Homework 1

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1 Problem 1

Using the hint, we parameterize our variable θ, and let

$$z = e^{i\theta}.$$

Obviously we pick our contour then to be the unit circle. Since θ goes from 0 to 2π, this contour is exactly one full unit circle. Now we use our parameterization to rewrite our integral

$$dz = ie^{i\theta}d\theta,$$

$$d\theta = -i\frac{1}{z}d.z$$

Also we know that

$$sin\theta = \frac{1}{2i}(z - \frac{1}{z})$$

The integral then becomes

$$\int_\gamma \frac{-i\frac{1}{z}dz}{2 - \frac{1}{2i}(z - \frac{1}{z})}. \tag{1}$$

After factoring out $2i$ and multiply a z on the top and bottom at the same time, equation (1) then becomes

$$\int_\gamma \frac{2dz}{4iz - z^2 + 1}. \tag{2}$$

It is nice since the poles now occur whenever the denominator is zero. And we observe that the denominator is a quadratic equation, using the root-finding formula we find the roots are

$$z_+ = 2i - \sqrt{3}i,$$

and

$$z_- = 2i + \sqrt{3}i.$$

Only z_+ lies inside the contour, therefore the integral is evaluated as

$$2\pi i \cdot Res(f(z_+)) = 2\pi i \cdot \frac{2}{4i - 2(2 - \sqrt{3})i} = \frac{4\pi}{2\sqrt{3}} = \boxed{\frac{2\pi}{\sqrt{3}}}.$$

2 Problem 2

Recall in class we found out that

$$\int_{-\infty}^{\infty} e^{iax} f(x) = 2\pi i \sum Res(e^{iaz} f(z)). \tag{3}$$

Using the Euler formula for the e^{iaz}, we have

$$\int_{-\infty}^{\infty} cos(ax) f(x) = Re[2\pi i \sum Res(e^{iaz} f(z))]. \tag{4}$$

So in the upper half plane, where e^{ibx} (in our case it's b in stead of a) behaves well, we find the poles are

$$x_+ = ai,$$

and

$$x_- = -ai.$$

Only x_+ lies in the UHP, therefore

$$Res(\frac{e^{ibx}}{z^2 + a^2})|_{ai} = \frac{e^{-ab}}{2ai}.$$

So the original integral becomes

$$Re[2\pi i \cdot \frac{e^{-ab}}{2ai}] = \boxed{\frac{\pi e^{-ab}}{a}}.$$

3 Problem 3 (not-brave method)

Given function

$$G(\omega) = \int_{-2}^{2} \frac{1}{2\pi} \frac{\sqrt{4-\epsilon^2}}{\omega - \epsilon + i\delta} d\epsilon, \delta \to 0^+, \tag{5}$$

we immediately notice that when $|\omega| > 2$, there exist no poles since the denominator is never zero. When $|\omega| \leq 2$, there exist poles. And the residues at those poles are

$$\frac{1}{2\pi} \frac{\sqrt{4-\omega^2}}{-1} = \frac{-1}{2\pi} \sqrt{4-\omega^2},$$

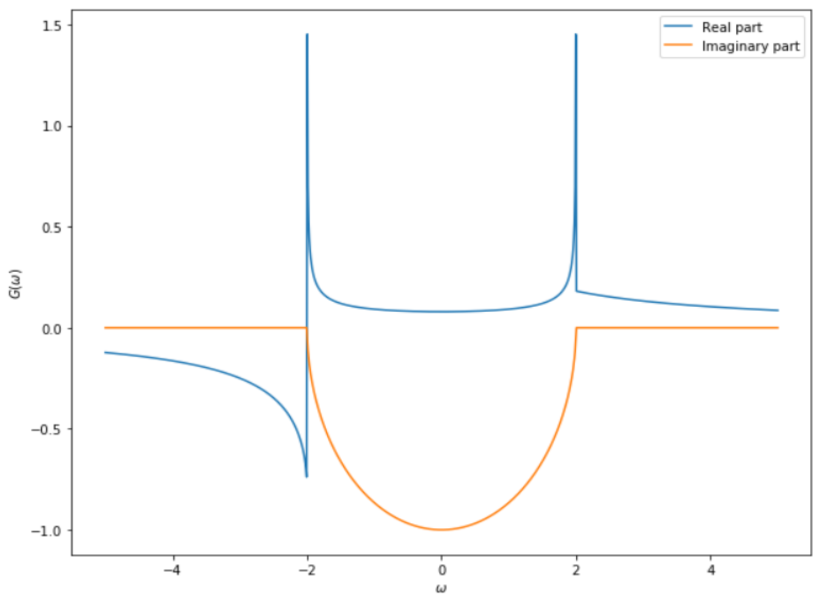
which goes to the calculation of the imaginary part of the integral (the result is just this without a factor of π). The real part then comes from principal value integration, which we derived in class

$$PV \int_{-2}^{2} f(\epsilon) d\epsilon = \int_{0}^{2} \frac{g(\omega+\epsilon) - g(\omega-\epsilon)}{\epsilon} \approx \int_{0}^{2} 2g\prime(\omega),$$

where

$$f(\epsilon) = \frac{g(\epsilon)}{\epsilon - \omega} = \frac{\sqrt{4-\epsilon^2}}{\omega - \epsilon}.$$

Combining the real and imaginary parts, I have have my plot below.



4 Problem 4

Given function

$$G(z) = \frac{1}{1+e^{-z}},$$

the poles occur when $e^{-z} = -1$, that is

$$z = i(2n+1)\pi,$$

where n is integer-valued. Hence the residues are

$$\frac{1}{-e^{-i(2n+1)\pi}} = \frac{1}{-(-1)} = \boxed{1}.$$

So this function $G(z)$ has simple poles at integers with residue 1, which is very convenient for us because we can now use the technique derived during class. Let $f(z)$ be

$$\sum_{-\infty}^{\infty} e^{-z\tau}\frac{1}{z+c}.$$

If we carefully draw our contour, and let it deform infinitely large, there will be no poles in the plane of $f(z)$ except at $-c$, assuming τ makes the numerator well-behaved, which is our case. Then we shall have

$$\int_\gamma G(z)f(z)dz = 0 = 2\pi i \sum f(z) + 2\pi i[Res(G(z=-c))].$$

Cancelling out factor $2\pi i$, we are left with a simple expression for the summation

$$\sum f(z) = -[Res(G(z=-c))] = -Res(\frac{e^{-z\tau}}{(1+e^{-z})(1+z)})|_{-c} = \boxed{-\frac{e^{c\tau}}{1+e^{c}}}.$$

In this case where $c = 1$, and $\tau = 0.5$, the sum then becomes

$$\boxed{-\frac{\sqrt{e}}{1+e}}. \tag{6}$$

The exact result calculated using wolfram-alpha is **-0.4434**, which takes roughly **1500** numerical steps of my code to reach.