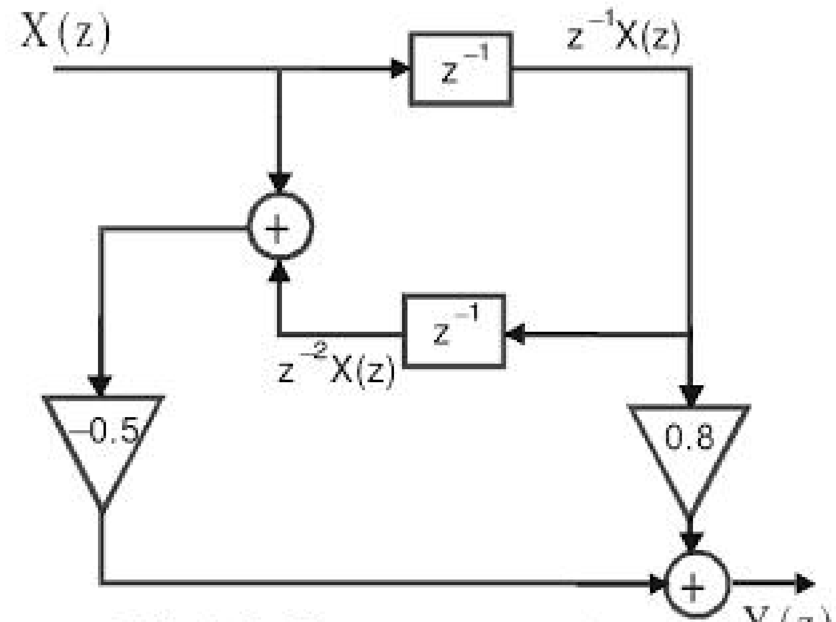
$$H(z) = \sum_{n=0}^{\infty} h(n)\,z^{-n} = \sum_{n=0}^{2} h(n)\,z^{-n}$$

$$= h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} = -0.5 + 0.8\,z^{-1} - 0.5\,z^{-2}$$

Let, $H(z) = \dfrac{Y(z)}{X(z)} = -0.5 + 0.8\,z^{-1} - 0.5\,z^{-2}$

$$\therefore Y(z) = -0.5\,X(z) + 0.8\,z^{-1}\,X(z) - 0.5\,z^{-2}\,X(z)$$

$$= -0.5\,[\,X(z) + z^{-2}\,X(z)\,] + 0.8\,z^{-1}\,X(z)$$



FigQ3.22 : Linear phase realization.

The linear phase structure is drawn using the above equation as shown in fig Q3.22.

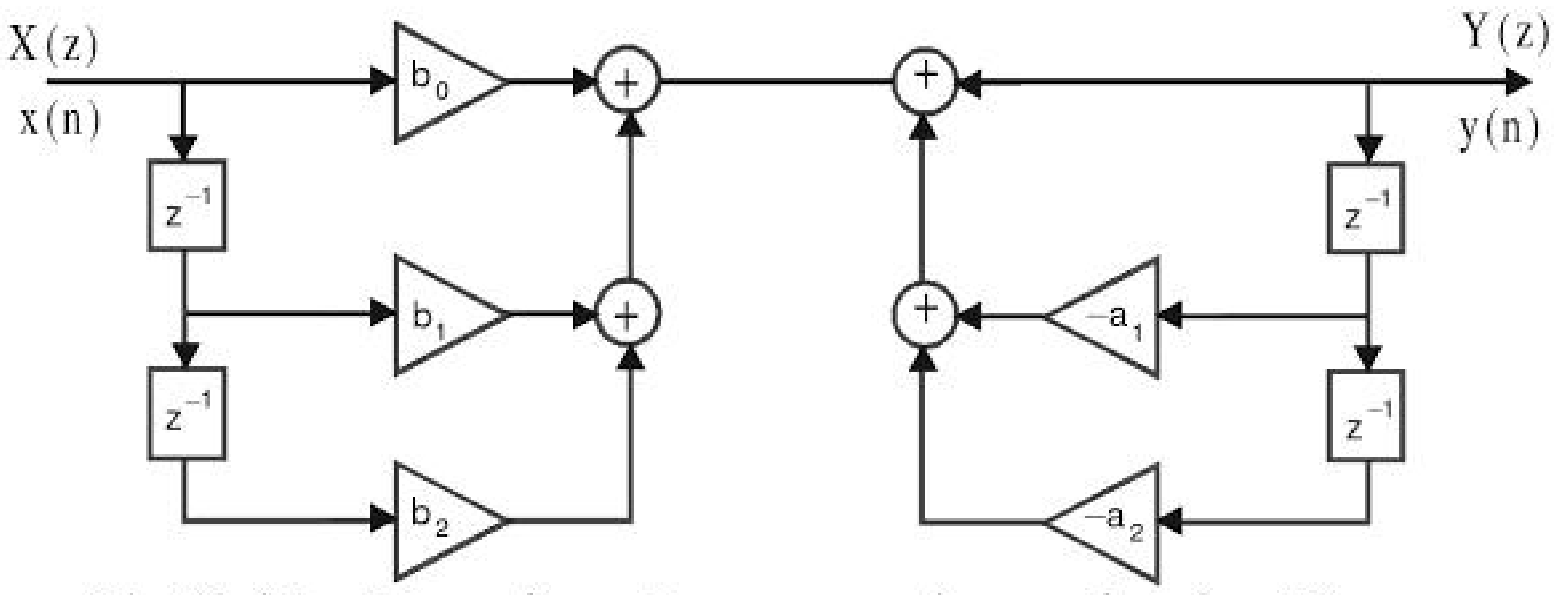
Q3.23 ***The transfer function of an IIR system has 'Z' number of zeros and 'P' number of poles. How many number of additions, multiplications and memory locations are required to realize the system in direct form-I and direct form-II.***

The realization of IIR system with Z zeros and P poles in direct form-I and II structure, involves Z+P number of additions and Z+P+1 number of multiplications. The direct form-I structure requires Z+P memory locations whereas the direct form-II structure requires only P number of memory locations.

Q3.24 ***What are the factors that influence the choice of structure for realization of an LTI system?***

The factors that influence the choice of realization structure are computational complexity, memory requirements, finite word length effects, parallel processing and pipelining of computations.

Q3.25 ***Draw the direct form-I structure of second-order IIR system with equal number of poles and zeros.***



FigQ3.25 : Direct form-I structure of second-order IIR system.

Q3.26 ***An LTI system is described by the difference equation, $y(n) = a_1 y(n-1) + x(n) + b_1 x(n-1)$. Realize it in direct form-I structure and convert to direct form-II structure.***

Solution

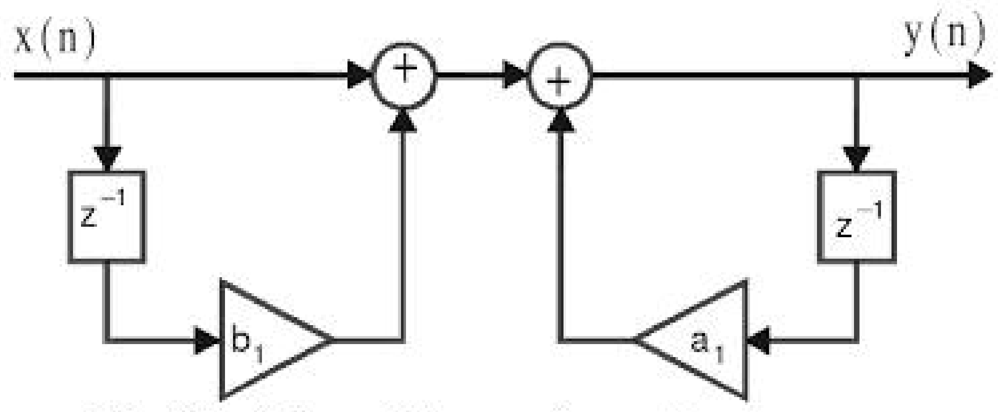
Given that, $y(n) = a_1 y(n-1) + x(n) + b_1 x(n-1)$.

Using the given equation the direct form-I structure is drawn as shown in fig Q3.26a.

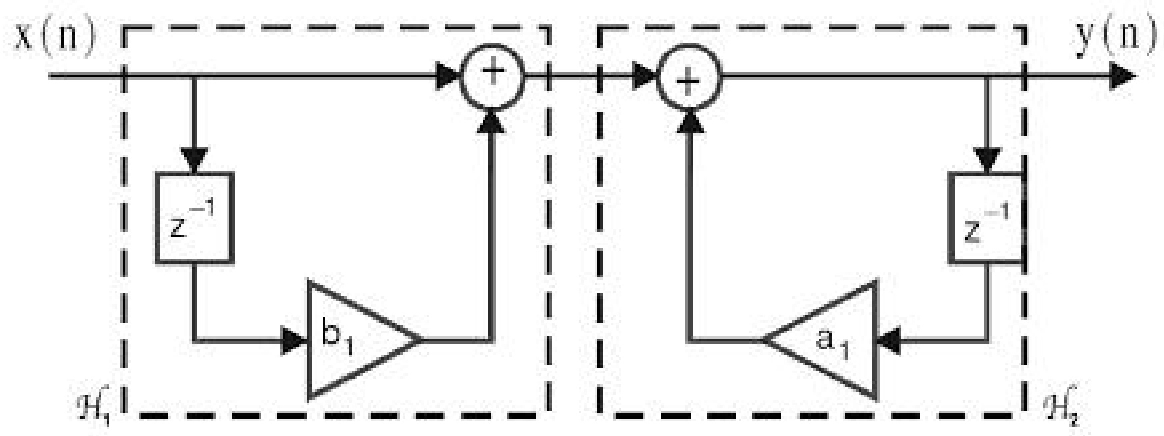
Direct form-I structure can be considered as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig Q3.26b.

By linearity property, order of cascading can be changed as shown in fig Q3.26c.

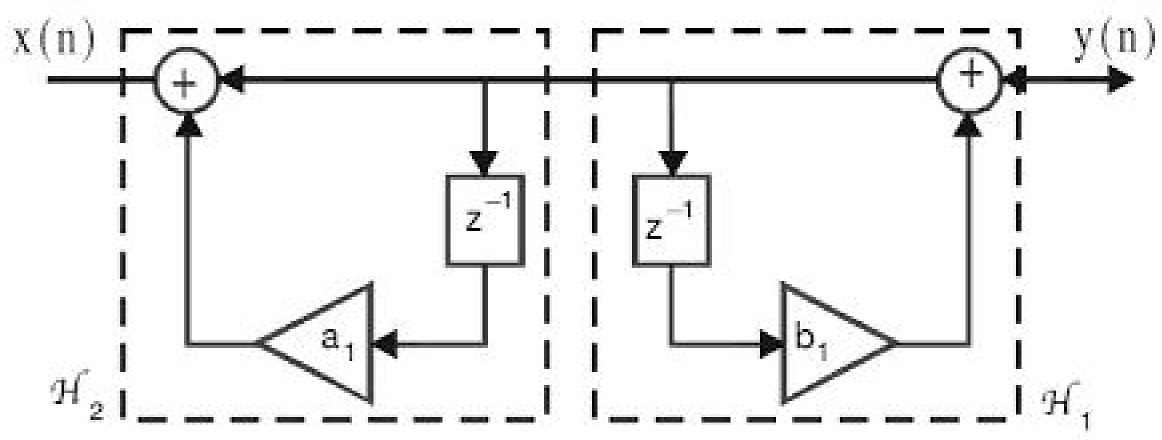
In fig Q3.26c, we can observe that the input to the delay in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of delays will be same. Hence the delays can be combined to get direct form-II structure as shown in fig Q3.26d.



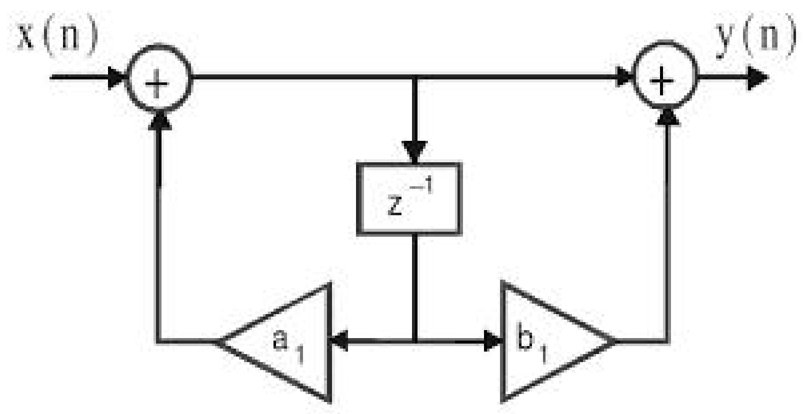
FigQ3.26a : Direct form-I structure.



FigQ3.26b : Direct form-I structure as cascade of two systems .



FigQ3.26c : Direct form-I structure after interchanging the order of cascading.



FigQ3.26d : Direct form-II structure.

Q3.27 ***What is the advantage in cascade and parallel realization of IIR systems ?***

In digital implementation of LTI system the coefficients of the difference equation governing the system are quantized. While quantizing the coefficients the value of poles may change. This will end up in a frequency response different to that of desired frequency response.

These effects can be avoided or minimized, if the LTI system is realized in cascade or parallel structure. [i.e, The sensitivity of frequency response characteristics to quantization of the coefficients is minimized]

Q3.28 ***Compare the direct form-I and II structures of an IIR systems, with M zeros and N poles.***

Direct form-I	*Direct form-II*
1. Separate delay for input and output.	1. Same delay for input and output.
2. M + N + 1 multiplications are involved.	2. M + N + 1 multiplications are involved.
3. M + N additions are involved.	3. M + N additions are involved.
4. M + N delays are involved.	4. N delays are involved.
5. M + N memory locations are required.	5. N memory locations are required.
6. Noncanonical structure.	6. Canonical structure.

Program 3.4

Write a MATLAB program to perform convolution of signals, $x_1(n)= (0.4)^n u(n)$ and $x_2(n)=(0.5)^n u(n)$, using Z-transform, and then to perform deconvolution using the result of convolution to extract $x_1(n)$ and $x_2(n)$.

```
%*** Program to perform convolution and deconvolution using z-transform

clear all;
syms n z
x1n=0.4^n;
x2n=0.5^n;

X1z=ztrans(x1n);
X2z=ztrans(x2n);
X3z=X1z*X2z;                    %product of z-transform of inputs
con12=iztrans(X3z);
disp('Convolution of x1(n) and x2(n) is');
simplify(con12)                 % convolution output

decon X1z=X3z/X1z;
decon x1n=iztrans(decon X1z);
disp('The signal x1(n) obtained by deconvolution is');
simplify(decon x1n)

decon X2z=X3z/X2z;
decon x2n=iztrans(decon X2z);
disp('The signal x2(n) obtained by deconvolution is');
simplify(decon x2n)
```

OUTPUT

```
 Convolution of x1(n) and x2(n) is
       ans =
             5*2^(-n)-4*2^n*5^(-n)
 The signal x1(n) obtained by deconvolution is
       ans =
            2^(-n)
 The signal x2(n) obtained by deconvolution is
       ans =
            2^n*5^(-n)
```

Program 3.5

Write a MATLAB program to find residues and poles of z-domain signal, $(3z^2+2z+1)/(z^2-3z+2)$

```
%*** Program to find partial fraction expansion of rational
% function of z

clear all
H=tf('z');
Ts=0.1;

b=[3 2 1 ];          %Numerator coefficients
a=[1 -3 2];          %Denominator coefficients

disp('The given transfer function is,');
H=tf([b], [a],Ts)

disp('The residues, poles and direct terms of given TF are,');
disp('(r - residue ; p - poles ; k - direct terms)');
[r,p,k]=residue(b,a)

disp('The num. and den. coefficients extracted from r,p,k,');
[b,a]=residue(r,p,k)
```

OUTPUT

```
The given transfer function is,
       Transfer function:
                         3 z^2 + 2 z + 1
                           ---------------
                          z^2 - 3 z + 2
Sampling time: 0.1
The residues, poles and direct terms of given TF are,
(r - residue ; p - poles ; k - direct terms)
       r =
           17
          -6
       p =
           2
           1
       k =
           3
The num. and den. coefficients extracted from r,p,k are,
       b =
              3      2      1
       a =
              1     -3      2
```

Program 3.6

Write a MATLAB program to find poles and zeros of z-domain signal, $(z^2+0.8z+0.8)/(z^2+0.49)$, and sketch the pole zero plot.

```
% Program to determine poles and zeros of rational function of z and
% to plot the poles and zeros in z-plane

clear all
syms z

num coeff=[1 0.8 0.8];             %find the factors of z^2+0.8z+0.8
disp('Roots of numerator polynomial z^2+0.8z+0.8 are zeros.');
zeros=roots(num coeff)

den coeff=[1 0 0.49];              %find the factors of z^2+0.49
disp('Roots of denominator polynomial z^2+0.49 are poles.');
poles=roots(den coeff)

H=tf('z');
Ts=0.1;

H=tf([num coeff],[den coeff],Ts);
zgrid on;
pzmap(H);                          %Pole-zero plot
```

OUTPUT

```
Roots of numerator polynomial z^2+0.8z+0.8 are zeros.
       zeros =
              -0.4000 + 0.8000i
              -0.4000 - 0.8000i

Roots of denominator polynomial z^2+0.49 are poles.
       poles =
              0 + 0.7000i
              0 - 0.7000i

The pole-zero plot is shown in fig P3.6.
```

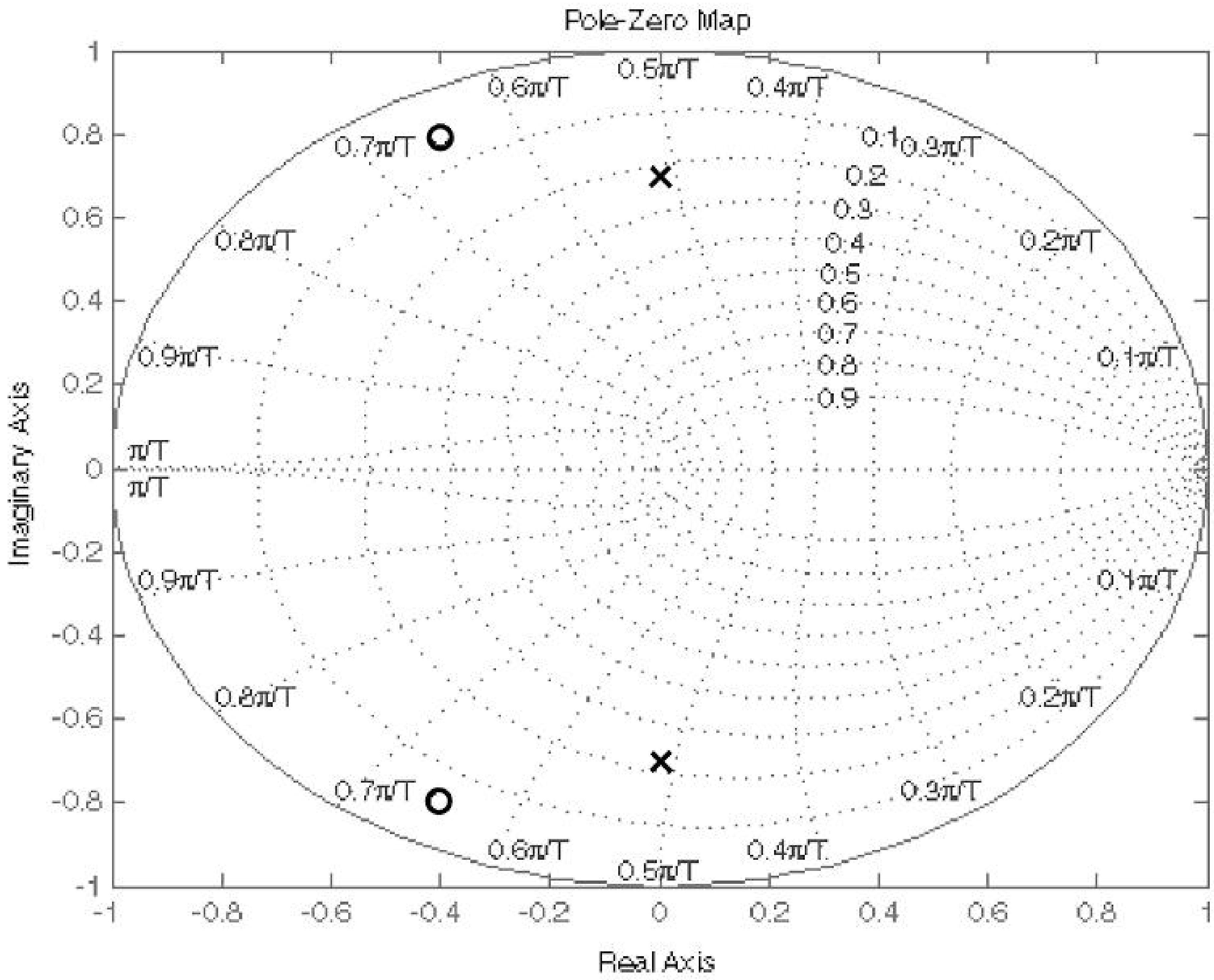


Fig P3.6 : Pole-Zero plot of program 3.6.

Program 3.7

Write a MATLAB program to compute and sketch the impulse response of discrete time system governed by transfer function, $H(z)=1/(1-0.8z^{-1}+0.16z^{2})$.

```
%******* Program to find impulse response of a discrete time system
clear all
syms z n

H=1/(1-0.8*(z^(-1))+0.16*(z^(-2)));
disp('Impulse response h(n) is');
h=iztrans(H);          %compute impulse response
simplify(h)

N=15;
b=[0 0 1];             %numerator coefficients
a=[1 -0.8 0.16];       %denominator coefficients
[H,n]=impz(b,a,N);     %compute N samples of impulse response

stem(n,H);             %sketch impulse response
xlabel('n');
ylabel('h(n)');
```

OUTPUT

```
Impulse response h(n) is
      ans =
            2^n*5^(-n)+2^n*5^(-n)*n
```

The sketch of impulse response is shown in fig P3.3.

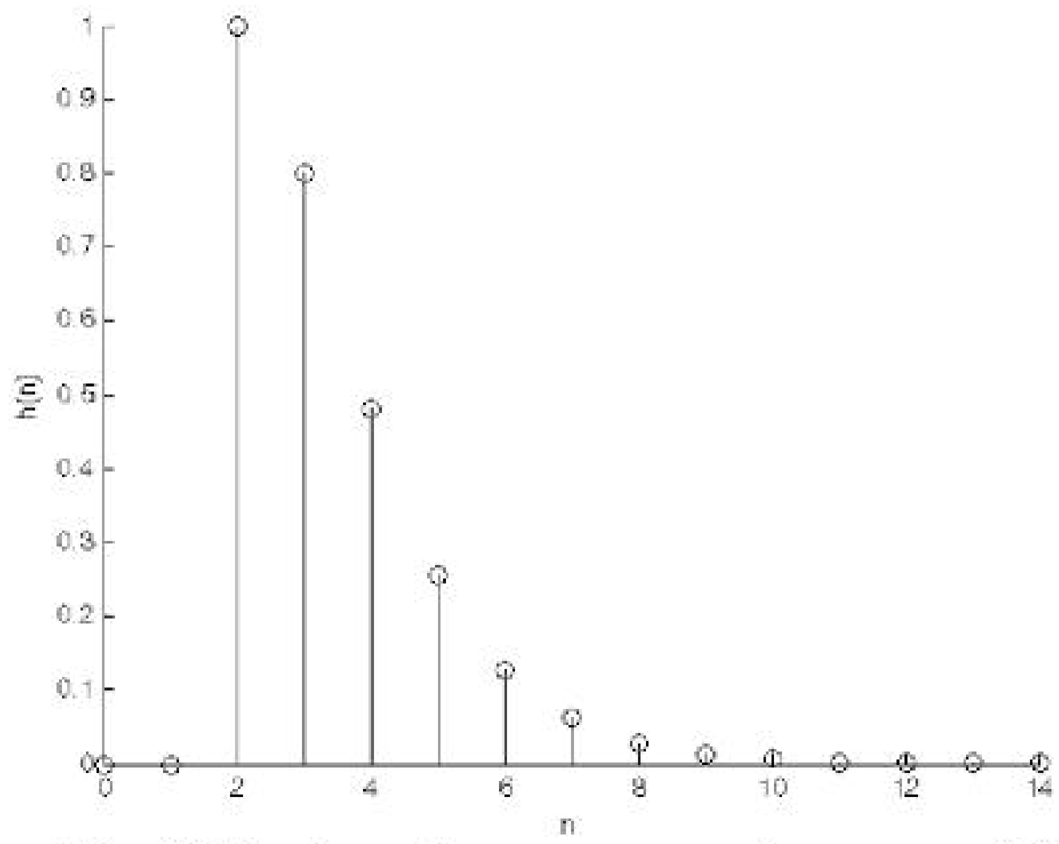


Fig P3.7 : Impulse response of program 3.7.

3.14 Exercises

I. Fill in the blanks with appropriate words

1. The ———— of X(z) is the set of all values of z, for which X(z) attains a finite value.
2. The transformation ——— maps the s-plane into z-plane.
3. The ——— of s-plane can be mapped into the ——— of the unit circle in z-plane.
4. The ratio of Ƶ-transform of output to Ƶ-transform of input is called _______ of the system.
5. In the mapping $z = e^{sT}$, the _______ poles of s-plane are mapped into _______ of unit circle in z-plane.
6. In impulse invariant mapping the _______ poles of s-plane are mapped into _______ of unit circle in z-plane.
7. In impulse invariant mapping the poles on the imaginary axis in s-plane are mapped on the _______ in z-plane.
8. In _______ transformation any strip of width 2p/T in s-plane is mapped into the entire z-plane.
9. The phenomena of high frequency components acquiring the identity of low frequency components is called _______.
10. For a causal LTI discrete time system the ROC should be _______ the circle of radius whose value corresponds to pole with _______ magnitude.
11. If X(z) is rational, then the ROC does not include _______ of X(z).
12. The sequences multiplied by u(–n) are _______ and defined for _______.
13. The inverse Ƶ-transform of transfer function is _______ of the system.
14. If Ƶ-transform of x(n) is X(z), then Ƶ-transform of x*(n) is _______.
15. The Ƶ-transform of a shifted signal, shifted by 'q' units of time is obtained by _______ to Ƶ-transform of unshifted signal.
16. In IIR systems, the ___________ structure will give direct relation between time domain and z-domain.
17. When number of delays is equal to order of the system, the structure is called ___________.
18. The direct form realization of IIR system with M zeros and N poles involves ___________ multiplications.
19. The direct form-II realization of N^{th} order IIR system requires ___________ delays and memory locations.
20. The direct form realization of N^{th} order FIR system involves ___________ additions.
21. ___________ realization is called realization with minimum number of multipliers

Answers

1. region of convergence	8. impulse invariant	15. multiplying z^q
2. s = (1/T) *ln* z	9. aliasing	16. direct form-I
3. left half, interior	10. outside, largest	17. canonic structure
4. transfer function	11. poles	18. M + N + 1
5. left half, interior	12. anticausal sequences, n £ 0	19. N
6. right half, exterior	13. impulse response	20. N – 1
7. unit circle	14. X*(z*)	21. linear phase

II. State whether the following statements are True/False

1. The Z-transform exists only for those values of z for which X(z) is finite.
2. When the input is an impulse sampled signal, the z-domain transfer function can be directly obtained from s-domain transfer function.
3. The jW axis in s-plane maps into the unit circle of z-plane in the clockwise direction.
4. The left half of s-plane maps into the interior of the unit circle in z-plane.
5. The system is unstable if all the poles of transfer function lies inside the unit circle in z-plane.
6. The Z-transform of impulse response gives the transfer function of LTI system.
7. If X(z) and H(z) are Z-transform of input and impulse response respectively, then the response of LTI system is given by inverse Z-transform of the product X(z) H(z).
8. For a stable LTI continuous time system the poles should lie on the right half of s-plane.
9. For a stable LTI discrete time system the poles should lie on the unit circle.
10. If $\mathcal{Z}\{x(n)\} = X(z)$, then $\mathcal{Z}\{n^m x(n)\} = -z\left(\frac{d}{dz}\right)^m X(z)$.
11. The direct form-I structure of IIR system employs same delay for input and output samples.
12. In direct form-II realization of IIR system, N memory locations are required to store delayed signals.
13. In parallel or cascade realization, the memory requirement depends on realization of individual sections.
14. Scaling multipliers has to be provided between individual sections of cascade structure.
15. The linear phase realization of N^{th} order FIR system for odd values of N involves N/2 multiplications.
16. For linear phase realization of FIR system, the impulse response should be symmetric.

Answers

1. True	5. False	9. False	13. True
2. True	6. True	10. False	14. True
3. False	7. True	11. False	15. False
4. True	8. False	12. True	16. True

III. Choose the right answer for the following questions

1. ***The impulse response,*** $h(n) = 1 \quad ; \ n = 0$

$$= -(1-b)\, b^{n-1} \ ; \ n \geq 1, \text{ can be represented as,}$$

a) $\delta(n)$ b) $u(n) - (1-b)\, b^{n-1}\, u(n-1)$

c) $\delta(n) - (1-b)\, b^{n-1}\, u(n-1)$ d) $u(n) - (1-b)\, b^{n-1}\, u(n)$

2. ***The Z-transform of*** $a^{-n}\, u(-n-1)$ ***is,***

a) $\dfrac{-z}{z-1/a}$ b) $\dfrac{z}{z-1/a}$ c) $\dfrac{z}{z-a}$ d) $\dfrac{-z}{z-a}$

3. ***The ROC of the sequence*** $x(n) = u(-n)$ ***is,***

a) $|z| > 1$ b) $|z| < 1$ c) no ROC d) $-1 < |z| < 1$

4. ***The inverse Z-transform of*** $\dfrac{3}{z-4}$, $|z| > 4$ ***is,***

a) $3(4)^n\, u(n-1)$ b) $3(4)^{n-1}\, u(n)$ c) $3(4)^{n-1}\, u(n+1)$ d) $3(4)^{n-1}\, u(n-1)$

5. ***ROC of x(n) contains,***

a) poles b) zeros c) no poles d) no zeros

6. ***The inverse Z-transform of*** $X(z) = e^{a/z}$, $|z| > 0$ ***is,***

a) $x(n) = \dfrac{-a^n}{n!}\, u(n)$ b) $x(n) = \dfrac{a^n}{n!}\, u(n)$ c) $x(n) = \dfrac{a^{n-1}}{n!}\, u(n-1)$ d) none of the above

7. ***The Z-transform of*** $x(n) = [u(n) - u(n-3)]$***, for ROC*** $|z| > 1$ ***is,***

a) $X(z) = \dfrac{z - z^{-3}}{z-1}$ b) $X(z) = \dfrac{z^{-2}}{(z-1)^2}$ c) $X(z) = \dfrac{z - 4z^{-2} + 3z^{-3}}{(z-1)^2}$ d) $X(z) = \dfrac{z - z^{-2}}{z-1}$

8. ***The system function*** $H(z) = \dfrac{z^3 - 2z^2 + z}{z^2 + 0.25z + 0.125}$ ***is,***

a) causal b) noncausal c) unstable but causal d) cannot be defined

9. ***If all the poles of the system function H(z) have magnitude smaller than 1, then the system will be,***

a) stable b) unstable c) BIBO stable d) a and c

10. ***If*** $x(n) = \{0.5, \ -0.25, \ 1\}$***, then Z-transform of the signal is,***

a) $\dfrac{z^2}{0.5z^2 - 0.25z + 1}$ b) $\dfrac{z^2}{z^2 - 0.5z + 0.25}$ c) $\dfrac{0.5z^2 - 0.25z + 1}{z^2}$ d) $\dfrac{2z^2 + 4z + 1}{z^2}$

11. ***The ROC of the signal*** $x(n) = a^n$ ***for*** $-5 < n < 5$ ***is,***

a) entire z-plane b) entire z-plane except z = 0 and z = ¥

c) entire z-plane except z = 0 d) entire z-plane except z = ¥

12. ***If Z-transform of x(n) is X(z) then Z-transform of x(−n) is,***

a) $-X(z)$ b) $X(-z)$ c) $-X(z^{-1})$ d) $X(z^{-1})$

13. The inverse Z-transform of X(z) can be defined as,

a) $x(n) = \frac{1}{2\pi}\oint_c X(z)\, z^{n-1}\, dz$ b) $x(n) = \frac{1}{2j}\oint_c X(z)\, z^{n-1}\, dz$

c) $x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz$ d) $x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{-n}\, dz$

14. The Z-transform is a,

a) finite series b) infinite power series c) geometric series d) both a and c

15. If the Z-transform of x(n) is X(z), then Z-transform of $(0.5)^n x(n)$ is,

a) X(0.5 z) b) $X(0.5^{-1} z)$ c) $X(2^{-1} z)$ d) X(2z)

16. The Z-transform of correlation of the sequences x(n) and y(n) is,

a) $X^*(z)\, Y^*(z^{-1})$ b) $X(z)\, Y(z^{-1})$ c) $X(z) * Y(z)$ d) $X(z^{-1})\, Y(z^{-1})$

17. The parseval's relation states that if $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$ then $\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)$ is,

a) $\frac{1}{2\pi}\oint_c X_1(z)\; X_2^*\left(\frac{1}{z}\right) z^{-1}\, dz$ b) $\frac{1}{2\pi}\oint_c X_1(z)\; X_2\left(\frac{1}{z^*}\right) z^{-1}\, dz$

c) $\frac{1}{2\pi j}\oint_c X_1(z)\; X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$ d) $\frac{1}{2\pi}\oint_c X_1(z)\; X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$

18. For a stable LTI discrete time system poles should lie —— and unit circle should be ——— .

a) outside unit circle, included in ROC b) inside unit circle, outside of ROC
c) inside unit circle, included in ROC d) outside unit circle, outside of ROC

19. An LTI system with impulse response, $h(n) = (-a)^n u(n)$ and $-a < -1$ will be,

a) stable system b) unstable system
c) anticausal system d) neither stable nor causal

20. If X(z) has a single pole on the unit circle, on negative real axis then, x(n) is,

a) signed constant sequence b) signed decaying sequence
c) signed growing sequence d) constant sequence

21. The Z-transform of $x(n) = -na^n u(-n-1)$ is,

a) $X(z) = \frac{az}{(z-a)^2}$ b) $\frac{az(z+a)}{(z-a)^3}$ c) $X(z) = \frac{az^{-1}}{(1-az^{-1})^2}$ d) both a and c

22. The ROC for $x(n) \xleftrightarrow[Z^{-1}]{Z} X(z)$ is R_1, then ROC of $a^n x(n) \xleftrightarrow[Z^{-1}]{Z} X\left(\frac{z}{a}\right)$ is,

a) $\frac{R_1}{a}$ b) aR_1 c) R_1 d) $\frac{1}{R_1}$

23. The Z-transform of a ramp function $x(n) = n\, u(n)$ is,

a) $X(z) = \frac{z}{(z-1)^2}$; ROC is $|z| > 1$ b) $X(z) = \frac{-z}{(z-1)^2}$; ROC is $|z| > 1$

c) $X(z) = \frac{z}{(z-1)^2}$; ROC is $|z| < 1$ d) $X(z) = \frac{-z}{(z-1)^2}$; ROC is $|z| < 1$

24. *By impulse invariant transformation, if x(nT) is sampled version of x(t), then $\mathcal{Z}\{x(nT)\}$ is,*

a) $\mathcal{L}\{x(nT)\}\big|_{z=e^{sT}}$ b) $\mathcal{L}^{-1}\{x(nT)\}\big|_{z=e^{-sT}}$ c) $\mathcal{L}\{x(nT)\}\big|_{z=e^{-sT}}$ d) $\mathcal{L}^{-1}\{x(nT)\}\big|_{z=e^{sT}}$

25. *The $\mathcal{Z}$-transform of $x(n)=\left[\sin\frac{\pi}{2}n\right]u(n)$ is,*

a) $\dfrac{z}{z+1}$ b) $\dfrac{z^2}{z^2+1}$ c) $\dfrac{1}{z+1}$ d) $\dfrac{z}{z^2+1}$

26. *The factor that influence the choice of realization of structure is,*

a) memory requirements
b) computational complexity
c) parallel processing and pipelining
d) all the above

27. *The structure that uses separate delays for input and output samples is,*

a) direct form-II
b) direct form-I
c) cascade form
d) parallel form

28. *The linear phase realization structure is used to represent,*

a) FIR systems
b) IIR systems
c) both FIR and IIR systems
d) all discrete time systems

29. *The effect of quantization of coefficients on the frequency response is minimized in,*

a) cascade realization
b) parallel realization
c) direct form structure
d) both a and b

30. *The direct form-I and II structures of IIR system will be identical in,*

a) all pole system
b) all zero system
c) both a and b
d) first-order and second-order systems

31. *The condition for symmetry of impulse response of FIR system is,*

a) h(n) = h(N–1)
b) h(n) = h(N+1)
c) h(n) = h(N–n)
d) h(n) = h(N–1–n)

32. *The linear phase realization is used in FIR systems in order to minimize,*

a) multipliers b) memory c) delays d) adders

33. *Which one of the following FIR system has linear phase response?*

a) y(n) = 0.4 x(n) + 0.1 x(n–1) + 0.5 x(n–2)
b) y(n) = 0.3 x(n) + x(n–1) + 3.0 x(n–2)
c) y(n) = 0.5 x(n) + 0.7 x(n–1)
d) y(n) = 0.6x(n) + 0.6 x(n–1)

34. *The quantization error increases, when the order of the system 'N' increases in case of,*

a) direct form realization
b) cascade or parallel form realization
c) all IIR systems
d) all FIR systems

35. *The number of memory locations required to realize the system, $H(z)=\dfrac{1+z^{-2}+2z^{-3}}{1+z^{-2}+z^{-4}}$ is,*

a) 8 b) 7 c) 2 d) 10

36. Number of multipliers and adders required for direct form realization of N^{th} order FIR system are,

a) N, N+1 **b)** N, N–1 **c)** N+1, N **d)** N–1, N+1

37. The realization of linear phase FIR system for odd values of 'N' needs,

a) $\frac{N}{2}$ multipliers **b)** $\frac{N+1}{2}$ multipliers **c)** N–1 multipliers **d)** N multipliers

Answers

1. c	7. d	13. c	19. a	25. d	31. d	37. b
2. a	8. b	14. b	20. a	26. d	32. a	
3. b	9. a	15. b	21. d	27. b	33. d	
4. d	10. c	16. b	22. a	28. a	34. a	
5. c	11. b	17. c	23. a	29. d	35. b	
6. b	12. d	18. c	24. a	30. c	36. b	

IV. Answer the following questions

1. Define one-sided and two-sided Z-transform.
2. What is region of convergence (ROC)?
3. State the final value theorem with regard to Z-transform.
4. State the initial value theorem with regard to Z-transform.
5. Define Z-transform of unit step signal.
6. What are the different methods available for inverse Z-transform?
7. When the z-domain transfer function of the system can be directly obtained from s-domain transfer function?
8. Define the transfer function of an LTI system.
9. Write the transfer function of N^{th} order LTI system.
10. What is impulse invariant transformation?
11. How is a point in s-plane mapped to z-plane in impulse invariant transformation?
12. Why is an impulse invariant transformation not considered to be one-to-one?
13. Give the importance of convolution and deconvolution operations using Z-transform.
14. Give the conditions for stability of an LTI discrete time system in z-plane.
15. Explain when an LTI dicrete time system will be causal.
16. Define ROC for various finite and infinite discrete time signals.
17. Explain the shifting property of a discrete time signal defined in the range $0 < n < ¥$ with an example.
18. What are all the properties of ROC of a rational function of z?
19. State and prove the convolution property of Z-transform.
20. State and prove the linearity property of Z-transform.
21. What are the various issues that are addressed by realization structures?
22. What are the basic elements used to construct the realization structures of discrete time system?
23. List the different types of structures for realization of IIR systems.

24. Draw the direct form-I structure of an N^{th} order IIR system with equal number of poles and zeros.
25. Draw the direct form-II structure of an N^{th} order IIR system with equal number of poles and zeros.
26. Explain the conversion of direct form-I structure to direct form-II structure with an example.
27. What are the difficulties in cascade realization?
28. Explain the realization of cascade structure of an IIR system.
29. Explain the realization of parallel structure of an IIR system.
30. What are the different types of structure for realization of FIR systems?
31. Draw the direct form structure of an N^{th} order FIR system.
32. What is the necessary condition for Linear phase realization of FIR system?
33. Draw the linear phase realization structure of an Nth order FIR system when 'N' is even.
34. Draw the linear phase realization structure of an Nth order FIR system when 'N' is odd.
35. Explain the realization of cascade structure of a FIR system.

V. Solve the following problems

E3.1 ***Determine the Z-transform and their ROC of the following discrete time signals.***

a) $x(n) = \{\underset{\uparrow}{4},\ 2,\ 8,\ 5\}$ **b)** $x(n) = \{3,\ 0,\ 0,\ 4,\ 45,\ \underset{\uparrow}{1}\}$

c) $x(n) = \{2,\ 1,\ 1,\ \underset{\uparrow}{2},\ 5,\ 8,\ 2\}$ **d)** $x(n) = -0.2^n\ u(n-1)$

e) $x(n) = (0.6)^n\ u(n) + (0.7)^n\ u(-n-1)$ **f)** $x(n) = (0.9)^{|n|}$

E3.2 ***Find the one-sided Z-transform of the following discrete time signals.***

a) $x(n) = n^2\ 5^n\ u(n)$ **b)** $x(n) = n(0.5)^{n+4}$ **c)** $x(n) = (0.5)^{n-2}\ [u(n) - u(n-2)]$

E3.3 ***Find the one-sided Z-transform of the discrete signals generated by mathematically sampling the following continuous time signals.***

a) $x(t) = 4t\,e^{-0.6t}\ u(t)$ **b)** $x(t) = 2\ t^3\ u(t)$

E3.4 ***Find the time domain initial value x(0) and final value x(¥) of the following z-domain functions.***

a) $X(z) = \dfrac{0.5}{\left(1-z^{-1}\right)^2\left(1+z^{-1}\right)}$ **b)** $X(z) = \dfrac{z^3}{(z-1)\left(z^2-0.2\right)}$

E3.5 ***Determine the inverse Z-transform of the following functions using contour integral method.***

a) $X(z) = \dfrac{(2z-1)z}{(z-1)^2}$ **b)** $X(z) = \dfrac{z^2+z}{(z-2)^2}$ **c)** $X(z) = \dfrac{(1-e^{-a})z}{(z-1)\ (z-e^{-a})}$

E3.6 ***Determine the inverse Z-transform of the following functions by partial fraction method.***

a) $X(z) = \dfrac{z^2}{(z+1)\ (z+2)^2}$ **b)** $X(z) = \dfrac{2z^2-z}{z^3-5z^2+8z-4}$ **c)** $X(z) = \dfrac{z(z^2+3)}{(z^2+1)^2}$

E3.7 ***Determine the inverse Z-transform of the function,*** $X(z) = \dfrac{2-z^{-1}}{\left[1-\frac{1}{4}z^{-1}\right]\left[1-\frac{1}{3}z^{-1}\right]}$

a) ROC : $|z| > \dfrac{1}{3}$, **b)** ROC : $|z| < \dfrac{1}{4}$, **c)** ROC : $\dfrac{1}{4} < |z| < \dfrac{1}{3}$.

E3.8 ***Determine the inverse Z-transform of the following function using power series method.***

$$X(z) = \frac{z}{2z^2 - 3z + 1}$$

a) ROC : $|z| < 0.5$, **b)** ROC : $|z| > 1$

E3.9 ***Determine the inverse Z-transform for the following functions using power series method.***

a) $X(z) = \dfrac{z^2 + z}{z^2 - 2z + 1}$; ROC : $|z| > 1$ **b)** $X(z) = \dfrac{1 - \frac{1}{3} z^{-1}}{1 + \frac{1}{3} z^{-1}}$; ROC : $|z| > \dfrac{1}{3}$

E3.10 ***Determine the transfer function and impulse response for the systems described by the following equations.***

a) $y(n) + 2y(n-1) - 3y(n-2) = x(n-1)$ **b)** $y(n) - \dfrac{7}{4}y(n-1) + \dfrac{5}{8}y(n-2) = 2x(n)$

c) $y(n) = 0.2x(n) - 5x(n-1) + 0.6y(n-1) - 0.08y(n-2)$

d) $y(n) - \dfrac{3}{2}y(n-1) = x(n) + \dfrac{2}{3}x(n-1)$

E3.11 ***A discrete time LTI system is characterized by the transfer function,*** $H(z) = \dfrac{z(6z - 8)}{\left(z - \frac{1}{2}\right)(z - 3)}$.

Specify the ROC of H(z) and determine h(n) for the system to be, (i) stable, (ii) causal.

E3.12 ***Determine the unit step response of the discrete time LTI system, whose input and output relation is described by the difference equation, $y(n) + 7\,y(n-1) = x(n)$, where the initial condition is, $y(-1) = 1$.***

E3.13 ***Determine the response of discrete time LTI system governed by the following difference equation, $4\,y(n) + 5\,y(n-1) + y(n-2) = x(n)$; with initial conditions, $y(-2) = -2$ and $y(-1) = 1$, for the input $x(n) = (0.5)^n\, u(n)$.***

E3.14 ***An LTI system has the impulse response h(n) defined by $h(n) = x_1(n-1) * x_2(n)$. The Z-transform of the two signals $x_1(n)$ and $x_2(n)$ are $X_1(z) = 2 - 4z^{-1}$ and $X_2(z) = 1 + 5z^{-2}$ respectively. Determine the output of the system for the input $\delta(n-1)$.***

E3.15 ***Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation,***

$$y(n) = -\frac{3}{4}\,y(n-1) + \frac{1}{2}\,y(n-2) + \frac{1}{4}\,y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

E3.16 ***Realize the direct form-I, II structures of the IIR system represented by the transfer function,***

$$H(z) = \frac{(z+5)}{(z+0.4)(z+0.5)(z+0.6)}$$

E3.17 ***Determine the direct form-I, II, cascade and parallel realization of the following LTI system.***

$$H(z) = \frac{(z^3 - 8z^2 + 13z - 5)}{(z - 0.75)(z^2 + z - 0.25)}$$

E3.18 ***Realize the cascade and parallel structures of the system governed by the difference equation,***

$$y(n) - \frac{3}{10}\,y(n-1) - \frac{1}{10}\,y(n-2) = x(n) + \frac{1}{9}\,x(n-1)$$

E3.19 ***Draw the direct form structure of the FIR systems described by the following equations,***

a) $y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}\,x(n-2) + \frac{1}{6}\,x(n-3) + \frac{1}{8}\,x(n-4)$

b) $y(n) = 0.2\,x(n) + 0.25\,x(n-1) + 0.3x(n-2) - 0.35x(n-3)$
$\qquad - 0.4\,x(n-4) - 0.45x(n-5) - 0.5x(n-6)$

E3.20 ***Realize the following FIR systems with minimum number of multipliers.***

a) $H(z) = 0.2 + 0.4z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4}$

b) $H(z) = \left(0.3 + \frac{1}{9}\,z^{-1} + 0.3\,z^{-2}\right)\left(0.5 - \frac{1}{7}\,z^{-1} + 0.5z^{-2}\right)$

c) $y(n) = -\frac{1}{8}\,x(n) + \frac{3}{4}\,x(n-1) + \frac{3}{2}\,x(n-2) + \frac{3}{4}\,x(n-3) - \frac{1}{8}x(n-4)$

Answers

E3.1

a) $X(z) = 4 + \frac{2}{z} + \frac{8}{z^2} + \frac{5}{z^3}$
ROC is entire z - plane except at $z = 0$.

b) $X(z) = 3z^5 + 4z^2 + 45z + 1$
ROC is entire z - plane except at $z = \infty$.

c) $X(z) = 2z^3 + 1z^2 + z + 2 + 5z^{-1} + 8z^{-2} + 2z^{-3}$
ROC is entire z - plane except at $z = 0$ and $z = \infty$.

d) $X(z) = \frac{-0.2}{z - 0.2}$; ROC is exterior of the circle of radius 0.2 in z - plane.

e) $X(z) = \frac{-0.1z}{(z-0.6)\,(z-0.7)}$; ROC is $0.6 < |z| < 0.7$

f) $X(z) = \frac{-0.21z}{(z-0.9)\,(z-1.11)}$; ROC is $0.9 < |z| < 1.11$

E3.2

a) $X(z) = \frac{5z(z+5)}{(z-5)^3}$ **b)** $X(z) = \frac{0.5^5\,z}{(z-0.5)^2}$ **c)** $X(z) = \frac{4z^2 - 1}{z(z-0.5)}$

E3.3

a) $X(z) = \frac{4\,zT\,e^{-0.6T}}{(z - e^{-0.6T})^2}$ **b)** $X(z) = \frac{2T^3\,z\left(z^2 + 4z + 1\right)}{(z-1)^4}$

E3.4

a) Initial value, $x(0) = 0.5$
Final value, $x(\infty) = \infty$

b) Initial value, $x(0) = 1$
Final value, $x(\infty) = 1.25$

E3.5

a) $x(n) = \left[n+2\right]\,u(n)$ **b)** $x(n) = (n+1)\,2^n\,u(n) + n\,2^{(n-1)}\,u(n-1)$

c) $x(n) = \left(1 - e^{-an}\right)\,u(n)$

E3.6

a) $x(n) = \left[(-2)^n - (-1)^n - n(-2)^n\right]u(n)$ **b)** $x(n) = \left[1 + (1.5n - 1)2^n\right]u(n)$

c) $x(n) = \left[\left[j(-j)^n - j^n\right] + \frac{n}{2j}\left[(-j)^n - j^n\right]\right]u(n)$

E3.7 **a)** $x(n) = \left[6\left(\frac{1}{4}\right)^n - 4\left(\frac{1}{3}\right)^n\right] u(n)$

b) $x(n) = \left[-6\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{3}\right)^n\right] u(-n-1)$

c) $x(n) = 6\left(\frac{1}{4}\right)^n u(n) + 4\left(\frac{1}{3}\right)^n u(-n-1)$

E3.8 **a)** $x(n)\{.....31,\ 15,\ 7,\ 3,\ \underset{\uparrow}{1}\}$

b) $x(n) = \left\{\underset{\uparrow}{0},\ \frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \frac{15}{16},\ \frac{31}{32},......\right\}$

E3.9 **a)** $\{\underset{\uparrow}{1},\ 3,\ 5,\ 7,......\}$

b) $x(n) = \left\{\underset{\uparrow}{1},\ -\frac{2}{3},\ \frac{2}{9},\ \frac{-2}{27},\ \frac{2}{81},\ ......\right\}$

E3.10 **a)** $H(z) = \dfrac{z}{z^2 + 2z - 3}$; $h(n) = \frac{1}{4}\left[1 - (-3)^n\right] u(n)$

b) $H(z) = \dfrac{2z^2}{z^2 - \frac{7}{4}z + \frac{5}{8}}$; $h(n) = \frac{1}{3}\left[-4\left(\frac{1}{2}\right)^n + 10\left(\frac{5}{4}\right)^n\right] u(n)$

c) $H(z) = \dfrac{0.2z^2 - 5z}{z^2 - 0.6z + 0.08}$; $h(n) = \left[24.8\ (0.2)^n - 24.6\ (0.4)^n\right] u(n)$

d) $H(z) = \dfrac{1 + \frac{2}{3}z^{-1}}{1 - \frac{3}{2}z^{-1}}$; $h(n) = \left(\frac{3}{2}\right)^n u(n) + \frac{2}{3}\left(\frac{3}{2}\right)^{n-1} u(n-1)$

E3.11 **i) Stable system**

ROC : $0.5 < |z| < 3$; $h(n) = 2\left(\frac{1}{2}\right)^n u(n) - 4(3)^n u(-n-1)$

ii) Causal system

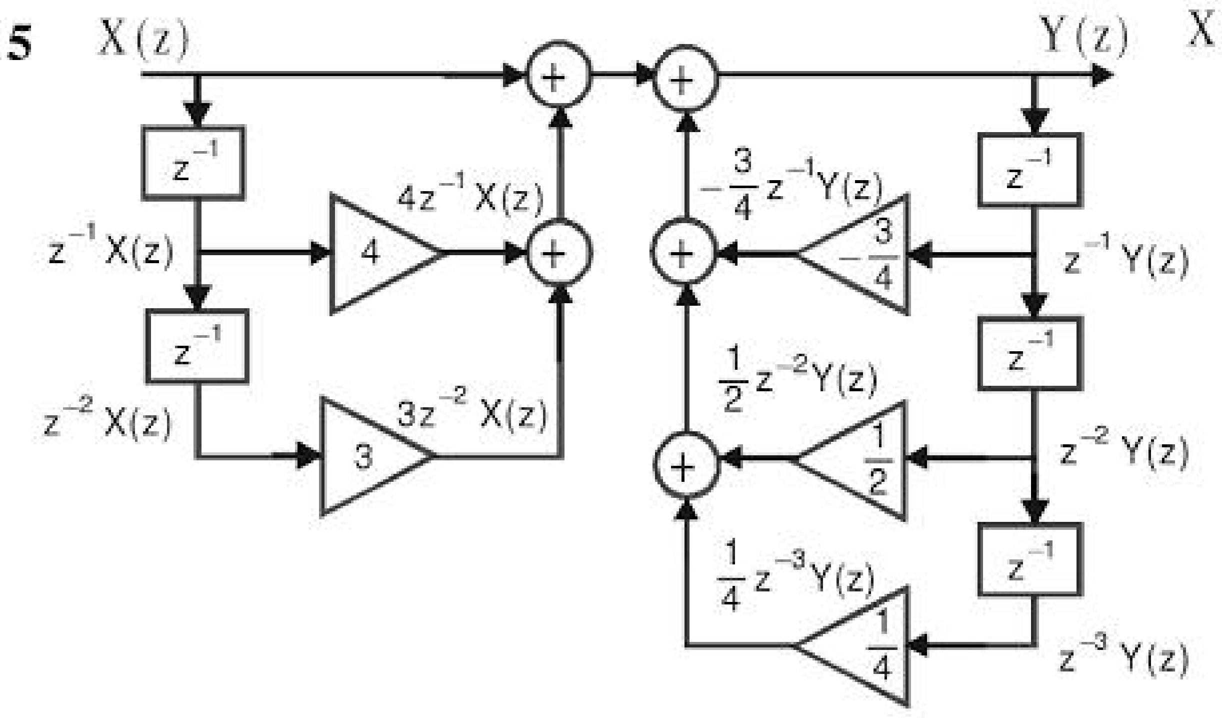
ROC : $|z| > 3$; $h(n) = 2\left(\frac{1}{2}\right)^n u(n) + 4(3)^n u(n)$

E3.12 $y(n) = \frac{1}{8}\left[1 - 49(-7)^n\right] u(n)$

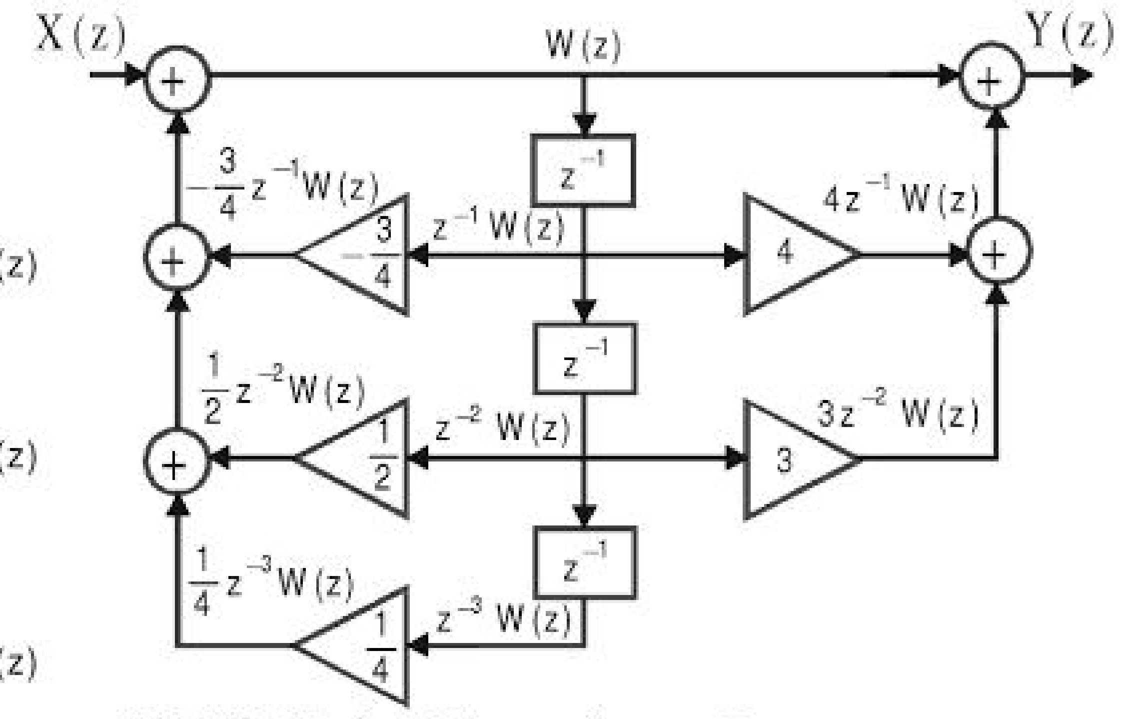
E3.13 $y(n) = \left[0.056(0.5)^n - 0.444(-1)^n - 0.111(-0.25)^n\right] u(n)$

E3.14 $y_1(n) = \{\underset{\uparrow}{0},\ 0,\ 2,\ -4,\ 10,\ -20\}$

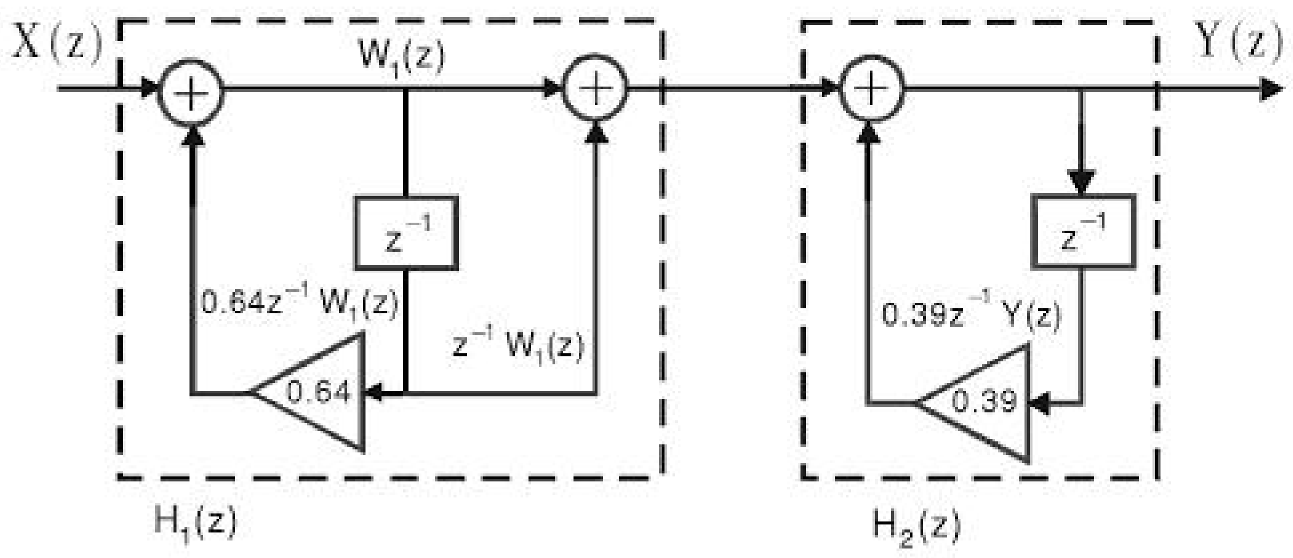
E3.15



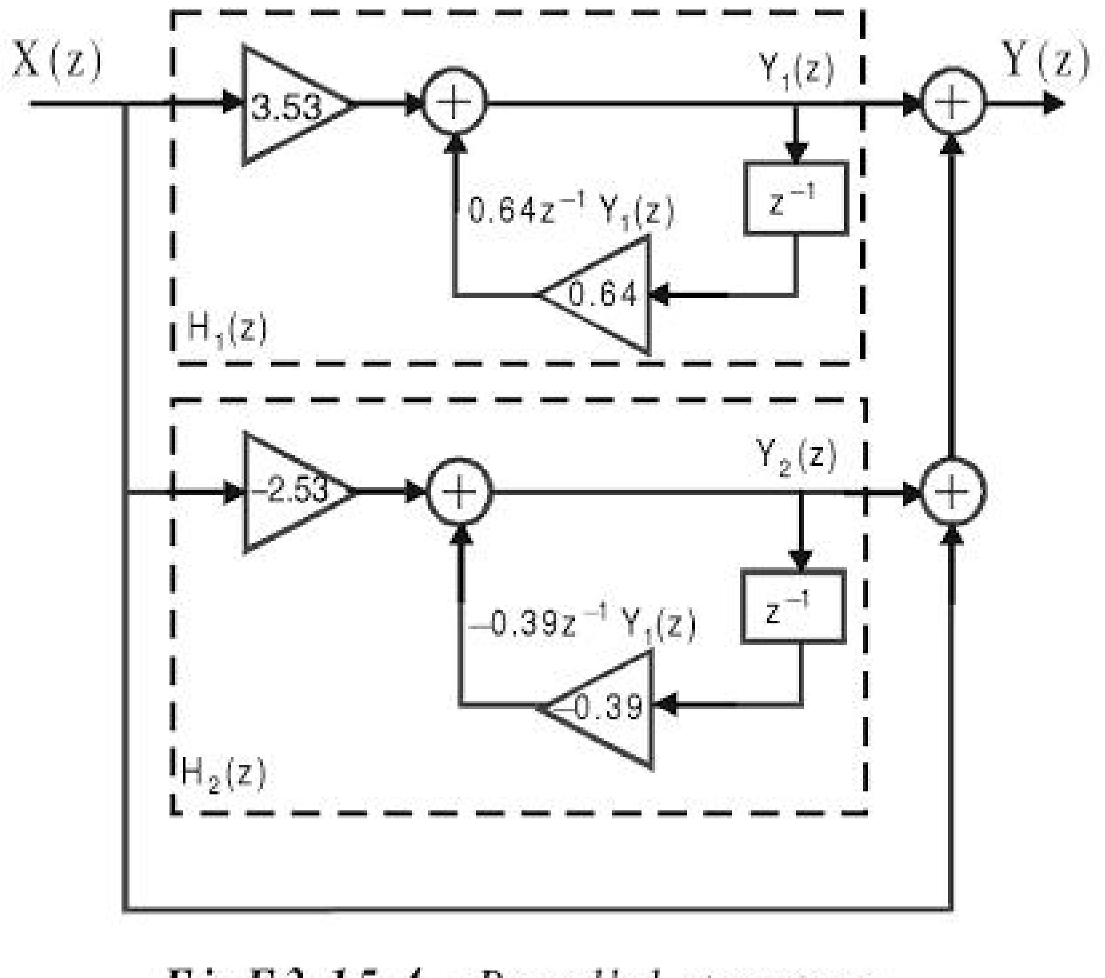
***FigE3.15.1 :** Direct form-I structure.*



***FigE3.15.2 :** Direct form-II structure.*

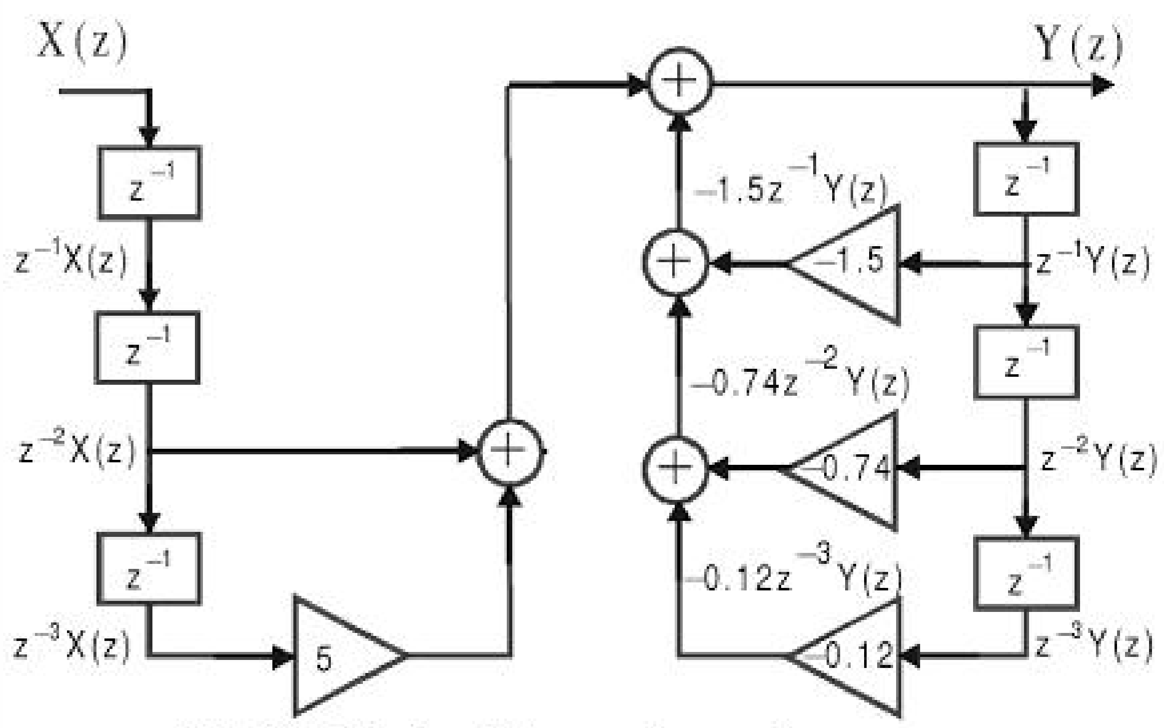


***FigE3.15.3 :** Cascade structure.*

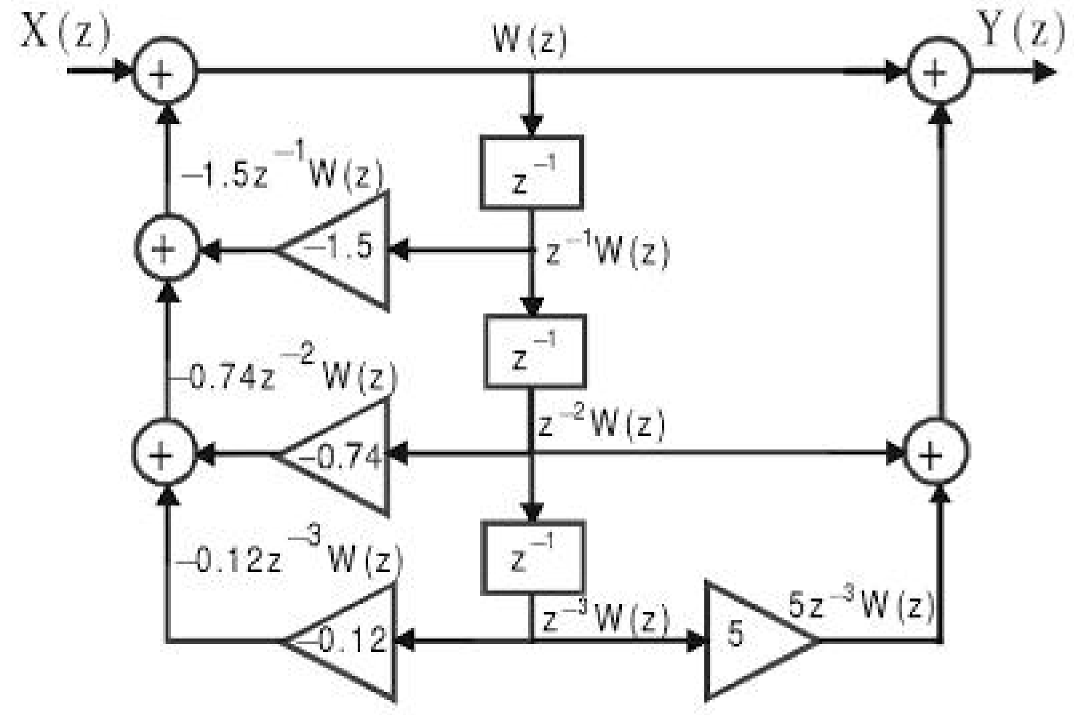


***FigE3.15.4 :** Parallel structure.*

E3.16

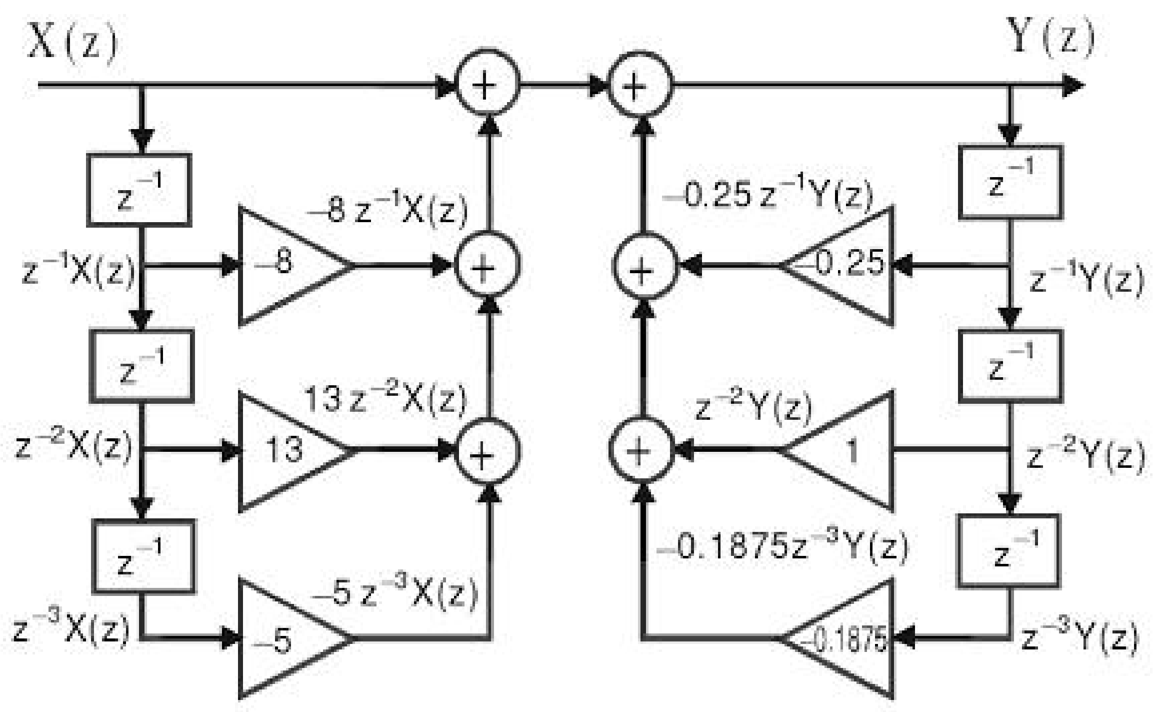


***FigE3.16.1 :** Direct form-I structure.*

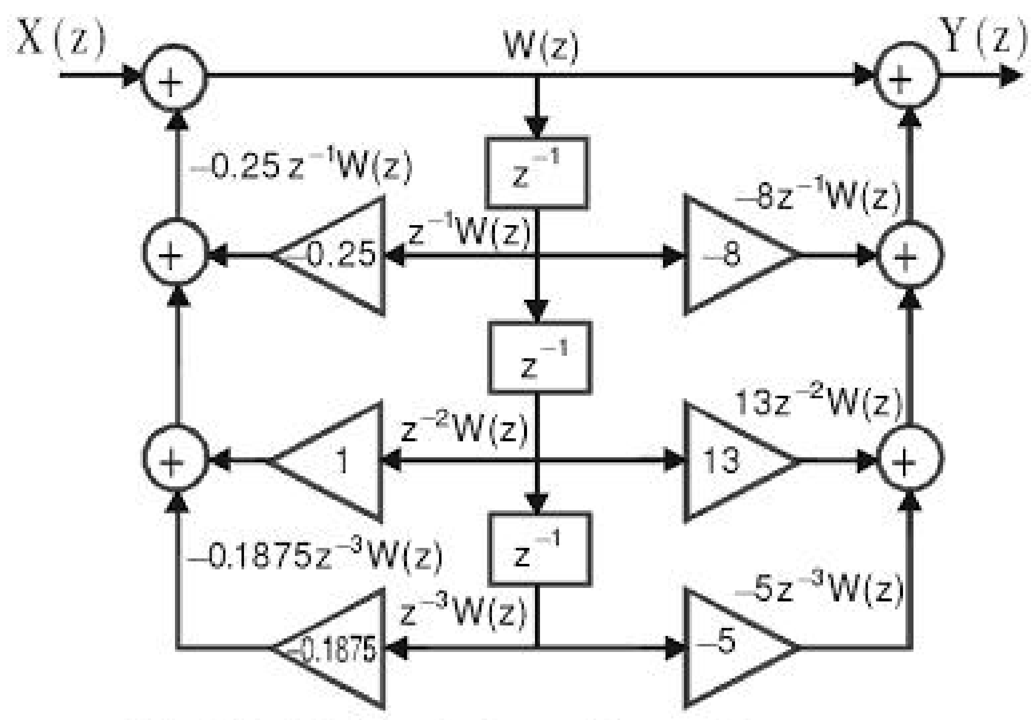


***FigE3.16.2 :** Direct form-II structure.*

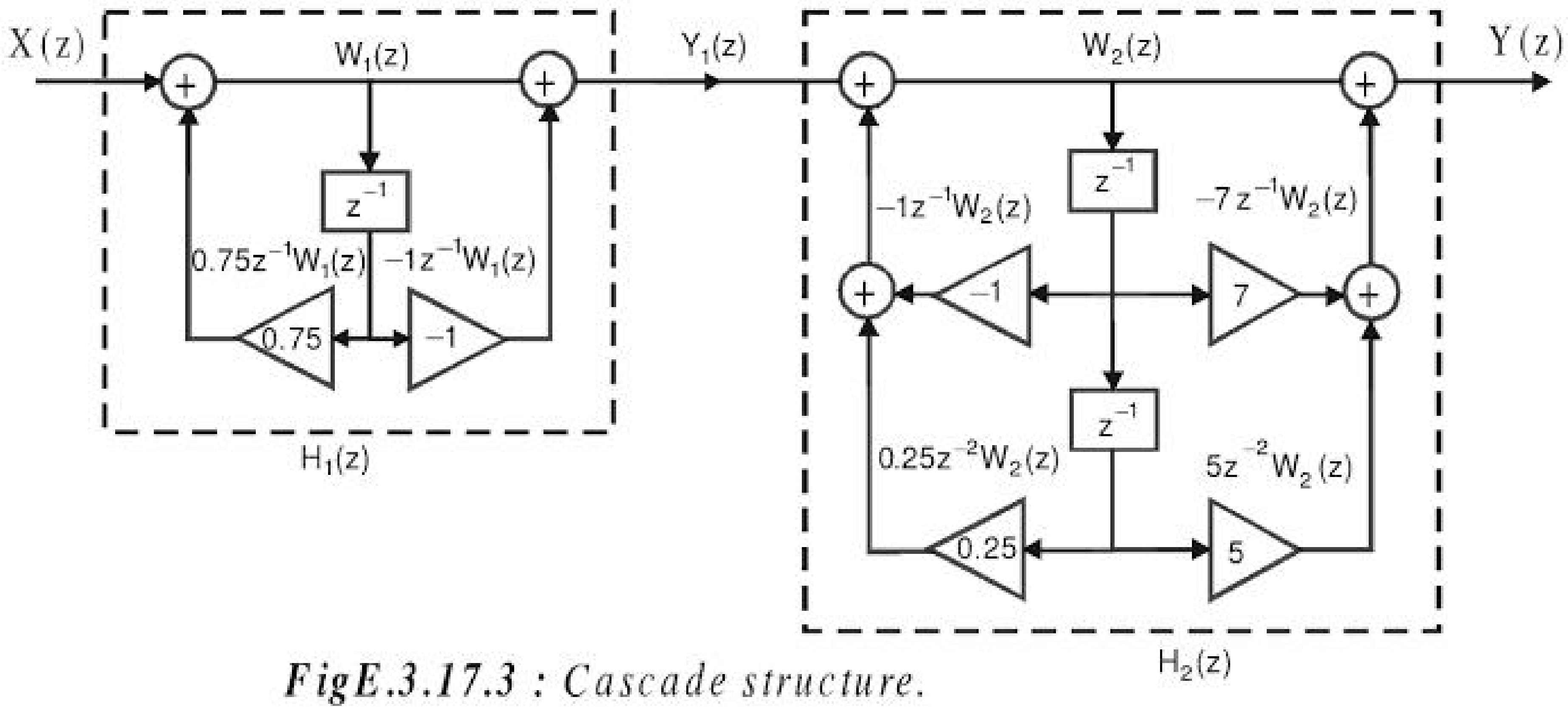
E3.17



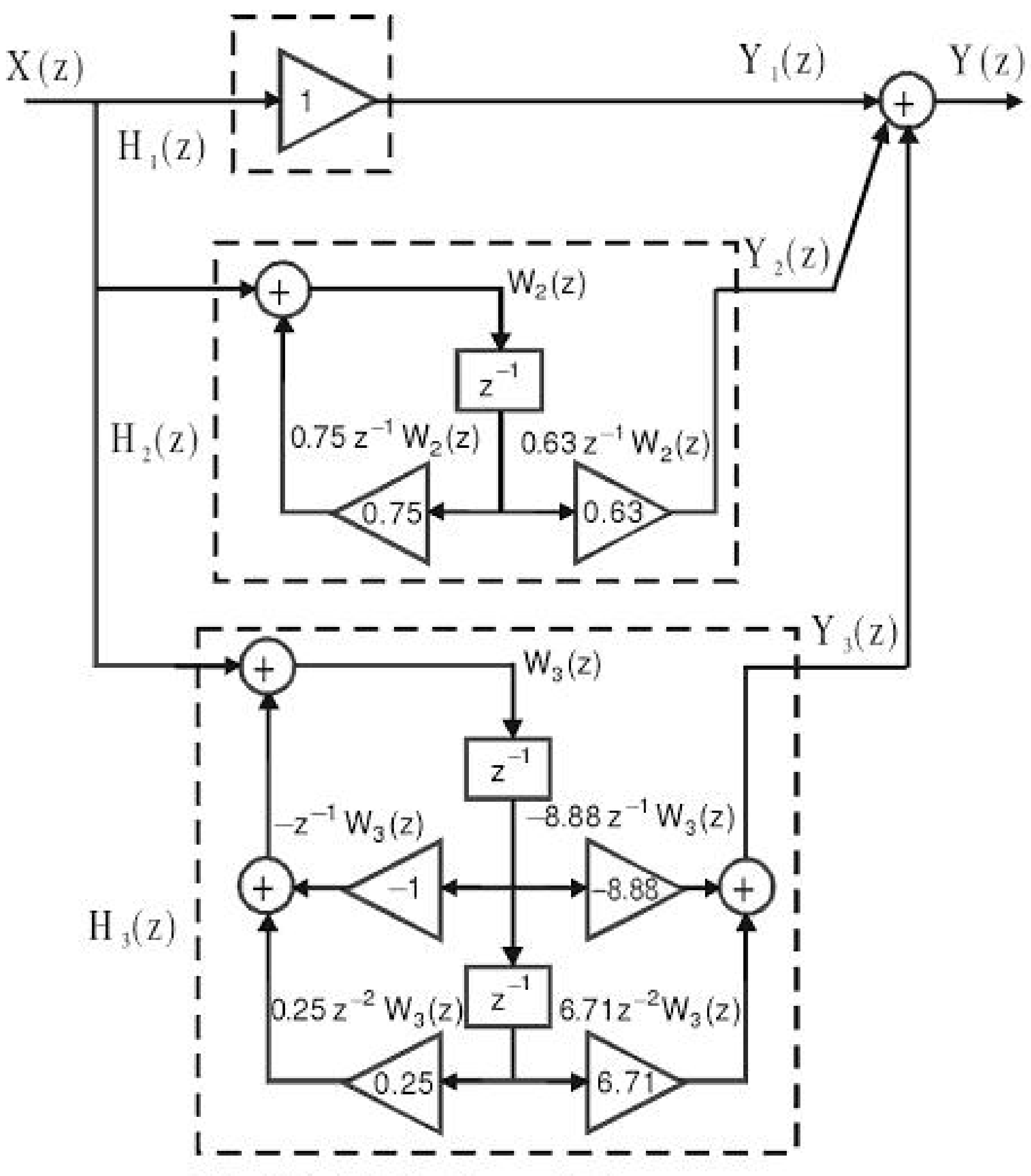
FigE3.17.1 : Direct form-I structure.



FigE3.17.2 : Direct form-II structure.

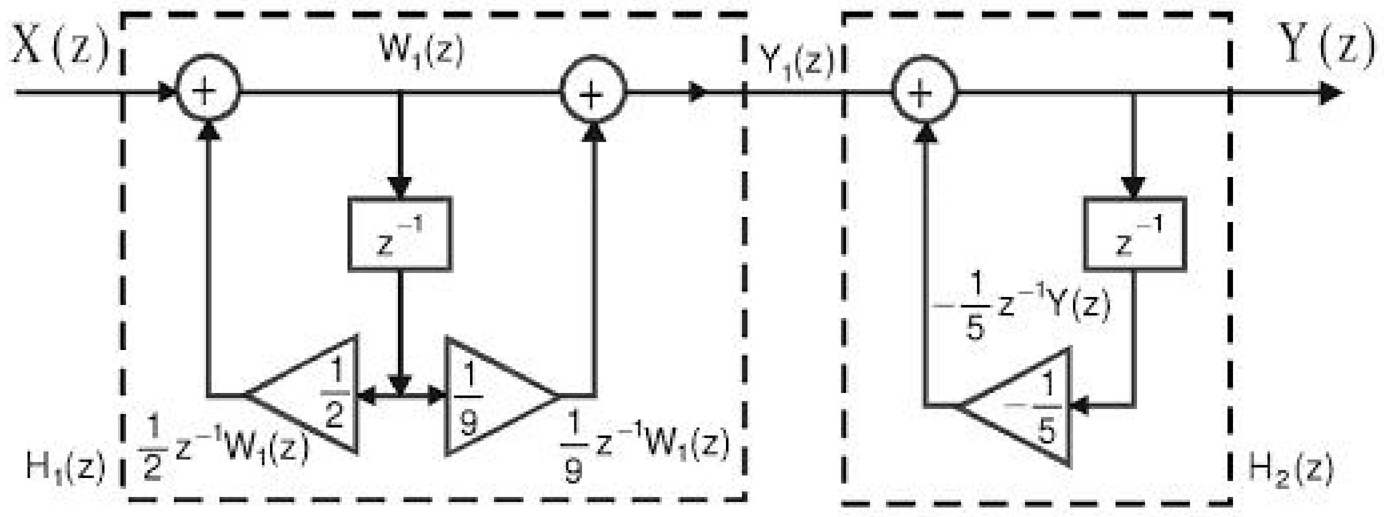


FigE.3.17.3 : Cascade structure.

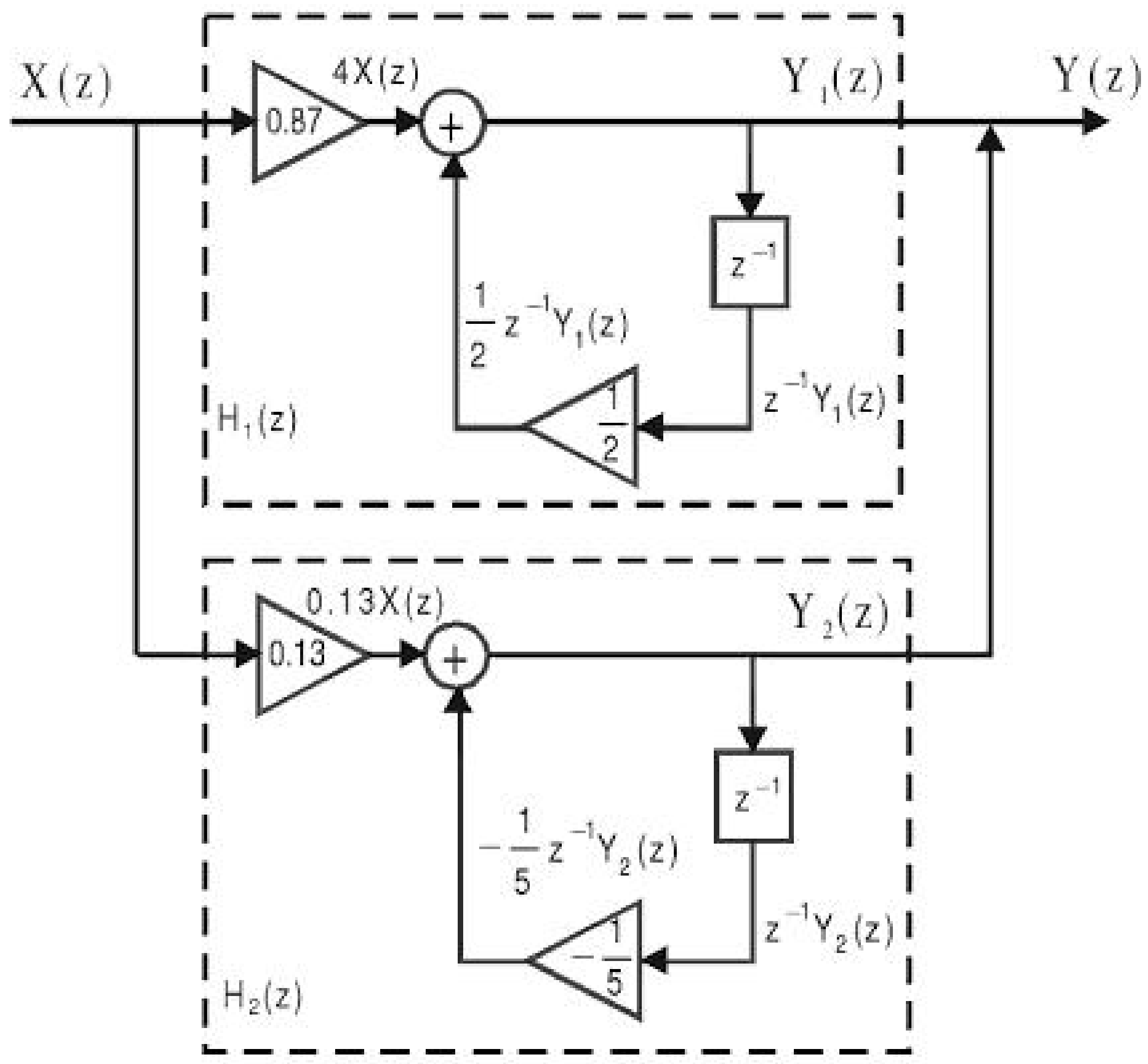


FigE3.17.4 : Parallel structure.

E3.18

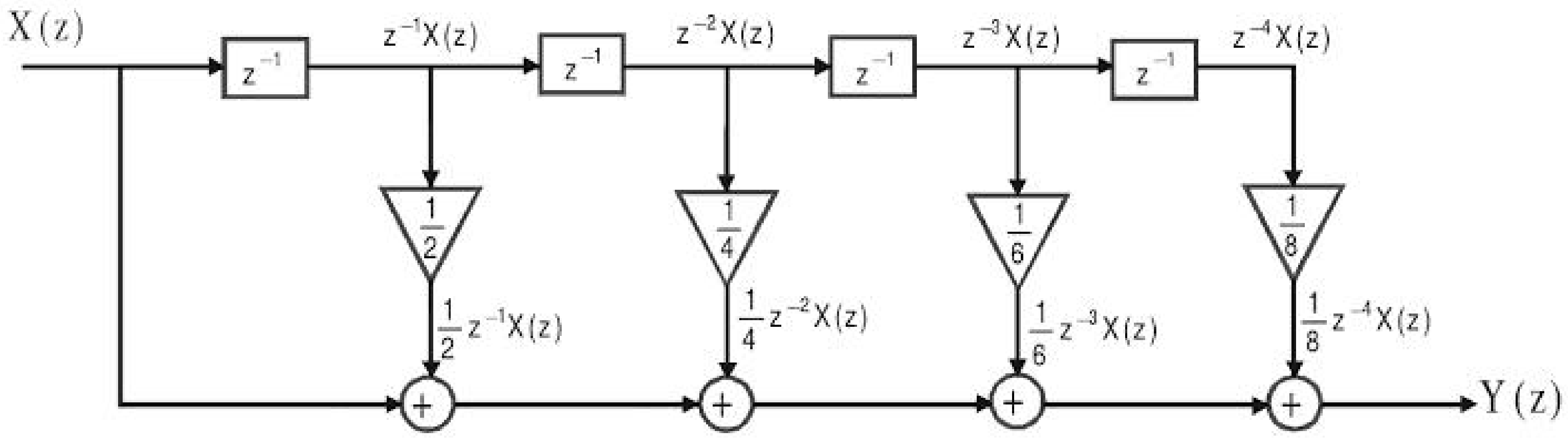


FigE3.18.1 : Cascade structure.



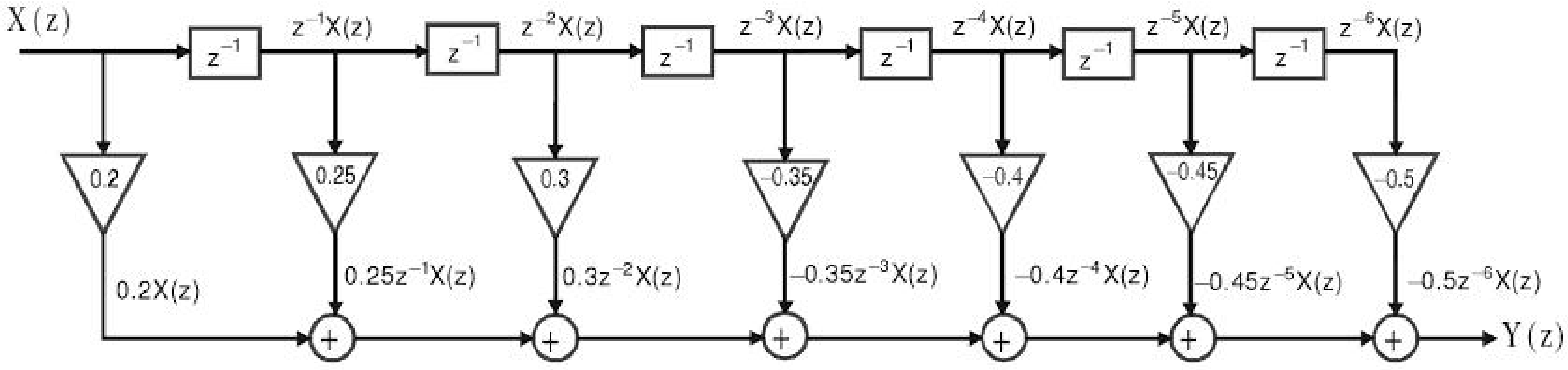
FigE3.18.2 : Parallel structure.

E3.19 a)



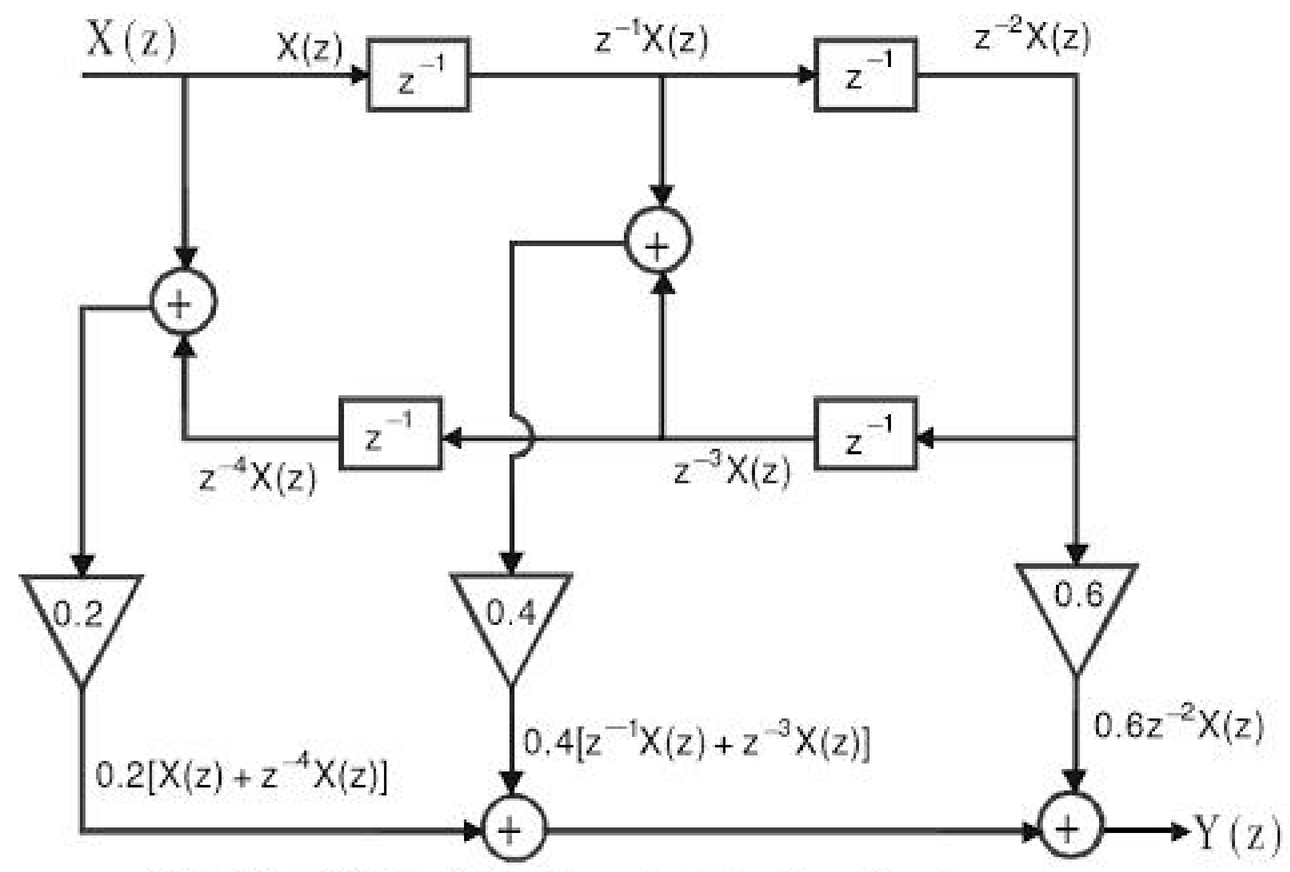
FigE3.19 a : Direct form structure.

E3.19 b)

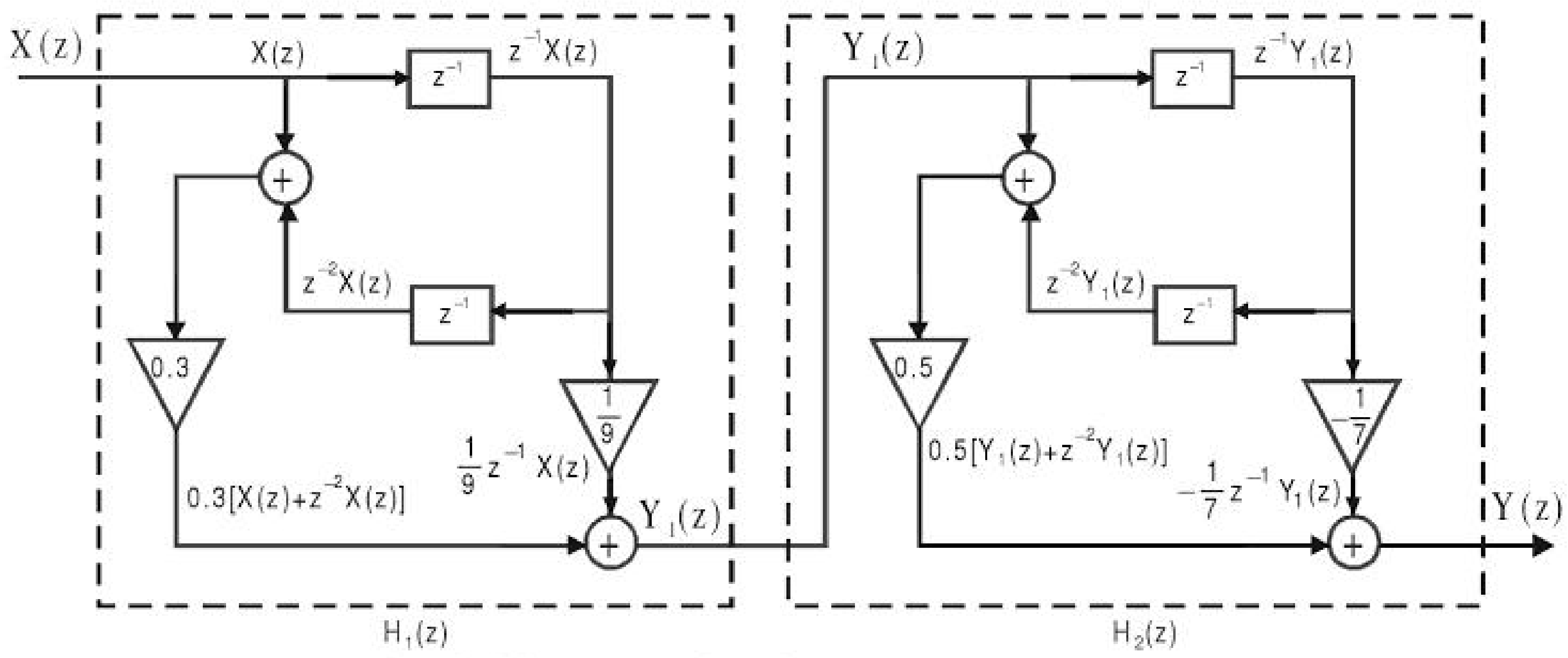


FigE3.19 b : Direct form structure.

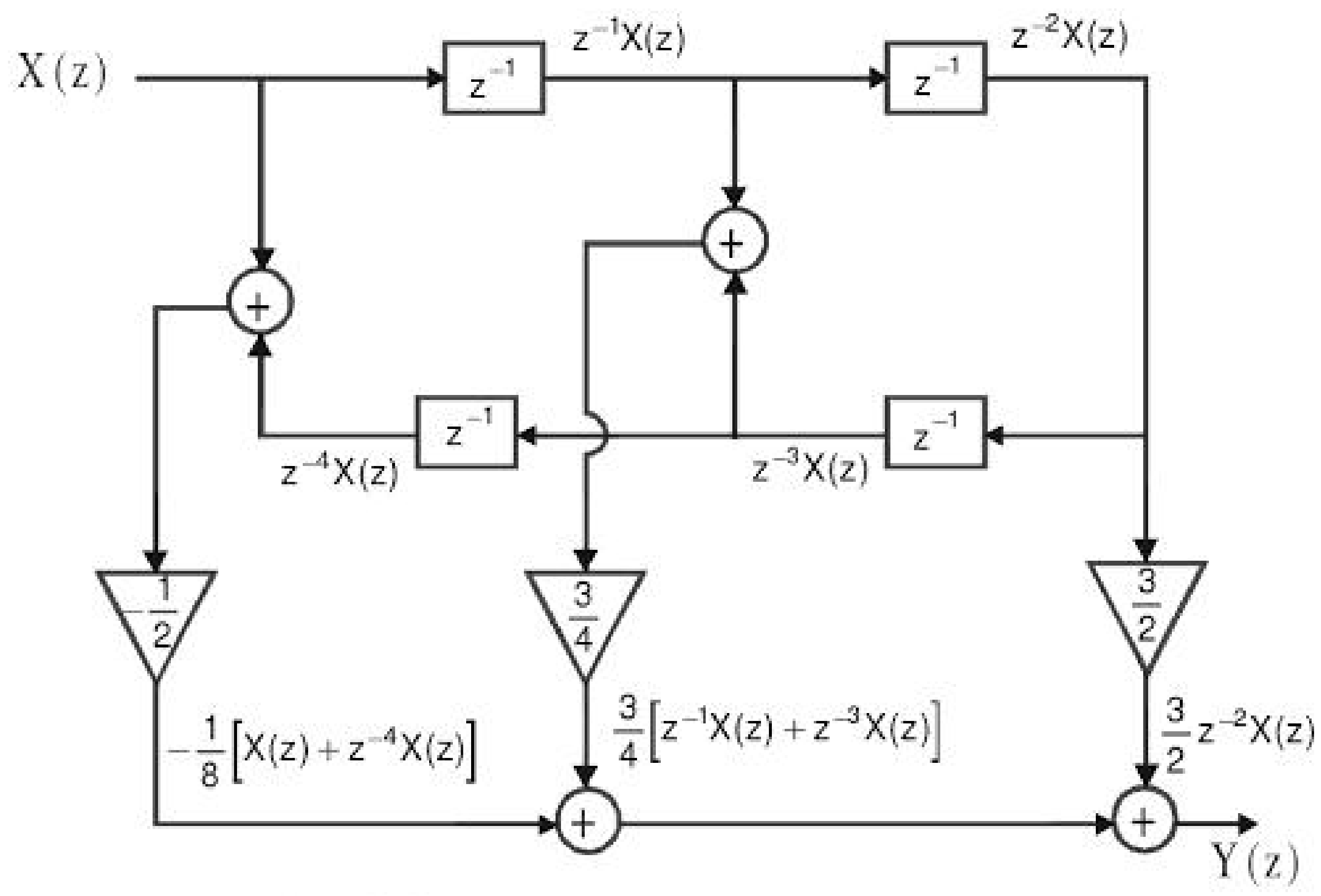
E3.20



FigE3.20a : *Linear phase structure.*



FigE3.20b: *Cascade of linear phase structure.*



FigE3.20c : *Linear phase structure.*

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4.2 Fourier Series of Discrete Time Signals (Discrete Time Fourier Series)

The Fourier series (or ***Discrete Time Fourier Series***, DTFS) of discrete time periodic signal x(n) with periodicity N is defined as,

$$\boxed{x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{N-1} c_k\, e^{j\omega_0 kn} = \sum_{k=0}^{N-1} c_k\, e^{j\omega_k n}} \quad .....(4.1)$$

where, c_k = Fourier coefficients; $w_0 = \frac{2\pi}{N}$ = Fundamental frequency of x(n)

$\omega_k = \omega_0 k = \frac{2\pi k}{N}$ = k^{th} harmonic frequency of x(n)

$c_k e^{j\omega_k n}$ = k^{th} harmonic component of x(n)

The Fourier coefficients, c_k for k = 0, 1, 2, ..., N - 1 can be evaluated using equation (4.2).

$$\boxed{c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \;;\; \text{for } k = 0,\, 1,\, 2,\, ...,\, N-1} \quad .....(4.2)$$

The ***Fourier coefficient*** c_k represents the amplitude and phase associated with the k^{th} frequency component. Hence we can say that the fourier coefficients provide the description of x(n) in the frequency domain.

<u>*Proof :*</u>

Consider the Fourier series representation of the discrete time signal x(n).

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}$$

Let us replace k by p.

$$\therefore\; x(n) = \sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}$$

Let us multiply the above equation by $e^{\frac{-j2\pi kn}{N}}$ on both sides.

$$x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

On evaluating the above equation for n = 0 to N – 1 and summing up the values we get,

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1}\sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

Let us interchange the order of summation in the right-hand side of the above equation and rearrange as shown below.

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} c_p \sum_{n=0}^{N-1} e^{\frac{j2\pi (p-k)n}{N}}$$

When p = k the right-hand side of the above equation reduces to c_k N.

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = c_k N$$

$$\therefore\; c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Note : The sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

$$\therefore\; \sum_{n=0}^{N-1} e^{\frac{j2\pi(p-k)n}{N}} = N\,;\; (p-k) = 0,\, \pm N,\, \pm 2N, ...$$
$$= 0\,;\; (p-k) \neq N$$

4.2.2 Properties of Discrete Time Fourier Series

The properties of discrete time Fourier series coefficients are listed in table 4.1. The proof of these properties are left as exercise to the readers.

Table 4.1 : Properties of Discrete Time Fourier Series Coefficients

Note : c_k are Fourier series coefficients of x(n) and d_k are Fourier series coefficients of y(n).

Property	Discrete time periodic signal	Fourier series coefficients
Linearity	$A\,x(n) + B\,y(n)$	$A\,c_k + B\,d_k$
Time shifting	$x(n-m)$	$c_k\, e^{\frac{j2\pi km}{N}}$
Frequency shifting	$e^{\frac{j2\pi nm}{N}}\, x(n)$	c_{k-m}
Conjugation	$x^*(n)$	c^*_{-k}
Time reversal	$x(-n)$	c_{-k}
Time scaling	$x(\frac{n}{m})$; for n multiple of m (periodic with period mN)	$\frac{1}{m}\, c_k$
Multiplication	$x(n)\, y(n)$	$\sum\limits_{m=0}^{N-1} c_m\, d_{k-m}$
Circular convolution	$\sum\limits_{m=0}^{N-1} x(m)\, y((n-m))_N$	$N\, c_K\, d_K$
Symmetry of real signals	x(n) is real	$c_k = c^*_{-k}$ $\lvert c_k\rvert = \lvert c_{-k}\rvert$ $\angle c_k = -\angle c_{-k}$ $Re\{c_k\} = Re\{c_{-k}\}$ $Im\{c_k\} = -Im\{c_{-k}\}$
Real and even	x(n) is real and even	c_k are real and even
Real and odd	x(n) is real and odd	c_k are imaginary and odd
Parseval's relation	Average power P of x(n) is defined as, $P = \frac{1}{N}\sum\limits_{n=0}^{N-1} \lvert x(n)\rvert^2$	Average power P in terms of Fourier series coefficients is, $P = \sum\limits_{k=0}^{N-1} \lvert c_k\rvert^2$
	$\frac{1}{N}\sum\limits_{n=0}^{N-1} \lvert x(n)\rvert^2 = \sum\limits_{k=0}^{N-1} \lvert c_k\rvert^2$	

Note : The average power in the signal is the sum of the powers of the individual frequency components. The sequence $|c_k|^2$ for k = 0, 1, 2,....., (N - 1) is the distribution of power as a function of frequency and so it is called the power density spectrum (or) power spectral density of the periodic signal.

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$$\therefore c_k = \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}e^{-j\frac{\pi k}{4}} - \frac{1}{\sqrt{2}}e^{-j\frac{3\pi k}{4}} - e^{-j\pi k} - \frac{1}{\sqrt{2}}e^{\left(-j\frac{8\pi k}{4}+j\frac{3\pi k}{4}\right)} + \frac{1}{\sqrt{2}}e^{\left(-j\frac{8\pi k}{4}+j\frac{\pi k}{4}\right)}\right]$$

$$= \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}e^{-j\frac{\pi k}{4}} - \frac{1}{\sqrt{2}}e^{-j\frac{3\pi k}{4}} - e^{-j\pi k} - \frac{1}{\sqrt{2}}e^{-j2\pi k}\, e^{j\frac{3\pi k}{4}} + \frac{1}{\sqrt{2}}e^{-j2\pi k}\, e^{j\frac{\pi k}{4}}\right] \qquad \boxed{e^{x+y} = e^x\, e^y}$$

$$= \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}e^{-j\frac{\pi k}{4}} - \frac{1}{\sqrt{2}}e^{-j\frac{3\pi k}{4}} - e^{-j\pi k} - \frac{1}{\sqrt{2}}e^{j\frac{3\pi k}{4}} + \frac{1}{\sqrt{2}}e^{j\frac{\pi k}{4}}\right]$$

For integer k,
$e^{-j2pk} = \cos 2pk - j \sin 2pk$
$= 1 - j0 = 1$

$$= \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}\left(e^{j\frac{\pi k}{4}} + e^{-j\frac{\pi k}{4}}\right) - \frac{1}{\sqrt{2}}\left(e^{j\frac{3\pi k}{4}} + e^{-j\frac{3\pi k}{4}}\right) - e^{-j\pi k}\right]$$

$$= \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}2\cos\frac{\pi k}{4} - \frac{1}{\sqrt{2}}2\cos\frac{3\pi k}{4} - (\cos\pi k - j\sin\pi k)\right]$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$= \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi k}{4} - \sqrt{2}\cos\frac{3\pi k}{4} - \cos\pi k\right]$$

For integer k, $\sin \pi k = 0$

When $k = 0$; $c_k = c_0 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 0}{4} - \sqrt{2}\cos\frac{3\pi\times 0}{4} - \cos\pi\times 0\right] = \frac{3}{8}\times 0 = 0$

When $k = 1$; $c_k = c_1 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 1}{4} - \sqrt{2}\cos\frac{3\pi\times 1}{4} - \cos\pi\times 1\right] = \frac{3}{8}\times 4 = 1.5$

When $k = 2$; $c_k = c_2 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 2}{4} - \sqrt{2}\cos\frac{3\pi\times 2}{4} - \cos\pi\times 2\right] = \frac{3}{8}\times 0 = 0$

When $k = 3$; $c_k = c_3 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 3}{4} - \sqrt{2}\cos\frac{3\pi\times 3}{4} - \cos\pi\times 3\right] = \frac{3}{8}\times 0 = 0$

When $k = 4$; $c_k = c_4 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 4}{4} - \sqrt{2}\cos\frac{3\pi\times 4}{4} - \cos\pi\times 4\right] = \frac{3}{8}\times 0 = 0$

When $k = 5$; $c_k = c_5 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 5}{4} - \sqrt{2}\cos\frac{3\pi\times 5}{4} - \cos\pi\times 5\right] = \frac{3}{8}\times 0 = 0$

When $k = 6$; $c_k = c_6 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 6}{4} - \sqrt{2}\cos\frac{3\pi\times 6}{4} - \cos\pi\times 6\right] = \frac{3}{8}\times 0 = 0$

When $k = 7$; $c_k = c_7 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 7}{4} - \sqrt{2}\cos\frac{3\pi\times 7}{4} - \cos\pi\times 7\right] = \frac{3}{8}\times 4 = 1.5$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{7} c_k\, e^{\frac{j2\pi kn}{8}} = \sum_{k=0}^{7} c_k\, e^{\frac{j\pi kn}{4}}$$

$$= c_0 + c_1 e^{\frac{j\pi n}{4}} + c_2 e^{\frac{j2\pi n}{4}} + c_3 e^{\frac{j3\pi n}{4}} + c_4 e^{\frac{j4\pi n}{4}} + c_5 e^{\frac{j5\pi n}{4}} + c_6 e^{\frac{j6\pi n}{4}} + c_7 e^{\frac{j7\pi n}{4}}$$

$$= 0 + 1.5 e^{\frac{j\pi n}{4}} + 0 + 0 + 0 + 0 + 0 + 1.5 e^{\frac{j7\pi n}{4}} = 1.5 e^{j\omega_0 n} + 1.5 e^{j7\omega_0 n} \; ; \; \text{where } \omega_0 = \frac{\pi}{4}$$

---

c) Given that, $x(n) = e^{\frac{j5\pi n}{2}}$

Test for Periodicity

Let, $x(n+N) = e^{\frac{j5\pi(n+N)}{2}} = e^{\left(\frac{j5\pi n}{2} + \frac{j5\pi N}{2}\right)}$

For periodicity $\frac{5\pi N}{2}$ should be integral multiple of 2p.

Let, $\dfrac{5\pi N}{2} = 2\pi \times M \quad \Rightarrow \quad N = \dfrac{4}{5}M$

Here, N is integer for M = 5, 10, 15, ...

Let, M = 5, \ N = 4

Here, x(n) is periodic with fundamental period N = 4, and fundamental frequency, $\omega_0 = \dfrac{2\pi}{N} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

Fourier Series

The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, 3, \ldots\ldots, N-1$$

Here, N = 4 and $x(n) = e^{\frac{j5\pi n}{2}}$

$$\therefore\ c_k = \frac{1}{4}\sum_{n=0}^{3} e^{\frac{j5\pi n}{2}}\, e^{\frac{-j2\pi kn}{4}} \quad ; \quad \text{for } k = 0, 1, 2, 3$$

$$= \frac{1}{4}\sum_{n=0}^{3} e^{\frac{j\pi n(5-k)}{2}} = \frac{1}{4}\left[e^0 + e^{\frac{j\pi(5-k)}{2}} + e^{\frac{j2\pi(5-k)}{2}} + e^{\frac{j3\pi(5-k)}{2}}\right]$$

$$= \frac{1}{4}\left[1 + e^{\frac{j\pi(5-k)}{2}} + e^{j\pi(5-k)} + e^{\frac{j3\pi(5-k)}{2}}\right]$$

$$= \frac{1}{4}\left[\begin{array}{l} 1 + \cos\dfrac{\pi(5-k)}{2} + j\sin\dfrac{\pi(5-k)}{2} + \cos\pi(5-k) + j\sin\pi(5-k) \\ \qquad + \cos\dfrac{3\pi(5-k)}{2} + j\sin\dfrac{3\pi(5-k)}{2} \end{array}\right]$$

When $k = 0;\ c_k = c_0 = \dfrac{1}{4}\left[1 + \cos\dfrac{5\pi}{2} + j\sin\dfrac{5\pi}{2} + \cos 5\pi + j\sin 5\pi + \cos\dfrac{15\pi}{2} + j\sin\dfrac{15\pi}{2}\right]$

$$= \frac{1}{4}\left[1 + 0 + j - 1 + j0 + 0 - j\right] = 0$$

When $k = 1;\ c_k = c_1 = \dfrac{1}{4}\left[1 + \cos 2\pi + j\sin 2\pi + \cos 4\pi + j\sin 4\pi + \cos 6\pi + j\sin 6\pi\right]$

$$= \frac{1}{4}\left[1 + 1 + j0 + 1 + j0 + 1 + j0\right] = 1$$

When $k = 2;\ c_k = c_2 = \dfrac{1}{4}\left[1 + \cos\dfrac{3\pi}{2} + j\sin\dfrac{3\pi}{2} + \cos 3\pi + j\sin 3\pi + \cos\dfrac{9\pi}{2} + j\sin\dfrac{9\pi}{2}\right]$

$$= \frac{1}{4}\left[1 + 0 - j - 1 + j0 + 0 + j\right] = 0$$

When $k = 3;\ c_k = c_3 = \dfrac{1}{4}\left[1 + \cos\pi + j\sin\pi + \cos 2\pi + j\sin 2\pi + \cos 3\pi + j\sin 3\pi\right]$

$$= \frac{1}{4}\left[1 - 1 + j0 + 1 + j0 - 1 + j0\right] = 0$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{3} c_k\, e^{\frac{j2\pi kn}{4}} = \sum_{k=0}^{3} c_k\, e^{\frac{j\pi kn}{2}}$$

$$= c_0 + c_1 e^{\frac{j\pi n}{2}} + c_2 e^{j\pi n} + c_3 e^{\frac{j3\pi n}{2}} = 0 + e^{\frac{j\pi n}{2}} + 0 + 0 = e^{\frac{j\pi n}{2}} = e^{j\omega_0 n}$$

Note : $x(n) = e^{\frac{j5\pi n}{2}} = e^{j\left(\frac{4\pi n}{2} + \frac{\pi n}{2}\right)} = e^{j2\pi n}\, e^{\frac{j\pi n}{2}} = e^{\frac{j\pi n}{2}} = e^{j\omega_o n}$

$\therefore$ *The given signal itself is in the Fourier series form.*

Example 4.2

Determine the Fourier series representation of the following discrete time signal and sketch the frequency spectrum.

$$x(n) = \{\ldots.. , \underline{1}, 2, -3, 1, 2, -3, 1, 2, -3, \ldots..\}$$

Solution

Given that, $x(n) = \{\ldots.. , 1, 2, -3, \underline{1}, 2, -3, 1, 2, -3, \ldots..\}$

Here the three samples 1, 2, −3 repeat again and again.

Therefore, x(n) is periodic with periodicity of N = 3, and fundamental frequency, $\omega_0 = \dfrac{2\pi}{N} = \dfrac{2\pi}{3}$.

Let, x(n) = {1, 2, −3} (considering one period). Now, the Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \frac{1}{3}\sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi kn}{3}}$$

$$= \frac{1}{3}\left[x(0) + x(1)\, e^{\frac{-j2\pi k}{3}} + x(2)\, e^{\frac{-j4\pi k}{3}}\right] = \frac{1}{3}\left[1 + 2\, e^{\frac{-j2\pi k}{3}} - 3\, e^{\frac{-j4\pi k}{3}}\right]$$

When $k = 0$; $c_k = c_0 = \dfrac{1}{3}[1 + 2 - 3] = 0$

When $k = 1$; $c_k = c_1 = \dfrac{1}{3}\left[1 + 2\, e^{\frac{-j2\pi}{3}} - 3\, e^{\frac{-j4\pi}{3}}\right]$

$$= \frac{1}{3}\left[1 + 2\cos\frac{2\pi}{3} - j2\sin\frac{2\pi}{3} - 3\cos\frac{4\pi}{3} + 3j\sin\frac{4\pi}{3}\right]$$

$$= \frac{1}{3}\left[1 - 2\times\frac{1}{2} - j2\times\frac{\sqrt{3}}{2} + 3\times\frac{1}{2} - 3j\times\frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{3}\left[\frac{3}{2} - j\frac{5\sqrt{3}}{2}\right] = \frac{1}{2} - j\frac{5\sqrt{3}}{6} = 0.5 - j1.443 \qquad \boxed{\frac{1.24}{\pi}\times\pi = 0.395\pi}$$

$$= 1.527\angle -1.24 \text{ rad} = 1.527\angle -0.395\pi = 1.527\, e^{-0.395\pi}$$

When $k = 2$; $c_k = c_2 = \dfrac{1}{3}\left[1 + 2\, e^{\frac{-j4\pi}{3}} - 3\, e^{\frac{-j8\pi}{3}}\right]$

$$= \frac{1}{3}\left[1 + 2\cos\frac{4\pi}{3} - j2\sin\frac{4\pi}{3} - 3\cos\frac{8\pi}{3} + 3j\sin\frac{8\pi}{3}\right]$$

$$= \frac{1}{3}\left[1 - 2\times\frac{1}{2} + j2\times\frac{\sqrt{3}}{2} + 3\times\frac{1}{2} + 3j\times\frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{3}\left[\frac{3}{2} + j\frac{5\sqrt{3}}{2}\right] = \frac{1}{2} + j\frac{5\sqrt{3}}{6} = 0.5 + j1.443 \qquad \boxed{\frac{1.24}{\pi}\times\pi = 0.395\pi}$$

$$= 1.527\angle 1.24 \text{ rad} = 1.527\angle 0.395\pi = 1.527\, e^{j0.395\pi}$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{2} c_k\, e^{\frac{j2\pi kn}{3}} = c_0 + c_1\, e^{\frac{j2\pi n}{3}} + c_2\, e^{\frac{j4\pi n}{3}}$$

$$= 0 + 1.527\, e^{-j0.395\pi}\, e^{\frac{j2\pi n}{3}} + 1.527\, e^{j0.395\pi}\, e^{\frac{j4\pi n}{3}}$$

$$= 1.527\, e^{-j0.395\pi}\, e^{j\omega_0 n} + 1.527\, e^{j0.395\pi}\, e^{j2\omega_0 n}$$

Frequency Spectrum

The frequency spectrum has two components : Magnitude spectrum and Phase spectrum.

The magnitude spectrum is obtained from magnitude of c_k and phase spectrum is obtained from phase of c_k.

Here, $c_k = \{c_0, \ c_1, \ c_2\} = \{0, \ 1.527 \angle -0.395\pi, \ 1.527 \angle 0.395\pi\}$

$\therefore$ Magnitude spectrum, $|c_k| = \{0, \ 1.527, \ 1.527\}$

Phase spectrum, $\angle c_k = \{0, \ -0.395\pi, \ 0.395\pi\}$

The sketch of magnitude and phase spectrum are shown in fig 1.

Here both the spectrum are periodic with period, N = 3.

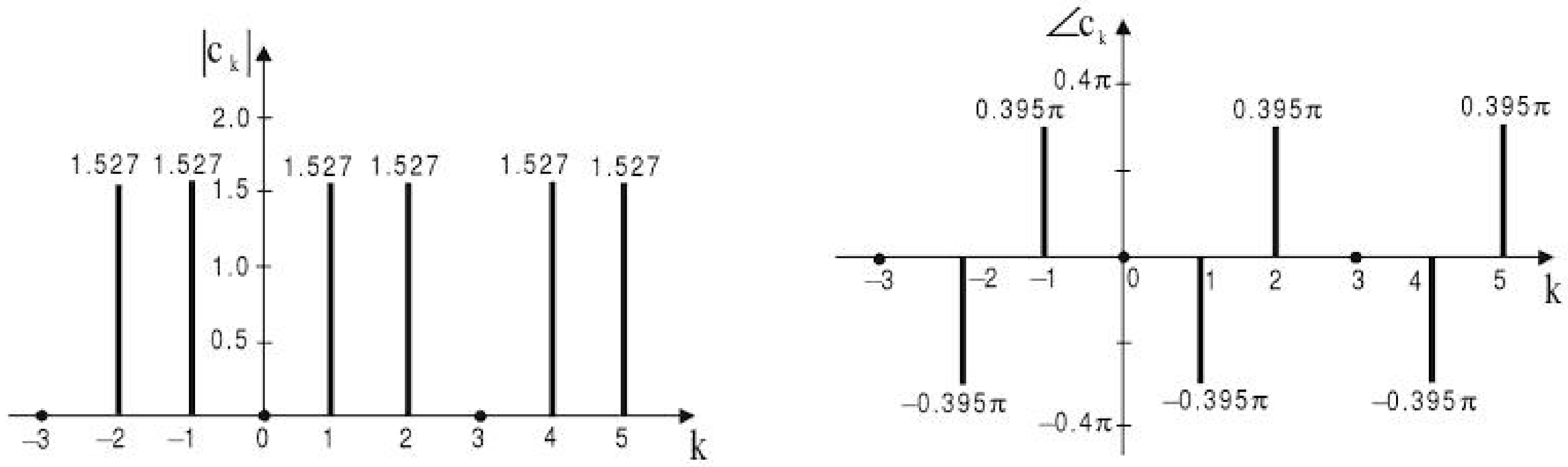

Fig 1a : Magnitude spectrum. *Fig 1b : Phase spectrum.*

Fig 1 : Frequency spectrum.

4.3 Fourier Transform of Discrete Time Signals (Discrete Time Fourier Transform)

4.3.1 Development of Discrete Time Fourier Transform From Discrete Time Fourier Series

Let $\tilde{x}(n)$ be a periodic sequence with period N. If the period N tends to infinity then the periodic sequence $\tilde{x}(n)$ will become a nonperiodic sequence x(n).

$$\therefore x(n) = \underset{N\to\infty}{\mathrm{Lt}} \ \tilde{x}(n)$$

Let c_k be Fourier coefficients of $\tilde{x}(n)$.

$$\therefore c_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}(n)\, e^{\frac{-j2\pi kn}{N}} \quad \Rightarrow \quad Nc_k = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{\frac{-j2\pi kn}{N}}$$

Since $\tilde{x}(n)$ is periodic, for even values of N, the summation index in the above equation can be changed from $n = -\left(\frac{N}{2}-1\right)$ to $+\frac{N}{2}$. (For odd values of N, the summation index is $n = -\frac{N}{2}$ to $+\frac{N}{2}$).

$$\therefore Nc_k = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}} \tilde{x}(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}} \tilde{x}(n)\, e^{-j\omega_k n} \quad .....(4.3)$$

$$\text{where,} \quad \omega_k = \frac{2\pi k}{N}$$

Let us define Nc_k as a function of $e^{j\omega_k}$.

$$\therefore \; X(e^{j\omega_k}) = Nc_k \qquad .....(4.4)$$

Now, using equation (4.3), the equation (4.4) can be expressed as shown below.

$$X(e^{j\omega_k}) = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}} \tilde{x}(n)\, e^{-j\omega_k n} \qquad .....(4.5)$$

Let, N ® ¥, in equation (4.5).

Now, $\tilde{x}(n)$ ® x(n), w_k ® w, and the summation index become -¥ to +¥.

Therefore, the equation (4.5) can be written as shown below.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(4.6)$$

The equation (4.6) is called Fourier transform of x(n), which is used to represent nonperiodic discrete time signal (as a function of frequency,w) in frequency domain.

Consider the Fourier series representation of $\tilde{x}(n)$ given below.

$$\tilde{x}(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}$$

Let us multiply and divide the above equation by $N/2\pi$.

$$\tilde{x}(n) = \frac{N}{2\pi} \times \frac{2\pi}{N} \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \frac{N}{2\pi} \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}\, \frac{2\pi}{N}$$

$$\boxed{\omega_k = \frac{2\pi k}{N}}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} Nc_k\, e^{\frac{j2\pi kn}{N}}\, \frac{2\pi}{N}$$

Using equation (4.4).

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{j\omega_k})\, e^{j\omega_k n}\, \frac{2\pi}{N} \qquad .....(4.7)$$

Let, N ® ¥, in equation (4.7).

Now, $\tilde{x}(n)$ ® x(n), w_k ® w, 2p / N ® dw, and summation becomes integral with limits 0 to 2p.

Therefore, the equation (4.7) can be written as shown below.

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad .....(4.8)$$

The equation (4.8) is called inverse Fourier transform of x(n), which is used to extract the discrete time signal from its frequency domain representation.

Since equation (4.6) extracts the frequency components of discrete time signal, the transformation using equation (4.6) is also called ***analysis*** of discrete time signal x(n). Since equation (4.8) integrates or combines the frequency components of discrete time signal, the inverse transformation using equation (4.8) is also called ***synthesis*** of discrete time signal x(n).

4.3.2 Definition of Discrete Time Fourier Transform

The Fourier transform (FT) of discrete-time signals is called ***Discrete Time Fourier Transform*** (i.e., DTFT). But for convenience the DTFT is also referred as FT in this book.

Let, x(n) = Discrete time signal

$X(e^{jw})$ = Fourier transform of x(n)

The Fourier transform of a finite energy discrete time signal, x(n) is defined as,

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}}$$

Symbolically the Fourier transform of x(n) is denoted as,

$\mathcal{F}\{x(n)\}$

where, $\mathcal{F}$ is the operator that represents Fourier transform.

$$\boxed{\therefore\; X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}}$$

The Fourier transform of a signal is said to exist if it can be expressed in a valid functional form. Since the computation of Fourier transform involves summing infinite number of terms, the Fourier transform exists only for the signals that are absolutely summable, i.e., given a signal x(n), the $X(e^{jw})$ exists only when,

$$\sum_{n=-\infty}^{+\infty} |x(n)| < \infty$$

4.3.3 Frequency Spectrum of Discrete Time Signal

The Fourier transform $X(e^{jw})$ of a signal x(n) represents the frequency content of x(n). We can say that, by taking Fourier transform, the signal x(n) is decomposed into its frequency components. Hence $X(e^{jw})$ is also called ***frequency spectrum*** of discrete time signal or ***signal spectrum***.

Magnitude and Phase Spectrum

The $X(e^{jw})$ is a complex valued function of w, and so it can be expressed in rectangular form as,

$X(e^{jw}) = X_r(e^{jw}) + jX_i(e^{jw})$

where, $X_r(e^{jw})$ = Real part of $X(e^{jw})$

$X_i(e^{jw})$ = Imaginary part of $X(e^{jw})$

The polar form of $X(e^{jw})$ is,

$X(e^{jw}) = |X(e^{jw})| \angle X(e^{jw})$

where, $|X(e^{jw})|$ = Magnitude spectrum

$\angle X(e^{jw})$ = Phase spectrum

The ***magnitude spectrum*** is defined as,

$$|X(e^{jw})|^2 = X(e^{jw})\, X^*(e^{jw}) \quad \text{or} \quad |X(e^{j\omega})| = \sqrt{X(e^{j\omega})\, X^*(e^{j\omega})}$$

where, $X^*(e^{jw})$ is complex conjugate of $X(e^{jw})$

Alternatively, $|X(e^{j\omega})|^2 = X(e^{j\omega})\; X^*(e^{j\omega})$

$$= \left[X_r(e^{j\omega}) + jX_i(e^{j\omega})\right]\left[X_r(e^{j\omega}) - jX_i(e^{j\omega})\right] = X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})$$

$$\boxed{\therefore\; |X(e^{j\omega})| = \sqrt{X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})}}$$

The ***phase spectrum*** is defined as,

$$\boxed{\angle X(e^{j\omega}) = \mathrm{Arg}[X(e^{j\omega})] = \tan^{-1}\left[\frac{X_i(e^{j\omega})}{X_r(e^{j\omega})}\right]}$$

4.3.4 Inverse Discrete Time Fourier Transform

Let, $x(n)$ = Discrete time signal
$X(e^{jw})$ = Fourier transform of $x(n)$

The ***inverse discrete time Fourier transform*** of $X(e^{jw})$ is defined as,

$$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad ; \text{for} \quad n = -\infty \text{ to } +\infty} \qquad .....(4.9)$$

Symbolically the inverse Fourier transform can be expressed as, $\mathcal{F}^{-1}\{X(e^{jw})\}$, where, $\mathcal{F}^{-1}$ is the operator that represents the inverse Fourier transform.

$$\boxed{\therefore\ x(n) = \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \ ; \ \text{for} \quad n = -\infty \text{ to } +\infty}$$

Since $X(e^{jw})$ is periodic with period 2p, the limits of integral in the above definition of inverse Fourier transform can be either "-p to +p", or "0 to 2p", or "any interval of 2p".

We also refer to $x(n)$ and $X(e^{jw})$ as a Fourier transform pair and this relation is expressed as,

$$\boxed{x(n) \overset{\mathcal{F}\ ®}{\underset{\mathcal{F}^{-1}}{\rightleftharpoons}} X(e^{jw})}$$

Alternate Method for Inverse Fourier Transform

The integral solution of equation (4.9)for the inverse Fourier transform is useful for analytic purpose, but sometimes it will be difficult to evaluate for typical functional forms of $X(e^{jw})$. An alternate and more useful method of determining the values of $x(n)$ follows directly from the definition of the Fourier transform.

Consider the definition of Fourier transform of $x(n)$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

Let us expand the above equation of $X(e^{jw})$ as shown below.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$= ..... + x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0)\, e^{0}$$
$$+ x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + ... \qquad .....(4.10.1)$$

Let us express the given function of $X(e^{jw})$ as a power series of e^{-jw} by long division as shown below.

$$X(e^{j\omega}) = ... + b_2\, e^{j2\omega} + b_1\, e^{j\omega} + a_0\, e^{0} + a_1\, e^{-j\omega} + a_2\, e^{-j2\omega} + ... \qquad .....(4.10.2)$$

On comparing the equations (4.10.1) and (4.10.2) we can say that the samples of signal $x(n)$ are simply the coefficients of e^{-jwn}.

4.3.5 Comparison of Fourier Transform of Discrete and Continuous Time Signals

1. The Fourier transform of a continuous time signal consists of a spectrum with a frequency range - ¥ to + ¥. But the Fourier transform of a discrete time signal is unique in the frequency range - p to + p (or equivalently 0 to 2p). Also Fourier transform of discrete time signal is periodic with period 2p. Hence the frequency range for any discrete-time signal is limited to -p to p (or 0 to 2p) and any frequency outside this interval has an equivalent frequency within this interval.

2. Since the continuous time signal is continuous in time the Fourier transform of continuous time signal involves integration but the Fourier transform of discrete time signal involves summation because the signal is discrete.

4.4 Properties of Discrete Time Fourier Transform

1. Linearity property

The linearity property of Fourier transform states that the Fourier transform of a linear weighted combination of two or more signals is equal to the similar linear weighted combination of the Fourier transform of the individual signals.

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{jw})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{jw})$ then by linearity property,

$\mathcal{F}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(e^{jw}) + a_2 X_2(e^{jw})$; where a_1 and a_2 are constants.

Proof :

By the definition of Fourier transform,

$$X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(4.11)$$

$$X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} \qquad .....(4.12)$$

$$\mathcal{F}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \left[a_1 x_1(n)\, e^{-j\omega n} + a_2 x_2(n)\, e^{-j\omega n}\right]$$

$$= \sum_{n=-\infty}^{+\infty} a_1 x_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{+\infty} a_2 x_2(n)\, e^{-j\omega n}$$

$$= a_1 \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} + a_2 \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n}$$

$$= a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

Using equations (4.11) and (4.12)

2. Periodicity

Let, $\mathcal{F}\{x(n)\} = X(e^{jw})$, then $X(e^{jw})$ is periodic with period 2p.

$\backslash$ $X(e^{j(w+2pm)}) = X(e^{jw})$; where m is an integer

Proof :

$$X(e^{j(\omega + 2\pi m)}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j(\omega + 2\pi m)n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}\, e^{-j2\pi mn}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = X(e^{j\omega})$$

Since m and n are integers, $e^{-j2pmn} = 1$

3. Time shifting or Fourier transform of delayed signal

Let, $\mathcal{F}\{x(n)\} = X(e^{jw})$, then $\mathcal{F}\{x(n-m)\} = e^{-jwm}\, X(e^{jw})$

Also, $\mathcal{F}\{x(n+m)\} = e^{jwm} X(e^{jw})$

This relation means that if a signal is shifted in time domain by m samples, its magnitude spectrum remains unchanged. However, the phase spectrum is changed by an amount $\pm$wm. This result can be explained if we recall that the frequency content of a signal depends only on its shape. Mathematically, we can say that delaying by m units in time domain is equivalent to multiplying the spectrum by e^{-jwm} in the frequency domain.

6. Frequency shifting

Let, $\mathcal{F}\{x(n)\} = X(e^{jw})$, then $\mathcal{F}\{e^{j\omega_0 n}\, x(n)\} = X(e^{j(\omega - \omega_0)})$

According to this property, multiplication of a sequence x(n) by $e^{j\omega_0 n}$ is equivalent to a frequency translation of the spectrum $X(e^{jw})$ by w_0.

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(4.16)$$

$$\therefore\ \mathcal{F}\{e^{j\omega_0 n}\, x(n)\} = \sum_{n=-\infty}^{+\infty} e^{j\omega_0 n}\, x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j(\omega - \omega_0)n} \qquad .....(4.17)$$

$$= X(e^{j(\omega - \omega_0)})$$

The equation (4.17) is similar to the form of equation (4.16)

7. Fourier transform of the product of two signals

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{jw})$

$\mathcal{F}\{x_2(n)\} = X_2(e^{jw})$

$$\text{Now, } \mathcal{F}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda \qquad .....(4.18)$$

The equation (4.18) is convolution of $X_1(e^{jw})$ and $X_2(e^{jw})$

This relation is the dual of time domain convolution. In other words, the Fourier transform of the product of two discrete time signals is equivalent to the convolution of their Fourier transform. [On the other hand, the Fourier transform of the convolution of two discrete time signals is equivalent to the product of their Fourier transform.]

Proof :

Let, $x_2(n)\, x_1(n) = x_3(n)$

$$\text{Now, } \mathcal{F}\{x_2(n)\, x_1(n)\} = \mathcal{F}\{x_3(n)\} = \sum_{n=-\infty}^{+\infty} x_3(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x_2(n)\, x_1(n)\, e^{-j\omega n} \qquad .....(4.19)$$

By the definition of inverse Fourier transform we get,

$$x_1(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Let, $w = 1$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, e^{j\lambda n}\, d\lambda \qquad .....(4.20)$$

On substituting for $x_1(n)$ from equation (4.20) in equation (4.19) we get,

$$\mathcal{F}\{x_1(n)\, x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, e^{j\lambda n}\, d\lambda\right] e^{-j\omega n}$$

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