Sum Over Unitaries

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1 Amplitude Amplification

Given a known state $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, and two mutually orthogonal subspaces that span $\mathcal{H}$, the 'good' subspace $\mathcal{H}_G$ and the 'bad' subspace $\mathcal{H}_B$, our goal is to evolve the initial state $|\psi\rangle$ into $\mathcal{H}_G$. We define a Hermitian P_G to be the projection operator $\mathcal{H} \to \mathcal{H}_G$. Then we have

$$|\psi\rangle = P_G |\psi\rangle + (1 - P_G) |\psi\rangle = \sin\theta |\psi_G\rangle + \cos\theta |\psi_B\rangle , \tag{1}$$

where $\theta = \arcsin(|P_G |\psi\rangle|)$, and $|\psi_G\rangle$ and $|\psi_B\rangle$ are the normalized projections of $|\psi\rangle$ into $\mathcal{H}_G$ and $\mathcal{H}_B$. Here notice that the probability of finding the system in 'good' state is $\sin^2\theta$.
Further define

$$|\psi^\perp\rangle = \cos(\theta) |\psi_G\rangle - \sin(\theta) |\psi_B\rangle , \tag{2}$$

such that $\langle\psi^\perp|\psi\rangle = 0$.
Multiply equation 1 by $\sin\theta$ and equation 2 by $\cos\theta$, we get

$$\sin(\theta) |\psi\rangle = \sin^2(\theta) |\psi_G\rangle + \sin(\theta)\cos(\theta) |\psi_B\rangle , \tag{3}$$

$$\cos(\theta) |\psi^\perp\rangle = \cos^2(\theta) |\psi_G\rangle + \sin(\theta)\cos(\theta) |\psi_B\rangle . \tag{4}$$

Adding equation 3 and equation 4, we get

$$|\psi_G\rangle = \sin(\theta) |\psi\rangle + \cos(\theta) |\psi^\perp\rangle . \tag{5}$$

Obviously, the 'bad' state can de written as

$$|\psi_B\rangle = \cos(\theta) |\psi\rangle - \sin(\theta) |\psi^\perp\rangle . \tag{6}$$

We define a new projection operator $P = |\psi\rangle\langle\psi|$, therefore

$$P |\psi_G\rangle = \sin(\theta) P |\psi\rangle + \cos(\theta) P |\psi^\perp\rangle = \sin(\theta) |\psi\rangle . \tag{7}$$

$$P |\psi_B\rangle = \cos(\theta) P |\psi\rangle - \sin(\theta) P |\psi^\perp\rangle = \cos(\theta) |\psi\rangle . \tag{8}$$

Define a new unitary operator $Q = -S_\psi S_G$, where

$$S_\psi = 1 - 2P,$$

$$S_G = 1 - 2P_G.$$

S_ψ flips the phase of the initial state, and S_G flips the phase in the 'good' subspace. Now we have

$$Q |\psi_G\rangle = -S_\psi S_G |\psi_G\rangle = (1 - 2P) |\psi_G\rangle = |\psi_G\rangle - 2\sin(\theta) |\psi\rangle = \cos(2\theta) |\psi_G\rangle - \sin(2\theta) |\psi_B\rangle \tag{9}$$

$$Q|\psi_B\rangle = -S_\psi S_G|\psi_B\rangle = -(1-2P)|\psi_B\rangle = -|\psi_B\rangle + 2\cos(\theta)|\psi_G\rangle = \sin(2\theta)|\psi_G\rangle + \cos(2\theta)|\psi_B\rangle\,. \tag{10}$$

Thus in the $|\psi_G\rangle$, $|\psi_B\rangle$ basis we have

$$Q = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}. \tag{11}$$

It is obvious that applying the Q operator n times on the state $|\psi\rangle$ gives

$$Q^n|\psi\rangle = \sin((2n+1)\theta)|\psi_G\rangle + \cos((2n+1)\theta)|\psi_B\rangle\,. \tag{12}$$

By applying Q to our state for n times (strategically chosen), we might be able to amplify the amplitude of $|\psi_G\rangle$ thus increase the probability of measuring the state in the 'good' subspace.

2 Grover's Algorithm

We are given a task to find out M solutions to a problem amongst an unsorted database containing N elements ($N \leqslant M$). Suppose there exists an oracle O that is able to tell whenever we find a solution. Precisely, the oracle is unitary operator that will flip a qubit whenever it sees a solution ($f(x) = 1$), and do nothing if it is not a solution ($f(x) = 0$).

$$O|x\rangle = (-1)^{f(x)}|x\rangle\,. \tag{13}$$

We define another unitary operator $S_\psi = 2P - 1$, where the projection operator $P := |\psi\rangle\langle\psi|$, and the Grover's algorithm is given as following:

1. Initialize the system to the state (by applying $H^{\otimes n}$)

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle$$

2. Perform the following subroutine, G, $r(N)$ times, which is asymptotically $O(\sqrt{N})$
   (a) Apply operator O
   (b) Apply operator S_ψ
3. Perform measurement.

Shown in figure 1, the subroutine G can be seen as a series rotations in the 2-D space spanned by $|\alpha\rangle$ and $|\beta\rangle$. Initially the state $|\psi\rangle$ is inclined at angle $\theta/2$ from $|\alpha\rangle$. The oracle O reflects the state about the state $|\alpha\rangle$, then the operator S_ψ reflects it about $|\psi\rangle$. By performing such a subroutine $r(N)$ times, we can make the probability of finding the state $|\psi\rangle$ in $|\beta\rangle$ approaching 1.
Note here we say the oracle O reflects the state about $|\alpha\rangle$. If we rename our 'good' subspace as $|\beta\rangle$ and 'bad' subspace as $|\alpha\rangle$, the oracle O becomes $1 - 2P_G$, i.e. reflection about the 'bad' subspace, and the subroutine G then becomes $G = (2P-1)(1-2P_G) = -(1-2P)(1-2P_G)$, same as our Q operator defined in the preceding section.

3 Oblivious Amplitude Amplification

The conventional amplitude amplification relies on using the projection operator P_ψ which is not possible if we do not know $|\psi\rangle$. Oblivious amplitude amplification can be performed to achieve similar

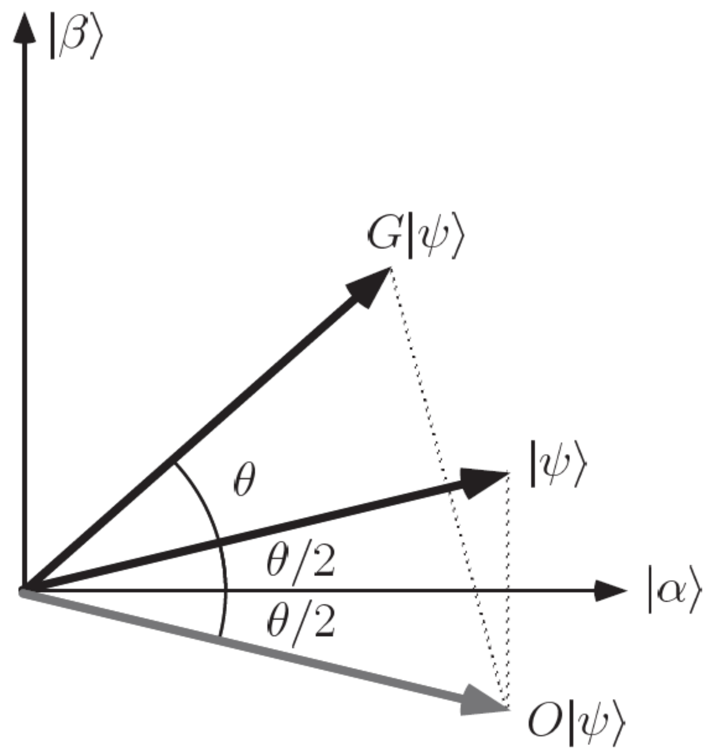

Figure 1: Geometrically interpretation of what the first iteration of the subroutine does

goal even when the initial state $|\psi\rangle$ is unknown.
Assume there exists a unitary operator W, such that

$$W|0\rangle|\psi\rangle = \frac{1}{s}|0\rangle U|\psi\rangle + \sqrt{1-\frac{1}{s^2}}\left|\Phi^{\perp}\right\rangle, \tag{14}$$

where $U = \sum_j \beta_j V_j$, $s = \sum_j \beta_j$, $|0\rangle$ represents the ancilla with all states in the 0 state, and $\left|\Phi^{\perp}\right\rangle$ is a state whose ancillary state is orthogonal to $|0\rangle$. Here U is derived from the time evolution operator $exp(-iHt)$ (not unitary) by rewriting it in Taylor series $\sum_{k=0}^{K}\frac{1}{k!}(-iHt/r)^k$ (not unitary). We use the form

$$H = \sum_{l=1}^{L}\alpha_l H_l,$$

where each H_l is unitary, to rewrite the Taylor series as

$$U = \sum_j \beta_j V_j, \tag{15}$$

where we have reached our goal that each V_j is unitary, and $\beta_j > 0$. Define

$$|\Psi\rangle = |0\rangle|\psi\rangle, |\Phi\rangle = |0\rangle U|\psi\rangle, \sin(\theta) = \frac{1}{s}. \tag{16}$$

Then equation 14 becomes

$$W|\Psi\rangle = \sin(\theta)|\Phi\rangle + \cos(\theta)\left|\Phi^{\perp}\right\rangle. \tag{17}$$

It is obvious that

$$W\left|\Psi^{\perp}\right\rangle = \cos(\theta)|\Phi\rangle - \sin(\theta)\left|\Phi^{\perp}\right\rangle. \tag{18}$$

Let $P = |0\rangle\langle 0| \otimes 1$, so $P\left|\Phi^\perp\right\rangle = 0$ and $P\left|\Psi^\perp\right\rangle = 0$. The former is true by definition, and the latter is proved by Childs' group shown in their paper as **Lemma 3.7 (2D Subspace Lemma)**. Using the same trick as we did in the first section, we obtain

$$W^\dagger |\Phi\rangle = \sin(\theta)|\Psi\rangle + \cos(\theta)\left|\Psi^\perp\right\rangle, \tag{19}$$

$$W^\dagger \left|\Phi^\perp\right\rangle = \cos(\theta)|\Psi\rangle - \sin(\theta)\left|\Psi^\perp\right\rangle. \tag{20}$$

Further define $R = 2P - 1$ and $S = -WRW^\dagger R$ as we did similarly in the first section, we obtain

$$S|\Phi\rangle = -WRW^\dagger|\Phi\rangle = -W(\sin(\theta)|\Psi\rangle - \cos(\theta)\left|\Psi^\perp\right\rangle) = \cos(2\theta)|\Phi\rangle - \sin(2\theta)\left|\Phi^\perp\right\rangle, \tag{21}$$

$$S\left|\Phi^\perp\right\rangle = WRW^\dagger\left|\Phi^\perp\right\rangle = W(\cos(\theta)|\Psi\rangle + \sin(\theta)\left|\Psi^\perp\right\rangle) = \sin(2\theta)|\Phi\rangle + \cos(2\theta)\left|\Phi^\perp\right\rangle. \tag{22}$$

Therefore

$$S = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}. \tag{23}$$

It is worth noticing that S acts as a rotation by 2θ in the subspace spanned by $|\Phi\rangle$ and $\left|\Phi^\perp\right\rangle$. For any integer n,

$$S^n W|\Psi\rangle = \sin((2n+1)\theta)|\Phi\rangle + \cos((2n+1)\theta)\left|\Phi^\perp\right\rangle. \tag{24}$$

When $n = 1$, equation 24 becomes

$$SW|\Psi\rangle = \sin(3\theta)|\Phi\rangle + \cos(3\theta)\left|\Phi^\perp\right\rangle. \tag{25}$$

If we then set $\sin(\theta) = 1/2 \rightarrow \sin(3\theta) = 0$, SW operator can be performed exactly with 100% success rate in a single step ($n = 1$) without having to know the initial state $|\psi\rangle$. This corresponds to the condition $s = 2$, and the truncated Taylor series $U = \sum_j \beta_j V_j$ is unitary.
For the general case, where the truncated Taylor series of time evolution operator is not unitary, and $s \neq 1$, we have

$$PSW = |0\rangle\left(\frac{3}{s}U - \frac{4}{s^3}UU^\dagger U\right)|\psi\rangle. \tag{26}$$