# MATH2089
# Numerical Methods
# Lecture 12

Boundary Value Problems

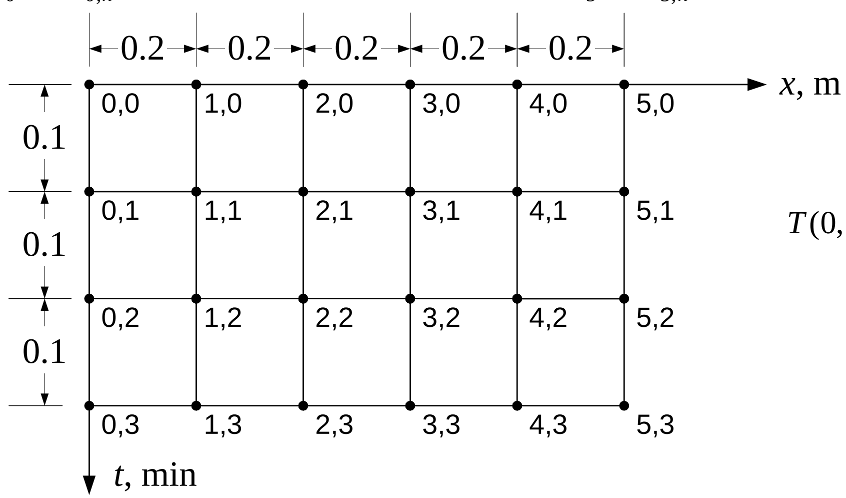
Parabolic Equations

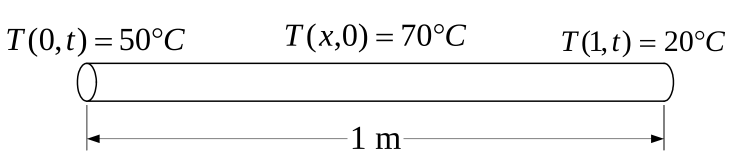
Elliptic and Hyperbolic Equations

# Example – Parabolic Equation

- A metal rod of length 1 m is initially at 70°C. The steady-state temperature of the left and right ends of the rod are given by 50°C and 20°C, respectively. Using $\alpha$ = 0.1 m2/min, $\Delta x$ = 0.2 m and $\Delta t$ = 0.1 min, determine the temperature distribution in the rod for $0 \leq t \leq 0.3$ min.

$T_0^k = T_{0,k} = 50°C \quad T_i^0 = T_{i,0} = 70°C \quad T_5^k = T_{5,k} = 20°C$



# Example (continue)

- Solution : Consider the parabolic equation

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} \;;\; 0 \le x \le 1\ \text{m}$$

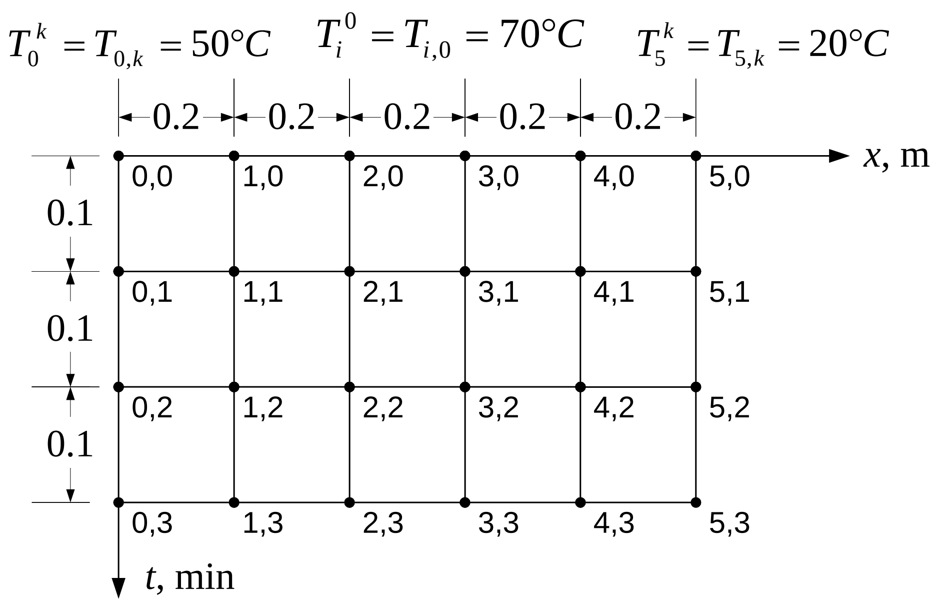
- Boundary conditions $T(0,t) = 50°\text{C},\ T(1,t) = 20°\text{C}$
- Initial condition $T(x,0) = 70°\text{C}$
- Check for stability

$$s = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{(0.1)(0.1)}{(0.2)^2} = 0.25 \le 0.5$$

# Example (continue)

- Assume $T_{0,\,0}$ and $T_{5,\,0}$ as the averages of the initial and steady-state values

$$T_0^0 = (70+50)/2 = 60°\text{C},$$
$$T_5^0 = (70+20)/2 = 45°\text{C}$$



# Example (continue)

- Using explicit FTCS (Forward Time Central Space) scheme

$$T_i^{k+1} = sT_{i+1}^k + \left(1-2s\right)T_i^k + sT_{i-1}^k$$

$$s = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{(0.1)(0.1)}{(0.2)^2} = 0.25 \le 0.5$$

- At $t_1$ = 0.1 min

$$T_1^1 = 0.25T_2^0 + 0.5T_1^0 + 0.25T_0^0 = 0.25(70) + 0.5(70) + 0.25(60) = 67.5°\text{C}$$

$$T_2^1 = 0.25T_3^0 + 0.5T_2^0 + 0.25T_1^0 = 0.25(70) + 0.5(70) + 0.25(70) = 70°\text{C}$$

$$T_3^1 = 0.25T_4^0 + 0.5T_3^0 + 0.25T_2^0 = 0.25(70) + 0.5(70) + 0.25(70) = 70°\text{C}$$

$$T_4^1 = 0.25T_5^0 + 0.5T_4^0 + 0.25T_3^0 = 0.25(45) + 0.5(70) + 0.25(70) = 63.75°\text{C}$$

# Example (continue)

- At $t_2$ = 0.2 min

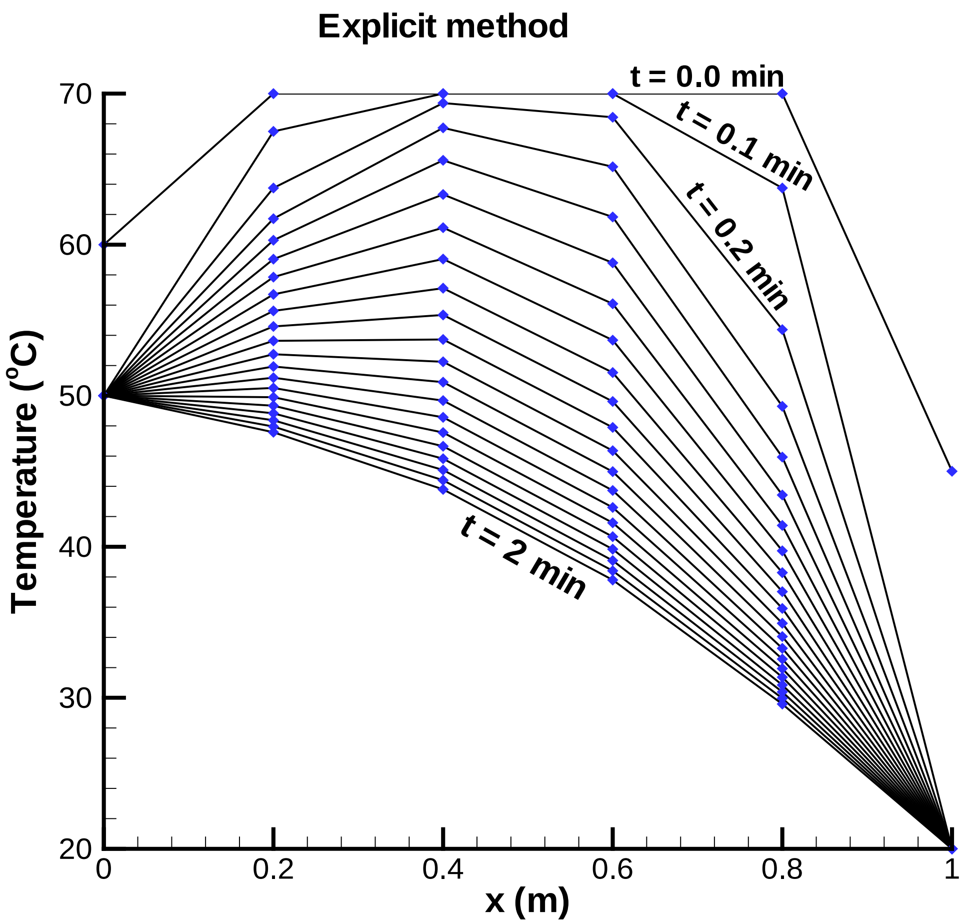
$$T_1^2 = 0.25T_2^1 + 0.5T_1^1 + 0.25T_0^1 = 0.25(70) + 0.5(67.5) + 0.25(50) = 63.75°\text{C}$$
$$T_2^2 = 0.25T_3^1 + 0.5T_2^1 + 0.25T_1^1 = 0.25(70) + 0.5(70) + 0.25(67.5) = 69.375°\text{C}$$
$$T_3^2 = 0.25T_4^1 + 0.5T_3^1 + 0.25T_2^1 = 0.25(63.75) + 0.5(70) + 0.25(70) = 68.4375°\text{C}$$
$$T_4^2 = 0.25T_5^1 + 0.5T_4^1 + 0.25T_3^1 = 0.25(20) + 0.5(63.75) + 0.25(70) = 54.375°\text{C}$$

- At $t_2$ = 0.3 min

$$T_1^3 = 0.25T_2^2 + 0.5T_1^2 + 0.25T_0^2 = 0.25(69.375) + 0.5(63.75) + 0.25(50) = 61.71875°\text{C}$$
$$T_2^3 = 0.25T_3^2 + 0.5T_2^2 + 0.25T_1^2 = 0.25(68.4375) + 0.5(69.375) + 0.25(63.75) = 67.734375°\text{C}$$
$$T_3^3 = 0.25T_4^2 + 0.5T_3^2 + 0.25T_2^2 = 0.25(54.375) + 0.5(66.4375) + 0.25(69.375) = 65.15625°\text{C}$$
$$T_4^3 = 0.25T_5^2 + 0.5T_4^2 + 0.25T_3^2 = 0.25(20) + 0.5(57.375) + 0.25(68.4735) = 49.296875°\text{C}$$

# Example (continue)

- Marching through time



# Example (continue)

- Using explicit FTCS (Forward Time Central Space) scheme with $\Delta t$ = 0.3 min

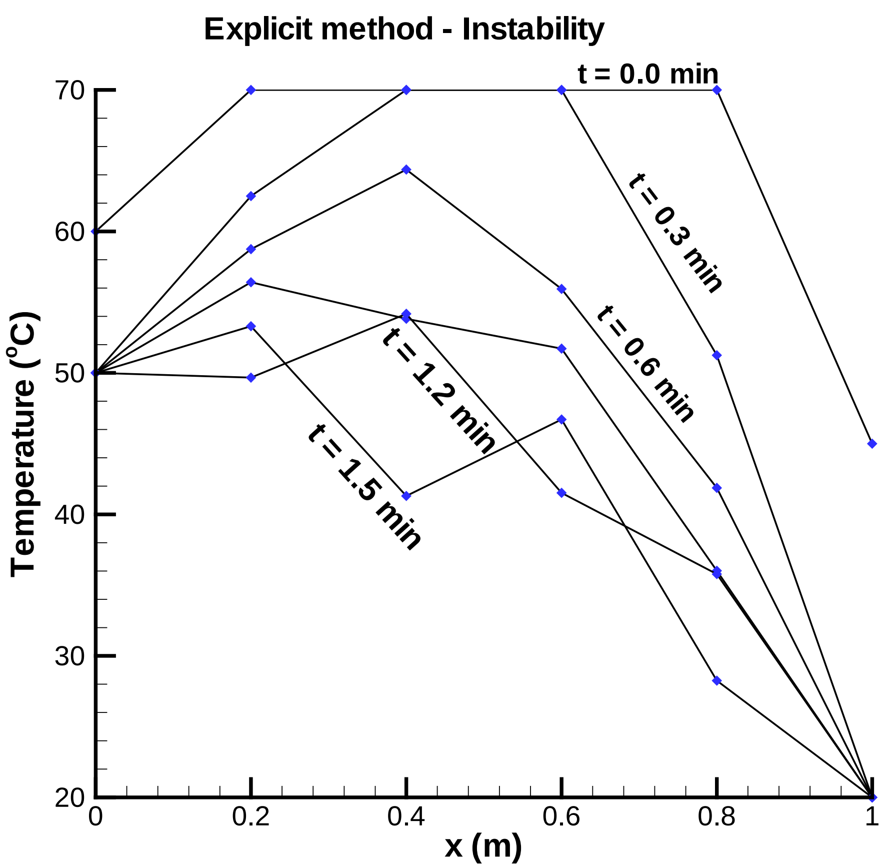
$$\frac{\alpha \Delta t}{(\Delta x)^2} = \frac{(0.1)(0.3)}{(0.2)^2} = 0.75 > 0.5$$

$$T_i^{k+1} = 0.75 T_{i+1}^k - 0.5 T_i^k + 0.75 T_{i-1}^k$$

- Repeat the same process

# Example (continue)

- Marching through time



# Example (continue)

- How about semi-implicit method?
- With $\Delta t$ = 0.3 min

$$s = \alpha \Delta t / (\Delta x)^2 = (0.1)(0.3)/(0.2)^2 = 0.75, \quad \theta = 0.5$$

$$-0.5sT_{i+1}^{k+1} + (1+s)T_i^{k+1} - 0.5sT_{i-1}^{k+1} = 0.5sT_{i+1}^{k} + (1-s)T_i^{k} + 0.5sT_{i-1}^{k}$$

$$-0.375T_{i-1}^{k+1} + 1.75T_i^{k+1} - 0.375T_{i+1}^{k+1} = 0.375T_{i-1}^{k} + 0.25T_i^{k} + 0.375T_{i+1}^{k}$$

- At $t_1$ = 0.3 min

$$T_0^0 = 60°\text{C},\ T_5^0 = 45°\text{C},\ T_0^1 = 50°\text{C},\ T_5^1 = 20°\text{C},\ T_1^0 = T_2^0 = T_3^0 = T_4^0 = 70°\text{C}$$

# Example (continue)

- Finite difference equations

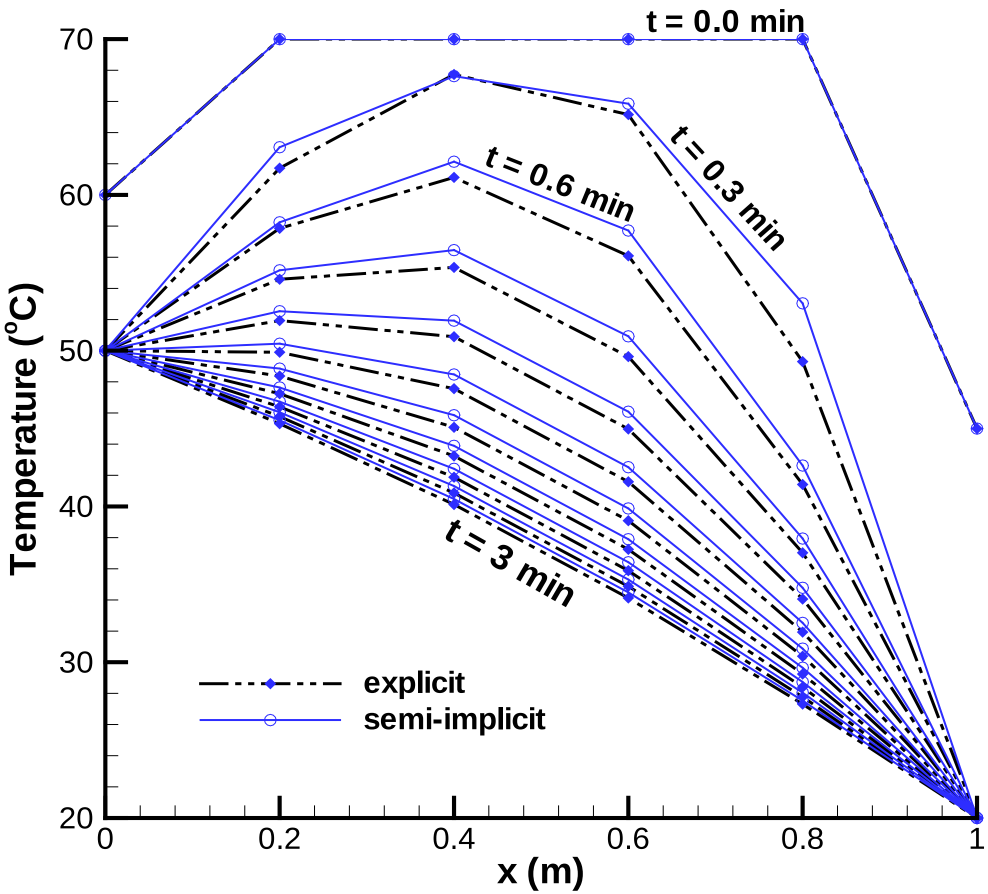
$$-0.375T_0^1 + 1.75T_1^1 - 0.375T_2^1 = 0.375T_0^0 + 0.25T_1^0 + 0.375T_2^0$$
$$-0.375T_1^1 + 1.75T_2^1 - 0.375T_3^1 = 0.375T_1^0 + 0.25T_2^0 + 0.375T_3^0$$
$$-0.375T_2^1 + 1.75T_3^1 - 0.375T_4^1 = 0.375T_2^0 + 0.25T_3^0 + 0.375T_4^0$$
$$-0.375T_3^1 + 1.75T_4^1 - 0.375T_5^1 = 0.375T_3^0 + 0.25T_4^0 + 0.375T_5^0$$

- In matrix form

$$\begin{bmatrix} 1.75 & -0.375 & 0 & 0 \\ -0.375 & 1.75 & -0.375 & 0 \\ 0 & -0.375 & 1.75 & -0.375 \\ 0 & 0 & -0.375 & 1.75 \end{bmatrix} \begin{Bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{Bmatrix} = \begin{Bmatrix} 85 \\ 70 \\ 70 \\ 68.125 \end{Bmatrix}$$

# Example (continue)

- Marching through time



# Partial Differential Equations – General terminology

- Differential equations involving more than one independent variable are called partial differential equations (PDEs)
- Linear second-order partial-differential equations can be classified into three categories depending on the values of the coefficients in this general formulation:

$$a\frac{\partial^2\phi}{\partial x^2}+b\frac{\partial^2\phi}{\partial x\partial y}+c\frac{\partial^2\phi}{\partial y^2}=G\left(x,y,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y}\right)$$

$$i.e.,\quad a\frac{\partial^2\phi}{\partial x^2}+b\frac{\partial^2\phi}{\partial x\partial y}+c\frac{\partial^2\phi}{\partial y^2}+d\frac{\partial\phi}{\partial x}+e\frac{\partial\phi}{\partial y}+f\phi=h$$

# Category

- The value of $b^2 - 4ac$ determine the type
- For example:
- $< 0$: Elliptic – Laplace equation (steady state with two spatial dimensions) $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$
- $= 0$: Parabolic – Heat conduction equation (time variation with one spatial dimension) $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$
- $> 0$: Hyperbolic – Wave equation (time variation with one spatial dimension) $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

# Boundary Conditions

- A PDE must have both initial and boundary conditions!
- The boundary conditions can be specified as:
  - Dirichlet condition

$$\phi(x,y) = f(x,y) \quad \text{or} \quad \phi(x,y) = \text{constant}$$

  - Neumann condition

$$\left.\frac{d\phi}{dn}\right|_{x,y} = f(x,y) \quad \text{or} \quad \left.\frac{d\phi}{dn}\right|_{x,y} = \text{constant}$$

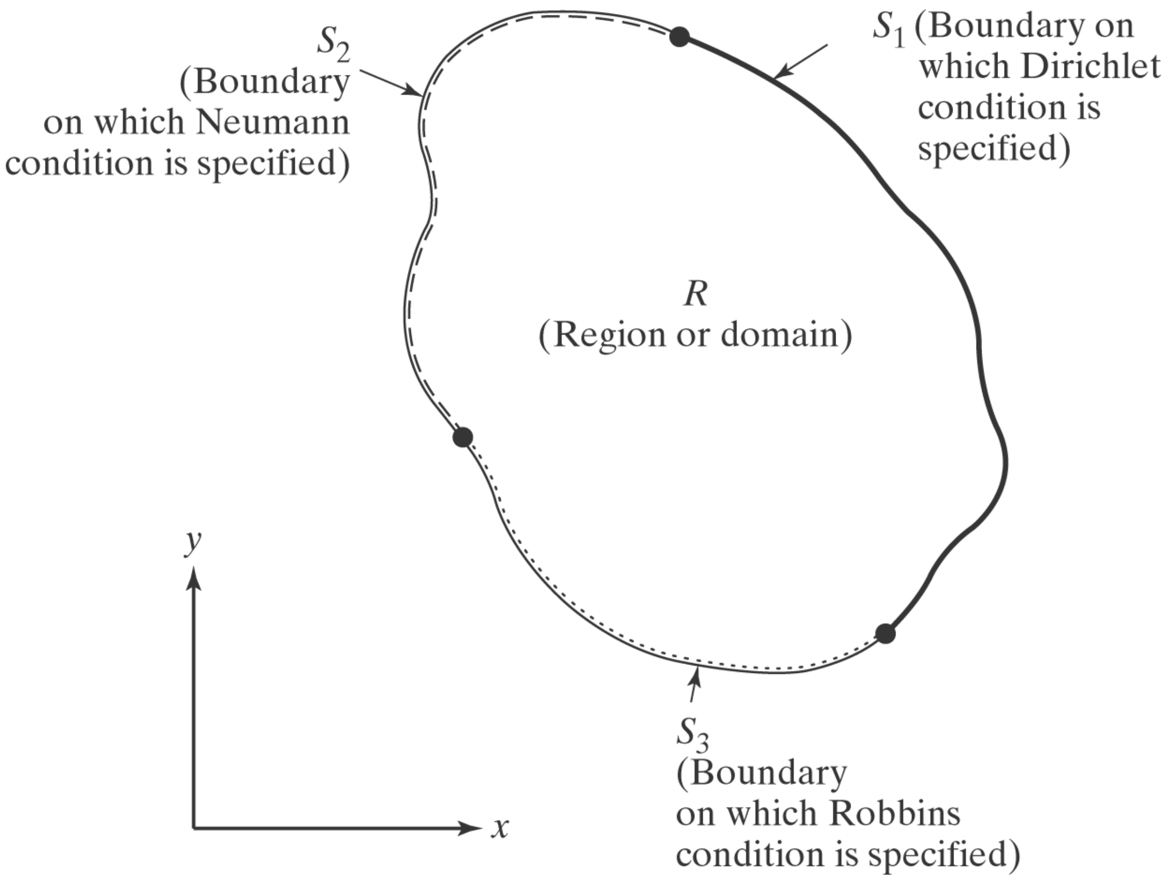
n is the direction normal to the boundary

  - Mixed condition

$$\left.\frac{d\phi}{dn}\right|_{x,y} + c\phi(x,y) = f(x,y) \quad \text{or} \quad \left.\frac{d\phi}{dn}\right|_{x,y} + c\phi(x,y) = \text{constant}$$

# Boundary Conditions (continue)

- Schematic description:

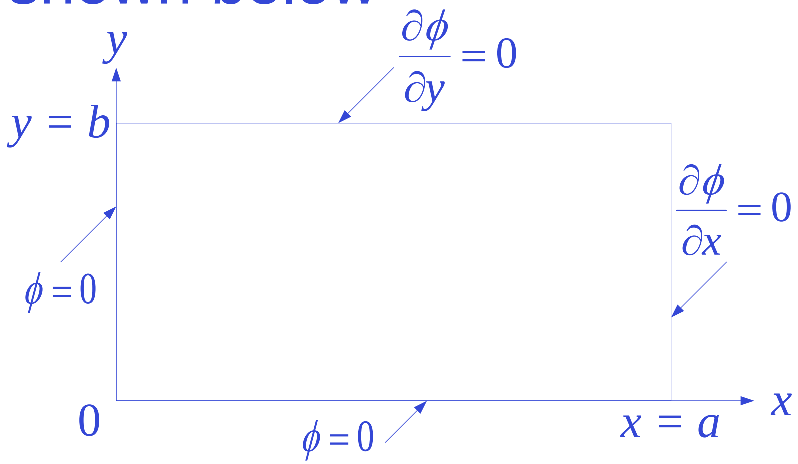


# Elliptic Partial Differential Equations

- Laplace equation: $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$
- Poisson equation: $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y)$$
- Note: function $f(x, y)$ is called source or non-homogeneous term

# Description of Problem

- Consider a rectangular region with the boundary conditions as shown below

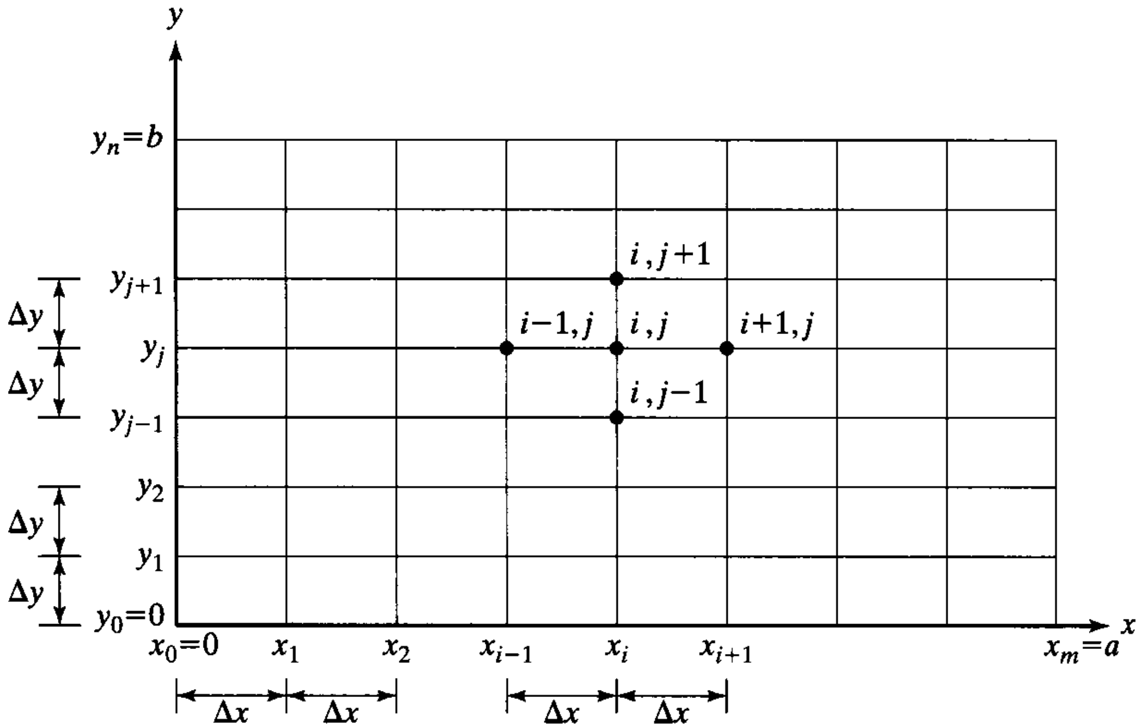



- The rectangular domain of integration is divided into *m* equal parts along the *x*-direction and *n* equal parts along the *y*-direction, so that the step sizes are given by

$$\Delta x = a / m, \quad \Delta y = b / n$$

# Description of Problem (continue)

- Let us define the coordinates of the mesh points which are denoted by $(x_i, y_i)$




Finite difference grid

# Approximation

- Employ the central-difference formula to derive the finite-difference equations for an interior grid point $(i,j)$:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y)$$

$$\Rightarrow \quad \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{(\Delta x)^2} + \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{(\Delta y)^2} = f(x, y)$$

Central difference along x

Central difference along y

# Boundary Conditions

$$\phi(0,y)=\phi_{0,j}=0,\quad j=0,1,2,\ldots,n$$

Along x = 0

$$\phi(x,0)=\phi_{i,0}=0,\quad i=0,1,2,\ldots,m$$

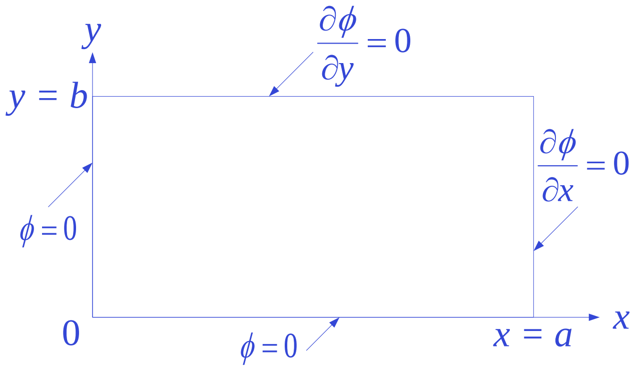
Along y = 0

$$\frac{\partial\phi}{\partial x}(a,y)=\left.\frac{\partial\phi}{\partial x}\right|_{m,j}=0,\quad j=0,1,2,\ldots,n$$

Along x = a

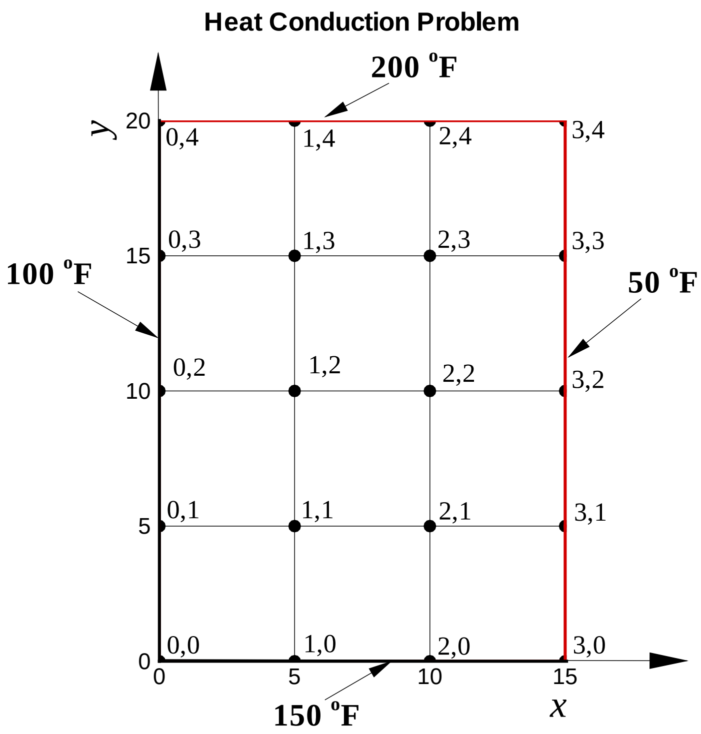
$$\frac{\partial\phi}{\partial y}(x,b)=\left.\frac{\partial\phi}{\partial y}\right|_{i,n}=0,\quad i=0,1,2,\ldots,m$$

Along y = b



# Example - Elliptic Equation

- Determine the steady-state temperature distribution in a rectangular plate of size 15 mm × 20 mm by solving the Laplace equation using $\Delta x = \Delta y = 5$ mm. Temperatures on the four sides of the plate are specified as indicated below.



# Example (continue)

- Solution : We approximate the temperature equation of the form:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- Applying central difference formula:

$$\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{(\Delta y)^2} = 0$$

- With $\Delta x = \Delta y$,

$$(\Delta x)^2 = (\Delta y)^2 \quad \Rightarrow \quad T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j} = 0$$

# Example (continue)

- Boundary conditions

$$T_{0,j} = 100\ °\text{F}, \quad T_{3,j} = 50\ °\text{F}, \quad j = 0,1,2,3,4$$

$$T_{i,0} = 150\ °\text{F}, \quad T_{i,4} = 200\ °\text{F}, \quad i = 0,1,2,3$$

or

$$T_{0,0} = T_{0,1} = T_{0,2} = T_{0,3} = T_{0,4} = 100\ °\text{F}, \quad T_{3,0} = T_{3,1} = T_{3,2} = T_{3,3} = T_{3,4} = 50\ °\text{F},$$

$$T_{0,0} = T_{1,0} = T_{2,0} = T_{3,0} = 150\ °\text{F}, \quad T_{0,4} = T_{1,4} = T_{2,4} = T_{3,4} = 200\ °\text{F}$$

# Example (continue)

- Finite difference equations:

$$(1,1) \Rightarrow \quad T_{0,1} + T_{2,1} + T_{1,0} + T_{1,2} - 4T_{1,1} = 0 \Rightarrow \quad 100 + T_{2,1} + 150 + T_{1,2} - 4T_{1,1} = 0$$

$$(2,1) \Rightarrow \quad T_{1,1} + T_{3,1} + T_{2,0} + T_{2,2} - 4T_{2,1} = 0 \Rightarrow \quad T_{1,1} + 50 + 150 + T_{2,2} - 4T_{2,1} = 0$$

$$(1,2) \Rightarrow \quad T_{0,2} + T_{2,2} + T_{1,1} + T_{1,3} - 4T_{1,2} = 0 \Rightarrow \quad 100 + T_{2,2} + T_{1,1} + T_{1,3} - 4T_{1,2} = 0$$

$$(2,2) \Rightarrow \quad T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3} - 4T_{2,2} = 0 \Rightarrow \quad T_{1,2} + 50 + T_{2,1} + T_{2,3} - 4T_{2,2} = 0$$

$$(1,3) \Rightarrow \quad T_{0,3} + T_{2,3} + T_{1,2} + T_{1,4} - 4T_{1,3} = 0 \Rightarrow \quad 100 + T_{2,3} + T_{1,2} + 200 - 4T_{1,3} = 0$$

$$(2,3) \Rightarrow \quad T_{1,3} + T_{3,3} + T_{2,2} + T_{2,4} - 4T_{2,3} = 0 \Rightarrow \quad T_{1,3} + 50 + T_{2,2} + 200 - 4T_{2,3} = 0$$

# Example (continue)

- In matrix form:

$$\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{21} \\ T_{12} \\ T_{22} \\ T_{13} \\ T_{23} \end{Bmatrix} = \begin{Bmatrix} -250 \\ -200 \\ -100 \\ -50 \\ -300 \\ -250 \end{Bmatrix}$$

# Example (continue)

- Using Gauss-Siedel:

$$T_{1,1}^{k+1} = (250 + T_{2,1}^{k} + T_{1,2}^{k})/4$$
$$T_{2,1}^{k+1} = (200 + T_{1,1}^{k+1} + T_{2,2}^{k})/4$$
$$T_{1,2}^{k+1} = (100 + T_{1,1}^{k+1} + T_{1,3}^{k} + T_{2,2}^{k})/4$$
$$T_{2,2}^{k+1} = (50 + T_{1,2}^{k+1} + T_{2,1}^{k+1} + T_{2,3}^{k})/4$$
$$T_{1,3}^{k+1} = (300 + T_{1,2}^{k+1} + T_{2,3}^{k})/4$$
$$T_{2,3}^{k+1} = (250 + T_{1,3}^{k+1} + T_{2,2}^{k+1})/4$$

$$\Rightarrow \begin{Bmatrix} T_{11} \\ T_{21} \\ T_{12} \\ T_{22} \\ T_{13} \\ T_{23} \end{Bmatrix} = \begin{Bmatrix} 116.045 \\ 103.002 \\ 111.18 \\ 95.9627 \\ 132.712 \\ 119.668 \end{Bmatrix}$$

- With a suitable initial values of $T_{i,j}^{0}$

$$T_{i,j}^{0} = (100 + 200 + 150 + 50)/4$$

# Example (continue)

- Solution from finite difference equation can be obtained within 18 iterations of Gauss-Seidel method in $10^{-6}$ of convergence criterion with previous initial values
- Note that the accuracy of the result can be improved by using smaller step size $\Delta x$ and $\Delta y$