MATH2089
Numerical Methods
Lecture 4

Part 1: Systems of Linear Equations

$\boxed{\text{Elimination Methods,}}$

LU Factorization,

Iterative Methods,

Special Linear Systems

Solving Systems of Equations

- Linear systems are likely to be the most widely applied numerical procedure when real-world problems are to be solved
- Linear systems are used in statistical analysis and in many engineering applications
- Methods of numerically solving *ordinary-differential* and *partial-differential* equations depend on these systems

Solving Systems of Equations (continue)

- For example, a set of linear algebraic equations can be expressed in a general form as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

where the coefficients a_{ij} and the constant b_i are known, x_j are the <u>unknowns</u> which are required to be determined and n is the number of equations (i = 1,2,…., n; j = 1,2,…., n)

Solving Systems of Equations (continue)

- In matrix form, $[A]\vec{x} = \vec{b}$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Solving Systems of Equations (continue)

- These linear systems can be solved by either *direct* or *indirect* (*iterative*) methods

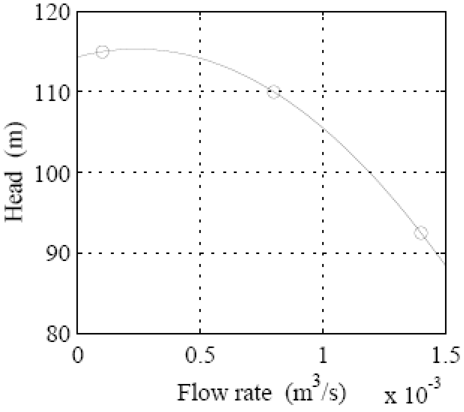
Direct methods	Iterative methods
Gauss elimination	*Jacobi*
Gauss-Jordan	*Gauss-Seidel*
LU decomposition	*Relaxation*

Example 1

- Many practical engineering applications and mathematical models of social sciences lead to a system of linear algebraic equations

  <u>Pump curve fitting</u>

  Objective: Find the coefficients of the quadratic equation that approximates the pump curve data



Example 1 (continue)

- Model equation: $h = c_1q^2 + c_2q + c_3$. Write the model equation for three points on the curve. This gives a system of linear equations with three unknowns: c_1, c_2 and c_3

Points from the pump curve:

$q\,(\mathrm{m}^3/\mathrm{s})$	1×10^{-4}	8×10^{-4}	1.4×10^{-3}
$h\,(\mathrm{m})$	115	110	92.5

Substitute each pair of data points into the model equation

$$\begin{array}{c} \\ \\ \end{array}\begin{aligned} & \qquad\quad q^2 \qquad\qquad\qquad q \end{aligned}$$

$$\begin{aligned}
115 &= 1 \times 10^{-8}\, c_1 + 1 \times 10^{-4}\, c_2 + c_3, \\
110 &= 64 \times 10^{-8}\, c_1 + 8 \times 10^{-4}\, c_2 + c_3, \\
92.5 &= 196 \times 10^{-8}\, c_1 + 14 \times 10^{-4}\, c_2 + c_3,
\end{aligned}$$

Rewrite in matrix form as

$$\begin{bmatrix} 1 \times 10^{-8} & 1 \times 10^{-4} & 1 \\ 64 \times 10^{-8} & 8 \times 10^{-4} & 1 \\ 196 \times 10^{-8} & 14 \times 10^{-4} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 115 \\ 110 \\ 92.5 \end{bmatrix}.$$

Example 1 (continue)

Using more compact symbolic notation

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 \times 10^{-8} & 1 \times 10^{-4} & 1 \\ 64 \times 10^{-8} & 8 \times 10^{-4} & 1 \\ 196 \times 10^{-8} & 14 \times 10^{-4} & 1 \end{bmatrix},$$

$$x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \qquad b = \begin{bmatrix} 115 \\ 110 \\ 92.5 \end{bmatrix}.$$

For any three (q, h) pairs

$$A = \begin{bmatrix} q_1^2 & q_1 & 1 \\ q_2^2 & q_2 & 1 \\ q_3^2 & q_3 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad b = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}.$$

Example 2

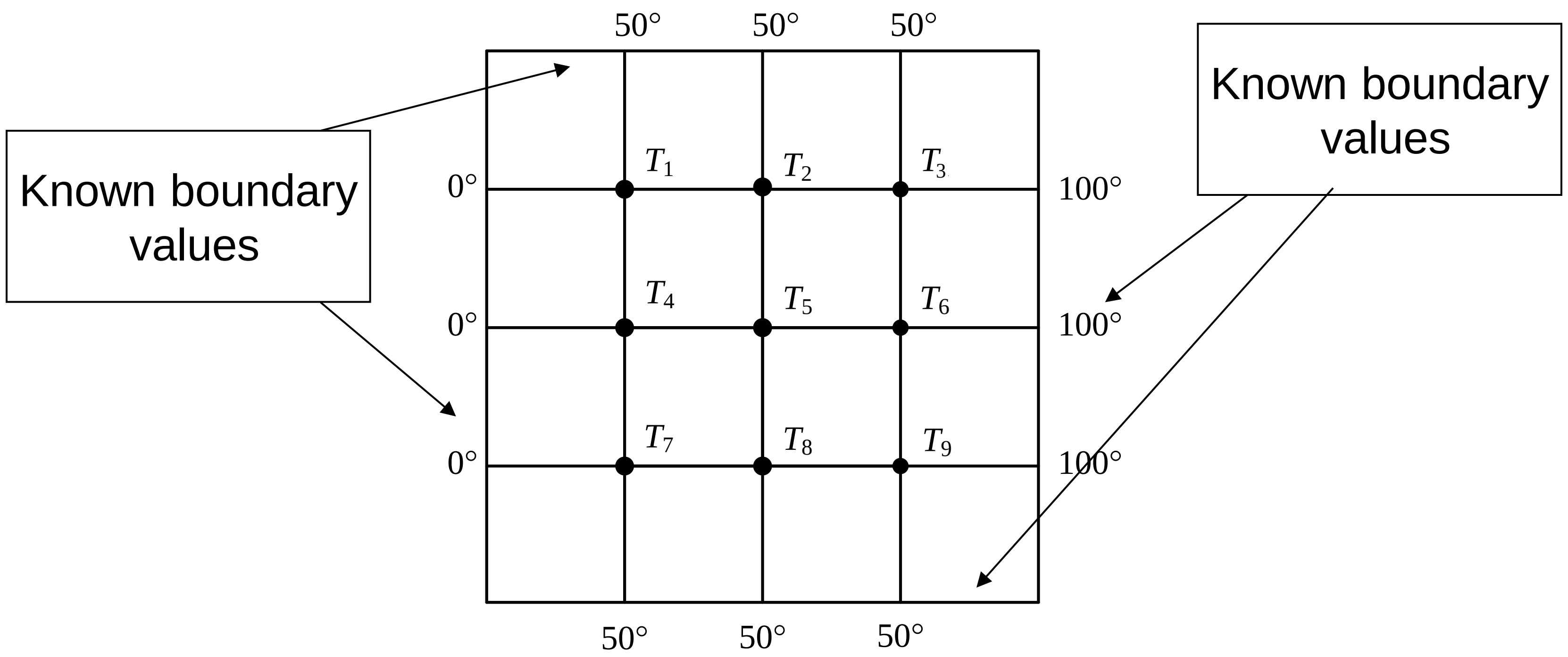
Steady-state temperature in a plate

A flat thin plate is 2 ft by 2 ft by 1 in. The edges are kept at constant temperatures.

Objective: find the temperature in the interior of this plate?

Let us estimate these temperatures by making a grid of points with each 0.5 ft apart, and let the temperatures be at the grid points. Temperature at each grid point satisfies a differential equation that can be simplified into a system of linear equations with constant coefficients

Example 2 (continue)



The problem is reduced to solving n linear equations where n depends on the size of the grid. Solve then for each of the discrete temperatures within the grid

Example 2 (continue)

The system of linear equations are:

$$-4T_1 + T_2 + T_4 = -50$$
$$T_1 - 4T_2 + T_3 + T_5 = -50$$
$$T_2 - 4T_3 + T_6 = -150$$
$$T_1 - 4T_4 + T_5 + T_7 = 0$$
$$T_2 + T_4 - 4T_5 + T_6 + T_8 = 0$$
$$T_3 + T_5 - 4T_6 + T_9 = -100$$
$$T_4 - 4T_7 + T_8 = -50$$
$$T_5 + T_7 - 4T_8 + T_9 = -50$$
$$T_6 + T_8 - 4T_9 = -150$$

General Equation

Brief Review of Matrix Algebra

- Matrix $m \times n$, $\quad [A]_{m\times n} = [a_{ij}] : i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n$
- Square matrix $m = n$
- Diagonal matrix, $a_{ij} \begin{cases} = 0, & i \neq j \\ \neq 0, & i = j \end{cases}$
- Identity matrix, $[I]$

$$I_n = [\delta_{ij}]_{n\times n} \quad where \quad \delta_{ij} = \begin{cases} 1 & when\ i = j, \\ 0 & when\ i \neq j. \end{cases}$$

- Zero matrix, [0]
- Symmetric matrix, $a_{ij} = a_{ji}$
- Transpose of matrix, $[A]_{m\times n} = [A]^T_{n\times m}$

Determinant of a Square Matrix

- $\det[A] = \sum_{j=1}^{n} a_{ij} \beta_{ij}$ for specific row *i*
- $\det[A] = \sum_{i=1}^{n} a_{ij} \beta_{ij}$ for specific column *j*
- $\beta_{ij} = \text{cofactor of } a_{ij} = (-1)^{i+j} M_{ij}$

  M_{ij} *is the minor of* a_{ij}

Example

$$M_{32} \text{ of } [A]_{3\times 3} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Basic Matrix Operations

$$[A]=[B] \quad \text{if} \quad a_{ij}=b_{ij}$$

$$[C]=[A]\pm[B]=[B]\pm[A]$$

Matrix Product

$$[C]_{m\times p}=[A]_{m\times n}[B]_{n\times p} \quad \rightarrow \quad c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$$

$$\left([A][B]\right)[C]=[A]\left([B][C]\right)$$

$$\left([A]+[B]\right)[C]=[A][C]+[B][C]$$

$$[C]^T=\left([A][B]\right)^T=[B]^T[A]^T$$

$$[C]^{-1}=\left([A][B]\right)^{-1}=[B]^{-1}[A]^{-1}$$

$$[I][A]=[A][I] \quad =[A]$$

$$[A]^{-1}[A]=[A][A]^{-1}=[I]$$

if $[A]$ is nonsingular,

inverse of a nonsingular matrix

Vector Norm

- A norm or length is a measure of the size of a vector or matrix
- The Euclidean norm of a vector $\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ is defined as

$$\|\vec{x}\| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$$

- In general, the L_p norm of a vector $\vec{x}$ is defined as

$$L_p = \left\{ \sum_{i=1}^{n} |x_i|^p \right\}^{\frac{1}{p}} \text{ and } L_\infty = \max_i |x_i|$$

Vector Norm (continue)

- The norm of a vector $\vec{x}$ and $\|\vec{x}\|$ has the following properties:
  - $\|\vec{x}\| \geq 0$ for any $\vec{x}$ and $\|\vec{x}\| = 0$ if and only if $\vec{x} = 0$
  - $\|k\vec{x}\| = |k| \|\vec{x}\|$ for any real or complex number k
  - $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ for any two vector $\vec{x}$ and $\vec{y}$ of the same order (triangle inequality)
  - $\|\vec{x}\vec{y}\| \leq \|\vec{x}\| \|\vec{y}\|$

Example: Find the norm L_1, L_2 and L_∞ of the vector $\vec{x} = \begin{Bmatrix} 2 \\ -5 \\ 3 \end{Bmatrix}$

Solution: $L_1 = \sum_{i=1}^{3} |x_i| = 2 + 5 + 3 = 10$ $\quad L_\infty = \max_i |x_i| = 5$

$$L_2 = \left\{ \sum_{i=1}^{3} |x_i|^2 \right\}^{\frac{1}{2}} = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38} = 6.1644$$

Matrix Norm

- The norm of a matrix is useful for defining the condition number of a matrix which can be used to quantify the degree of ill conditioning of a set of linear equations

$$\left\|[A]\right\| = \max \frac{\left\|\vec{y}\right\|}{\left\|\vec{x}\right\|} = \max \frac{\left\|[A]\vec{x}\right\|}{\left\|\vec{x}\right\|}$$

- For two square matrices [A] and [B], $\left\|[A][B]\right\| \le \left\|[A]\right\| \left\|[B]\right\|$
- For any matrix [A] and vector $\vec{x}$, $\left\|[A]\vec{x}\right\| \le \left\|[A]\right\| \left\|\vec{x}\right\|$

$$\left\|[A]\right\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} \left|a_{ij}\right| = \text{maximum column sum}$$

$$\left\|[A]\right\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} \left|a_{ij}\right| = \text{maximum row sum}$$

The Euclidean norm of an $m \times n$ matrix $\left\|[A]\right\|_e = \left|\sum\sum a_{ij}^2\right|^{\frac{1}{2}}$

Linearly Independent Equations

- None of equations can be expressed as a linear combination of other equations in the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 3; \\ 2x_1 + x_2 - x_3 &= 0; \\ 3x_1 + 2x_2 + 2x_3 &= 15. \end{aligned}$$

Solution:

$$\begin{aligned} x_1 &= 1, \\ x_2 &= 2, \\ x_3 &= 4. \end{aligned}$$

- For linearly dependent,

$$\begin{aligned} x_1 - x_2 + x_3 &= 3; \\ 2x_1 + x_2 - x_3 &= 0; \\ 8x_1 + x_2 - x_3 &= 6. \Rightarrow \because R_3 \leftarrow 2R_1 + 3R_2 \end{aligned}$$

Determinant of [A] is zero, $\left\| \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 8 & 1 & -1 \end{bmatrix} \right\| = 0$

Linearly Independent Equations (continue)

- If equations are linearly dependent, the coefficient matrix [A] will be **singular** and the solution cannot be found!
- Equations might become linearly dependent during numerical computations (due to round-off errors) → an accurate solution cannot thus be found

Ill Conditioned Equations

- Small changes in the coefficients a_{ij} may lead to very large variations in the solution

Example: $\begin{aligned} x_1 - x_2 &= 5 \\ kx_1 - x_2 &= 4 \end{aligned}$ Determinant $\rightarrow \begin{vmatrix} 1 & -1 \\ k & -1 \end{vmatrix} = (k-1)$

Solution: $x_1 = \dfrac{1}{1-k}, \quad x_2 = \dfrac{5k-4}{1-k}$

Ill Conditioned Equations (continue)

- The set of equations are ill conditioned when k is nearly unity
- This is an example of ill conditioned equation introduced through the coefficient matrix
- The constant vector $\vec{b}$ can also cause ill conditioning of equations

Value of k	Solution		Determinant of the coefficient matrix, $(k-1)$
	$x_1 = \frac{1}{(1-k)}$	$x_2 = \frac{(5k-4)}{(1-k)}$	
1.0000	No solution		0.0000
0.9997	3333.3333	3328.3333	−0.0003
0.9998	5000.0000	4995.0000	−0.0002
0.9999	10000.0000	9995.0000	−0.0001
1.0001	−10000.0000	−10005.0000	+0.0001
1.0002	−5000.0000	−5005.0000	+0.0002
1.0003	−3333.3333	−3338.3333	+0.0003

Quantification of the Degree of Ill Conditioning

- **Method 1**: Condition number, $\text{cond}(A) = \|A\| \|A^{-1}\|$

  $\text{cond}(A) \geq 1$, , cond(A) of [I] = 1, Larger condition number $\rightarrow$ the equations are poorly conditioned or ill conditioned

- **Method 2**: Determinant, $D(A) = \dfrac{|\det[A]|}{A_1 A_2 \cdots A_n}$ |

  $$A_i = \left\{a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2\right\}^{\frac{1}{2}}$$

  $D(A)$ = 0 if [A] is singular and =1 if [A] is diagonal

  $D(A) \rightarrow 1$, the better the condition of the equations

Example

Consider the matrix $[A]=\begin{bmatrix} 1 & -1 \\ k & -1 \end{bmatrix}$

$$\det(A)=(k-1)$$

$$A_1=\left\{(1)^2+(-1)^2\right\}^{\frac{1}{2}}=\sqrt{2},\quad A_2=\left\{k^2+(-1)^2\right\}^{\frac{1}{2}}=\sqrt{k^2+1}$$

$$D(A)=\frac{\left|\det[A]\right|}{A_1A_2\cdots A_n}=\frac{\left|k-1\right|}{\sqrt{2}\sqrt{\left(k^2+1\right)}}$$

$D(A)$ = 0 if k = 1

$D(A)$ = 5.00025 $\times$ 10^{-5} if k = 0.9999

$D(A)$ = 4.99975 $\times$ 10^{-5} if k = 1.0001

$D(A)$ = 1.00010 $\times$ 10^{-5} if k = 0.9998

Example (continue)

- If a set of equations is ill conditioned, the resulting solution will be inaccurate
- Without a measure of ill conditioning, it is not easy to examine the solution and determine whether it is in error
- Simple tests to identify a set of ill conditioned equations:
  - The diagonal elements a_{ii} are smaller than the off-diagonal elements a_{ii} $(i \neq j)$
  - A smaller change in a_{ii} results in significantly larger changes in the solution
  - A smaller change in b_i result in significantly larger changes in the solution vector x_i.
  - $[A][A]^{-1}$ or $[A]^{-1}[A] \neq [I]$
  - $\det[A] \det[A]^{-1} \neq 1$

Gauss Elimination

- An matrix is called *upper-triangular* provided that the elements satisfy $a_{ij} = 0$ whenever $i > j$
- If $[A]$ is an upper-triangular matrix, then $Ax = B$ is said to be an upper-triangular system of linear equations

$$
\begin{array}{rcrcrcccrcrcl}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & a_{1,3}x_3 & + & \cdots & + & a_{1,n-1}x_{n-1} & + & a_{1,n}x_n & = & b_1 \\
 & & a_{2,2}x_2 & + & a_{2,3}x_3 & + & \cdots & + & a_{2,n-1}x_{n-1} & + & a_{2,n}x_n & = & b_2 \\
 & & & & a_{3,3}x_3 & + & \cdots & + & a_{3,n-1}x_{n-1} & + & a_{3,n}x_n & = & b_3 \\
 & & & & & & & & \vdots & & \vdots & & \vdots \\
 & & & & & & & & a_{n-1,n-1}x_{n-1} & + & a_{n-1,n}x_n & = & b_{n-1} \\
 & & & & & & & & & & a_{n,n}x_n & = & b_n
\end{array}
$$

Gauss Elimination (continue)

- *Forward elimination* is used to change matrix [A], so that it becomes upper triangular
- The solution then can be determined in a simple manner using a process known as *back substitution*
- The operations used in reducing the equations to a triangular form are known as elementary operations
  - Any equation can be multiplied (or divided) by a nonzero scalar
  - Any equation can be added (or subtracted from) another equation
  - The positions of any two equations in the set can be interchanged

Gauss Elimination (continue)

- Introducing the concept of augmented matrix, the matrix $[A]$ can be defined as the $A \times (A + 1)$ matrix

$$[A'] = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & . & . & . & a_{1n} & b_1 \\ a_{21} & a_{22} & . & . & . & a_{2n} & b_2 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} & b_n \end{array}\right]$$

Gauss Elimination (continue)

Forward elimination

$$\left.\begin{array}{r} \text{Row 1} \rightarrow R_1 \\ \text{Row 2} \rightarrow R_2 \\ \text{Row 3} \rightarrow R_3 \end{array}\right. \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

a_{11} is Pivot element

R_1 is Pivot equation $\rightarrow$

$R_2^{(1)} = R_2 - (a_{21}/a_{11})R_1 \rightarrow$

$R_3^{(1)} = R_3 - (a_{31}/a_{11})R_1 \rightarrow$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right]$$

$$\Downarrow$$

$a_{22}^{(1)}$ is Pivot element

R_2 is Pivot equation $\rightarrow$

$R_3^{(2)} = R_3^{(1)} - (a_{32}^{(1)}/a_{22}^{(1)})R_2^{(1)} \rightarrow$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(2)} & b_3^{(2)} \end{array}\right]$$

Back substitution $\Downarrow$

$$\left\{\begin{array}{l} x_3 = b_3^{(2)} / a_{33}^{(2)} \\ x_2 = (b_2^{(1)} - a_{23}^{(1)} x_3) / a_{22}^{(1)} \\ x_1 = (b_1 - a_{12}x_2 - a_{13}x_3) / a_{11} \end{array}\right\}$$

Gauss Elimination (continue)

- The general formula for the elements $a_{ij}^{(k)}$ at the end of the *k*th elimination step can be expressed as

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)} \quad , i=k+1,\ k+2,..n$$

$$j=k,\ k+1,\ k+2,..\ n+1;\ k=1,2,..,n\text{-}1$$

Example

Consider the augmented matrix $\left[\begin{array}{ccc|c} 0.3 & 0.52 & 1 & -0.01 \\ 0.5 & 1 & 1.9 & 0.67 \\ 0.1 & 0.3 & 0.5 & -0.44 \end{array}\right]$

Solution:

$$\left[\begin{array}{ccc|c} 0.3 & 0.52 & 1 & -0.01 \\ 0 & 0.1333 & 0.2333 & 0.6867 \\ 0 & 0.1267 & 0.1667 & -0.4367 \end{array}\right] \begin{array}{l} \\ R_2-(0.5/0.3)R_1 \\ R_3-(0.1/0.3)R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 0.3 & 0.52 & 1 & -0.01 \\ 0 & 0.1333 & 0.2333 & 0.6867 \\ 0 & 0 & -0.055 & -1.089 \end{array}\right] \begin{array}{l} \\ \\ R_3^{(1)}-(0.1267/0.1333)R_2^{(1)} \end{array}$$

$$\left\{\begin{array}{l} x_3=-1.089/(-0.055)=19.8 \\ x_2=(0.6867-0.2333x_3)/0.1333=29.5 \\ x_1=(-0.01-0.52x_2-x_3)/0.3=-14.9 \end{array}\right\}$$

Check the answers by

$$0.3(-14.9)+0.52(-29.5)+(19.8)=-0.01$$
$$0.5(-14.9)+(-29.5)+1.9(19.8)=0.67$$
$$0.1(-14.9)+0.3(-29.5)+0.5(19.8)=-0.44$$

Gauss-Jordan Elimination

- **Normalization:** diagonal element a_{ii} = 0
- **Elimination:** all off-diagonal elements = 0 for $[A] \rightarrow$ Identity matrix $[I]$
- The elements above the diagonal are made zero at the same time that zeros are created below the diagonal

Normalization **Elimination**

$$R_1^{(1)} = R_1 / a_{11} \rightarrow \left[\begin{array}{ccc|c} 1 & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \Rightarrow \begin{array}{l} R_2^{(1)} = R_2 - a_{21} R_1^{(1)} \rightarrow \\ R_3^{(1)} = R_3 - a_{31} R_1^{(1)} \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right]$$

$$R_2^{(2)} = R_2^{(1)} / a_{22}^{(1)} \left[\begin{array}{ccc|c} 1 & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ 0 & 1 & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right] \Rightarrow \begin{array}{l} R_1^{(2)} = R_1^{(1)} - a_{12}^{(1)} R_2^{(2)} \\ \\ R_3^{(2)} = R_3^{(1)} - a_{32}^{(1)} R_2^{(2)} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & a_{13}^{(2)} & b_1^{(2)} \\ 0 & 1 & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(2)} & b_3^{(2)} \end{array}\right]$$

$$R_3^{(3)} = R_3^{(2)} / a_{33}^{(2)} \left[\begin{array}{ccc|c} 1 & 0 & a_{13}^{(2)} & b_1^{(2)} \\ 0 & 1 & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & 1 & b_3^{(3)} \end{array}\right] \Rightarrow \begin{array}{l} R_1^{(3)} = R_1^{(2)} - a_{13}^{(2)} R_3^{(3)} \\ R_2^{(3)} = R_2^{(2)} - a_{23}^{(2)} R_3^{(3)} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1^{(3)} \\ 0 & 1 & 0 & b_2^{(3)} \\ 0 & 0 & 1 & b_3^{(3)} \end{array}\right]$$

Example

Consider the augmented matrix $\left[\begin{array}{ccc|c} 0.3 & 0.52 & 1 & -0.01 \\ 0.5 & 1 & 1.9 & 0.67 \\ 0.1 & 0.3 & 0.5 & -0.44 \end{array}\right]$

Solution:

$$R_1^{(1)} = R_1/0.3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1.733 & 3.333 & -0.03333 \\ 0.5 & 1 & 1.9 & 0.67 \\ 0.1 & 0.3 & 0.5 & -0.44 \end{array}\right] \Rightarrow \begin{array}{l} R_2^{(1)} = R_2 - 0.5R_1^{(1)} \rightarrow \\ R_3^{(1)} = R_3 - 0.1R_1^{(1)} \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1.733 & 3.333 & -0.03333 \\ 0 & 0.1335 & 0.2333 & 0.6867 \\ 0 & 0.1267 & 0.1667 & -0.4367 \end{array}\right]$$

$$R_2^{(2)} = R_2^{(1)}/0.1335 \left[\begin{array}{ccc|c} 1 & 1.733 & 3.333 & -0.03333 \\ 0 & 1 & 1.75 & 5.15 \\ 0 & 0.1267 & 0.1667 & -0.4367 \end{array}\right] \Rightarrow \begin{array}{l} R_1^{(2)} = R_1^{(1)} - 1.733R_2^{(2)} \\ \\ R_3^{(2)} = R_3^{(1)} - 0.1267R_2^{(2)} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0.3 & -8.96 \\ 0 & 1 & 1.75 & 5.15 \\ 0 & 0 & -0.055 & -1.089 \end{array}\right]$$

$$R_3^{(3)} = R_3^{(2)}/(-0.055) \left[\begin{array}{ccc|c} 1 & 0 & 0.3 & -8.96 \\ 0 & 1 & 1.75 & 5.15 \\ 0 & 0 & 1 & 19.8 \end{array}\right] \Rightarrow \begin{array}{l} R_1^{(3)} = R_1^{(2)} - 0.3R_3^{(3)} \\ R_2^{(3)} = R_2^{(2)} - 1.75R_3^{(3)} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -14.9 \\ 0 & 1 & 0 & -29.5 \\ 0 & 0 & 1 & 19.8 \end{array}\right]$$

Comparison of Methods

Gauss Elimination

$$\underbrace{\frac{n^3}{3}+O(n^2)}_{\text{Forward elimination}}+\underbrace{\frac{n^2}{2}+O(n)}_{\text{Back substitution}}\xrightarrow{\text{as n increase}}\frac{n^3}{3}+O(n^2)$$

Gauss-Jordan Elimination

$$\frac{n^3}{2}+n^2-\frac{n}{2}\xrightarrow{\text{as n increase}}\frac{n^3}{2}+O(n^2)$$

Gauss-Jordan ~ more 50% operation than Gauss elimination

- Efficiency of Gaussian elimination is usually measured by counting the total number of multiplications and divisions involved
- Direct methods can lead to exact answers. The answers are just close approximations to the exact answer due to round-off error. Where there are many equations, the effect of rounding-off may cause large effect, i.e. ill-condition
- Dividing by zero may occur during elimination process. Zeros may be created in the diagonal position even if they are not present in the original matrix of coefficients

Some Considerations

Pitfalls of Elimination Methods

- Round-odd errors
- Division of zero
- Ill-conditioned systems
- Singular systems

Techniques for Improving Solutions

- Use more significant figures
- Pivoting
- Scaling

Pivoting

- **Partial pivoting:** rows can be switched so that the largest element is the pivot element
- **Complete pivoting:** rows and columns can be switched so that the largest element is the pivot element. It is rarely used

Example: Solve

$$\begin{aligned}
2x_2 \qquad\qquad + \quad x_4 &= 0 \\
2x_1 + 2x_2 + 3x_3 + 2x_4 &= -2 \\
4x_1 - 3x_2 \qquad\qquad + \quad x_4 &= -7 \\
6x_1 + \quad x_2 - 6x_3 - 5x_4 &= 6
\end{aligned}$$

Pivoting (continue)

Solution: The augmented matrix $\left[\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{array}\right]$

cannot be solved using Gauss elimination

By interchanging $R_1 \leftrightarrow R_4$ $\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$

Pivoting (continue)

Forward Elimination:

Eliminate a_{i1} , $i > 1$ $\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$

By interchanging $R_2^{(1)} \leftrightarrow R_3^{(1)}$ $\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$

Pivoting (continue)

Forward Elimination:

Eliminate $a_{i2}^{(1)}$, $i > 2$

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 2.1818 & 3.3636 & -5.9999 \end{array}\right]$$

Eliminate $a_{i3}^{(2)}$, $i > 3$

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 0 & 1.5600 & -3.1199 \end{array}\right]$$

Pivoting (continue)

Backward Elimination:

$$x_4 = \frac{-3.1199}{1.5600} = -1.9999$$

$$x_3 = \frac{-9.0001 - 5.6364 x_4}{6.8182} = 0.33325$$

$$x_2 = \frac{-11 - 4.3333 x_4 - 4 x_3}{-3.6667} = 1.0000$$

$$x_1 = \frac{6 + 5x_4 + 6x_3 - x_2}{6} = -0.50000$$

Note: Pivoting will guarantee a nonzero divisor and will add the advantage of giving improved arithmetic precision

Exact answers: $\{x_1 \quad x_2 \quad x_3 \quad x_4\} = \{-\tfrac{1}{2} \quad 1 \quad \tfrac{1}{3} \quad -2\}$

Scaling

- It is the operation of dividing some of the equations by the largest coefficient of the equation when they have much larger coefficients than others

  Example: Solve $\begin{bmatrix} 3 & 2 & 100 \\ -1 & 3 & 100 \\ 1 & 2 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 105 \\ 102 \\ 2 \end{Bmatrix}$ by carrying out three digits to emphasize round-off and using partial pivoting

Scaling (continue)

Solution: Without scaling

$$\left[\begin{array}{ccc|c} 3 & 2 & 100 & 105 \\ 0 & 3.67 & 133 & 137 \\ 0 & 1.33 & -34.3 & -33 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 2 & 100 & 105 \\ 0 & 3.67 & 133 & 137 \\ 0 & 0 & -82.5 & -82.6 \end{array}\right] \rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.05 \\ 0.929 \end{Bmatrix}$$

Exact solution

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{Bmatrix}$$

Scaling (continue)

Solution: With scaling

$$\begin{bmatrix} 0.03 & 0.02 & 1 & | & 1.05 \\ -0.01 & 0.03 & 1.00 & | & 1.02 \\ 0.50 & 1.00 & -0.50 & | & 1.00 \end{bmatrix}$$ using partial pivoting

$$\begin{bmatrix} 0.50 & 1.00 & -0.50 & | & 1.00 \\ -0.01 & 0.03 & 1.00 & | & 1.02 \\ 0.03 & 0.02 & 1.00 & | & 1.05 \end{bmatrix}$$

$$\begin{bmatrix} 0.50 & 1.00 & -0.50 & | & 1.00 \\ 0 & 0.05 & 0.99 & | & 1.04 \\ 0 & -0.04 & 1.03 & | & 0.99 \end{bmatrix} \rightarrow \begin{bmatrix} 0.50 & 1.00 & -0.50 & | & 1.00 \\ 0 & 0.05 & 0.99 & | & 1.04 \\ 0 & 0 & 1.82 & | & 1.82 \end{bmatrix} \rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{Bmatrix}$$

Scaling (continue)

- Whenever the coefficients in one column are widely different from those in another column, scaling is beneficial
- When all values are about the same order of magnitude, scaling should be avoided, for the additional round-off error incurred during the scaling operation itself may adversely affect the accuracy