Lecture 10: Sinusoidal Steady-State Analysis and Power Analysis

ELEC1111 Electrical and Telecommunications Engineering

Never Stand Still

Faculty of Engineering

School of Electrical Engineering and Telecommunications

Objectives

- Analysis of AC circuits under steady-state using:
  - Nodal analysis based on KCL
  - Mesh analysis based on KVL
  - Superposition theory
  - Source transformation
  - Thévenin and Norton equivalent circuits

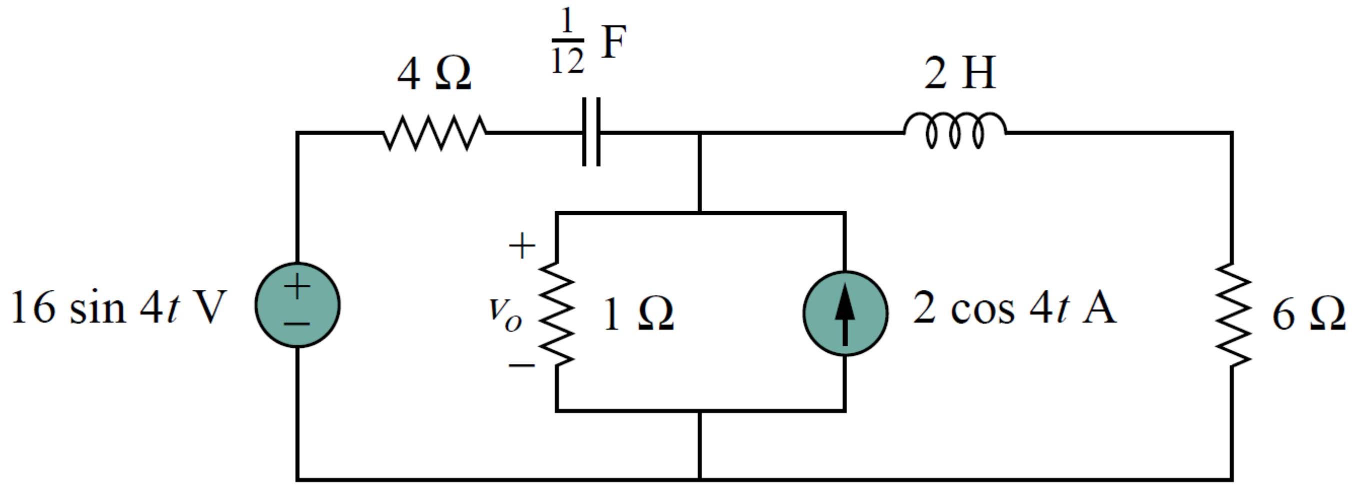
Sinusoidal steady-state analysis

- Steps to analyze AC circuits
  1. Transform the circuit to the frequency domain.
  2. Solve the problem using circuit techniques learnt previously.
     - Analysis is similar to the DC circuit analysis but uses complex numbers.
  3. Transform the resulting phasor to the time domain.

Nodal Analysis

- An example of nodal analysis in the frequency domain.

Calculate the voltage v_0 across the 1Ω resistor



Nodal Analysis

- Firstly we need to transform the sources to phasors

$$\omega = 4 \text{ rad/sec}$$

$$16\sin(\omega t) = 16\angle -90^{\circ} = -j16$$

$$2\cos(\omega t) = 2\angle 0^{\circ}$$

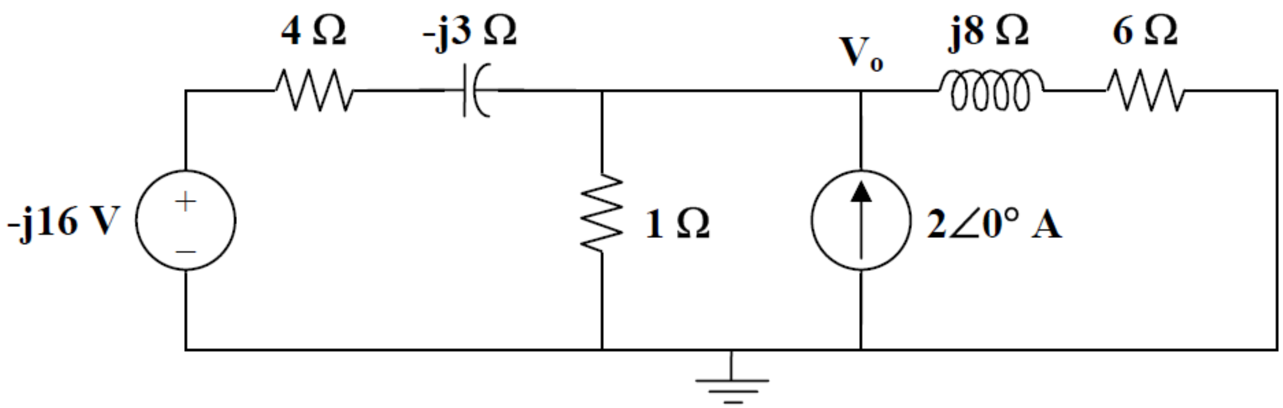
- Then we need to calculate the impedances at the angular frequency of the sources.

$$2H \rightarrow j\omega L = j\cdot 2\cdot 4 = j8\,\Omega$$

$$\frac{1}{12}F \rightarrow \frac{1}{j\omega C} = \frac{1}{j4\frac{1}{12}} = \frac{3}{j} = -j3\,\Omega$$

Nodal Analysis

- The equivalent circuit in the frequency domain is



- Applying nodal analysis

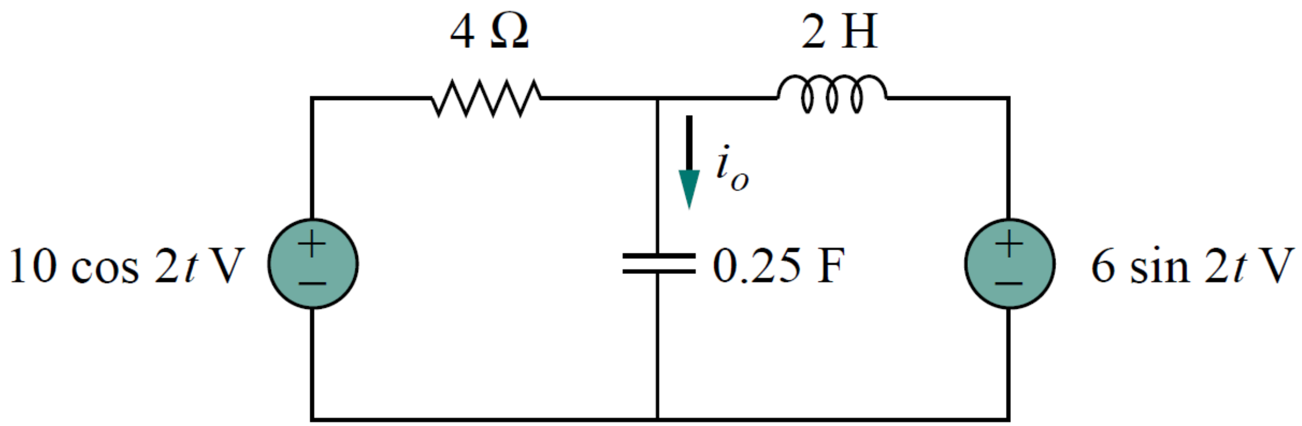
$$\frac{-j16-\mathbf{V}_0}{4-j3}+2=\frac{\mathbf{V}_0}{1}+\frac{\mathbf{V}_0}{6+j8} \Rightarrow \frac{-j16}{4-j3}+2=\left(1+\frac{1}{4-j3}+\frac{1}{6+j8}\right)\mathbf{V}_0 \Rightarrow$$

$$\mathbf{V}_0=\frac{3.92-j2.56}{1.22+j0.04}=\frac{4.682\angle\text{-}33.15^\circ}{1.2207\angle 1.88^\circ}=3.835\angle-35.02^\circ\ \text{V}$$

$$\text{v}_0(t)=3.835\cos(4t-35.02^\circ)\ \text{V}$$

Mesh Analysis

- Example of Mesh Analysis in the frequency domain
  - Calculate current i_0 through the capacitor



- Again we start by transforming the sources to phasors

$$\omega = 2 \text{ rad/sec}$$

$$6\sin(\omega t) = 6\angle -90^\circ = -j6$$

$$10\cos(\omega t) = 10\angle 0^\circ$$

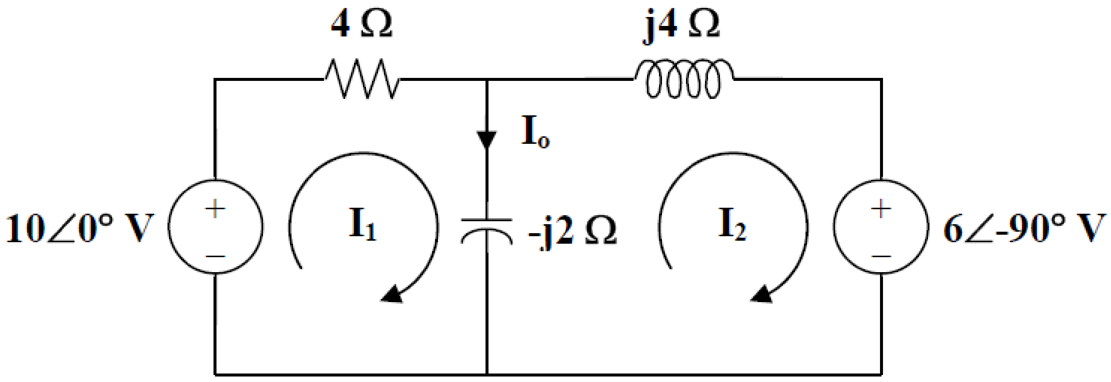
Mesh Analysis

- Transform the capacitance and inductance in the frequency domain

$$C = 0.25F \rightarrow \frac{1}{j \cdot 2 \cdot 0.25} = \frac{2}{j} = -j2\,\Omega$$

$$L = 2H \rightarrow j \cdot 2 \cdot 2 = j4\,\Omega$$

- The equivalent circuit in the frequency domain is



Mesh Analysis

- The equations for the two loops are written as:

$$\text{Loop 1:}\ -10+(4-j2)\mathbf{I}_1+j2\mathbf{I}_2=0 \Rightarrow (2-j)\mathbf{I}_1+j\mathbf{I}_2=5$$

$$\text{Loop 2:}\quad j2\mathbf{I}_1+(j4-j2)\mathbf{I}_2-j6=0 \Rightarrow \mathbf{I}_1+\mathbf{I}_2=3$$

- The equations can be written in a matrix form as

$$\begin{pmatrix} 2-j & j \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{pmatrix}=\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\Delta=2(1-j) \qquad \Delta_1=5-j3 \qquad \Delta_2=1-j3$$

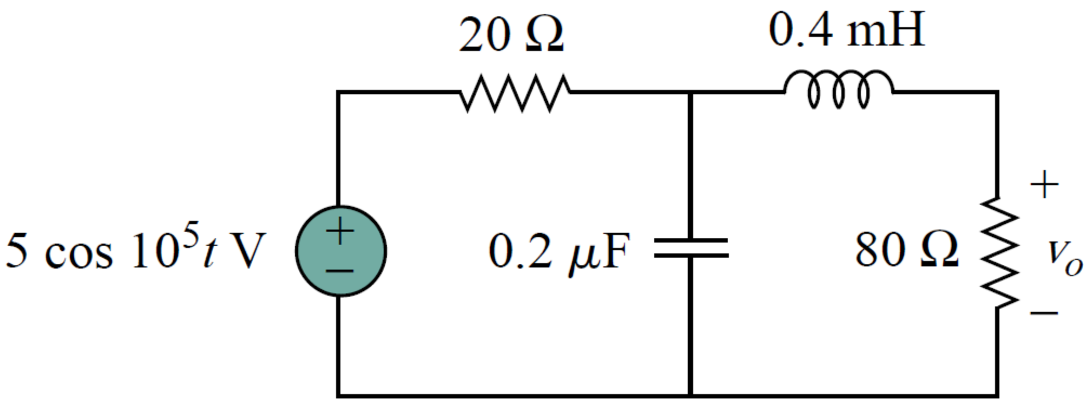
- The current is then equal to

$$\mathbf{I}_0=\mathbf{I}_1-\mathbf{I}_2=\frac{\Delta_1-\Delta_2}{\Delta}=\frac{4}{2(1-j)}=1+j=1.414\angle 45^\circ\ \text{A}$$

$$i_0(t)=1.414cos(2t+45^\circ)\ \text{A}$$

Source Transformation

- An example of source transformation



- Firstly transform the inductance and capacitance to the frequency domain

$$L = 0.4mH \rightarrow j\omega L = j(10^5)(0.4 \times 10^{-3}) = j40\,\Omega$$

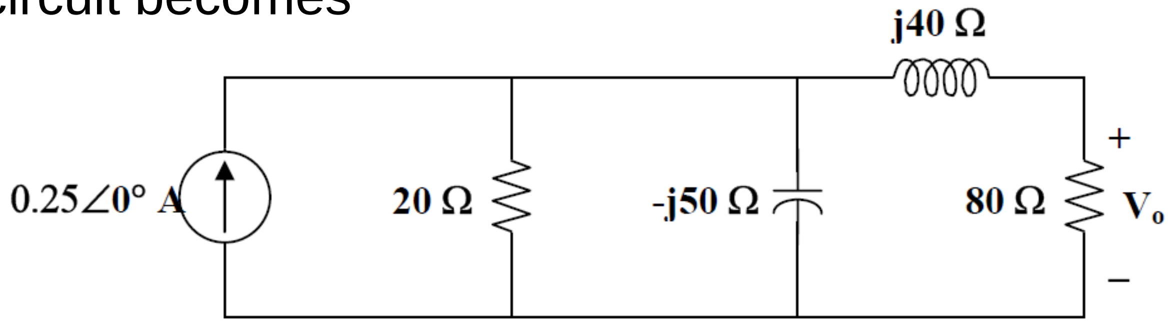
$$C = 0.2uF \rightarrow \frac{1}{j\omega C} = \frac{1}{j(10^5)(0.2 \times 10^{-6})} = -j50\,\Omega$$

Source Transformation

- After transforming the voltage source to the equivalent current source

$$I_s = \frac{V_s}{Z_s} = \frac{5}{20} = 0.25 \text{ A}$$

- The circuit becomes



- The impedance parallel to the source is

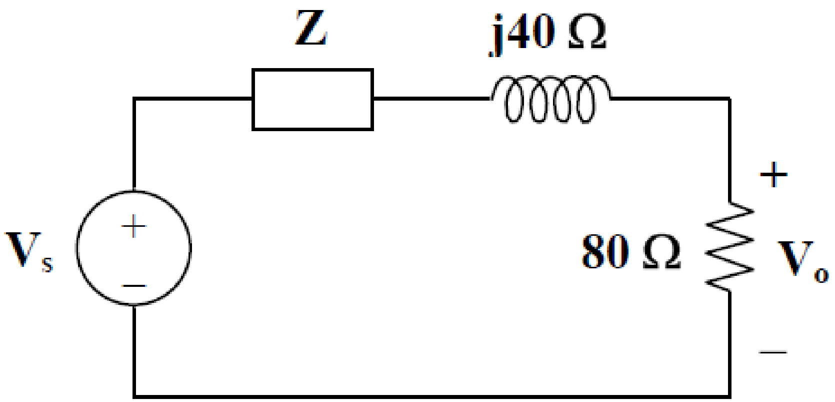
$$\mathbf{Z} = 20 \,||\, -j50 = \frac{-j100}{2 - j5}$$

Source Transformation

- Transforming the current source back to the equivalent voltage source

$$\mathbf{V}_s = (0.25\angle 0^\circ)\mathbf{Z} = \frac{-j25}{2-j5}$$

- And the equivalent circuit as



Source Transformation

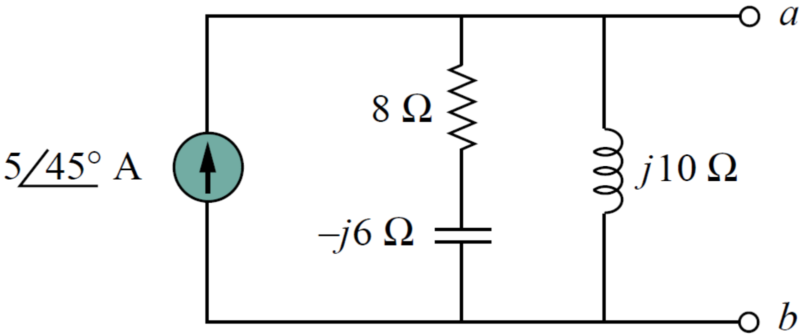
- The voltage we are looking for can now be calculated from a simple voltage division as

$$\mathbf{V}_0 = \frac{80}{\mathbf{Z}+80+j40}\mathbf{V}_s = \frac{80}{\frac{-j100}{2-j5}+80+j40}\cdot\frac{-j25}{2-j5} = \ldots = 36.15\angle-40.6^\circ$$

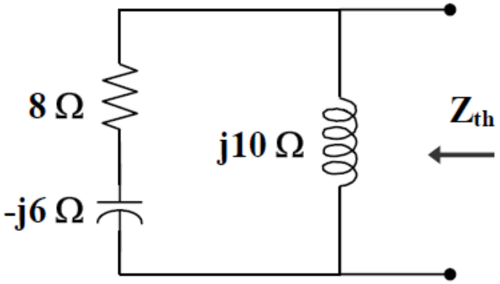
$$v_0(t) = 36.15cos(10^5 t - 40.6^\circ)\text{ V}$$

Thévenin and Norton circuits

- Calculation of the Thévenin equivalent circuit



- Firstly we will calculate the Thévenin equivalent impedance

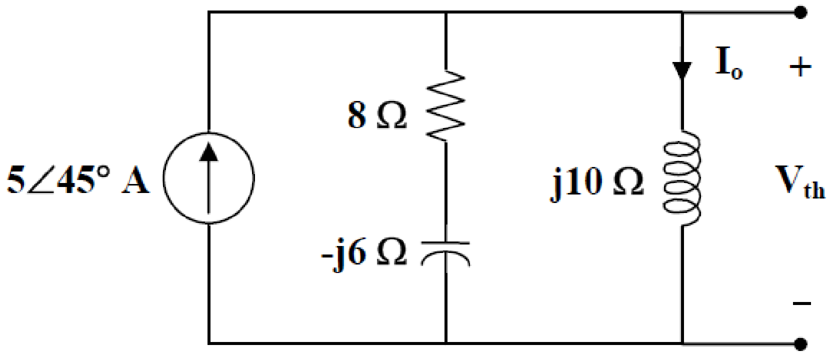


$$\mathbf{Z}_{Th} = j10 \,\|\, (8 - j6) = \frac{(j10)(8 - j6)}{8 + j4} = 5(2 + j)\ \Omega$$

$$11.18\angle 26.56^{\circ}\ \Omega$$

Thévenin and Norton circuit

- In order to calculate the Thévenin voltage we analyze the circuit



$$\mathbf{I}_0 = \frac{8 - j6}{8 - j6 + j10}(5\angle 45^\circ) = \frac{4 - j3}{4 + j2}(5\angle 45^\circ)$$

$$V_{th} = j10\mathbf{I}_0 = j10\frac{4 - j3}{4 + j2}(5\angle 45^\circ) = \ldots = 55.9\angle 71.56^\circ \text{ V}$$

Summary

- Analysis of AC circuits under steady-state using
  - Nodal analysis based on KCL
  - Mesh analysis based on KVL
  - Superposition theory
  - Source transformation
  - Thévenin and Norton equivalent circuits

AC Power Analysis

The objectives of this lecture are to:

- Define instantaneous and average power
- Calculate the requirements for maximum average power transfer
- Define the effective or RMS value of power

Instantaneous and Average Power

- **Instantaneous power** is the power at any instant in time and describes the rate at which an element absorbs energy. It is the product of instantaneous voltage and instantaneous current in an element.

$$p(t) = v(t)i(t)$$

- Assuming sinusoidal voltages and currents at the circuit terminals are

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

- Then the instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

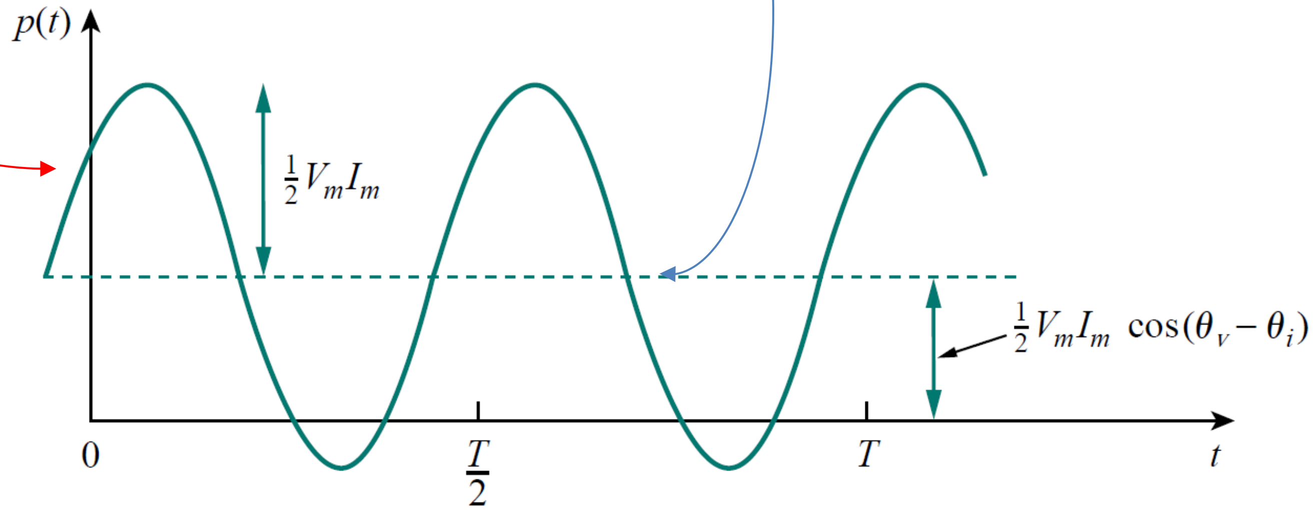
$$\boxed{\cos A \cos B = \tfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)}$$

Instantaneous and Average Power

- Therefore can re-write as

$$\boxed{p(t)=\boxed{\frac{1}{2}V_m I_m \cos(\theta_v-\theta_i)}+\frac{1}{2}V_m I_m \cos(2\omega t+\theta_v+\theta_i)}$$

- Two distinct parts can be identified
  - A constant part, not dependent on time
  - A varying part with double the frequency of the voltage and current



Instantaneous and Average Power

- Average power, in Watts (W), is the average value of the instantaneous power over a period.

$$P = \frac{1}{T}\int_0^T p(t)\,dt$$

- Integrating the equation for the instantaneous power over a period yields a constant value as the integral of the sinusoidal term is equal to zero.
- The average power for sinusoidal voltages and currents does not depend on time while the instantaneous power varies with time.

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

Instantaneous and Average Power

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

- How can average power be represented using phasor quantities ?

$$\frac{1}{2}\left(\mathbf{VI}^*\right) = \frac{1}{2} V_m I_m \angle(\theta_v - \theta_i)$$

$$= \frac{1}{2} V_m I_m \left(\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)\right)$$

$$\mathbf{V} = V_m e^{j\theta_v} = V_m \angle \theta_v$$
$$I = I_m e^{j\theta_i} = I_m \angle \theta_i$$
$$I^* = I_m e^{-j\theta_i} = I_m \angle -\theta_i$$

- The average power is then

$$P = \frac{1}{2}\mathrm{Re}\left(\mathbf{VI}^*\right) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

Instantaneous and Average Power

- Two special cases of load (Z=R+jX):
  - In a **resistive** load (Z=R) the voltage is always in phase with the current

$$\theta_v = \theta_i \;\rightarrow\; P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R$$

  - A resistive load absorbs power at all times

  - When the load is **purely reactive** (Z=jX)

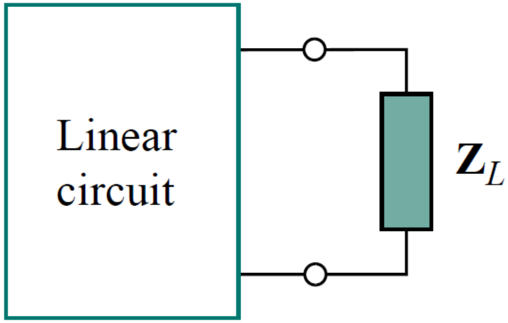
$$\theta_v - \theta_i = \pm 90^\circ \;\rightarrow\; P = \frac{1}{2} V_m I_m \cos(\pm 90^\circ) = 0$$

  - A purely reactive circuit absorbs no average power

Maximum Average Power Transfer

- The problem of maximizing the average power transfer can be equally applied to AC circuits
- Representing a source with the Thévenin equivalent circuit and the load as an impedance

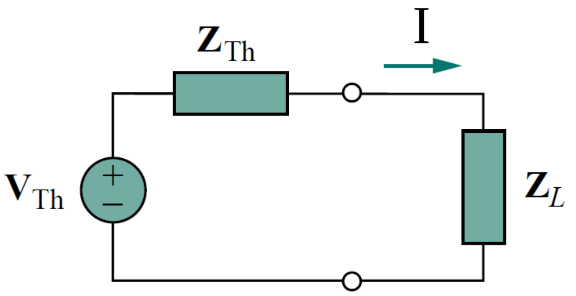
$$Z_{th} = R_{th} + jX_{th} \qquad Z_L = R_L + jX_L$$



- The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{th}}{\mathbf{Z}_{th} + \mathbf{Z}_L} = \frac{\mathbf{V}_{th}}{R_{th} + jX_{th} + R_L + jX_L}$$

$$P = \frac{1}{2}\left|\mathbf{I}\right|^2 R_L = \frac{1}{2}\frac{\left|\mathbf{V}_{th}\right|^2 R_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$$



Maximum Average Power Transfer

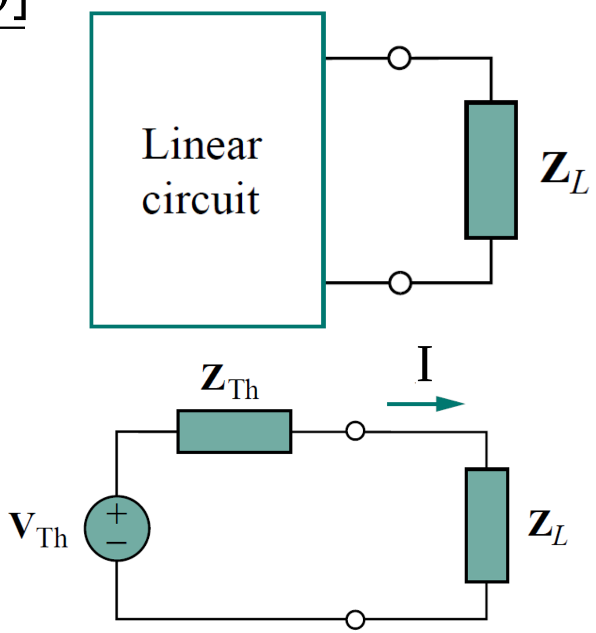
- Differentiating the power equation and equating to zero in order to calculate the maximum we find

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{th}|^2 \left[(R_{th} + R_L)^2 + (X_{th} + X_L)^2 - 2R_L(R_{th} + R_L) \right]}{2\left[(R_{th} + R_L)^2 + (X_{th} + X_L)^2 \right]^2}$$

$$\frac{\partial P}{\partial X_L} = -\frac{|\mathbf{V}_{th}|^2 R_L (X_{th} + X_L)}{\left[(R_{th} + R_L)^2 + (X_{th} + X_L)^2 \right]^2}$$

$$\frac{\partial P}{\partial R_L} = 0 \rightarrow R_L = \sqrt{R_{th}^2 + (X_{th} + X_L)^2}$$

$$\frac{\partial P}{\partial X_L} = 0 \rightarrow X_L = -X_{th}$$



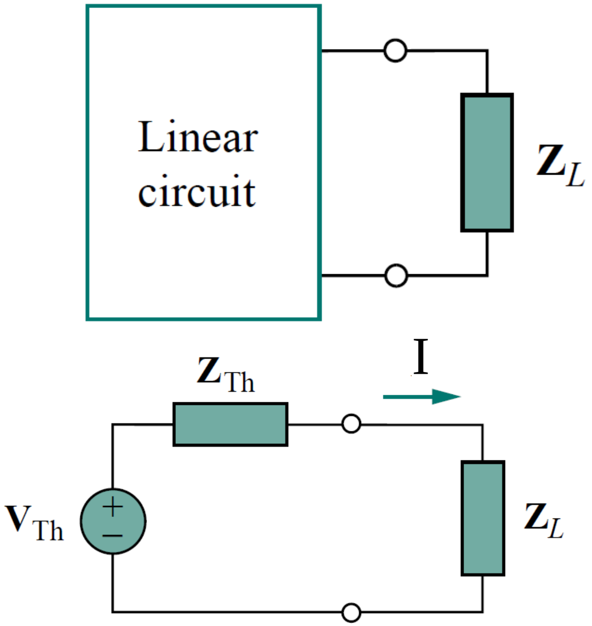
Maximum Average Power Transfer

- And finally

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{th} - jX_{th} = \mathbf{Z}_{th}^*}$$

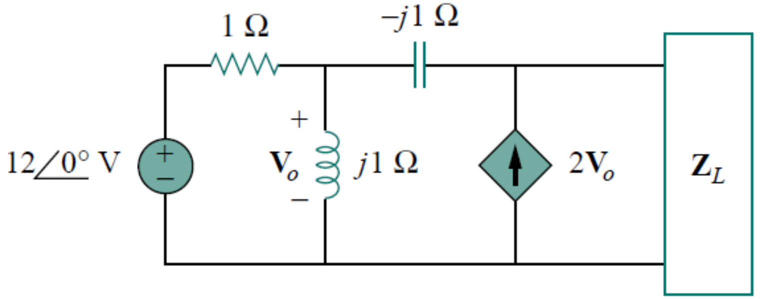
- For maximum average power transfer the load impedance must be equal to the complex conjugate of the Thévenin Impedance.
- The maximum average power transferred is

$$P_{\max} = \frac{|\mathbf{V}_{th}|^2}{8R_{th}}$$



Maximum Average Power Transfer (example)

- Considering the circuit of the figure, find the load value that absorbs the maximum power and the value of the maximum power.

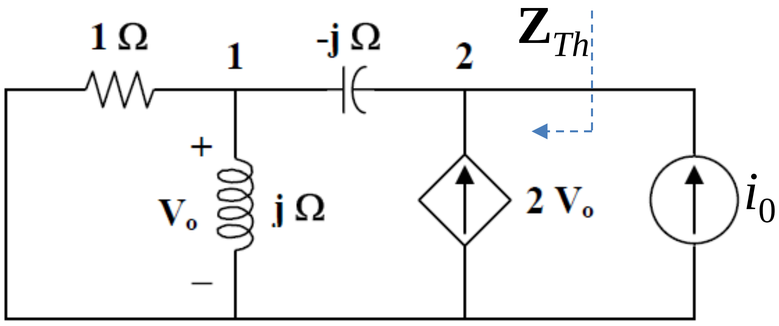


- We need to calculate the Thévenin equivalent of the circuit at the load terminals.
- At node 1

$$\frac{\mathbf{V}_o}{1} + \frac{\mathbf{V}_o}{j} = \frac{\mathbf{V}_2 - \mathbf{V}_o}{-j} \rightarrow \mathbf{V}_o = j\mathbf{V}_2$$

- At node 2

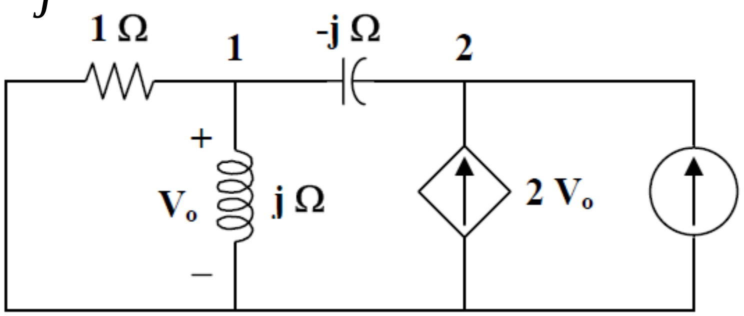
$$i_0 + 2\mathbf{V}_0 = \frac{\mathbf{V}_2 - \mathbf{V}_o}{-j} \rightarrow i_0 = j\mathbf{V}_2 - (2 + j)\mathbf{V}_o$$



Maximum Average Power Transfer (example)

- Substituting the equations

$$i_0 = j\mathbf{V}_2 - (2+j)j(\mathbf{V}_2) = (1-j)\mathbf{V}_2 \Rightarrow \mathbf{V}_2 = \frac{i_0}{1-j}$$

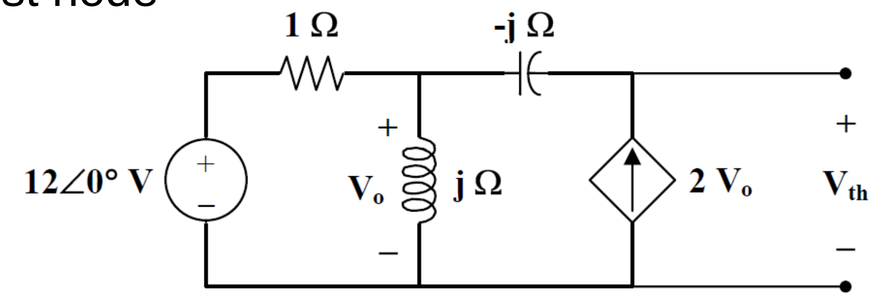


$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_2}{i_0} = \frac{1}{1-j} = \frac{1+j}{2} = 0.5 + j0.5$$

$$\boxed{\mathbf{Z}_L = \mathbf{Z}^*_{Th} = 0.5 - j0.5\,\Omega}$$

condition for maximum average power transfer

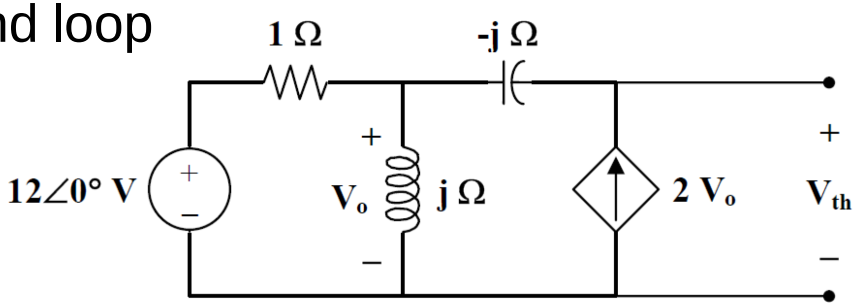
- The Thévenin voltage has to be calculated next.
  - Using Nodal analysis at the first node



$$2\mathbf{V}_o + \frac{12-\mathbf{V}_o}{1} = \frac{\mathbf{V}_o}{j} \Rightarrow \mathbf{V}_o = \frac{-12}{1+j}$$

Maximum Average Power Transfer (example)

- Considering the mesh on the second loop



$$\mathbf{V}_{Th} = -j \times 2\mathbf{V}_0 + \mathbf{V}_0$$

$$\mathbf{V}_{Th} = (1 - j2)\frac{-12}{1 + j} = 6 + j18$$

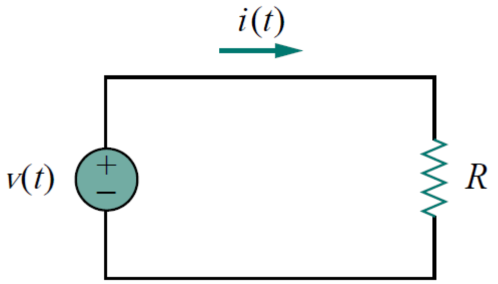
- The formula to calculate the maximum power was given earlier

$$P_{\max} = \frac{|\mathbf{V}_{th}|^2}{8R_{th}} = \frac{6^2 + 18^2}{8(0.5)} = 90\ \mathrm{W}$$

Effective or Root Mean Square (RMS) Value

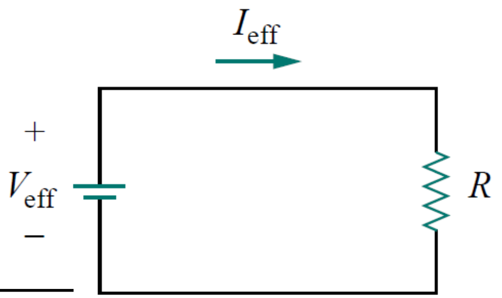
- The effective value of a periodic current is the DC current that delivers the same average power to a resistor as the periodic current.
- In the AC circuit the average power absorbed by the resistor is

$$P = \frac{1}{T}\int_0^T i^2 R dt = \frac{R}{T}\int_0^T i^2 dt$$



- While the power in the DC circuit is equal to

$$P = RI_{eff}^2$$



$$I_{eff} = I_{RMS} = \sqrt{\frac{1}{T}\int_0^T i^2 dt} \qquad V_{eff} = V_{RMS} = \sqrt{\frac{1}{T}\int_0^T v^2 dt}$$

Effective or RMS Value

- For a sinusoid the calculations yield:

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t dt} = \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1+\cos 2\omega t)dt} = \frac{I_m}{\sqrt{2}}$$

- The average power can be written in terms of the RMS values

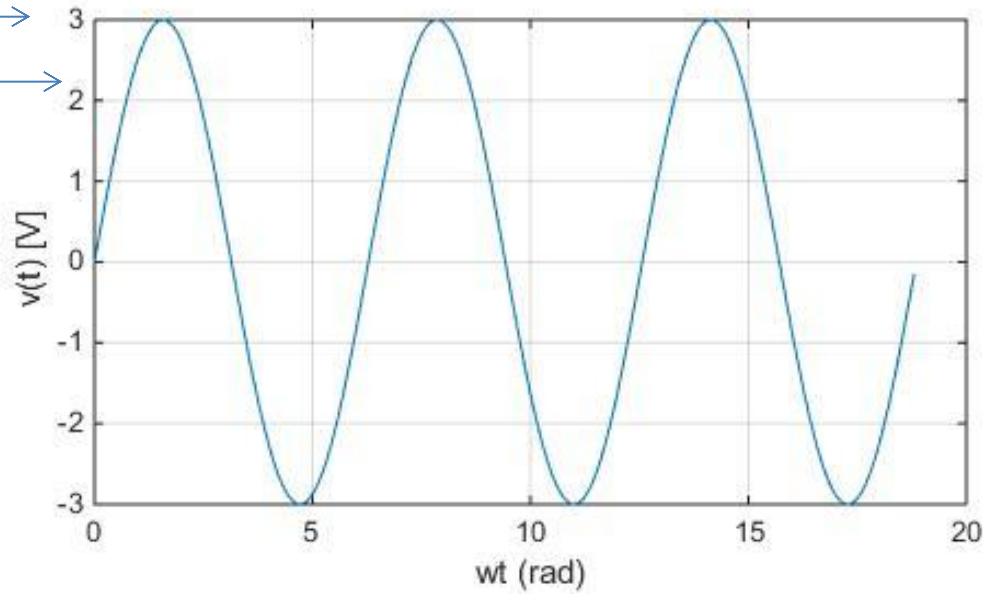
$$P = \frac{1}{2}V_m I_m \cos(\theta_u - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_u - \theta_i)$$

$$= V_{rms} I_{rms} \cos(\theta_u - \theta_i)$$

- RMS values are commonly used. The power industry provides magnitudes in terms of their RMS values and analog voltmeters and ammeters are designed to read RMS values.

Sinusoid Example

- For a sinusoid the calculations yield:
- v(t)=3sin(ωt) V
  - V_m=3V
  - $V_{RMS} = 3/\sqrt{2}$=2.12V



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Instantaneous and Average Power - revisited

- If we calculate the average power in the frequency domain through phasors, then

$$Note: \angle(\theta_v - \theta_i) = e^{j(\theta_v - \theta_i)} = \cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)$$

$$\frac{1}{2}\left(\mathbf{VI}^*\right) = \frac{1}{2}V_m I_m \angle(\theta_v - \theta_i)$$

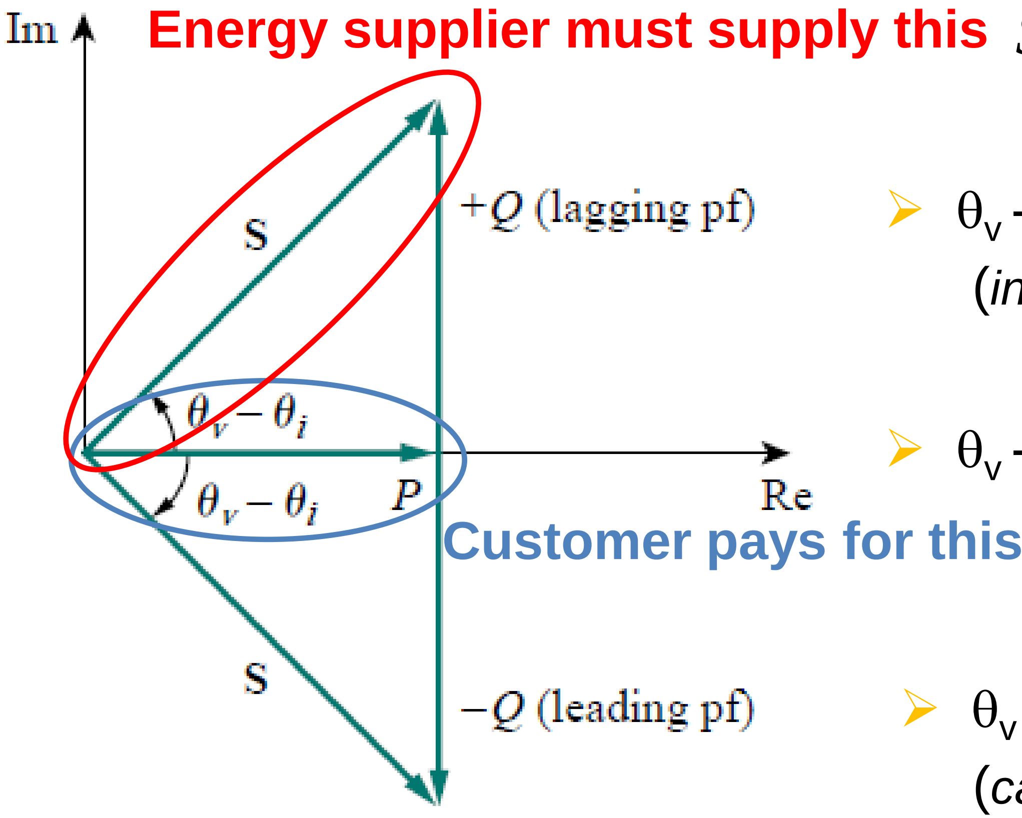
$$= \frac{1}{2}V_m I_m \left(\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)\right)$$

- The average power is then

What about this ?

$$P = \frac{1}{2}\mathrm{Re}\left(\mathbf{VI}^*\right) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

Instantaneous and Average Power - revisited



$S = P + jQ$

- $\theta_v - \theta_i > 0$ (*inductive Z=R+jX*)
- $\theta_v - \theta_i = 0$ (*resistive Z=R*)
- $\theta_v - \theta_i < 0$ (*capacitive Z=R-jX*)

Apparent Power and Power Factor

- Previously we calculated the average power as

$$P = V_{rms} I_{rms} \cos(\theta_u - \theta_i)$$

- This equation can be decomposed in two terms. The first is the product of the RMS values of the current and voltage and is known as apparent power

$$S = V_{rms} I_{rms}$$

- The second term is the cosine of the angle difference between the current and voltage phasors and (in purely sinusoidal waveforms) is called the power factor (pf).
- Apparent power is measured in VA (volt-ampere)

Apparent Power and Power Factor

- The power factor is dimensionless and is the ratio of the average and the apparent power.

$$\text{power factor} = \frac{P}{S} = cos(\theta_u - \theta_i)$$

- It is also equal to the angle of the load impedance.
- The values of the power factor range between **zero (0)** and **unity (1)**.
  - For purely resistive load, the angle between current and voltage is 0, hence the power factor is 1 and the apparent power is equal to the average power .
  - For purely reactive loads, the angle can be either -90º or 90º. In both cases the power factor is equal to zero and the differentiation is based on whether the current leads the voltage (leading, capacitive load) or the current lags the voltage phasor (lagging, inductive).

Complex Power

- Complex power contains all the information of the power absorbed by a given load. It is defined as the product of the RMS voltage phasor and the conjugate of the RMS current phasor.

$$S = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$$

$$S = V_{rms} I_{rms} \angle(\theta_u - \theta_i)$$
$$= V_{rms} I_{rms} \cos(\theta_u - \theta_i) + jV_{rms} I_{rms} \sin(\theta_u - \theta_i)$$

- The complex power may be expressed in terms of the load impedance based on the substitution

$$\mathbf{V}_{rms} = \mathbf{Z}\mathbf{I}^*_{rms}$$

$$S = I^2_{rms} Z = \frac{V^2_{rms}}{Z^*}$$

Note: Complex conjugate of I is I*

I=a+jb → I*=a-jb

I=Ae$^{j\theta}$ → I*=Ae$^{-j\theta}$

Complex Power

- The real part of the complex power defines the average, **active** or real power

$$P = \mathrm{Re}(S) = I_{rms}^2 R$$

- and the imaginary part defines the **reactive** power.

$$Q = \mathrm{Im}(S) = I_{rms}^2 X$$

- Comparing to the equation derived from the phasor we conclude that

$$P = V_{rms} I_{rms} \cos(\theta_u - \theta_i)$$
$$Q = V_{rms} I_{rms} \sin(\theta_u - \theta_i)$$

Complex Power

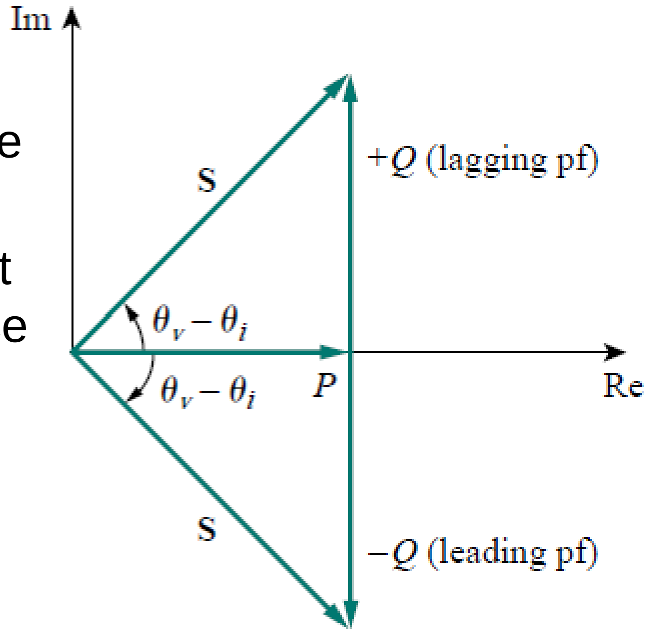
- The real power P is the average power delivered to the load. It is the actual power dissipated to the load (its unit is Watt).

- The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of reactive power is the VAr (Volt-ampere reactive)

  - For resistive loads $Q = 0$ (power factor = unity) $\rightarrow$ current in phase voltage
  - For inductive loads $Q > 0$ (lagging power factor)$\rightarrow$current lags voltage
  - For capacitive loads $Q < 0$ (leading power factor) $\rightarrow$current leads voltage

Complex Power

- By introducing complex power we can derive the real and reactive power directly from the current and voltage phasors.
- All the necessary quantities (power and power factor) can be represented in the form of the **power triangle**.
  - When S lies in the first quadrant, we have an inductive load and lagging pf.
  - When S lies in the fourth quadrant, we have a capacitive load and leading power factor
  - When S lies in the second or third quadrant the load impedance has negative resistance denoting an active circuit.



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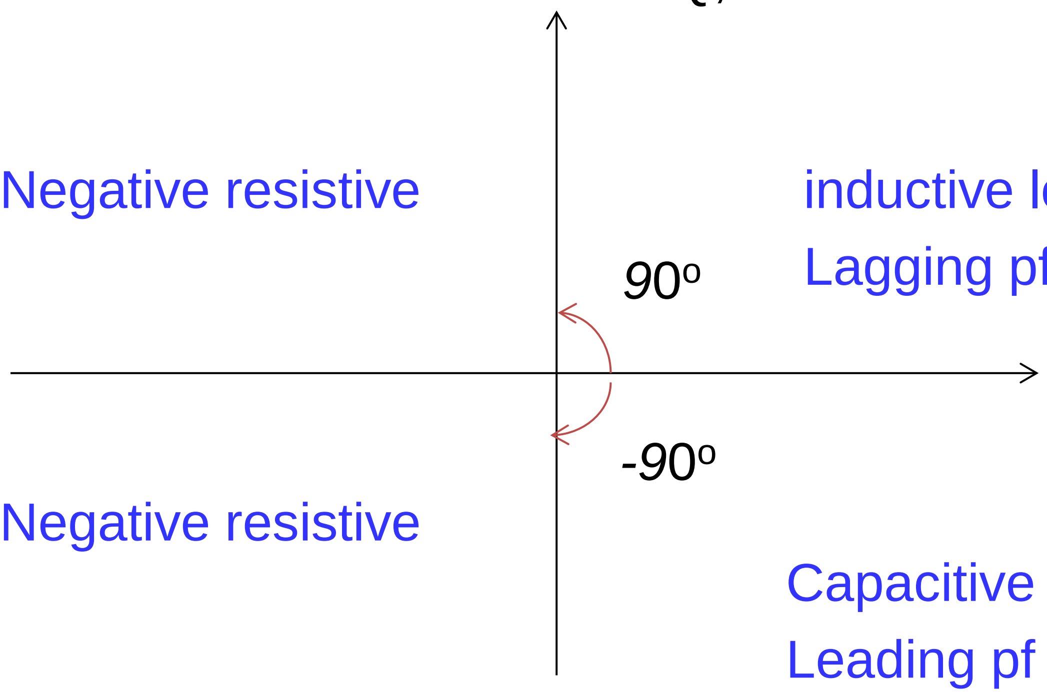
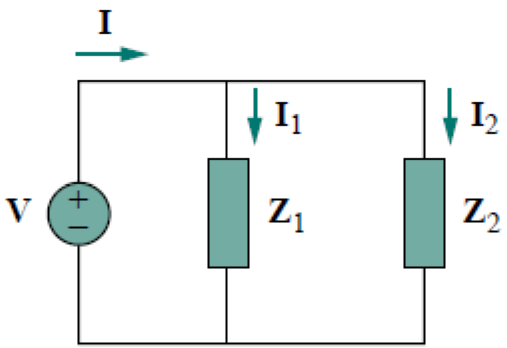
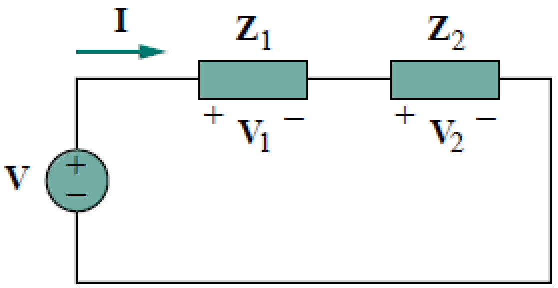
Apparent Power and Power Factor



Conservation of AC Power

- Through the application of KCL and KVL we can easily deduce the law of power conservation in AC circuits, stating that the total power supplied by the source is equal to the total power delivered to the load.
- The complex, real and reactive power of the sources is equal to the respective sums of the complex, real and reactive powers of the individual loads.
  - The last is not valid for apparent power which needs to be calculated from the total complex power.





Conservation of AC Power

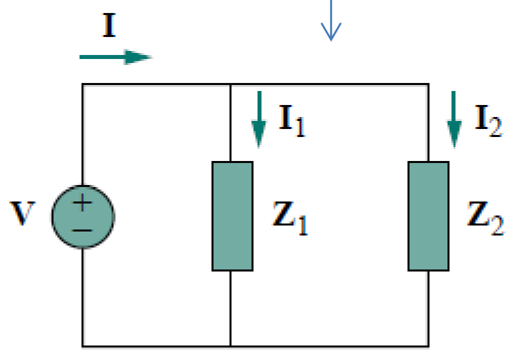
- Considering the series connected load and KVL

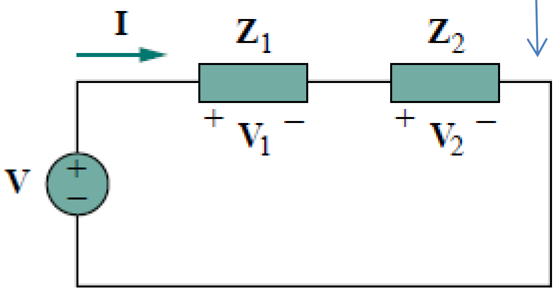
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$$

And the complex power supplied by the source is

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}(\mathbf{V}_1 + \mathbf{V}_2)\mathbf{I}^* = \frac{1}{2}\mathbf{V}_1\mathbf{I}^* + \frac{1}{2}\mathbf{V}_2\mathbf{I}^* = \mathbf{S}_1 + \mathbf{S}_2$$

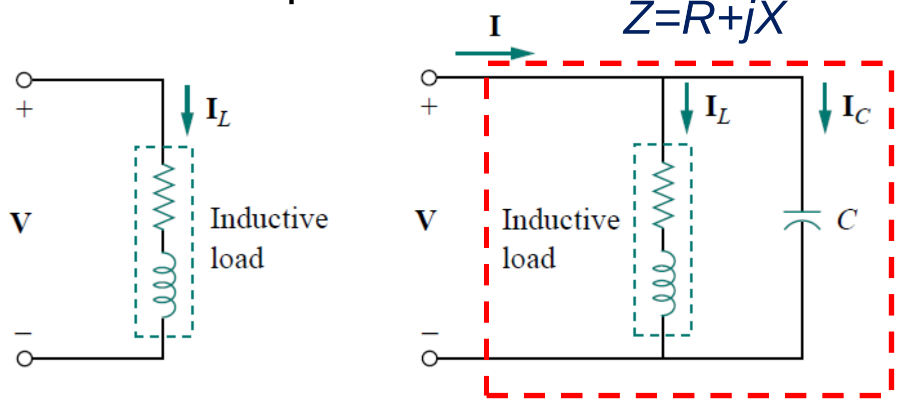
- The same can be derived for the parallel connected circuit and KCL





Power factor correction

- Reactive power does not have an active component yet it increases the current in the circuit leading to additional losses. It is therefore beneficial to have a power factor close to unity. As this is not the case for most loads, we can modify the power factor seen by the sources.
- **Power factor (pf) correction** is the process of modifying the power factor of a load (usually close to unity) without altering the voltage and current to the original load.
  - As most loads are inductive, power factor correction is usually done by means of a capacitive element.



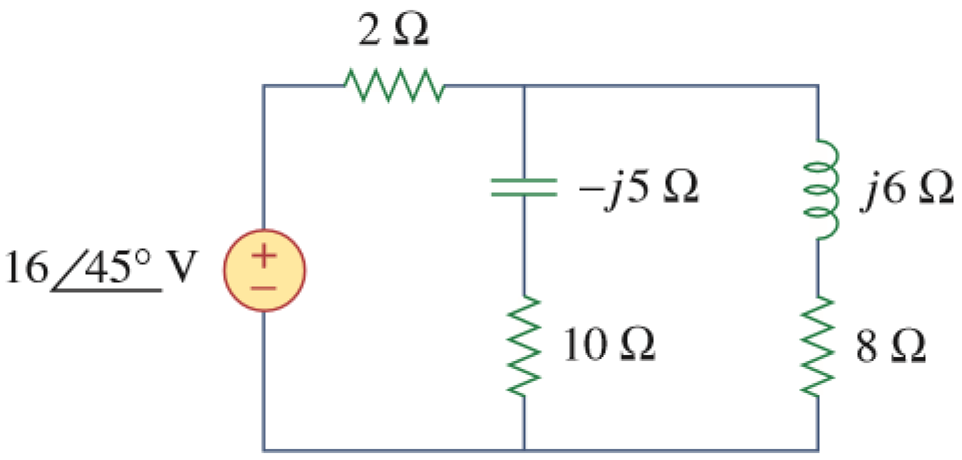
pf=1 means X=0, Z=R
i.e. purely resistive

Summary

- Instantaneous and Average Power
  - Time and frequency domain
  - Purely resistive and purely inductive loads
- Maximum average power transfer
- Effective or RMS value
- Reactive Power
- Apparent Power
- Complex Power
- Power Factor
- Power Factor Correction

EXAMPLE 1

- For the following circuit, find:
  - Power factor
  - Average power delivered by the source
  - Reactive power
  - Apparent power
  - Complex power
  - The capacitance value of a capacitor Cx such what when Cx connected in parallel with the combination of 8+j6Ω, the circuit have a unity power factor. Assume that $\omega=10^3$ rad/s



Fundamentals of Electric Circuits",

Alexander and Sadiku, McGraw-Hill.