Homework 4

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1 Problem 1

Given the retarded Green's function for a Fermi liquid

$$g^R(k,\omega) = \frac{z}{\omega - z(\epsilon_k - \mu) + i\gamma} + g_{inc}(k,\omega), \tag{1}$$

with $0 \leq z \leq 1$ and γ proportional to T^2, the spectral function at $\omega = 0$ is then

$$\begin{aligned} A(k,\omega=0) &= -\frac{1}{\pi}\operatorname{Im}\{g^R(k,\omega=0)\} \qquad (2)\\ &= -\frac{1}{\pi}\operatorname{Im}\left\{\frac{-z^2(\epsilon_k-\mu) - iz\gamma}{z^2(\epsilon_k-\mu)+\gamma^2}\right\} \\ &= -\frac{1}{\pi}\frac{-z\gamma}{z^2(\epsilon_k-\mu)^2+\gamma^2} \\ &= \frac{z\gamma}{\pi((z(\epsilon_k-\mu))^2)+\gamma^2} \end{aligned} \tag{3}$$

For non-interacting systems, $z = 1$, and $\gamma \to 0^+$, so we get the density of states at $\omega = 0$

$$\begin{aligned} \rho(\omega=0) &= \frac{1}{\pi}\sum_k \frac{z\gamma}{\pi((z(\epsilon_k-\mu))^2)+\gamma^2} \qquad (4)\\ &= \frac{\gamma/z}{\pi}\sum_k \frac{1}{(\epsilon_k-\mu)^2+(\frac{\gamma}{z})^2} \\ &\approx \boxed{\frac{\gamma/z}{\pi}\sum_k \frac{1}{(\epsilon_k-\mu)^2}} \\ &= \frac{1}{z}\rho_0(\omega=0), \end{aligned}$$

where

$$\rho_0(\omega=0) = \frac{\gamma}{\pi}\sum_k \frac{1}{(\epsilon_k-\mu)^2}. \tag{5}$$

2 Problem 2

2.1 a

The local DOS for a spin-1/2 problem is

$$A(\omega) = 2(1-w_1)\rho_{bethe}(\omega) + w_1\rho_{bethe}(\omega-1), \tag{6}$$

as we let $t^* = U = 1$.

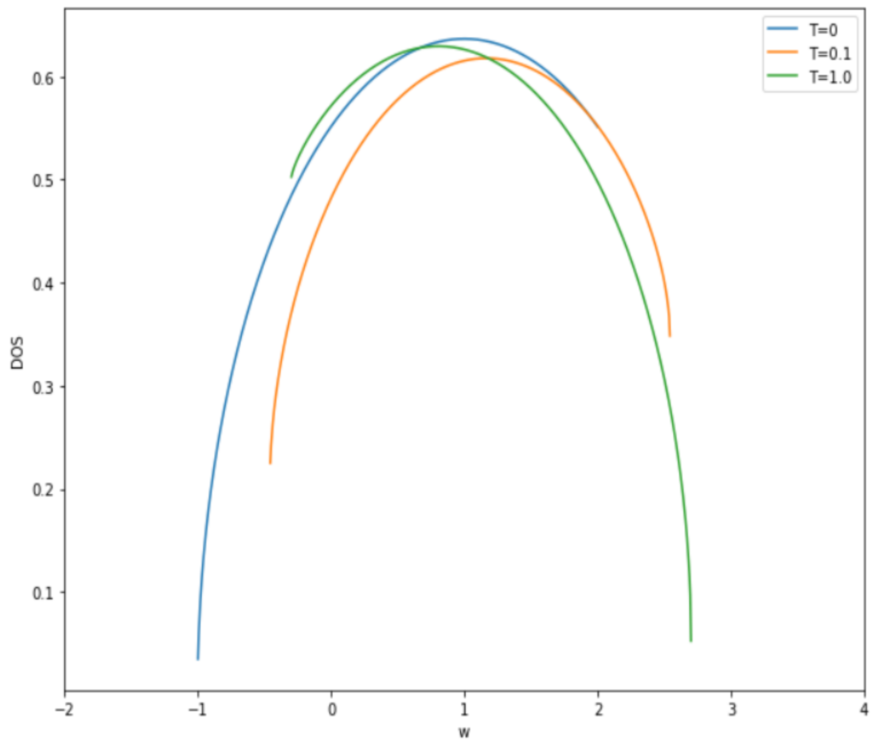

Figure 1: The chemical potential found for $T = 0.1$ is -0.54199204, and the chemical potential at $T = 1$ is -0.70044287. The graph is plotted with ω located at 0, as I subtracted the chemical potential from the ω for nonzero temperatures. The plot for $T = 0$ is unchanged.

2.2 b

The electron density for this part is carried out using

$$n_c = 2T \sum_n (G(i\omega_n) - \frac{1}{i\omega_n + \mu}) + 2f(-\mu), \tag{7}$$

where f is the Fermi-Dirac distribution, after appropriate regularization. If we use the same number of positive and negative matsubara frequencies for the sum, we will end up getting a zero imaginary part. I chose to use 10000 negative and 10000 positive matsubara frequencies. (If I used 10001 matsubara frequencies, the ratio of this result versus the result I got from using 10000 n's is bigger than 0.9999) For $T = 0.01$, the numerical result gives $\boxed{0.17311912}$. For $T = 0.1$, $n_c = \boxed{0.34062638}$. And for $T = 1.0$, $n_c = \boxed{0.74426769}$. (*The analytic results disagree with the numerical ones, so I don't think I got them correctly.)

2.3 c

$$g^R(\omega) = \int d\omega' \frac{A(\omega')}{\omega - \omega' + i\delta} = \int d\omega' \frac{\rho_{bethe}(\omega')}{\omega + \mu - \Sigma(\omega) - \omega'} \tag{8}$$

$$= \frac{1 - w_1}{2\pi} \int_{-2}^{2} d\omega' \frac{2\sqrt{4 - \omega'^2}}{\omega - \omega' + i\delta} + \frac{w_1}{2\pi} \int_{-1}^{3} d\omega' \frac{2\sqrt{4 - (\omega' - 1)^2}}{\omega - (\omega' - 1) + i\delta} = \frac{1}{2}(\omega + \mu - \Sigma(\omega) - \sqrt{(\omega + \mu - \Sigma(\omega))^2 - 4})$$

If we shift ω' by 1, the second part of the first integral will have a range of -2 to 2, cancelling the w_1 term too. It becomes

$$g^R = \frac{1}{\pi} \int_{-2}^{2} \frac{\sqrt{4 - \omega'^2}}{\omega - \omega' + i\delta} d\omega' = \omega - \sqrt{\omega^2 - 4} \tag{9}$$
$$= \frac{1}{2}(\omega + \mu - \Sigma(\omega) - \sqrt{(\omega + \mu - \Sigma(\omega))^2 - 4}).$$

(I just used the results derived in the lecture notes). Therefore

$$\Sigma(\omega) = \boxed{\frac{1}{4}(4\mu + 3\sqrt{\omega^2 - 4} - \omega)}. \tag{10}$$

(Since it doesn't contain w_1, I think it is incorrect, therefore I didn't graph it. Since the real part doesn't look to approach zero when ω is small, using Kramers-Kronig, the imaginary part isn't approaching zero as well. So based on this wrong relation, it is **not** a fermi liquid.)

3 Problem 3

Given that

$$n = \frac{\mathrm{Tr}\{e^{-\beta H} c^+ c\}}{z}, \tag{11}$$

where $z = \mathrm{Tr}\{e^{-\beta H}\}$, and

$$G(\tau) = \frac{-\mathrm{Tr}\{e^{-\beta H} T_\tau c(\tau) c^+(0)\}}{z}, \tag{12}$$

we are to figure out the following.

3.1 a

$$\lim_{\tau \to 0^+} G(\tau) = \frac{-\mathrm{Tr}\{e^{-\beta H}\} c c^+}{z} \tag{13}$$
$$= -\frac{\mathrm{Tr}\{e^{-\beta H}\}(1 - c^+ c)}{z}$$
$$= -\frac{\mathrm{Tr}\{e^{-\beta H}\}}{z} + \frac{\mathrm{Tr}\{e^{-\beta H}\} c^+ c}{z}$$
$$= \boxed{-1 + n}.$$

3.2 b

$$\lim_{\tau \to 0^-} G(\tau) = \frac{\mathrm{Tr}\{e^{-\beta H}\} c^+ c}{z} \tag{14}$$
$$= \boxed{n}.$$

3.3 c

$$\lim_{\tau \to \beta^+} G(\tau) = - \lim_{\tau \to 0+} G(\tau) = \boxed{1 - n}. \tag{15}$$

Since G is anti-periodic with period β.

3.4 d

$$\lim_{\tau \to \beta^-} G(\tau) = - \lim_{\tau \to 0^-} = \boxed{-n}. \tag{16}$$