Homework 3

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1 Problem 1

Given

$$H = -\mu c^+ c, \tag{1}$$

and let

$$S(U) = exp\bigg(- i \int \theta(t)\theta(t_0 - t)Uc^+ cdt\bigg), \tag{2}$$

be a phase factor of the time-ordered product of the Green's function.

1.1 1(a)

The generalized partition function is given

$$Z(U) = Tr\{Te^{-i\int \theta(t)\theta(t_0-t)Uc^+(t)c(t)dt}e^{-\beta H}\} \tag{3}$$

For the case of spinless fermions, we only have two possible states $|0\rangle$ and $|1\rangle$, therefore

$$\begin{aligned} Z(U) &= \langle 0| e^{-i\int \theta(t)\theta(t_0-t)Uc^+(t)c(t)dt}e^{-\beta H} |0\rangle \\ &+ \langle 1| e^{-i\int \theta(t)\theta(t_0-t)Uc^+(t)c(t)dt}e^{-\beta H} |1\rangle \\ &= e^{-i\int \theta(t)\theta(t_0-t)Uc^+(t)c(t)dt} \langle 0|0\rangle \\ +e^{-i\int \theta(t)\theta(t_0-t)Uc^+(t)c(t)dt} &\langle 1| e^{-\beta(-\mu N)} |1\rangle \\ &= 1 + e^{-i\int_0^{t_0} Udt} \langle 1| e^{-\beta(-1)\mu} |1\rangle \\ &= \boxed{1 + e^{-iUt_0}e^{\beta\mu}}. \end{aligned} \tag{4}$$

1.2 1(b)

See attachment.

2 Problem 2

Given the Hamiltonian in real space

$$H = -t\sum_{ij\sigma}(c^+_{i\sigma}c_{j\sigma} + c^+_{j\sigma}c_{i\sigma}) + U\sum_i(c^+_{i\uparrow}c_{i\uparrow}c^+_{i\downarrow}c_{i\downarrow}). \tag{5}$$

2.1 2(a)

Using

$$c_{j\sigma} = \frac{1}{\sqrt{N}} \sum_k e^{ik\cdot r_j} c_{k\sigma}$$
$$c^+_{j\sigma} = \frac{1}{\sqrt{N}} \sum_k e^{-ik\cdot r_j} c^+_{k\sigma}. \tag{6}$$

We Fourier-transform the Hamiltonian into momentum space

$$\begin{aligned}
H = -t\sum_{ij\sigma} \frac{1}{N} \sum_{kk'} \left(e^{-ik\cdot r_i} e^{ik'\cdot r_j} c^+_{k\sigma} c_{k'\sigma} + e^{-ik\cdot r_j} e^{ik'\cdot r_i} c^+_{k'\sigma} c_{k\sigma} \right) \\
+U \sum_i \frac{1}{N^2} \sum_{k_1k_2k_3k_4} e^{-ik_1\cdot r_i} e^{ik_2\cdot r_i} e^{-ik_3\cdot r_i} e^{ik_4\cdot r_i} c^+_{k_1\uparrow} c_{k_2\uparrow} c^+_{k_3\downarrow} c_{k_4\downarrow} \\
= -t\sum_{k\sigma\delta} \frac{1}{N} \left(e^{-ik\cdot r_i} e^{ik\cdot(r_i+\delta)} c^+_{k\sigma} c_{k\sigma} + e^{-ik\cdot(r_i+\delta)} e^{ik\cdot r_i} c^+_{k\sigma} c_{k\sigma} \right) + \frac{U}{N} \sum_{k_1k_2k_3} c^+_{k_1\uparrow} c_{k_2\uparrow} c^+_{k_3\downarrow} c_{k_1-k_2+k_3\downarrow} \\
= -\frac{t}{2} \sum_{\sigma k\delta} \frac{1}{N} \left(e^{ik\cdot\delta} + e^{-ik\cdot\delta} \right) c^+_{k\sigma} c_{k\sigma} + \frac{U}{N} \sum_{k_1k_2k_3} c^+_{k_1+k_2-k_3\uparrow} c_{k_2\uparrow} c^+_{k_3\downarrow} c_{k_1\downarrow} \\
= \boxed{\sum_{k\sigma} \epsilon_k c^+_{k\sigma} c_{k\sigma} + \frac{U}{N} \sum_{k_1k_2k_3} c^+_{k_1+k_2-k_3\uparrow} c_{k_2\uparrow} c^+_{k_3\downarrow} c_{k_1\downarrow}},
\end{aligned} \tag{7}$$

with $\epsilon_k = -t\sum_\delta \cos(k\cdot\delta) = -\frac{t}{2}\sum_\delta (e^{ik\cdot\delta} + e^{-ik\cdot\delta})$.

2.2 2(b)

$$i\frac{d}{dt}c_{k\sigma}(t) = -e^{iHt}[H, c_{k\sigma}]e^{-iHt}$$

$$= -e^{iHt}\Big[\sum_{k\sigma}\epsilon_k n_{k\sigma}, c_{k\sigma}\Big]e^{-iHt}] - e^{iHt}\Big[\sum_{k_1k_2k_3} c^+_{k_1\uparrow}c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k_4\downarrow}, c_{k\sigma}\Big]$$

$$= e^{iHt}\sum_{k\sigma}\epsilon_k c_{k\sigma}e^{-iHt}\delta_{kk}$$

$$-\sum_{k_1k_2k_3} e^{iHt}\Big(c^+_{k_1\uparrow}c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow} - c_{k\sigma}c^+_{k_1\uparrow}c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow}\Big)e^{-iHt}$$

$$+c^+_{k_1\uparrow}c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k\sigma}c_{k_1-k_2+k_3\downarrow} + (\delta_{kk_1}\delta_{\sigma\uparrow} - c^+_{k_1\uparrow}c_{k\sigma})c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow}$$

$$= e^{iHt}\epsilon_k c_{k\sigma}e^{-iHt}$$

$$+\sum_{k_1k_2k_3} c^+_{k_1\uparrow}c_{k_2\uparrow}(\delta_{kk_3}\delta_{\sigma\downarrow} - c_{k\sigma}c^+_{k_3\downarrow})c_{k_1-k_2+k_3\downarrow}$$

$$+\delta_{kk_1}\delta_{\sigma\uparrow}c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow}$$

$$+c^+_{k_1\uparrow}c_{k_2\uparrow}c_{k\sigma}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow}$$

$$= e^{iHt}\epsilon_k c_{k\sigma}e^{-iHt}$$

$$+\sum_{k_1k_2k_3} c^+_{k_1\uparrow}c_{k_2\uparrow}\delta_{kk_3}\delta_{\sigma\downarrow}c_{k_1-k_2+k_3\downarrow} - \delta_{kk_1}\delta_{\sigma\uparrow}c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow}$$

$$-c^+_{k_1\uparrow}c_{k_2\uparrow}c_{k\sigma}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow} + c^+_{k_1\uparrow}c_{k_2\uparrow}c_{k\sigma}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow}$$

$$= e^{iHt}\epsilon_k c_{k\sigma}e^{-iHt} + \sum_{k_1k_2k_3} c^+_{k_1\uparrow}c_{k_2\uparrow}\delta_{kk_3}\delta_{\sigma\downarrow}c_{k_1-k_2+k_3\downarrow} + \delta_{kk_1}\delta_{\sigma\uparrow}c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k_1-k_2+k_3\downarrow}$$

$$= \boxed{e^{iHt}\epsilon_k c_{k\sigma}e^{-iHt} + \sum_{k_1k_2} c^+_{k_1\uparrow}c_{k_2\uparrow}c_{k_1-k_2+k\downarrow} - \sum_{k_2k_3} c_{k_2\uparrow}c^+_{k_3\downarrow}c_{k-k_2+k_3\downarrow}} \tag{8}$$

2.3 2(c)

Let

$$g_\sigma(k, t-t') = -i < T[c_{k\sigma}(t), c^+_{k_\sigma}(t')] >, \tag{9}$$

we are to find the equation satisfied by

$$\Big[i\frac{d}{dt} - \epsilon_k\Big]g_\sigma(k, t-t') =? \tag{10}$$

Just plug the definition into the diff eqn, we get

$$\Big[i\frac{d}{dt} - \epsilon_k\Big]g_\sigma(k, t-t') = \Big[i\frac{d}{dt} - \epsilon_k\Big](-i)\Big(\theta(t-t') < c_{k\sigma}(t)c^+_{k\sigma}(t') > -\theta(t-t') < c^+_{k\sigma}(t')c_{k\sigma}(t) > \Big)$$

$$= \delta(t-t')\Big(c_{k\sigma}(t), c^+_{k\sigma}(t')\Big)_+$$

$$+i\frac{U}{N}\Big(\theta(t-t') < \Big(c^+_{k_1\uparrow}(t)c_{k_2\uparrow}(t)c_{k_1-k_2+k\downarrow}(t) + c_{k_2\uparrow}(t)c^+_{k_3\downarrow}(t)c_{k-k_2+k_3\downarrow}(t)\Big)c^+_{k\sigma}(t') >$$

$$-\theta(t'-t) < c^+_{k\sigma}(t')\Big(c^+_{k_1\uparrow}(t)c_{k_2\uparrow}(t)c_{k_1-k_2+k\downarrow}(t) + c_{k_2\uparrow}(t)c^+_{k_3\downarrow}(t)c_{k-k_2+k_3\downarrow}(t)\Big) > \Big) \tag{11}$$

Define

$$g^{\sigma_1\sigma_2;\sigma_1'\sigma_2'}(k_1t_1k_2t_2;k_1't_1'k_2't_2') = (-i)^2 < T[c_{k_1\sigma_1}(t_1)c_{k_2\sigma_2}(t_2)c^+_{k_2'\sigma_2'}(t_2')c^+_{k_1'\sigma_1'}(t_1')] > \tag{12}$$

and use this to simplify equation 11, we get

$$\boxed{\delta(t-t') + \sum_{k_1k_2}\frac{iU}{N}g^{\uparrow\downarrow;\uparrow\sigma}(k_2t, k_1-k_2+kt;k_1t,kt') + \sum_{k_2k_3}\frac{iU}{N}g^{\uparrow\downarrow;\uparrow\sigma}(k_2t, k-k_2+k_3t;k_3t,kt')} \tag{13}$$

3 Problem 3

Recall in class we short-handed

$$\rho_n = \rho_m e^{-\beta(E_n-E_m-\mu(N_n-N_m))} = \rho_m e^{-\beta(\omega-\mu(N_n-N_m))} = \rho_m e^{-\beta(\omega-\mu)} \tag{14}$$

with $N_n - N_m = 1$. As $T \to 0$, and $\omega > \mu$

$$e^{-\beta(\omega-\mu)} = e^{\frac{\mu-\omega}{T}} \to e^{-\frac{+}{0}} \to 0 \tag{15}$$

Since the Green's function $\tilde{g}^<(k,\omega)$ is proportional to ρ_n, $\boxed{\tilde{g}^<(k,\omega)=0}$ when $\omega > \mu$. Similarly,

$$\rho_m = \rho_n e^{\beta(\omega-\mu)} = \rho_n e^{-\beta(\mu-\omega)} \tag{16}$$

therefore

$$\tilde{g}^>(k,\omega) = -e^{-\beta(\mu-\omega)}\tilde{g}^<(k,\omega) \tag{17}$$

and when $\mu > \omega$, we get

$$e^{-\frac{+}{0}} \to 0 \to \boxed{\tilde{g}^>(k,\omega)=0} \tag{18}$$

Recall that

$$\tilde{g}^>(k,\omega) = -2i\pi\delta(\omega-\epsilon_k)(1-f(\omega) \tag{19}$$

as $T=0$, $\mu=\epsilon_k$, and as $\omega<\mu$, $f(\omega)=1$, therefore

$$\tilde{g}^>(k,\omega) = -2i\pi\delta(\omega-\epsilon_k)(1-1) = 0 \tag{20}$$

Similarly, when $\omega > \mu$, at $T=0$

$$\tilde{g}^<(k,\omega) = 2i\pi\delta(\omega-\epsilon_k)(0) = 0 \tag{21}$$

At $T=0$, the Fermi-Dirac distribution takes values

$$f(\omega) = \begin{cases} 1 & \omega < \mu \\ 0 & \omega > \mu \end{cases} \tag{22}$$

as $\mu = \epsilon_F$. The spectral function then can be defined as

$$A(k,\omega) = \frac{i}{2\pi}\left(\tilde{g}^>(k,\omega) - \tilde{g}^<(k,\omega)\right) \tag{23}$$

and

$$\boxed{\tilde{g}^>(k,\omega) = \lim_{\omega\to\mu^+}\int d\omega' \frac{A(k,\omega)}{\omega-\omega'}} \tag{24}$$

$$\boxed{\tilde{g}^<(k,\omega) = \lim_{\omega\to\mu^-}\int d\omega' \frac{A(k,\omega)}{\omega-\omega'}} \tag{25}$$

4 Problem 4

Recall that

$$A(k,\omega) = \frac{i}{2\pi}(G^R - G^A) \tag{26}$$

Given

$$I = \lim_{\delta \to 0^-} \int d\omega e^{-\omega\delta} F(\omega) A(k,\omega) f(\omega) \tag{27}$$

first plug the spectral into the definition and get

$$= \int_{\gamma_A} d\omega F(\omega) \frac{i}{2\pi} G^R(\omega) f(\omega) - \int_{\gamma_B} d\omega F(\omega) \frac{i}{2\pi} G^A(\omega) f(\omega) \tag{28}$$

Recall that in HW 1 problem 4, the poles of the function

$$G(z) = \frac{1}{1+e^{-z}}$$

are $z = i\pi(2n+1)$, with residues 1. Obviously the function $G(z)$ is the Fermi-Dirac function $f(\omega)$ here. Using the conclusion we landed for that problem

$$\int_\gamma G(z) f(z) dz = 0 = 2\pi i \sum f(z) + 2\pi i [Res(G(z=-c))] \tag{29}$$

equation 28 becomes

$$= \frac{i}{2\pi} Res\left(\frac{F(\omega)G^R(\omega)}{e^{\beta\omega}+1}\right) - \frac{i}{2\pi} Res\left(\frac{F(\omega)G^A(\omega)}{e^{\beta\omega}+1}\right) \tag{30}$$

the poles are at the fermionic matsubara frequencies $\omega = i\omega_n$, and taking the derivative of the denominator, we get

$$= 2\pi i \left(\frac{i}{2\pi} \frac{F(i\omega_n)G^R(i\omega_n)}{\beta e^{\beta i\omega_n}} - \frac{i}{2\pi} \frac{F(i\omega_n)G^A(i\omega_n)}{\beta e^{\beta i\omega_n}}\right) \tag{31}$$

since $e^{\beta i\omega_n} = \pm 1$, and $T = \frac{1}{\beta}$, we get

$$I = \mp\left(T\sum_n F(i\omega_n)G^R(i\omega_n) - T\sum_n F(i\omega_n)G^A(i\omega_n)\right) = \boxed{\mp T\sum_n F(i\omega_n)G(k,i\omega_n)} \tag{32}$$