

# A Study of Sampled-data Integrator Controller and Its Application in Speed control of DC Motors

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**Abstract**—Sampled-data control methods are well known as excellent tool for digital control systems, because that systematically treats the inter-sampling behavior in the controller design process. In practical applications, integrator controllers are very useful for servo-control and tracking control purpose. But unfortunately, research about the sampled-data integrator controller is few and far. In the paper, we study a designing method of sampled-data integrator controller. That may provide a simple way for practical engineer to use sampled-data control methods in applications when an integrator is needed in controllers. In addition, in order to show the validity of sampled-data integrator controllers, we applied the proposed method to speed control of DC motors, and show the method is very useful under the condition when load torque suddenly changes. A simulation model is constructed using Matlab/Simulation. Some interesting results have been obtained compared with the conventional digital LQG control method.

## I. INTRODUCTION

THE importance of digital control in computer controlled systems is well recognized. But conventional digital control methods [1],[2] have not provided some available solutions to solve the problem how to control the values between sampling intervals. Recent years, sampled-data control methods give a systematical design process considering the values between sampling intervals and abstract so many interesting. Since then, many papers have been published, but almost study from theories [3]-[9]. The consideration about sampled-data integrator controller, that is very useful in practical applications, is not really investigated. Also it is useful to show some effects in applications using sampled-data control methods, but such studies are not yet enough. In addition, in comparison to the conventional control method, such as LQG [2], is also very interesting.

In the paper, we study a designing method of sampled-data controllers with an integrator. That may provide a simple way for practical engineers to use sampled-data control methods in applications when an integrator is needed in controllers. In addition, in order to show the validity of sampled-data integrator controller, we use the proposed method to the

Manuscript received October 31, 2008.

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## II. DC MOTOR MODEL AND INTEGRATOR CONTROLLERS

### A. DC Motor Model

A simple DC motor model is well known as follows. Electric parts:

$$\begin{aligned} E_a &= L \frac{di_a}{dt} + Ri_a + k_e \omega_m \\ \frac{di_a}{dt} &= -\frac{R}{L} i_a - \frac{k_e}{L} \omega_m + \frac{1}{L} E_a \end{aligned}$$

where  $E_a$  is the input voltage,  $i_a$  is the current of the motor,  $L$  is the inductance,  $R$  is the resistance,  $k_e$  is constant number of EMF,  $\omega_m$  is the rotor speed, respectively.

Mechanic parts:

$$\begin{aligned} \tau &= J \frac{d\omega_m}{dt} + D\omega_m, \quad (\tau = k_T i_a) \\ \frac{d\omega_m}{dt} &= \frac{k_T}{J} i_a - \frac{D}{J} \omega_m \end{aligned}$$

where  $\tau$  is the torque,  $J$  is the inertial moment,  $D$  is the viscous resistance, respectively.

Take voltage  $E_a$  as the input, the current and the speed as the states, and the speed as the output, a state space equation can be obtained as follows.

$$\begin{bmatrix} \dot{i}_a(t) \\ \dot{\omega}_m(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{k_e}{L} \\ \frac{k_T}{J} & -\frac{D}{J} \end{bmatrix} \begin{bmatrix} i_a(t) \\ \omega_m(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t) \quad (1)$$

$$= Ax(t) + bu(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i_a(t) \\ \omega_m(t) \end{bmatrix} = cx(t) \quad (2)$$

### B. Extended error plant with an integrator

Because we consider a tracking control of speed, so an integrator is introduced, and the extended error system including the state of the motor and an integrator can be obtained as follows.

$$e(t) = r(t) - y(t) \quad (3)$$

$$\dot{w}(t) = r(t) - y(t) = r(t) - cx(t) \quad (4)$$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \quad (5)$$

where  $e$  is the error and  $w$  is the integration of the error  $e$ . Let  $x_s$ ,  $w_s$ ,  $u_s$  note the final values of states, integrator and input, respectively, we have

$$\begin{bmatrix} \dot{x}_s \\ \dot{w}_s \end{bmatrix} = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u_s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r = 0. \quad (6)$$

Define

$$\hat{x}(t) = x(t) - x_s, \quad \hat{w}(t) = w(t) - w_s, \quad \hat{u}(t) = u(t) - u_s,$$

we have

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{w}}(t) \end{bmatrix} &= \begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \\ &= \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} \hat{u}(t) \\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix} \begin{bmatrix} x_s \\ w_s \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u_s \\ &= \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} \hat{u}(t) \end{aligned} \quad (7)$$

Eq.6 has been used in the last equation.

Define

$$x_1 = \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix}, \quad A_1 = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad (8)$$

we have

$$\dot{x}_1(t) = A_1 x_1(t) + b_1 \hat{u}(t). \quad (9)$$

### C. Control systems with integrators

It is well known that a simple way is introducing an integrator for speed tracking control in a control system as shown in Fig.1

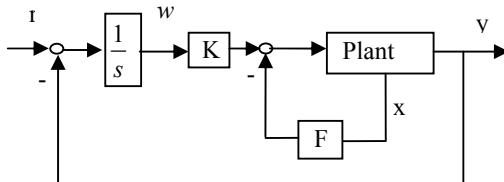


Fig.1 Integrator control system

$$u(t) = -Fx(t) + Kw(t) = -[F \quad -K] \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \quad (10)$$

In order to use the LQR method to solve the integrator controller, it is usual to use the extended error plant (6) and the extended state feedback controller

$$\hat{u}(t) = -F_e \begin{bmatrix} \hat{x}(t) \\ \hat{w}(t) \end{bmatrix} \quad (11)$$

From the relation between  $\hat{u}(t)$  and  $u(t)$ , we have

$$u(t) = -F_e \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + F_e \begin{bmatrix} x_s \\ w_s \end{bmatrix} + u_s. \quad (12)$$

From (10), (11) and (12), we have

$$[F \quad -K] = F_e. \quad (13)$$

### III. SAMPLED-DATA INTEGRATOR CONTROLLERS

#### A. Sampled-data LQG methods

Considering a give continuous system  
 $\dot{x}(t) = Ax(t) + bu(t),$

and a continuous performance function

$$J = \int_0^\infty \{x(t)^T Q_s x(t) + u(t)^T R_s u(t)\} dt,$$

then a sampled-data system and the performance function can be obtained as follows

$$\begin{aligned} x(nh+p) &= e^{Ap} x(nh) + \int_0^p e^{A(m-nh)} b dm \times u(nh) \\ &= [\varphi \quad \psi] \begin{bmatrix} x(nh) \\ u(nh) \end{bmatrix} \end{aligned}$$

where

$$\varphi = e^{Ap}, \quad \psi = \int_0^p e^{Am} b dm.$$

$$\begin{aligned} J &= \sum_{n=0}^{\infty} \int_{nh}^{nh+h} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q_s & 0 \\ 0 & R_s \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \\ &= \sum_{n=0}^{\infty} \int_{nh}^{nh+h} \begin{bmatrix} x^T(nh) & u^T(nh) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \varphi^T & 0 \\ \psi^T & I \end{bmatrix} \begin{bmatrix} Q_s & 0 \\ 0 & R_s \end{bmatrix} \begin{bmatrix} \varphi & \psi \\ 0 & I \end{bmatrix} \begin{bmatrix} x(nh) \\ u(nh) \end{bmatrix} dp \\ &= \sum_{n=0}^{\infty} \begin{bmatrix} x^T(nh) & u^T(nh) \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(nh) \\ u(nh) \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \{x^T(nh)Qx(nh) + 2x^T(nh)Su(nh) + u^T(nh)Ru(nh)\} \end{aligned}$$

where

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \int_0^h \begin{bmatrix} \varphi^T & 0 \\ \psi^T & I \end{bmatrix} \begin{bmatrix} Q_s & 0 \\ 0 & R_s \end{bmatrix} \begin{bmatrix} \varphi & \psi \\ 0 & I \end{bmatrix} dp. \quad (14)$$

This performance function contains a cross term, but it can be simplified as following.

Note

$$\begin{aligned} x^T Qx + 2x^T Su + u^T Ru \\ = x^T (Q - SR^{-1}S^T)x + (u + R^{-1}S^T x)^T R(u + R^{-1}S^T x) \end{aligned}$$

then

$$\begin{aligned} J &= \sum_{n=0}^{\infty} \{x^T(Q - SR^{-1}S^T)x + (u + R^{-1}S^T x)^T R(u + R^{-1}S^T x)\} \\ &= \sum_{n=0}^{\infty} \{x^T Q_1 x + u_1^T R u_1\} \end{aligned}$$

where

$$Q_1 = Q - SR^{-1}S^T, \quad u_1 = u + R^{-1}S^T x. \quad (15)$$

Let  $P$  is a solution of the following Riccati equation

$$P = A^T P A + Q_1 - A^T P b (R + b^T P b)^{-1} b^T P A, \quad (16)$$

we have the optimal input as

$$u = -\{(R + b^T P b)^{-1} b^T P (A - b R^{-1} S^T) + R^{-1} S^T\}x. \quad (17)$$

As a result, the design process is, first to determine the weight parameter  $Q_s$  and  $R_s$  in the continuous performance function, then calculate the  $Q$ ,  $S$  and  $R$  using above relationship (14) and (15), finally using the conventional digital LQR algorithm, the optimal sampled-data controller can be obtained.

### B. Sampled-data controller with integrators

In this section, we use the above results into the extended error system of the DC motor expressed in section II, which includes an integrator.

First, we define the continuous performance function related with the extended error plant as follows.

$$J = \int_0^{\infty} \{x_1^T(t) \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} x_1(t) + \hat{u}^T(t) R \hat{u}(t)\} dt \quad (18)$$

The sampling-data extended error system can be obtained as

$$\begin{bmatrix} x_1(nh+p) \\ \hat{u}(nh+p) \end{bmatrix} = \begin{bmatrix} \exp \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} p & \int_0^p (\exp \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} m) \begin{bmatrix} b \\ 0 \end{bmatrix} dm \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1(nh) \\ \hat{u}(nh) \end{bmatrix}$$

Then the above continuous performance function can be rewritten as

$$\begin{aligned} J &= \int_0^{\infty} [\bar{x}^T(t) \bar{w}^T(t) \hat{u}^T(t)] \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{w}(t) \\ \hat{u}(t) \end{bmatrix} dt \\ &= \sum_{n=0}^{\infty} \left\{ \int_{nh}^{nh+h} [\bar{x}^T(nh+p) \bar{w}^T(nh+p) \hat{u}^T(nh+p)] \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} \bar{x}(nh+p) \\ \bar{w}(nh+p) \\ \hat{u}(nh+p) \end{bmatrix} dp \right\} \\ &= \sum_{n=0}^{\infty} \left\{ [x_1^T(nh) \hat{u}^T(nh)] \int_0^h \left[ \exp \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} p \int_0^p (\exp \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} m) \begin{bmatrix} b \\ 0 \end{bmatrix} dm \right] \begin{bmatrix} b \\ 0 \end{bmatrix} dp \right] \begin{bmatrix} x_1(nh) \\ \hat{u}(nh) \end{bmatrix} \right\} \end{aligned}$$

Let

$$\begin{bmatrix} \alpha_1 & \beta_2^T & \alpha_2 \\ \beta_2 & \beta_1 & \alpha_3 \\ \alpha_2^T & \alpha_3 & R_1 \end{bmatrix} = \begin{bmatrix} \int_0^h \left[ \exp \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} p \int_0^p (\exp \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} m) \begin{bmatrix} b \\ 0 \end{bmatrix} dm \right] \begin{bmatrix} b \\ 0 \end{bmatrix} dm \right]^T \\ \times \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} \int_0^h \left[ \exp \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} p \int_0^p (\exp \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} m) \begin{bmatrix} b \\ 0 \end{bmatrix} dm \right] \begin{bmatrix} b \\ 0 \end{bmatrix} dm \right] dt \end{bmatrix}$$

$$(19)$$

we have

$$\begin{aligned} &= \sum_{n=0}^{\infty} \{x_1^T(nh) Q x_1(nh) + 2x_1^T(nh) N \hat{u}(nh) + \hat{u}^T(nh) R_1 \hat{u}(nh)\} \\ &= \sum_{n=0}^{\infty} \{x_1^T(nh) Q_1 x_1(nh) + u_1^T(nh) R_1 u_1(nh)\} \end{aligned} \quad (20)$$

where

$$\begin{aligned} x_1(nh) &= \begin{bmatrix} \bar{x}(nh) \\ \bar{w}(nh) \end{bmatrix}, \quad Q = \begin{bmatrix} \alpha_1 & \beta_2^T \\ \beta_2 & \beta_1 \end{bmatrix}, \quad N = \begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix}. \\ Q_1 &= Q - N R_1^{-1} N^T, \\ u_1(nh) &= \hat{u}(nh) + R_1^{-1} N^T x_1(nh) \end{aligned} \quad (21)$$

For the digital performance function (20) and the sample model of extended error plant (9), using the conventional digital LQR algorithm, the optimal control input can be obtained as follows.

$$u_1(nh) = -F_1 x_1(nh) = -(R_1 + b_1^T P_1 b_1)^{-1} b_1^T P_1 A_1 x_1(nh)$$

where,  $P_1$  satisfies the following Riccati equation.

$$P_1 = A_1^T P_1 A_1 + Q_1 - A_1^T P_1 b_1 (R_1 + b_1^T P_1 b_1)^{-1} b_1^T P_1 A_1$$

Substitute above equation into (21), we have

$$\begin{aligned} \hat{u}(nh) &= u_1(nh) - R_1^{-1} N^T x_1(nh) \\ &= -\{(R_1 + b_1^T P_1 b_1)^{-1} b_1^T P_1 A_1 + R_1^{-1} N^T\} x_1(nh) \\ &= -F_e x_1(nh) \end{aligned}$$

From (13), we can determine the feedback gain matrix  $F$  and  $K$  in the Fig.1 as follows.

$$[F \quad -K] = F_e = (R_1 + b_1^T P_1 b_1)^{-1} b_1^T P_1 A_1 + R_1^{-1} N^T \quad (22)$$

## IV. SIMULATION MODEL AND RESULTS

### A. Simulation model

Some software can be used to solve the sampled-data controller problem, but in order to understand the detail design process and compare with the conventional digital LQR controller, we use Matlab/Simlink, symbolic tool box, and control tool box to make the simulation model. Because we have derived the detail of design process at above section, so the simulation model will be easy to construct and understand. The parameters of a DC motor are select as follows.

$$\begin{aligned} R_a &= 1.2[\Omega], \quad L_a = 10^{-3}[H], \quad k_e = 0.0707, \\ J &= 4.705 \times 10^{-5}, \quad k_T = 0.00707 \end{aligned}$$

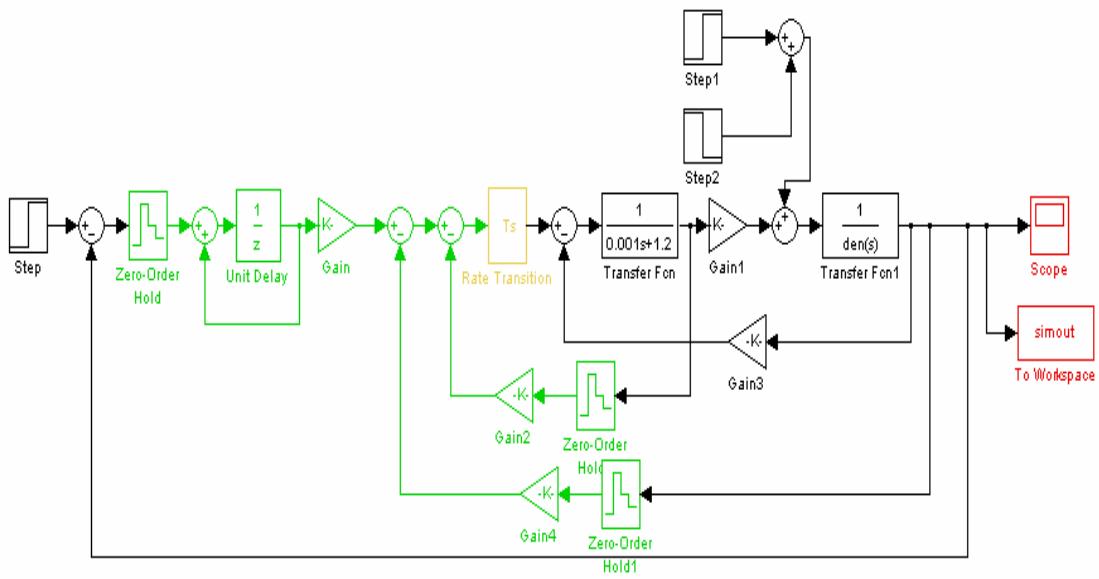


Fig. 2 Simulation model

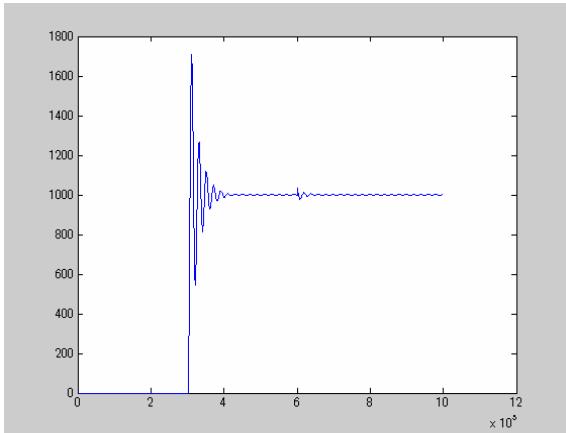


Fig.3 (a) Speed (Conventional LQG)

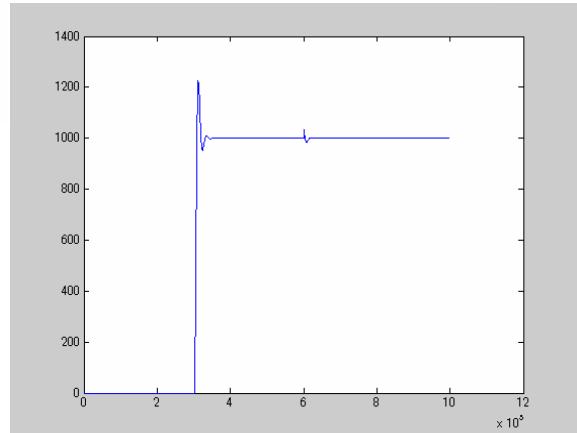


Fig.3 (b) Speed (Proposed sampled-data method)

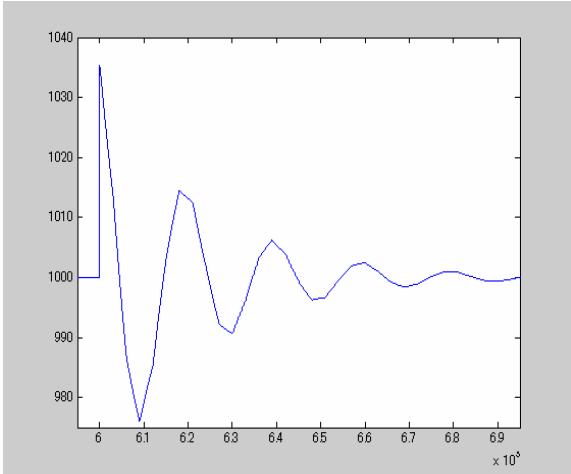


Fig.4 (a) Extended part of Fig.3 (a) (Load changed)

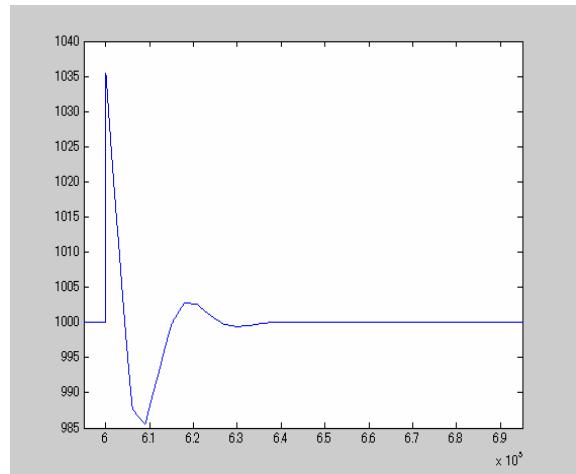


Fig.4 (b) Extended part of Fig.3 (b) (Load changed)

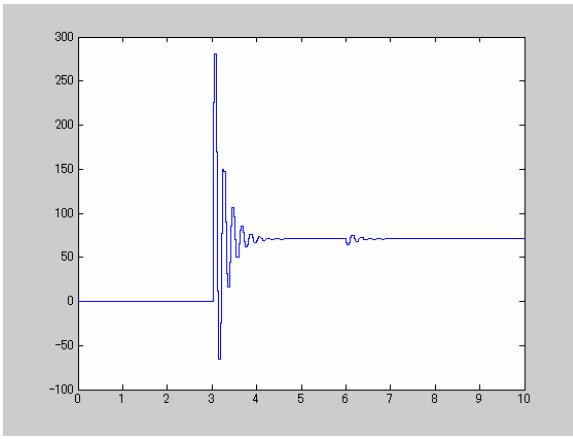


Fig.5. (a) Motor current wave (conventional LQG)

The digital controller has the sampling time 0.03(sec) and the simulation step of the DC motor has the sampling time  $10^{-5}$ (sec), the two parts are connected by “Rate transition” block. The commands of “lqr” and “dlqr” are used to design the sampled-data LQR controller and the conventional digital LQR controller, respectively. The total system is constructed in Simulink. The load is suddenly changed from 0 to 1[N.m] at 6[sec]. For conventional digital LQR design, the extended error plant (7) is sampled by sampling time 0.03(sec), the digital performance function is set as follow

$$J_d = \sum_{k=0}^{\infty} \{x_i(k)^T Q_d x_i(k) + \hat{u}(k)^T R_d \hat{u}(k)\},$$

and the weight parameters are selected as follows.

$$Q_d = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad R_d = 1$$

For the continuous performance function, we select the weight parameters as follows

$$Q_s = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_s = 1.$$

Fig.3 is the speed responses, it is apparent that the conventional LQG method shows high vibratory characteristics and the proposed sampled-data integrator controller shows suppressed vibratory characteristics. That means the interval vibration can be suppressed using sampled-data design method compared with the conventional LQG method. An important characteristic of the DC motor control is that the load is an external input and has a wide variable range. It is difficult to control the infection of such wide variable parameter. Fig.4 shows the proposed sampled-data integrator controller can also suppress the infection of such case. That is very useful in practical situations. Another important characteristic in the DC motor

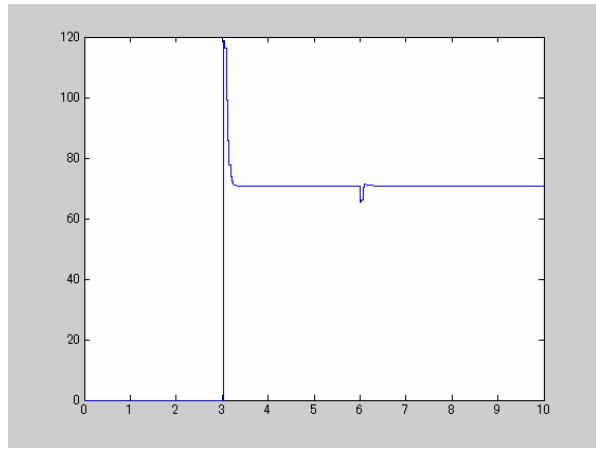


Fig.5 (b) Motor current wave (Proposed method)

control is to suppress the current of the motor as low as possible in order to save the electrical energy. Fig.5 also shows the proposed method has good performance.

## V. CONCLUSION

A sampled-data integrator controller is proposed in familiar way from conventional LQG design style. That provides a simple way for practical engineer to take the sampled-data control technique into practical applications. The simulation results show many benefits of the sampled-data integrator controllers, such as quickly response but low vibrations, suppressing interval vibrations even when the load is suddenly changed and low control signal.

The experimental verification will be a future work.

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