MATH2089
Numerical Methods
Lecture 8

Numerical Integration

Numerical Integration

- The function to be integrated can be in one of the following forms
  - A continuous function being linear, quadratic, higher order polynomial, exponential, etc.
  - For example

$$f(x) = a_0 x + a_1$$

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \ldots$$

  - Integration with respect to x

$$\int f(x) = \int a_0 x + a_1 = \frac{1}{2} a_0 x^2 + a_1 x + C$$

$$\int f(x) = \int a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \ldots$$

$$= \frac{1}{n+1} a_0 x^{n+1} + \frac{1}{n} a_1 x^n + \frac{1}{n-1} a_2 x^{n-1} + \frac{1}{n-2} a_3 x^{n-2} + \ldots + C$$

Some Engineering Application

- Newton's second law $F = m\dfrac{d^2s}{dt^2}$ $v(t) = \dfrac{ds(t)}{dt}$ $s = \int_0^t v(t)dt$
- Axial displacement of an elemental element under a load *P:* $\dfrac{du}{dx} = \dfrac{P}{EA}$ where E is the Young's modulus. At x = l, $u = \int_0^l \dfrac{P}{EA}dx$
- Mean value between the limits of *a* and *b*

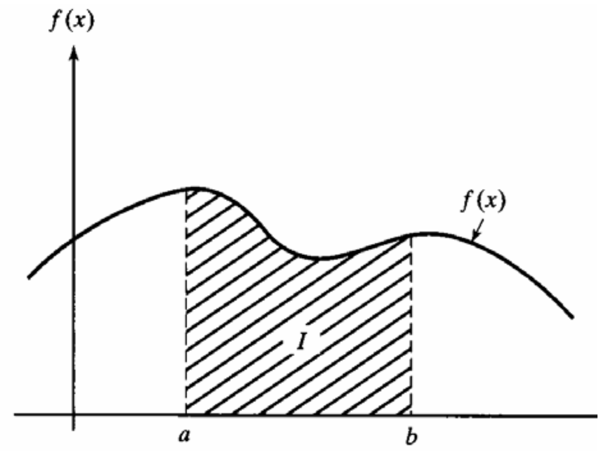
$$\frac{\int_a^b f(x)dx}{b-a}$$

- Determination of mass

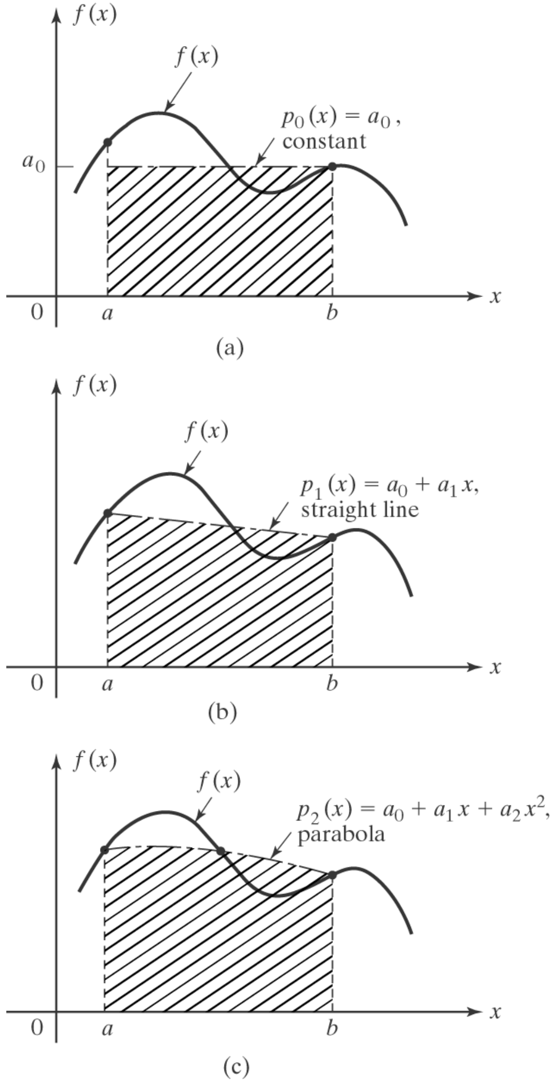
$$m = A\int_0^L \rho(x)dx$$

Newton-Cotes Integration Formulae

- Based on a set of discrete data, how we can best approximate the integration $I = \int_a^b f(x)dx$
- Integration is the area under the curve
- Newton-Cotes formulae replace a complicated function or tabular data by some approximating function which can be integrated easily
- We can construct polynomials which passes through the discrete points defined by $f(x)$



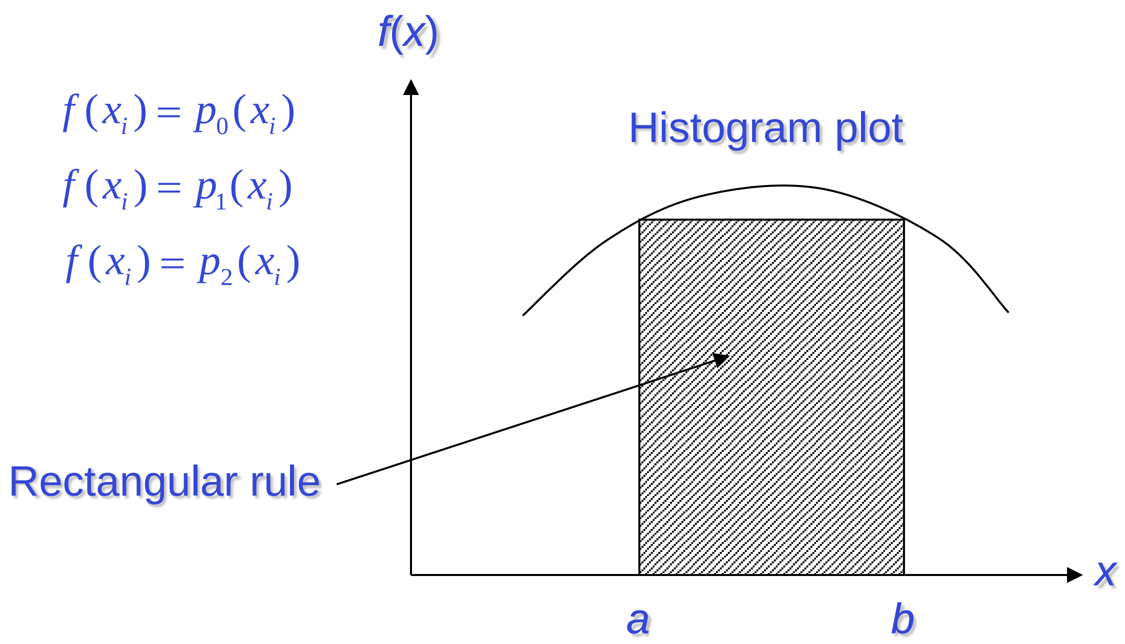
Numerical Integration



Different types of approximation of *f(x)*

Newton-Cotes Integration Formulae (continue)

- We can express the function $f(x)$ in terms of $p_m(x)$ which have the same values at finite number of points
- Hence, $f(x_i) = p_m(x_i)$
- This subsequently results in the common approximations:
  - Rectangular rule $f(x_i) = p_0(x_i)$
  - Trapezoidal rule $f(x_i) = p_1(x_i)$
  - Simpson 1/3 rule $f(x_i) = p_2(x_i)$

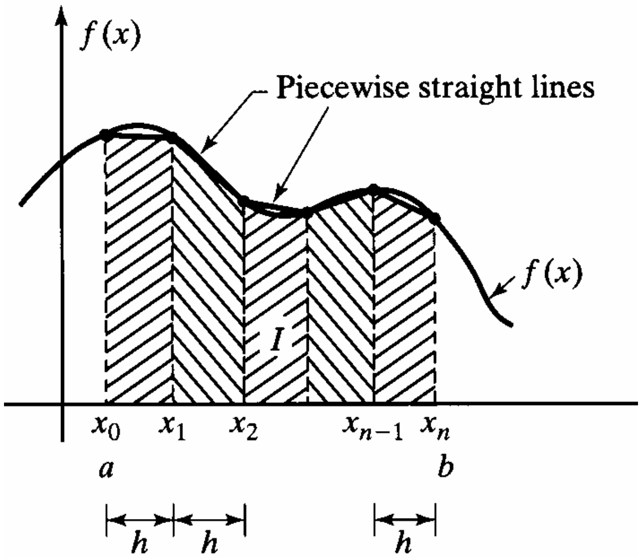


Trapezoidal Rule

- Dividing into a number of finite strips of width $h = \Delta x = (b-a)/n$ the integration is approximated by a series of straight-line segments within each strip of width *h*
- Area under the curve $f(x)$ in the interval $x_{i-1} \le x \le x_{i+1}$ is equal to the area of trapezoid. Noting that $p_1(x) = a_0 + a_1 x$

$$I = \int_a^b f(x)dx \approx \int_a^b p_1(x)dx = \int_a^b (a_0 + a_1 x)dx$$

- What's next? Need to determine a_0 and a_1!!!



Trapezoidal Rule (continue)

- Solve from two equations:

$$(x_{i-1}, f_{i-1}) \rightarrow f_{i-1} = a_0 + a_1 x_{i-1}$$

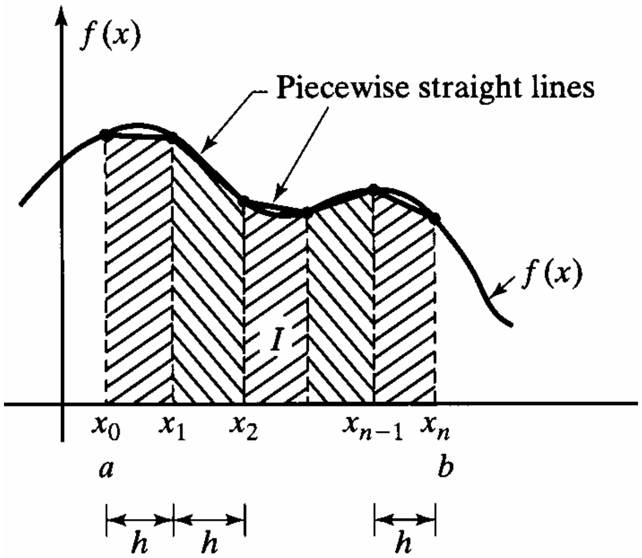
$$(x_i, f_i) \rightarrow f_i = a_0 + a_1 x_i$$

Question?

$$p_1(x) = f_{i-1} + \frac{f_i - f_{i-1}}{x_i - x_{i-1}}(x - x_{i-1})$$

$$I_i = \int_{x_{i-1}}^{x_i} (a_0 + a_1 x)dx = \int_{x_{i-1}}^{x_i} \left(f_{i-1} + \frac{f_i - f_{i-1}}{x_i - x_{i-1}}(x - x_{i-1})\right)dx$$

$$= (x_i - x_{i-1})\frac{(f_{i-1} + f_i)}{2}$$



Trapezoidal Rule (continue)

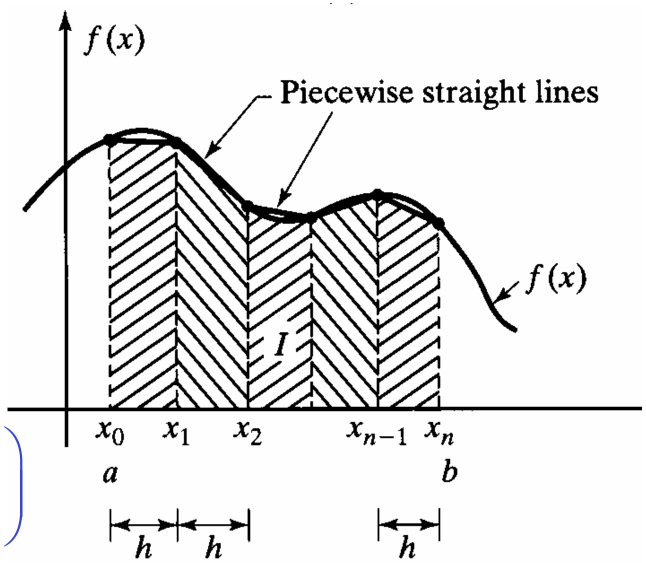
- Since $h = x_i - x_{i-1}$,

$$I_i = h\left(\frac{f_{i-1} + f_i}{2}\right)$$ = width x average height

- For the whole integration between *a* and *b*

$$I = \int_a^b f(x)dx \approx \sum_{i=1}^{n} I_i = h\sum_{i=1}^{n}\left(\frac{f_{i-1} + f_i}{2}\right)$$

Sum of shaded areas!!!



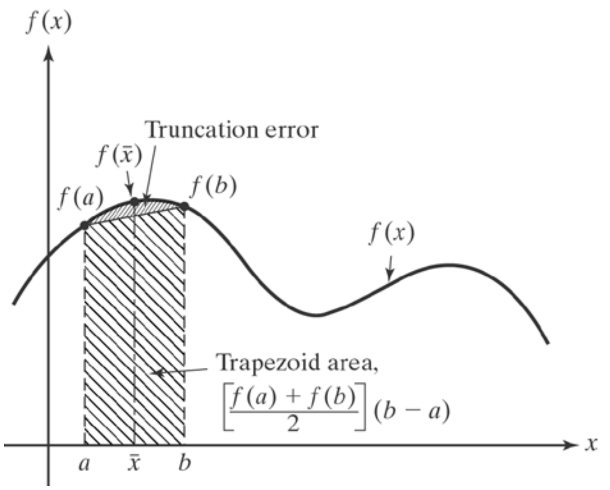
$$= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right) = \frac{h}{2}\left(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n\right)$$

Extensively used in engineering applications – simplicity in programming!!!

Truncation Error in Trapezoidal Rule

- Error:

$$E \approx \underbrace{\int_a^b f(x)dx}_{\text{Exact integral}} - \underbrace{\left[\frac{f(a)+f(b)}{2}\right](b-a)}_{\text{Approximate integral by Trapezoidal rule}}$$



- Using Taylor's expansion of $f(x)$ about $\bar{x}=(a+b)/2$

$$f(x) = f(\bar{x}) + \underbrace{(x-\bar{x})}_{x^*} f'(\bar{x}) + \frac{(x-\bar{x})^2}{2!} f''(\bar{x}) + \ldots$$

- *Exact integral* = Integrating by *x** from –*h*/2 to *h*/2
- *Approximate integral*: evaluate *f (a), f (b)* using the Taylor's series, $a-\bar{x}=-h/2, b-\bar{x}=h/2$

Truncation Error in Trapezoidal Rule (continue)

- For one segment,

$$h = \left(\frac{b-a}{2} \right)$$

- Local error is $E_a \approx -\frac{1}{12} h^3 f''(\bar{x}) = -\frac{1}{12} h^3 \overline{f}''$

- For multi-segment,

$$h = \left(\frac{b-a}{n} \right)$$

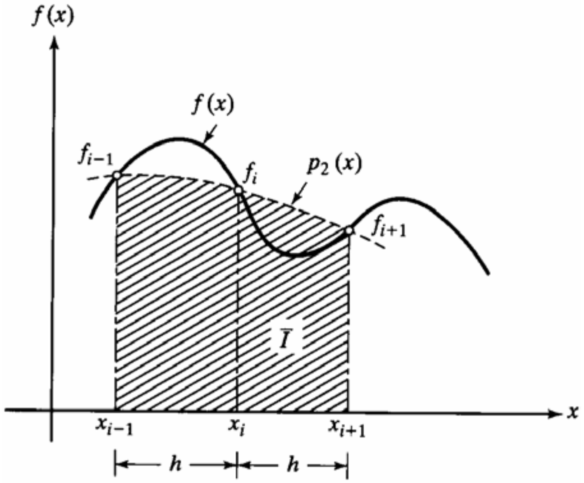
- Global error is $E \approx -\frac{1}{12} (b-a) h^2 \overline{f}'' = O\left(h^2\right)$

Simpson 1/3 and Composite rule

- Dividing into a number of finite strips of width $h = \Delta x = (b-a)/n$ the integration is approximated by a series of quadratic functions within each strip of width *h*
- Area under the curve $f(x)$ is equal to the area as shown

$$I = \int_a^b f(x)dx \approx \int_a^b p_2(x)dx; \quad p_2(x) = a_0 + a_1 x + a_2 x^2$$

- What's next? Need to determine a_0, a_1 and a_2!!!



Simpson 1/3 and Composite rule (continue)

- Solve 3 equations
- For simplicity, set $x_i = 0$, $x_{i-1} = -h$ and $x_{i+1} = h$

$$p_2(x_{i-1} = -h) = f_{i-1} = a_0 - a_1 h + a_2 h^2$$

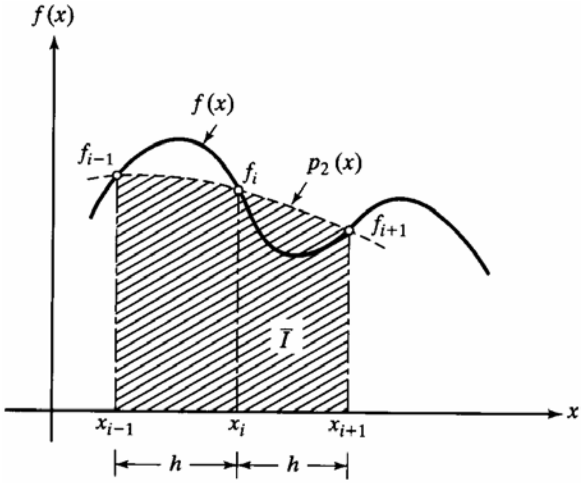
$$p_2(x_i = 0) = f_i = a_0$$

$$p_2(x_{i+1} = h) = f_{i+1} = a_0 + a_1 h + a_2 h^2$$

Question?

$$a_2 = \frac{f_{i-1} - 2f_i + f_{i+1}}{2h^2}$$

$$a_1 = \frac{f_{i+1} - f_{i-1}}{2h}, \quad a_0 = f_i$$



Simpson 1/3 and Composite rule (continue)

- Integration for two segments

$$\bar{I} = \int_{x_{i-1}}^{x_{i+1}} p_2(x)dx = \int_{-h}^{h} (a_0 + a_1 x + a_2 x^2)dx = \underbrace{(2h)}_{\text{width}} \underbrace{\frac{\left(f_{i-1} + 4f_i + f_{i+1}\right)}{6}}_{\text{average height}}$$

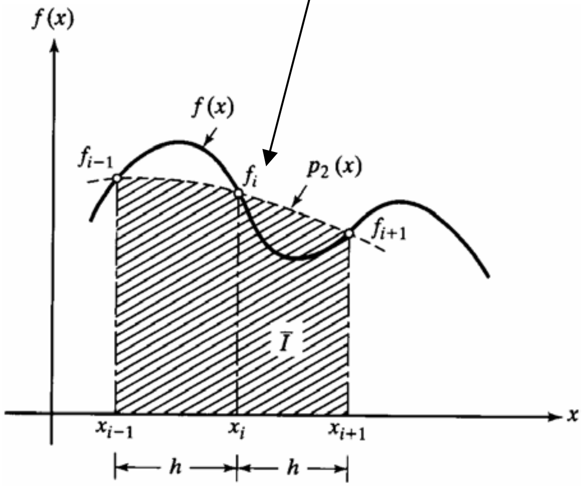
$$I = \int_a^b f(x)dx \approx \sum_{j=1}^{n/2} \left(\bar{I}\right)_j = \frac{h}{3}\sum_{j=1}^{n/2}\left(f_{2j-2} + 4f_{2j-1} + f_{2j}\right)$$

Need to have two sequential segments!!!

- Between *a* and *b*

$$= \frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + f_{n-2} + 2f_{n-1} + f_n]$$

$$= \frac{h}{3}\left[f_0 + 4\sum_{i=1,3,5,\ldots}^{n-1} f_i + 2\sum_{i=2,4,6,\ldots}^{n-2} f_i + f_n\right]$$



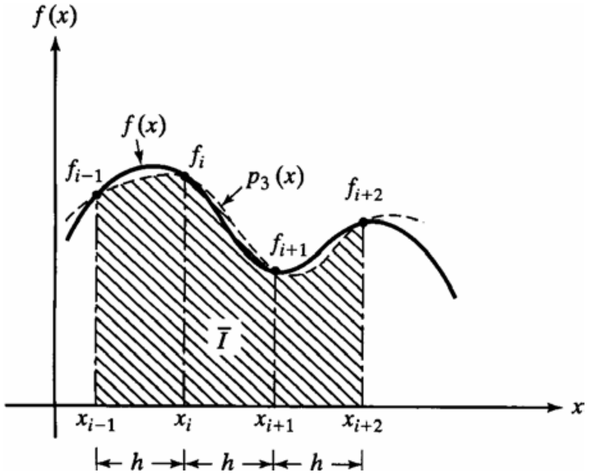
- Global error $E \approx -\frac{1}{180}h^4(b-a)\bar{f}^{(4)} \approx O\left(h^4\right)$

Simpson 3/8 rule

- Dividing into a number of finite strips of width $h = \Delta x = (b-a)/n$ the integration is approximated by a series of third-order polynomials within each strip of width *h*
- Area under the curve $f(x)$ is equal to the area as shown

$$I = \int_a^b f(x)dx \approx \int_a^b p_3(x)dx \qquad p_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

- What's next? Need to determine a_0, a_1, a_2 and a_3!!!

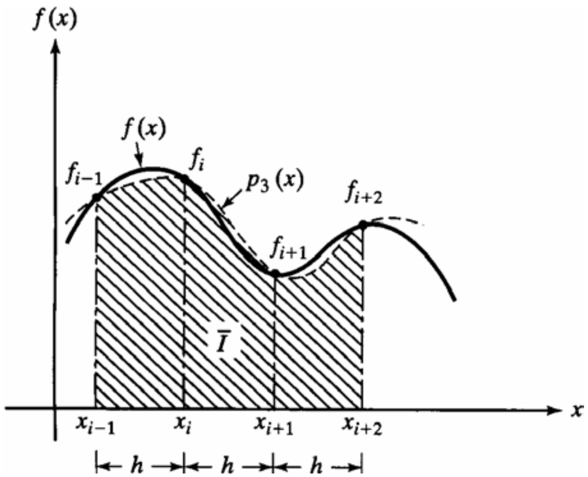


Simpson 3/8 rule (continue)

- Solve 4 equations
- Setting $x_i = 0$, $x_{i-1} = -h$, $x_{i+1} = h$ and $x_{i+2} = 2h$

$$p_2(x_{i-1} = -h) = f_{i-1}$$
$$p_2(x_i = 0) = f_i$$
$$p_2(x_{i+1} = h) = f_{i+1}$$
$$p_2(x_{i+2} = 2h) = f_{i+2}$$

$$p_2(x_{i-1} = -h) = f_{i-1} = a_0 - a_1 h + a_2 h^2 - a_3 h^3$$
$$p_2(x_i = 0) = f_i = a_0$$
$$p_2(x_{i+1} = h) = f_{i+1} = a_0 + a_1 h + a_2 h^2 + a_3 h^3$$
$$p_2(x_{i+2} = 2h) = f_{i+2} = a_0 + 2a_1 h + 4a_2 h^2 + 8a_3 h^3$$



$$a_2 = \frac{1}{6h^3}(-f_{i-1} + 3f_i - 3f_{i+1} + f_{i+2})$$
$$a_2 = \frac{1}{2h^2}(f_{i-1} - 2f_i + f_{i+1})$$
$$a_1 = \frac{1}{6h}(-2f_{i-1} - 3f_i + 6f_{i+1} - f_{i+2})$$
$$a_0 = f_i$$

Question?

Simpson 3/8 rule (continue)

- Integration for three segments

$$\overline{I} = \int_{x_{i-1}}^{x_{i+2}} p_3(x)dx = \int_{-h}^{2h} (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx = \underbrace{(3h)}_{\text{width}} \underbrace{\frac{\left(f_{i-1} + 3f_i + 3f_{i+1} + f_{i+2}\right)}{8}}_{\text{average height}}$$

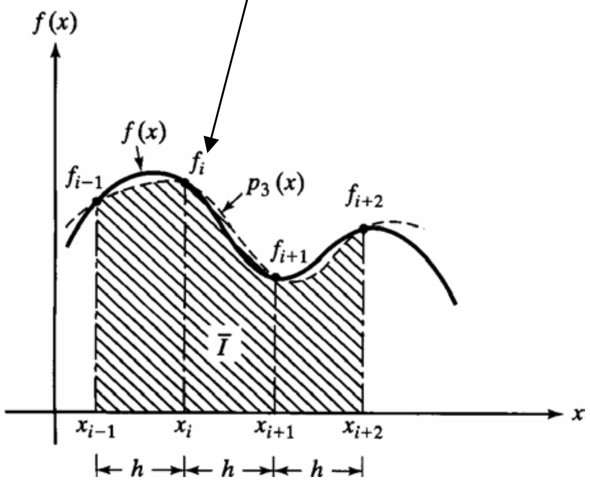
$$I = \int_a^b f(x)dx \approx \sum_{j=1}^{n/3} \left(\overline{I}\right)_j = \frac{3h}{8} \sum_{j=1}^{n/3} \left(f_{3j-3} + 3f_{3j-2} + 3f_{3j-1} + f_{3j}\right)$$

Need to have three sequential segments!!!

- Between *a* and *b*

$$= \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + \ldots + 3f_{n-2} + 3f_{n-1} + f_n\right]$$

$$= \frac{3h}{8}\left[f_0 + 3\sum_{i=1,4,7,\ldots}^{n-2} \left(f_i + f_{i+1}\right) + 2\sum_{i=3,6,9,\ldots}^{n-3} f_i + f_n\right]$$



- Global error $E \approx -\dfrac{1}{80} h^4 (b-a)\,\overline{f}^{(4)} \approx O\left(h^4\right)$

Example

- Determine the value of integral using Simpson 1/3 rule with n = 2, 4, 8 and Simpson 3/8 rule with n = 3, 9. Compare results with exact value.

$$I = \int_0^2 f(x)dx = \int_0^2 \left(1 + 3x - 5x^2 + 2x^3 + x^4\right) dx$$

$$= \left[x + \frac{3}{2}x^2 - \frac{5}{3}x^3 + \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_0^2 = \frac{136}{15} = 9.06667$$

Exact value

Example (continue)

- Table of discrete values for Simpson 1/3 rule

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$f(x)$	1	1.4727	1.5625	1.5977	2	3.2852	6.0625	11.035	19

- n = 2, h = 1

$$I_{n=2} = \tfrac{h}{3}(f(0) + 4f(1) + f(2)) = \tfrac{1}{3}(1 + 4(2) + 19) = 9.3333$$

$$E_t = (9.06667 - 9.3333) \times 100\% = -2.94\%$$

- n = 4, h = 0.5

$$I_{n=4} = \tfrac{0.5}{3}(f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)) = 9.08333$$

$$E_t = (9.06667 - 9.08333) \times 100\% = -0.184\%$$

- n = 8, h = 0.2

$$I_{n=8} = \tfrac{0.25}{3}(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)) = 9.0677 \qquad E_t = (9.06667 - 9.0677) \times 100\% = -0.011\%$$

Example (continue)

- Table of discrete values for Simpson 3/8 rule

x	0	2/9	4/9	2/3	8/9	10/9	4/3	14/9	16/9	2
$f(x)$	1	1.4441	1.5603	1.5679	1.7450	2.4281	4.0123	6.9512	11.757	19

- n = 3, h = 2/3

$$I_{n=2} = \tfrac{3}{8}\left(\tfrac{2}{3}\right)\left(f(0) + 3f(\tfrac{2}{3}) + 3f(\tfrac{4}{3}) + f(2)\right) = 9.1852$$

$$E_t = \left(9.06667 - 9.1852\right) \times 100\% = -1.31\%$$

- n = 9, h = 2/9

$$I_{n=9} = \tfrac{3}{8}\left(\tfrac{2}{9}\right)\left(f(0) + 3[f(\tfrac{2}{9}) + f(\tfrac{4}{9}) + f(\tfrac{8}{9}) + (\tfrac{10}{9}) + f(\tfrac{14}{9}) + (\tfrac{16}{9})] + 2[f(\tfrac{2}{3}) + f(\tfrac{4}{3})] + f(2)\right)$$
$$= 9.0681$$

$$E_t = \left(9.06667 - 9.0681\right) \times 100\% = -0.016\%$$

General Newton-Cotes Formulae

- Newton-Cotes formulae are derived by using a polynomial of order m to approximate the function $f(x)$

$$I = \int_a^b f(x)dx \approx \int_a^b p_m(x)dx \qquad p_m(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + c_{m-1}x^{m-1} + c_m x^m$$

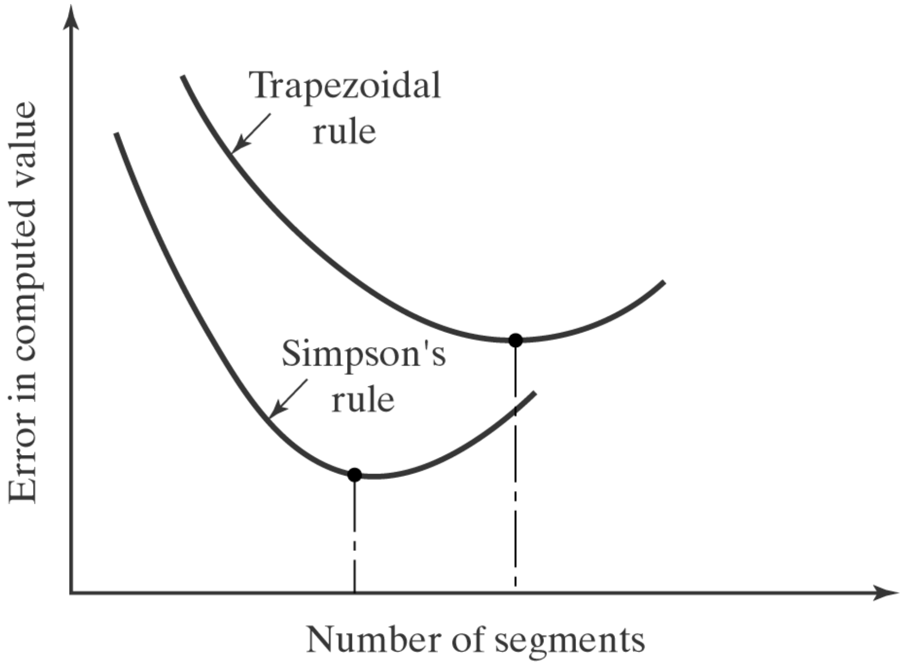
- A summary of some Newton-Cotes formulae

m	Name of formula	Formula $h=(b-a)/n$	Truncation error	Number of segments
0	Rectangular	hf_{i-1} / hf_i	$-\frac{1}{2}(b-a)h\bar{f}'$	0
1	Trapezoidal	$\frac{h}{2}(f_{i-1}+f_i)$	$-\frac{1}{12}(b-a)h^2\bar{f}''$	1
2	Simpson's 1/3	$\frac{h}{3}(f_{i-1}+4f_i+f_{i+1})$	$-\frac{1}{180}(b-a)h^4\bar{f}^{(4)}$	2
3	Simpson's 3/8	$\frac{3h}{8}(f_{i-1}+3f_i+3f_{i+1}+f_{i+2})$	$-\frac{1}{80}(b-a)h^4\bar{f}^{(4)}$	3

Comments

- Trapezoidal rule is exact for polynomials up to order one
- Simpson's rule is exact for polynomials up to order three
- Truncation errors of Simpson's 1/3 rule and 3/8 rule are the same
- Simpson's 3/8 rule is rarely use by itself
- If n is even (2,4,6,…), the Simpson's 1/3 rule can be used
- If n is odd (3,5,7,…), the Simpson 3/8 rule can be applied for the first three segments and the Simpson's 1/3 rule can be used for the remaining even number of segments

Comments



Variation of accuracy with increasing number of segments

Richardson's Extrapolation for Integration

- Represents another way of improving the accuracy of the estimates of the integration
- Truncation error $E = O(h^n) \approx ch^n$
- If I_1 and I_2 are the approximated integrals with a step size h_1 and h_2

$$\left.\begin{aligned} I &= I_1 + O(h_1^n) = I_1 + E_1 \approx I_1 + ch_1^n \\ I &= I_2 + O(h_2^n) = I_2 + E_2 \approx I_2 + ch_2^n \end{aligned}\right\} m \qquad I_1 + ch_1^n \approx I_2 + ch_2^n \quad \Rightarrow \quad c = \frac{I_2 - I_1}{h_1^n - h_2^n}$$

$$I \approx I_2 + \frac{I_2 - I_1}{\left\{\left(\dfrac{h_1}{h_2}\right)^n - 1\right\}}$$

- Using $h_2 = h_1 / 2$

$$I = I_2 + \frac{I_2 - I_1}{2^n - 1} + O(h_2^{n+2})$$

Richardson's Extrapolation for Integration (continue)

- Better estimate = more accurate + $\dfrac{1}{2^n - 1}$ (more accurate – less accurate) + $O(h^{n+2})$
- The more accurate is the one computed with the smaller value of *h*
- For example, apply it for trapezoidal rule $E = O(h^2)$

$$I \approx I_2 + \frac{I_2 - I_1}{2^2 - 1} = I_2 + \frac{I_2 - I_1}{3} + O(h_2^4)$$

Identical to the Simpson's 1/3 rule with h_2 with a truncation error of $O(h^4)$!!!

Integration with Unequal Segments

- It can be handled using a combination of different integration rules
- For practical cases, use trapezoidal and Simpson rules. The use of Simpson rule increase the accuracy of the integral rule

Example

- The turning-moment ($f(x)$)and crank-angle (x) data of an internal combustion engine are shown in the discrete table as shown below. Determine the area under the turning-moment diagram.

x	0	0.1	0.25	0.33	0.5	0.7	0.9	1.1	1.25	1.4
$f(x)$	0.85	2.8	3.25	3	2.85	3.85	5.75	4.9	3	0.84

- Determine $I = \int_0^{1.4} f(x)dx$ for unequal segments

Example (continue)

- Apply trapezoidal rule for 1st - 4th segments

$$I_{1-4} = h_1\left(\frac{f_0 + f_1}{2}\right) + h_2\left(\frac{f_1 + f_2}{2}\right) + h_3\left(\frac{f_2 + f_3}{2}\right) + h_4\left(\frac{f_3 + f_4}{2}\right)$$

$$I_{1-4} = 0.1\left(\frac{0.85 + 2.8}{2}\right) + 0.15\left(\frac{2.8 + 3.25}{2}\right) + 0.08\left(\frac{3.25 + 3}{2}\right) + 0.17\left(\frac{3 + 2.85}{2}\right) = 1.3835$$

- Apply Simpson's 3/8 rule for 5th - 7th segments

$$I_{5-7} = \frac{3h_{5-7}}{8}\left(f_4 + 3f_5 + 3f_6 + f_7\right) = \frac{3(0.2)}{8}\left(2.85 + 3(3.85 + 5.75) + 4.9\right) = 2.7413$$

- Apply Simpson's 1/3 rule for 8th - 9th segments

$$I_{7-9} = \frac{h_{8-9}}{3}\left(f_7 + 4f_8 + f_9\right) = \frac{0.15}{3}\left(4.9 + 4(3) + 0.84\right) = 0.887$$

- Then, $I = \int_0^{1.4} f(x)dx = I_{1-4} + I_{5-7} + I_{7-9} = 1.3835 + 2.7413 + 0.887 = 5.0118$

Integration of Improper Integrals

- When the limits of integration are infinite

$$\int_a^{\infty} f(x)dx, \int_{-\infty}^{b} f(x)dx, \int_{-\infty}^{\infty} f(x)dx$$

  - Integral is evaluated with successively increasing values of *n* until any further increase of *n* does not improve the value of the integral significantly

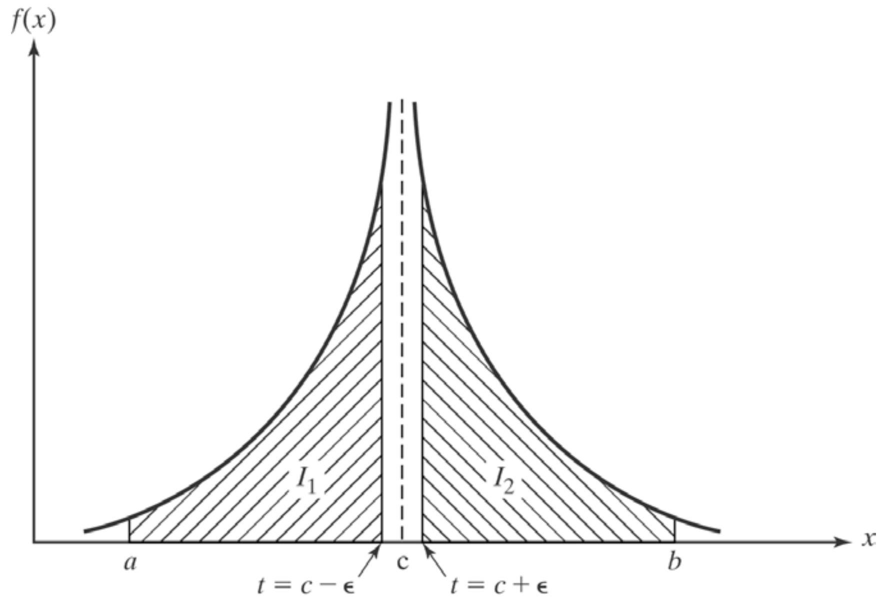
- When $f(x)$ is singular or infinite discontinuous at some points in the range of integration

$$\int_0^4 \frac{1}{\sqrt{4-x}}dx, \quad \int_{-2}^{0} \frac{1}{\sqrt{4-x^2}}dx, \quad \int_{-2}^{26} \frac{1}{(x+1)^{2/3}}dx$$

  - Start with a small value of ε, the process converges if the integral is not significantly affected by a further reduction in the value of ε

Integration of Improper Integrals (continue)

- For example,



$$I = \int_{-2}^{26} \frac{1}{(x+1)^{2/3}}\,dx = I_1 + I_2 = \int_{-2}^{-1-\varepsilon} \frac{1}{(x+1)^{2/3}}\,dx + \int_{-1+\varepsilon}^{26} \frac{1}{(x+1)^{2/3}}\,dx$$

Integration of Improper Integrals (continue)

- An example of integration by parts,

$$I = \int_0^5 \frac{e^{2x}}{\sqrt{x}} dx = 2e^{2x}\sqrt{x}\Big|_0^5 - \int_0^5 4\sqrt{x}e^{2x} dx$$

- An example of the transformation: *x* = sin *y* or *dx* = cos *y* *dy*

$$I = \int_0^5 \frac{1+x^3+x^5}{\sqrt{x^2-1}} dx = \int_0^{\pi/2} \left(1+\sin^3 y+\sin^5 y\right) dy$$

Integration of Improper Integrals - Example

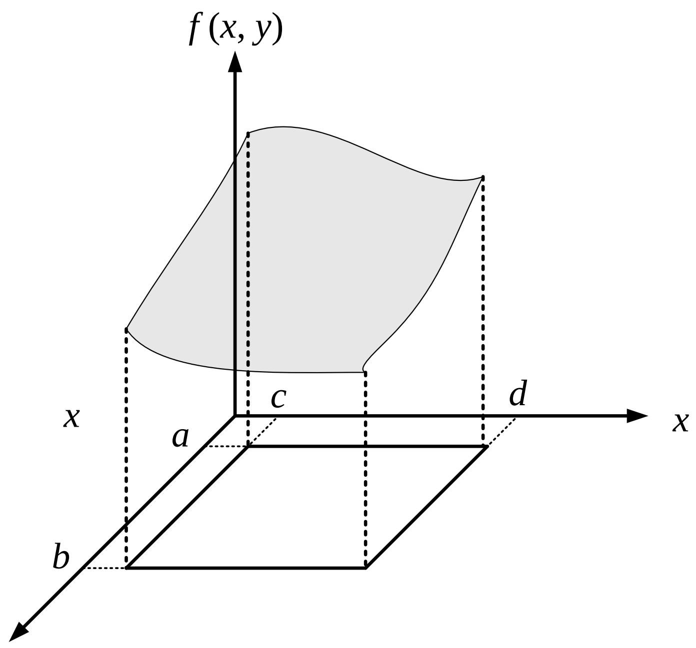
Determine the value of the integral $\int_0^{\infty} e^{-x^2}dx$ using h=0.2 within an accuracy of 10^{-4} .

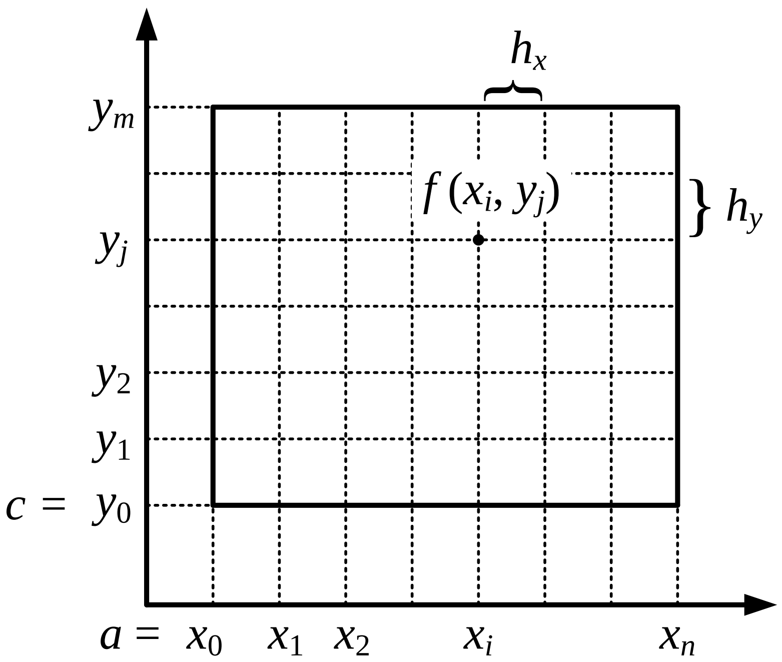
n	x	$f(x)$
	0	1
	0.2	0.96079
2	0.4	0.85214
	0.6	0.69768
4	0.8	0.52729
	1	0.36788
6	1.2	0.23693
	1.4	0.14086
8	1.6	0.077305
	1.8	0.039164
10	2	0.018316
	2.2	0.0079071
12	2.4	0.0031511
	2.6	0.0011592
14	2.8	0.00039367
	3	0.00012341
16	3.2	3.5713E-05

Multiple Integration

- Consider the case as shown below

$$I=\iint_A f(x,y)dA=\int_a^b\left(\int_c^d f(x,y)dy\right)dx=\int_c^d\left(\int_a^b f(x,y)dx\right)dy$$





Multiple Integration (continue)

- Divide the segments of integral
  - into n segments of equal width (h_x) along *x*-axis
  - into m segments of equal width (h_y) along *y*-axis

- We can hold y direction constant while integrating with respect to x direction (vice versa in the second case). Hence, collapsing the integration to the usual one-dimensional form.

$$I = \int_c^d \left(\int_a^b f(x,y)dx \right) dy = \int_c^d I_x(y)dy \quad \text{or} \quad I = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_a^b I_y(x)dx$$

Multiple Integration (continue)

- Applying the trapezoidal rule, for example,

$$I_x(y_0) = \frac{h_x}{2}\Big(f(x_0, y_0) + 2(f(x_1, y_0) + f(x_2, y_0) + \cdots + f(x_{n-1}, y_0)) + f(x_n, y_0)\Big)$$

$$I_x(y_1) = \frac{h_x}{2}\Big(f(x_0, y_1) + 2(f(x_1, y_1) + f(x_2, y_1) + \cdots + f(x_{n-1}, y_1)) + f(x_n, y_1)\Big)$$

$$\vdots$$

$$I_x(y_j) = \frac{h_x}{2}\Big(f(x_0, y_j) + 2(f(x_1, y_j) + f(x_2, y_j) + \cdots + f(x_{n-1}, y_j)) + f(x_n, y_0)\Big)$$

$$\vdots$$

$$I_x(y_m) = \frac{h_x}{2}\Big(f(x_0, y_m) + 2(f(x_1, y_m) + f(x_2, y_m) + \cdots + f(x_{n-1}, y_m)) + f(x_n, y_m)\Big)$$

- Then, the integral for the x-direction is

$$I = \int_c^d I_x(y)dy = \frac{h_y}{2}\Big(I_x(y_0) + 2(I_x(y_1) + I_x(y_2) + \cdots + I_x(y_{m-1})) + I_x(y_m)\Big)$$

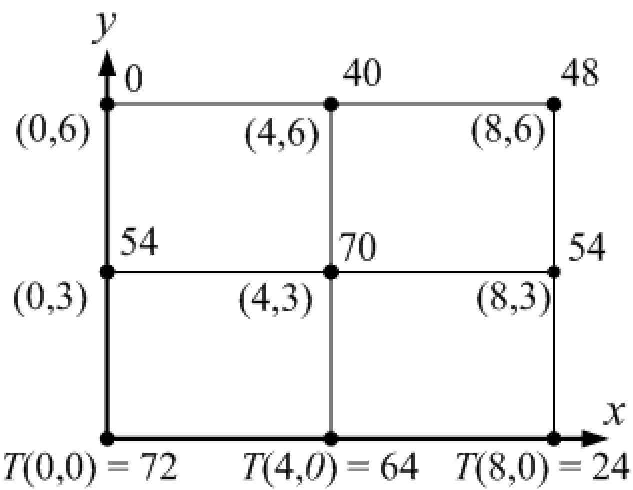
Multiple Integration - Example

Suppose that the temperature of a rectangular heated plate is described by the following function:

$$T(x,y) = 2xy + 2x - x^2 - 2y^2 + 72$$

If the plate is 8-m long (x dimension) and 6-m wide (y dimension), compute the average temperature using trapezoidal and Simpson's 1/3 rules with $h_x = 4$, $h_y = 3$.

The calculation domain



x	y	$T(x,y)$
0	0	72
4	0	64
8	0	24
0	3	54
4	3	70
8	3	54
0	6	0
4	6	40
8	6	48