MATH2089
Numerical Methods
Lecture 11

Boundary Value Problems

Parabolic Equations

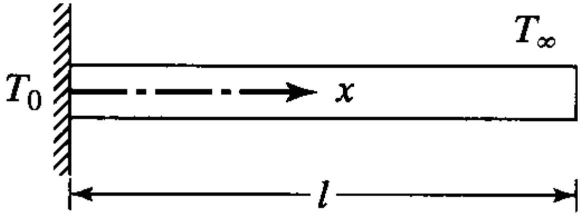
Methods of Solutions

Boundary Value Problems

- Conditions for the equation are not all given at the initial point but are specified at two different values for the independent variable
- These values are usually at the endpoints (or boundaries) of some domain of interest
- Example
  - A cooling fin extends from a hot surface

$$kA\frac{d^2T}{dx^2} - hP\left(T - T_\infty\right) = 0$$

  - Boundary conditions: $T(x = 0) = T_0$ and $-k\frac{dT(x=l)}{dx} = h\left(T(x=l) - T_\infty\right)$



Main Focus

- This lecture is concerned with solving boundary-value problems by replacing the derivatives with **finite difference** approximations
- When this is accomplished, the solution is obtained by solving a set of simultaneous equations

Finite Difference Methods

- FDA (finite difference approximation) of the differential equation is obtained at a number of mesh points in the interval of integration
- This leads to converting the differential equation to a set of simultaneous algebraic equations
- For example, consider

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = S(x), \quad x_0 \le x \le x_N$$

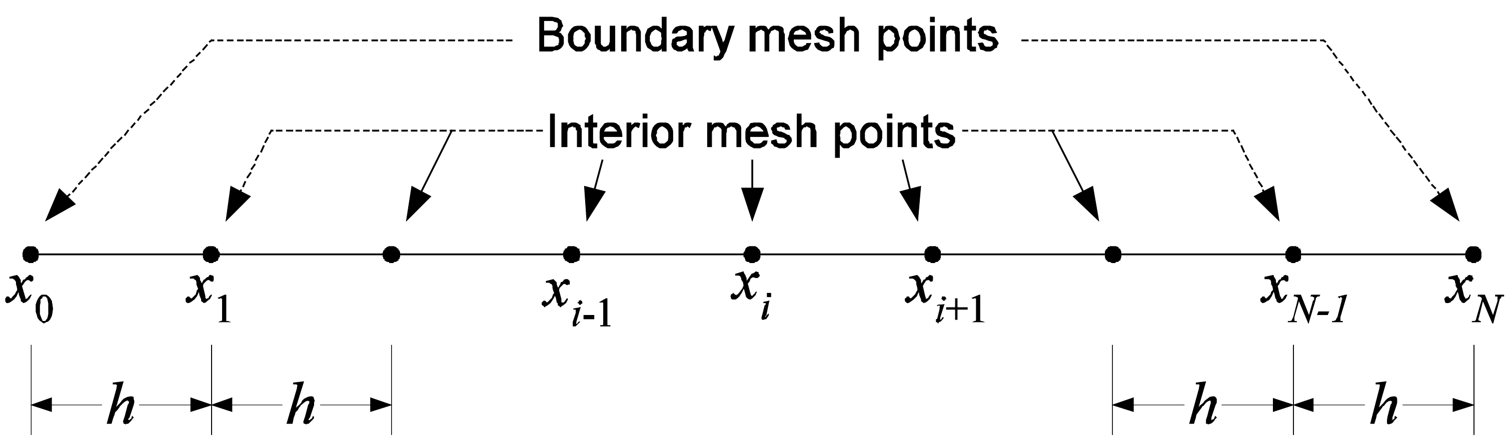
subject to boundary conditions (end points)

$$y(x_0) = Y_0 \qquad y(x_N) = Y_N$$

Finite Difference Methods (continue)

- Consider the problem as having discrete points along *x*, the interval of integration can be divided into *N* equal parts of width

$$h = (x_0 - x_N)/N \quad \Rightarrow \quad x_i = x_0 + ih$$



Finite Difference Methods (continue)

- Employing central difference

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2), \quad \frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2)$$

- Hence, at the interior node *I*

$$P(x_i)\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + Q(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + R(x_i)y_i = S(x_i), \quad i = 1, 2, \ldots, N-1$$

- Let $P(x_i) = a_{2,i}, \quad Q(x_i) = a_{1,i}, \quad R(x_i) = a_{0,i}, \quad S(x_i) = g_i$

$$\underbrace{\left(\frac{a_{2,i}}{h^2} - \frac{a_{1,i}}{2h}\right)}_{A_i} y_{i-1} + \underbrace{\left(a_{0,i} - \frac{2a_{2,i}}{h^2}\right)}_{B_i} y_i + \underbrace{\left(\frac{a_{2,i}}{h^2} + \frac{a_{1,i}}{2h}\right)}_{C_i} y_{i+1} = \underbrace{g_i}_{D_i}, \quad i = 1, 2, \ldots, N-1$$

Finite Difference Methods (continue)

- Simplifying,

$$A_i y_{i-1} + B_i y_i + C_i y_{i+1} = D_i, \quad i = 1, 2, \ldots, N-1$$

- Applying the boundary conditions $y(x_0) = Y_0$ and $y(x_N) = Y_N$ we can write

$$\begin{cases} i = 1, & B_1 y_1 + C_1 y_2 = D_1 - A_1 Y_0 \\ i = 2, & A_2 y_1 + B_2 y_2 + C_2 y_3 = D_2 \\ i = 3, & A_3 y_2 + B_3 y_3 + C_3 y_4 = D_3 \\ & \vdots \\ i = N-2, & A_{N-2} y_{N-3} + B_{N-2} y_{N-2} + C_{N-2} y_{N-1} = D_{N-2} \\ i = N-1, & A_{N-1} y_{N-2} + B_{N-1} y_{N-1} = D_{N-1} - C_{N-1} Y_N \end{cases}$$

Finite Difference Methods (continue)

- System of (*N*-1) equations with (*N*-1) unknowns y_i
- Set of simultaneous algebraic equations which can be expressed in the matrix form

$$[A]\vec{y} = \vec{b}$$

$$\begin{bmatrix} B_1 & C_1 & 0 & 0 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & 0 & 0 & 0 & 0 \\ 0 & A_3 & B_3 & C_3 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & A_{N-3} & B_{N-3} & C_{N-3} & 0 \\ 0 & 0 & 0 & 0 & A_{N-2} & B_{N-2} & C_{N-2} \\ 0 & 0 & 0 & 0 & 0 & A_{N-1} & B_{N-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-3} \\ y_{N-2} \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} D_1 - A_1 Y_0 \\ D_2 \\ D_3 \\ \vdots \\ D_{N-3} \\ D_{N-2} \\ D_{N-1} - C_{N-1} Y_N \end{bmatrix}$$

Finite Difference Methods (continue)

- What type of matrix is it?
- Tri-Diagonal matrix which can be solved using the TDMA (Tri-Diagonal Matrix Algorithm)
- Can be resolved via two steps
- Step 1, eliminate A_i $B_1^{(1)} = B_1, \quad B_i^{(1)} = B_i - \frac{A_i}{B_{i-1}^{(1)}} C_{i-1}, \quad D_i^{(1)} = D_i - \frac{A_i}{B_{i-1}^{(1)}} D_{i-1}^{(1)}$

$$\left[\begin{array}{ccccccc|c}
B_1 & C_1 & & & & & & D_1^{(1)} \\
0 & B_2^{(1)} & C_2 & & & & & D_2^{(1)} \\
 & 0 & B_3^{(1)} & C_3 & & & & D_3^{(1)} \\
 & & \ddots & \ddots & \ddots & & & \vdots \\
 & & & 0 & B_{N-3}^{(1)} & C_{N-3} & & D_{N-3}^{(1)} \\
 & & & & 0 & B_{N-2}^{(1)} & C_{N-2} & D_{N-2}^{(1)} \\
 & & & & & 0 & B_{N-1}^{(1)} & D_{N-1}^{(1)}
\end{array}\right] \qquad i = 2, 3, \ldots, N-1$$

Finite Difference Methods (continue)

- Step 2, backward substitution

$$y_{N-1} = \frac{D_{N-1}^{(1)}}{B_{N-1}^{(1)}},$$

$$y_i = \frac{D_i^{(1)} - C_i y_{i+1}}{B_i^{(1)}}, \quad i = N-2, N-3, \ldots, 2, 1$$

Example

- Formulate the boundary-value problem for an infinitely long hollow cylinder whose inner surface (r = 5″) is maintained at 200°F, while the outer surface (r = 10″) is maintained at 65°F. The radial heat transfer in a thick hollow cylinder is given by

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0$$

- <u>Solution</u>

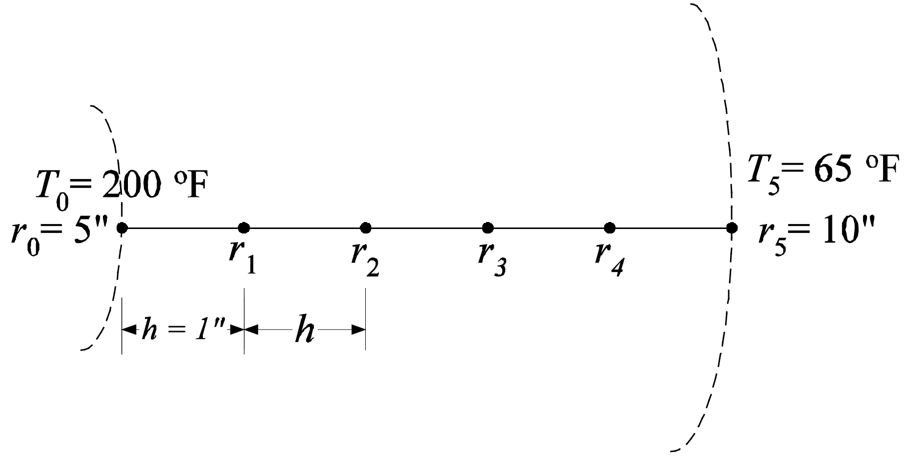
Using $h = 1''$, interval $5'' \leq x \leq 10''$ can be divided into $N = 5$

Example (continue)

- Employ the central difference formulae

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \quad \Rightarrow$$

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} + \frac{1}{r_i}\frac{T_{i+1} - T_{i-1}}{2h} = 0$$



Example (continue)

- Rearrange

$$\left(\frac{1}{h^2}-\frac{1}{2hr_i}\right)T_{i-1}-\left(\frac{2}{h^2}\right)T_i+\left(\frac{1}{h^2}+\frac{1}{2hr_i}\right)T_{i+1}=0$$

A_i B_i C_i D_i

- $i = 1$, $A_1=0$, $B_1=-2/h^2$,

$$C_1=\left(\frac{1}{h^2}+\frac{1}{2hr_1}\right) \qquad D_1=-\left(\frac{1}{h^2}-\frac{1}{2hr_i}\right)*T_0$$

Example (continue)

- Discrete form of equation

$$\left(\frac{1}{h^2}-\frac{1}{2hr_i}\right)T_{i-1}-\left(\frac{2}{h^2}\right)T_i+\left(\frac{1}{h^2}+\frac{1}{2hr_i}\right)T_{i+1}=0$$

A_i B_i C_i D_i

- i = 2,3

$$A_i=\left(\frac{1}{h^2}-\frac{1}{2hr_i}\right) \qquad C_i=\left(\frac{1}{h^2}+\frac{1}{2hr_i}\right)$$

$$B_i=-\left(\frac{2}{h^2}\right) \qquad D_i=0$$

Example (continue)

- Discrete form of equation

$$\left(\frac{1}{h^2}-\frac{1}{2hr_i}\right)T_{i-1}-\left(\frac{2}{h^2}\right)T_i+\left(\frac{1}{h^2}+\frac{1}{2hr_i}\right)T_{i+1}=0$$

A_i B_i C_i D_i

- i = 4, B_4=-2/h^2, C_4=0,

$$A_4=\left(\frac{1}{h^2}-\frac{1}{2hr_4}\right) \qquad D_4=-\left(\frac{1}{h^2}+\frac{1}{2hr_i}\right)*T_5$$

Example (continue)

- Finite difference approximation at the interior node points i = 1, 2, 3 and 4 are

$$
\begin{array}{ll}
i=1, \quad 11T_0 - 24T_1 + 13T_2 = 0 & \quad -24T_1 + 13T_2 = -11(200) \\
i=2, \quad 13T_1 - 28T_2 + 15T_3 = 0 & \quad 13T_1 - 28T_2 + 15T_3 = 0 \\
i=3, \quad 15T_2 - 32T_3 + 17T_4 = 0 \quad \Rightarrow & \quad 15T_2 - 32T_3 + 17T_4 = 0 \\
i=4, \quad 17T_3 - 36T_4 + 19T_5 = 0 & \quad 17T_3 - 36T_4 = -19(65)
\end{array}
$$

- Expressed in matrix form

$$
\begin{bmatrix} -24 & 13 & 0 & 0 \\ 13 & -28 & 15 & 0 \\ 0 & 15 & -32 & 17 \\ 0 & 0 & 17 & -36 \end{bmatrix}
\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}
=
\begin{Bmatrix} -2200 \\ 0 \\ 0 \\ -1235 \end{Bmatrix}
$$

Example (continue)

- Solved by TDMA
- Step 1, eliminate A_i

$$\begin{bmatrix} -24 & 13 & 0 & 0 \\ 0 & -20.958 & 15 & 0 \\ 0 & 0 & -21.264 & 17 \\ 0 & 0 & 0 & -22.409 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} -2200 \\ -1191.7 \\ -852.88 \\ -1916.84 \end{Bmatrix}$$

- Step 2, backward substitution

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 164.53 \\ 134.51 \\ 108.49 \\ 85.538 \end{Bmatrix}$$

Boundary Conditions

- There are different types of boundary conditions that can be imposed
- ***Dirichlet condition***

$$y(x_0) = f(x) \quad \text{or} \quad y(x_0) = \text{constant}$$

- ***Neumann condition***: derivative in the *n* direction

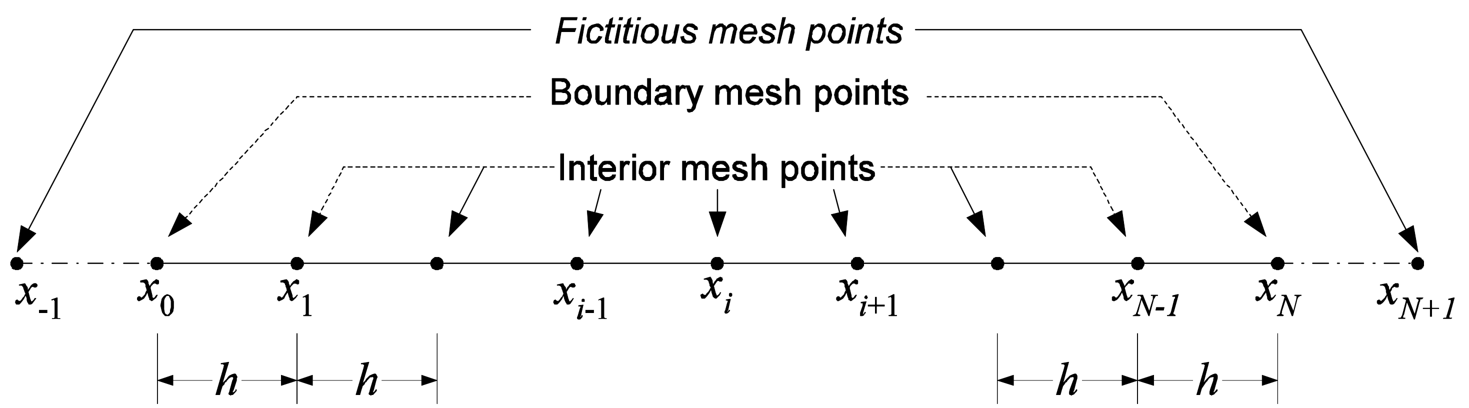
$$\left.\frac{dy}{dn}\right|_{x_0} = f(x) \quad \text{or} \quad \left.\frac{dy}{dn}\right|_{x_0} = \text{constant}$$

- ***Mixed condition***

$$\left.\frac{dy}{dn}\right|_{x_0} + cy(x_0) = f(x) \quad \text{or} \quad \left.\frac{dy}{dn}\right|_{x_0} + cy(x_0) = \text{constant}$$

FDA of Boundary Conditions

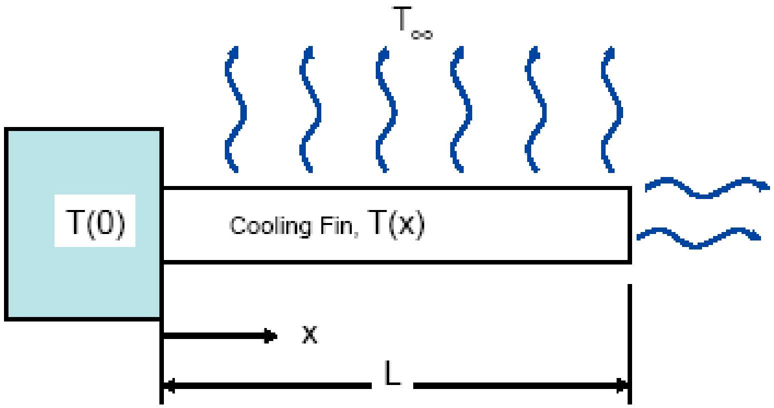
- Leads to an additional equation at x_0, hence the system consists of N simultaneous equations in tri-diagonal form in the N unknowns
- How to employ central difference formulae at end-points? Use fictitious points as seen below



Example

- A cooling fin extends from a hot furnace wall as shown below. Assuming that heat flows only in the *x*-direction, the thermal equilibrium relation leads to the following governing equation

$$kA\frac{d^2T}{dx^2} - hP\left(T - T_\infty\right) = 0$$



$$T\left(0\right) = T_0$$

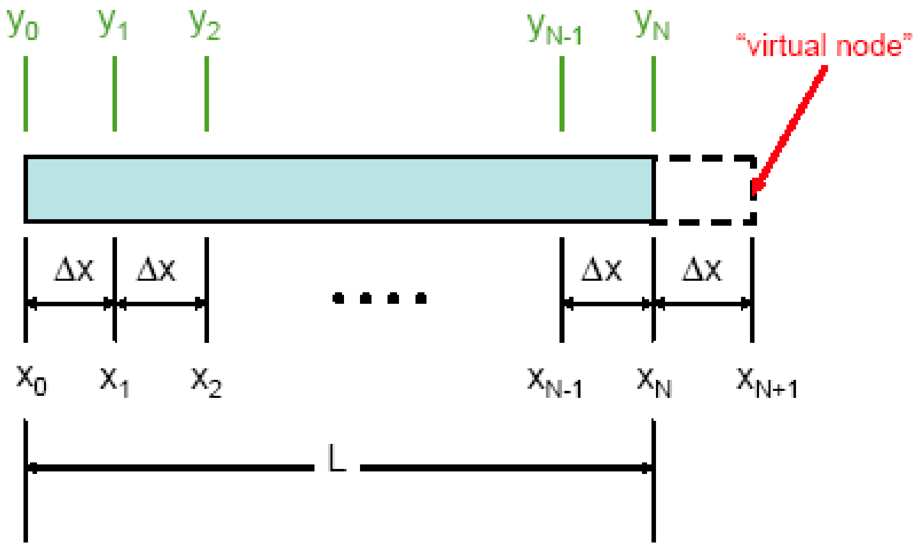
$$-k\frac{dT\left(L\right)}{dx} = h\left(T\left(L\right) - T_\infty\right)$$

Example (continue)

- Let $Y = T - T_\infty$, $k/h = A/P$, $T_0 - T_\infty = 10$, the boundary conditions are

$$\frac{d^2Y}{dx^2} - Y = 0, \quad Y(0) = T_0 - T_\infty = 10, \quad -\frac{dY}{dx} = Y(1)$$

- Consider a finite difference solution



Example (continue)

- Divide the fin into N equal segments of length $\Delta x = L/N$
- Temperature difference at each node is denoted at y_i, $i = 1, 2, \ldots N$
- First and second-derivative at each node can be expressed to order $O(\Delta x^2)$ using centered difference expressions

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2\Delta x}, \quad y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$$

Example (continue)

- At node 1 (Note y_0 = 10)

$$y_1'' = y_1 \Rightarrow \frac{y_0 - 2y_1 + y_2}{\Delta x^2} = y_1 \Rightarrow \left(2 + \Delta x^2\right) y_1 - y_2 = y_0$$

- Similarly, at nodes 2, 3, …, N-1, we get

$$y_i'' = y_i \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{\Delta x^2} = y_i \Rightarrow -y_{i-1} + \left(2 + \Delta x^2\right) y_i - y_{i+1} = 0$$

- End point requires *special* treatment because of the convective boundary condition at $x = L$

Example (continue)

- By introducing a fictitious node at N+1, with temperature y_{N+1}, leads to an approximation at node N

$$y''_N = y_N \Rightarrow \frac{y_{N-1} - 2y_N + y_{N+1}}{\Delta x^2} = y_N \Rightarrow -y_{N-1} + \left(2 + \Delta x^2\right) y_N - y_{N+1} = 0$$

- To obtain the relationship for y_{N+1}, we employ the natural boundary condition at $x = L$. Together with the central difference expression

$$-y'_N = y_N \Rightarrow -\frac{y_{N+1} - y_{N-1}}{2\Delta x} = y_N \Rightarrow y_{N+1} = y_{N-1} - 2\Delta x y_N$$

$$-\frac{dY}{dx} = Y(1)$$

Example (continue)

- Eliminating y_{N+1} using

$$-y'_N = y_N \Rightarrow -\frac{y_{N-1} - y_{N+1}}{2\Delta x} = y_N \Rightarrow y_{N+1} = y_{N-1} - 2\Delta x y_N$$

we get

$$-2y_{N-1} + \left(2 + 2\Delta x + \Delta x^2\right) y_N = 0$$

- Equations for N unknowns can be summarized as

$$\left(2 + \Delta x^2\right) y_1 - y_2 = y_0$$

$$-y_{i-1} + \left(2 + \Delta x^2\right) y_i - y_{i+1} = 0 \quad i = 2,3,\ldots,N-1$$

$$-2y_{N-1} + \left(2 + 2\Delta x + \Delta x^2\right) y_N = 0$$

Shooting Method

- An alternative procedure for solving a two-point boundary value problem involves its conversion to an ***Initial value problem*** (IVP) by the determination of sufficient additional conditions at one boundary
- ***(IVP)***: values of the dependent variables or their derivatives are known at the initial value of the independent variable
- Missing initial conditions are determined in a way which causes the given conditions at the other boundary to be satisfied

Procedure

- The steps involved are:
  - split the second (or higher) order equation into two (or more) equivalent first order equations
  - estimate values for the missing initial conditions
  - integrate the equations as an initial value problem
  - compare the solution at the final boundary with the given final boundary condition(s); if they don't agree, then
  - adjust the estimated values of the missing initial condition(s)
  - repeat the integration until the process converges

Non-linear Boundary Value Problems

- Consider the equation

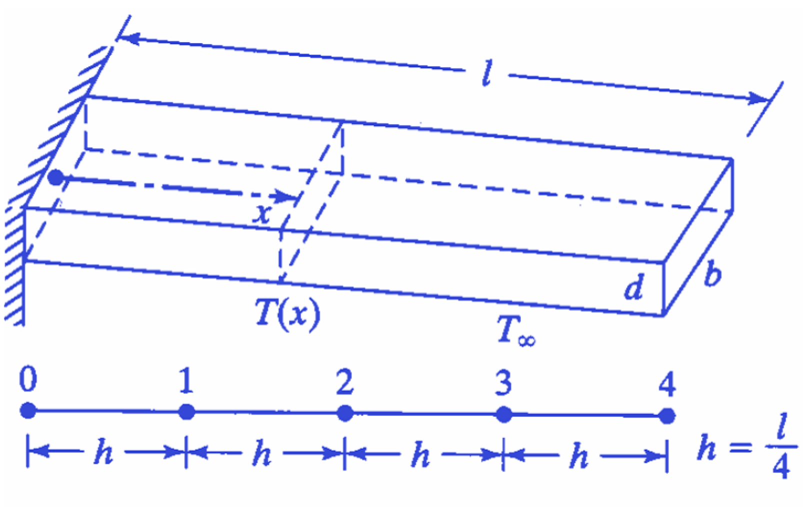
$$\frac{d^2T}{dx^2} = \frac{\sigma\varepsilon P}{kA}\left(T^4 - T_\infty^4\right)$$

- What is the physical significance?
- Employing an energy balance, conduction balances with the radiation at the surface of the material
- How do you solve it?

Example

- Find the temperature distribution in a rectangular fin given by

$$\frac{d^2T}{dx^2} = \frac{\sigma\varepsilon P}{kA}\left(T^4 - T_\infty^4\right)$$



$$c = \frac{\sigma\varepsilon P}{kA} = 1.9\times10^{-9}$$

$$T(0) = 1000K$$

$$T(l) = 350K \quad l = 2\text{m}$$

$$T_\infty = 500K$$

Example (continue)

- Divide the fin into equal segments of *h*
- Second-derivative at each node can be expressed to order O(Δx²) using centered difference expressions

$$T_i'' = \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}$$

- The set of equations to be solved for *i* = 1,2,3

$$i = 1, \quad \frac{T_0 - 2T_1 + T_2}{h^2} - cT_1^4 + cT_\infty^4 = 0$$

$$i = 2, \quad \frac{T_1 - 2T_2 + T_3}{h^2} - cT_2^4 + cT_\infty^4 = 0$$

$$i = 3, \quad \frac{T_2 - 2T_3 + T_4}{h^2} - cT_3^4 + cT_\infty^4 = 0$$

Example (continue)

- Non-linear equations must be solved through the iterative Newton-Rhapson method

- Rearranging,

$$f_1(T_1,T_2,T_3) = \frac{T_0 - 2T_1 + T_2}{h^2} - cT_1^4 + cT_\infty^4 = 0$$

$$f_2(T_1,T_2,T_3) = \frac{T_1 - 2T_2 + T_3}{h^2} - cT_2^4 + cT_\infty^4 = 0$$

$$f_3(T_1,T_2,T_3) = \frac{T_2 - 2T_3 + T_4}{h^2} - cT_3^4 + cT_\infty^4 = 0$$

Example (continue)

- Step 1: start with an initial guess $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}$
- Step 2: evaluate the function values

$$f_1^{(1)} = f_1(T_1^{(1)}, T_2^{(1)}, T_3^{(1)})$$
$$f_2^{(1)} = f_2(T_1^{(1)}, T_2^{(1)}, T_3^{(1)})$$
$$f_3^{(1)} = f_3(T_1^{(1)}, T_2^{(1)}, T_3^{(1)})$$

- Step 3: find the partial derivatives of the function f_i and solve the equations as

$$\begin{bmatrix} \frac{\partial f_1^{(1)}}{\partial T_1} & \frac{\partial f_1^{(1)}}{\partial T_2} & \frac{\partial f_1^{(1)}}{\partial T_3} \\ \frac{\partial f_2^{(1)}}{\partial T_1} & \frac{\partial f_2^{(1)}}{\partial T_2} & \frac{\partial f_2^{(1)}}{\partial T_3} \\ \frac{\partial f_3^{(1)}}{\partial T_1} & \frac{\partial f_3^{(1)}}{\partial T_2} & \frac{\partial f_3^{(1)}}{\partial T_3} \end{bmatrix} \begin{Bmatrix} \Delta T_1^{(1)} \\ \Delta T_2^{(1)} \\ \Delta T_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} -f_1^{(1)} \\ -f_2^{(1)} \\ -f_3^{(1)} \end{Bmatrix}$$

Example (continue)

- Step 4: find the new solution as

$$
\begin{aligned}
T_1^{(i+1)} &= T_1^{(i)} + \Delta T_1^{(i)} \\
T_2^{(i+1)} &= T_2^{(i)} + \Delta T_2^{(i)} \\
T_3^{(i+1)} &= T_3^{(i)} + \Delta T_3^{(i)}
\end{aligned}
$$

- Step 5: check the convergence by evaluating the function values

$$
\begin{aligned}
f_1^{(i+1)} &= f_1(T_1^{(i+1)}, T_2^{(i+1)}, T_3^{(i+1)}); \quad \left| f_1^{(i+1)} \right| \le \varepsilon \\
f_2^{(i+1)} &= f_2(T_1^{(i+1)}, T_2^{(i+1)}, T_3^{(i+1)}); \quad \left| f_2^{(i+1)} \right| \le \varepsilon \\
f_3^{(i+1)} &= f_3(T_1^{(i+1)}, T_2^{(i+1)}, T_3^{(i+1)}); \quad \left| f_3^{(i+1)} \right| \le \varepsilon
\end{aligned}
$$

Example (continue)

- Step 1: start with an initial guess

$$T_0 = 1000K$$
$$T_4 = 350K$$
$$T_\infty = 500K$$

$$T_1^{(1)} = 800K, T_2^{(1)} = 700K, T_3^{(1)} = 600K$$

- Step 2: evaluate the function values

$$f_1^{(1)}(T_1^{(1)}, T_2^{(1)}, T_3^{(1)}) = \frac{T_0 - 2T_1^{(1)} + T_2^{(1)}}{h^2} - c\left(T_1^{(1)}\right)^4 + cT_\infty^4 = 0$$

$$f_2^{(1)}(T_1^{(1)}, T_2^{(1)}, T_3^{(1)}) = \frac{T_1^{(1)} - 2T_2^{(1)} + T_3^{(1)}}{h^2} - c\left(T_2^{(1)}\right)^4 + cT_\infty^4 = 0$$

$$f_3^{(1)}(T_1^{(1)}, T_2^{(1)}, T_3^{(1)}) = \frac{T_2^{(1)} - 2T_3^{(1)} + T_4}{h^2} - c\left(T_3^{(1)}\right)^4 + cT_\infty^4 = 0$$

Example (continue)

- Step 3: find the partial derivatives of the function f_i, for example,

$$\frac{\partial f_1^{(1)}}{\partial T_1} = -\frac{2}{h^2} - 4c\left(T_1^{(1)}\right)^3 \qquad \frac{\partial f_1^{(1)}}{\partial T_2} = \frac{1}{h^2} \qquad \frac{\partial f_1^{(1)}}{\partial T_3} = 0$$

and so forth for other partial derivatives, Use Gaussian elimination to evaluate the matrix

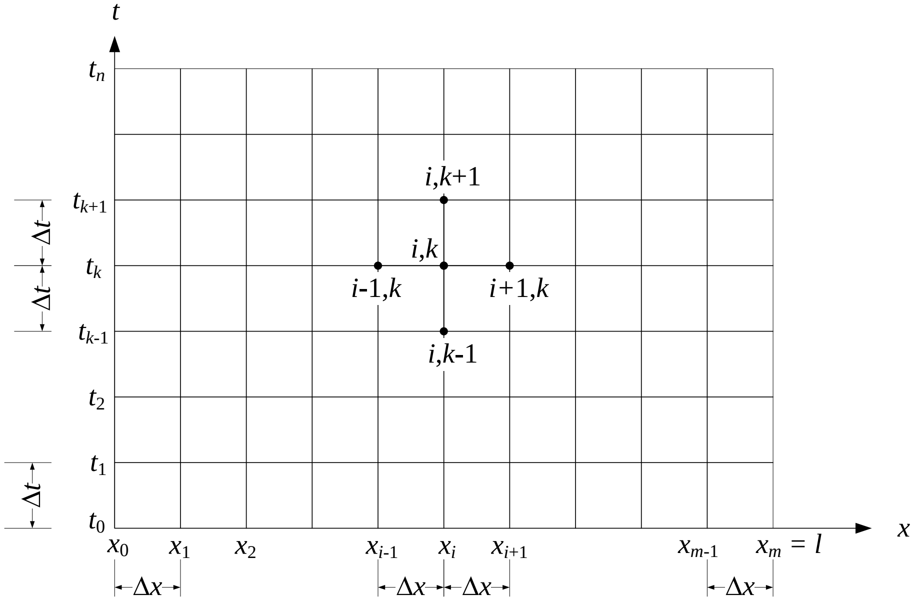
$$\begin{bmatrix} \frac{\partial f_1^{(1)}}{\partial T_1} & \frac{\partial f_1^{(1)}}{\partial T_2} & \frac{\partial f_1^{(1)}}{\partial T_3} \\ \frac{\partial f_2^{(1)}}{\partial T_1} & \frac{\partial f_2^{(1)}}{\partial T_2} & \frac{\partial f_2^{(1)}}{\partial T_3} \\ \frac{\partial f_3^{(1)}}{\partial T_1} & \frac{\partial f_3^{(1)}}{\partial T_2} & \frac{\partial f_3^{(1)}}{\partial T_3} \end{bmatrix} \begin{Bmatrix} \Delta T_1^{(1)} \\ \Delta T_2^{(1)} \\ \Delta T_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} -f_1^{(1)} \\ -f_2^{(1)} \\ -f_3^{(1)} \end{Bmatrix}$$

Parabolic Equations

- The one-dimensional time varying heat conduction equation is an example of a parabolic equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

- To derive the finite difference equation, the grid shown below is considered



Parabolic Equations (continue)

- This problem can be solved finite difference approximations in two approaches: explicit and implicit methods

Explicit Method

- Forward Time Centred Space (FTCS) scheme

$$\underbrace{\frac{T_i^{k+1}-T_i^{k}}{\Delta t}}_{\text{forward difference}}=\alpha\underbrace{\frac{T_{i+1}^{k}-2T_i^{k}+T_{i-1}^{k}}{(\Delta x)^2}}_{\text{central difference}}$$

- Rearranging,

$$T_i^{k+1}=\left(\frac{\alpha\Delta t}{(\Delta x)^2}\right)T_{i+1}^{k}+\left(1-\frac{2\alpha\Delta t}{(\Delta x)^2}\right)T_i^{k}+\left(\frac{\alpha\Delta t}{(\Delta x)^2}\right)T_{i-1}^{k}$$

- Let $s=\alpha\Delta t/(\Delta x)^2$

$$T_i^{k+1}=sT_{i+1}^{k}+\left(1-2s\right)T_i^{k}+sT_{i-1}^{k}$$

Explicit Method (continue)

- The solution will be stable if all the coefficients are positive

$$1-2s=\left(1-\frac{2\alpha\Delta t}{(\Delta x)^2}\right)\geq 0 \quad \textit{or} \quad s=\frac{\alpha\Delta t}{(\Delta x)^2}\leq\frac{1}{2}$$

- Disadvantages:
  - The stability condition imposes a restriction on the step size Δx and Δt
  - Expression of T_i^{k+1} depends only on T_{i-1}^k, T_i^k and T_{i+1}^k. It should depend on all values of T^k

Implicit Method

- Using central difference formula

$$\left.\frac{\partial T}{\partial t}\right|_{i,k+\frac{1}{2}} = \frac{T_i^{k+1} - T_i^k}{\Delta t}$$

- Using a weighted average of the central difference values

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,k+\frac{1}{2}} = \theta \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,k+1} + (1-\theta)\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,k}$$

where θ = weighting factor and

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,k+1} = \frac{\left(T_{i+1}^{k+1} - 2T_i^{k+1} + T_{i-1}^{k+1}\right)}{(\Delta x)^2} \qquad \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,k} = \frac{\left(T_{i+1}^{k} - 2T_i^{k} + T_{i-1}^{k}\right)}{(\Delta x)^2}$$

Implicit Method (continue)

- We get

$$\frac{T_i^{k+1}-T_i^k}{\Delta t}=\theta\alpha\left(\frac{T_{i+1}^{k+1}-2T_i^{k+1}+T_{i-1}^{k+1}}{(\Delta x)^2}\right)+(1-\theta)\alpha\left(\frac{T_{i+1}^{k}-2T_i^{k}+T_{i-1}^{k}}{(\Delta x)^2}\right)$$

- Rearranging,

$$T_i^{k+1}-T_i^k=\theta\frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{i+1}^{k+1}-2T_i^{k+1}+T_{i-1}^{k+1}\right)+(1-\theta)\frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{i+1}^{k}-2T_i^{k}+T_{i-1}^{k}\right)$$

- Let $s=\alpha\Delta t/(\Delta x)^2$

$$T_i^{k+1}-T_i^k=\theta s\left(T_{i+1}^{k+1}-2T_i^{k+1}+T_{i-1}^{k+1}\right)+(1-\theta)s\left(T_{i+1}^{k}-2T_i^{k}+T_{i-1}^{k}\right)$$

Implicit Method (continue)

$$-\theta s T_{i+1}^{k+1} + (1+2\theta s)T_i^{k+1} - \theta s T_{i-1}^{k+1} = (1-\theta)sT_{i+1}^{k} + (1-2(1-\theta)s)T_i^{k} + (1-\theta)sT_{i-1}^{k}$$

- Fully-implicit (θ = 1)

$$-sT_{i+1}^{k+1} + (1+2s)T_i^{k+1} - sT_{i-1}^{k+1} = T_i^{k}$$

- Semi-implicit (θ = 0.5) – Crank-Nicolson Method

$$-0.5sT_{i+1}^{k+1} + (1+s)T_i^{k+1} - 0.5sT_{i-1}^{k+1} = 0.5sT_{i+1}^{k} + (1-s)T_i^{k} + 0.5sT_{i-1}^{k}$$

- Explicit (θ = 0)

$$T_i^{k+1} = sT_{i+1}^{k} + \left(1-2s\right)T_i^{k} + sT_{i-1}^{k}$$