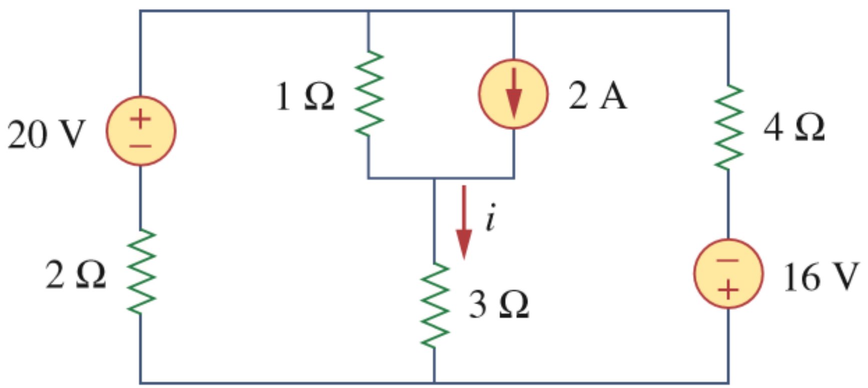
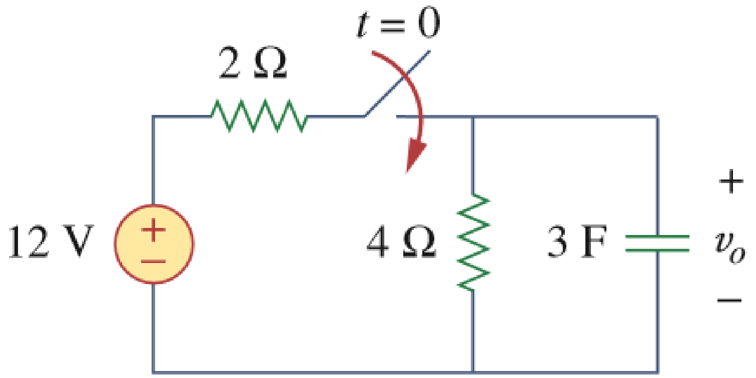
Weeks 1-6 Revision

Basis methods for circuit analysis

- Ohm's Law
- Voltage/Current Division Laws
- Kirchhoff's Current Law (KCL)
- Kirchhoff's Voltage Law (KVL)
- Nodal Analysis
- Mesh Analysis
- Superposition
- Source Transformation
- Thévenin's Theorem
- Norton's Theorem
- Maximum Power Transfer
- Capacitor, Inductor, Transient analysis

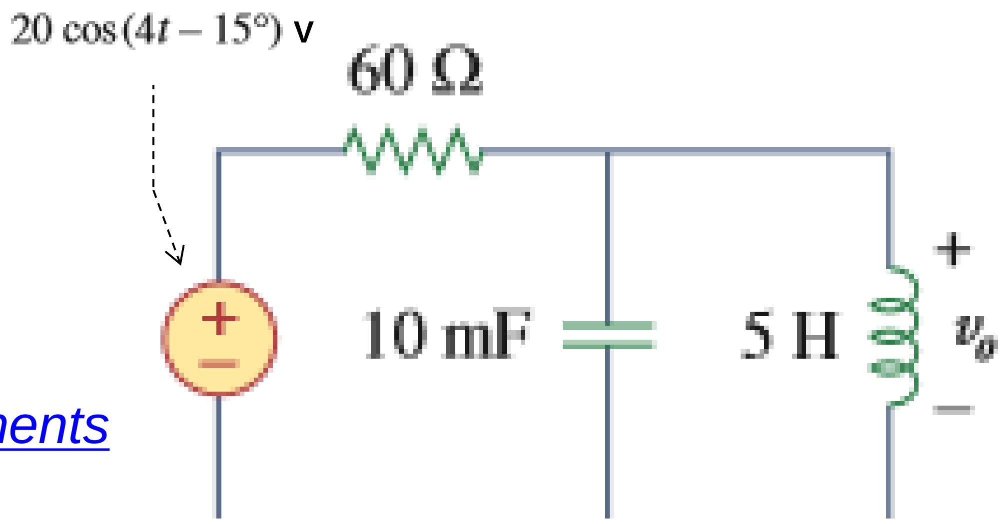




Weeks 7-12 Contents

AC signals (sinusoid signal)

- Phasor
- AC Power



Other electrical/electronic elements

- OpAmp
- Transformers

Combination logic, digital circuit analysis

The basic laws & analysis methods in weeks 1-4 can be applied in all electrical circuits and will be used to analyse the circuit with AC input source

Lecture 9: Sinusoids and Phasors

ELEC1111 Electrical and Telecommunications Engineering

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Objectives

- Introduce sinusoids and phasors for AC circuit analysis
- Derive phasor relationships for circuit elements
- Introduce Impedance and Admittance
- Derive Kirchhoff's current and voltage laws in the frequency domain

Sinusoids

- Time varying signals with the form of a sine or cosine function
  - Common in nature:
    - Pendulum motion
    - Vibration of a string
    - Natural response of second order systems
  - Easy to generate and transmit
    - Electrical machines (generators)
    - Oscillators
  - Used in Fourier analysis,
  - Easy to handle mathematically

$$v(t) = V_m \sin \omega t \qquad \text{period } T = \frac{2\pi}{\omega}$$

Sinusoids

$$v(t) = V_m \sin \omega t$$

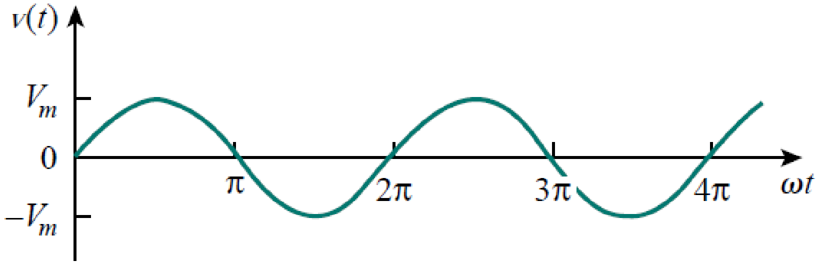
- Sinusoids are periodic functions, satisfying

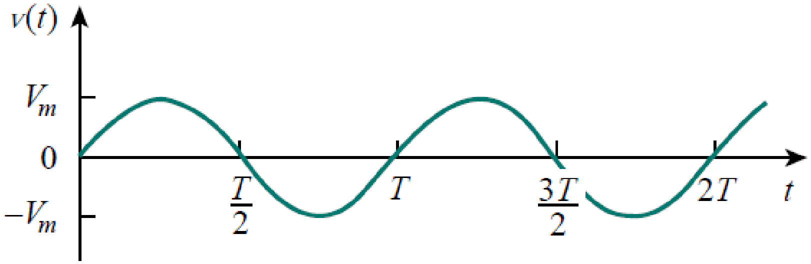
$$v(t + nT) = v(t)$$

for all *t* and all integer values of *n.*

$$v(t + nT) = V_m \sin \omega (t + nT) = V_m \sin \omega (t + n\tfrac{2\pi}{\omega})$$

$$= V_m \sin(\omega t + 2\pi n) = V_m \sin(\omega t) = v(t)$$





Sinusoids

- Can be expressed in either a sine or a cosine form.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

- Sine to cosine form transformation and vice-versa

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$

$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$

$$\sin(\omega t \pm 90^\circ) = \pm cos \omega t$$

$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

How to Sum Sinusoids

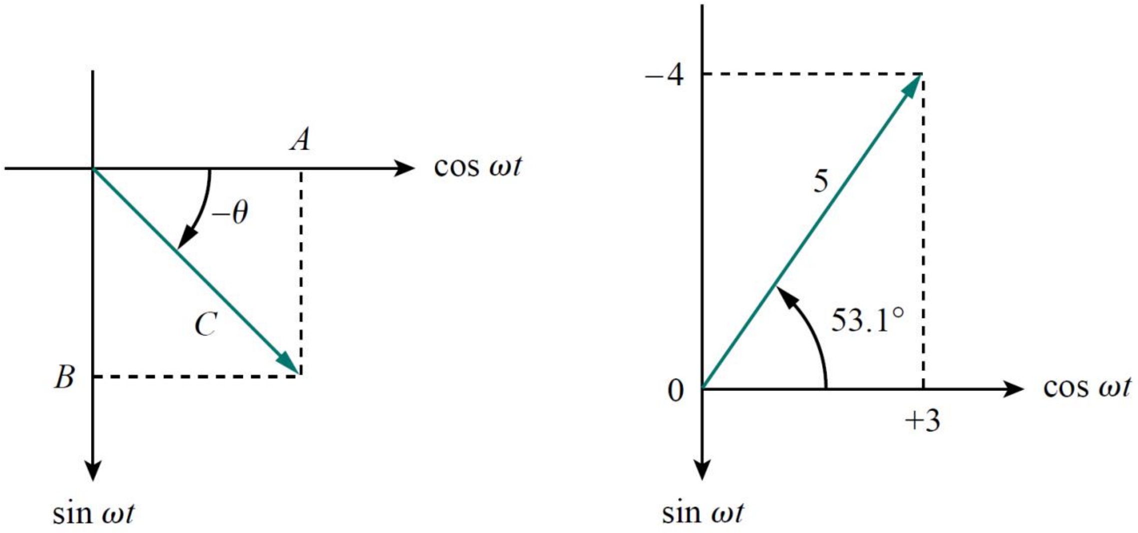
- Sum DC voltages: $V_o = V_1 + V_2 = 3V + 5V = 8V$

- Sum Sinusoid voltages:

$$v_o(t) = v_1(t) + v_2(t) = 3\sin(4\omega t + \pi/3) + 5\sin(4\omega t + \pi/4) = ?$$

Sinusoids

- Sinusoids can be added and subtracted using a graphical approach



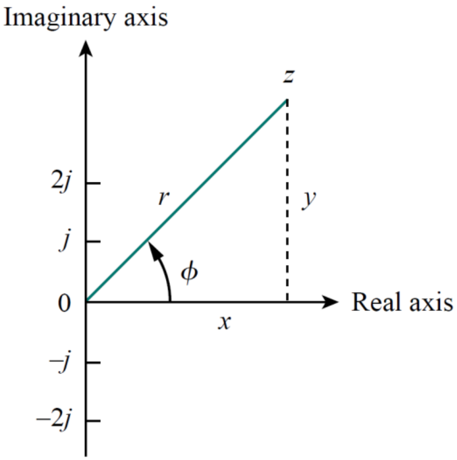
$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \theta)$$

$$C = \sqrt{A^2 + B^2} \quad \theta = \tan^{-1}\frac{B}{A}$$

Phasors

- A phasor is a complex number that represents the amplitude and the phase of a sinusoid
  - A convenient way to express sinusoids.

- A complex number can be written in three forms
  - Rectangular $z = x + jy$
  - Polar $z = r\angle\phi$
  - Exponential $z = re^{j\phi}$



Phasors

- Rectangular to polar

$$r = \sqrt{x^2 + y^2} \qquad \phi = tan^{-1}\frac{y}{x}$$

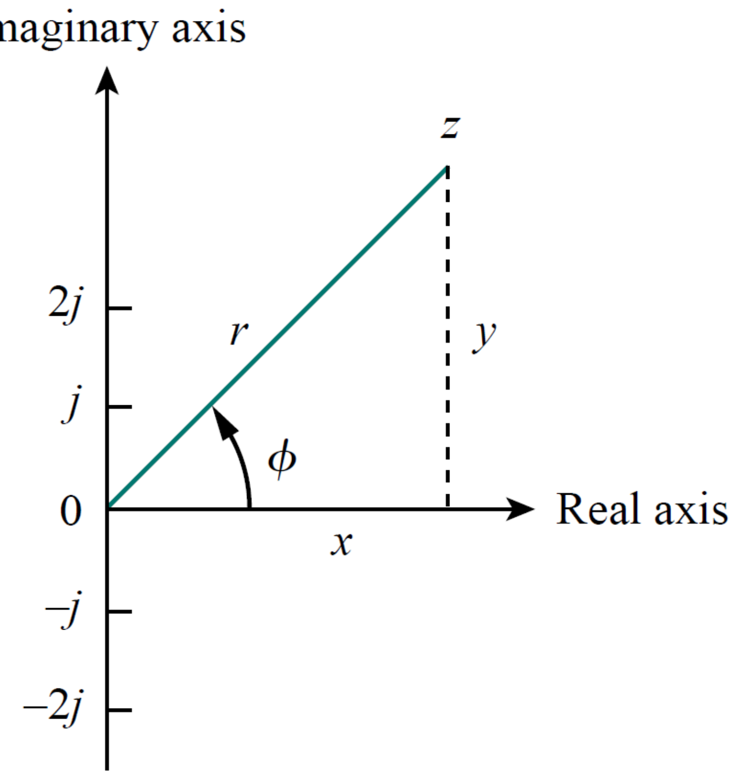
- Polar to rectangular

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)$$

- Euler's Identity

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$



Example

- Show the following numbers in phasor diagram

$$\mathbf{V}_1 = 5\angle 30^o$$

$$\mathbf{V}_2 = 5\angle -60^o$$

$$\mathbf{V}_3 = 3 + j2$$

$$\mathbf{V}_4 = -3 + j2$$

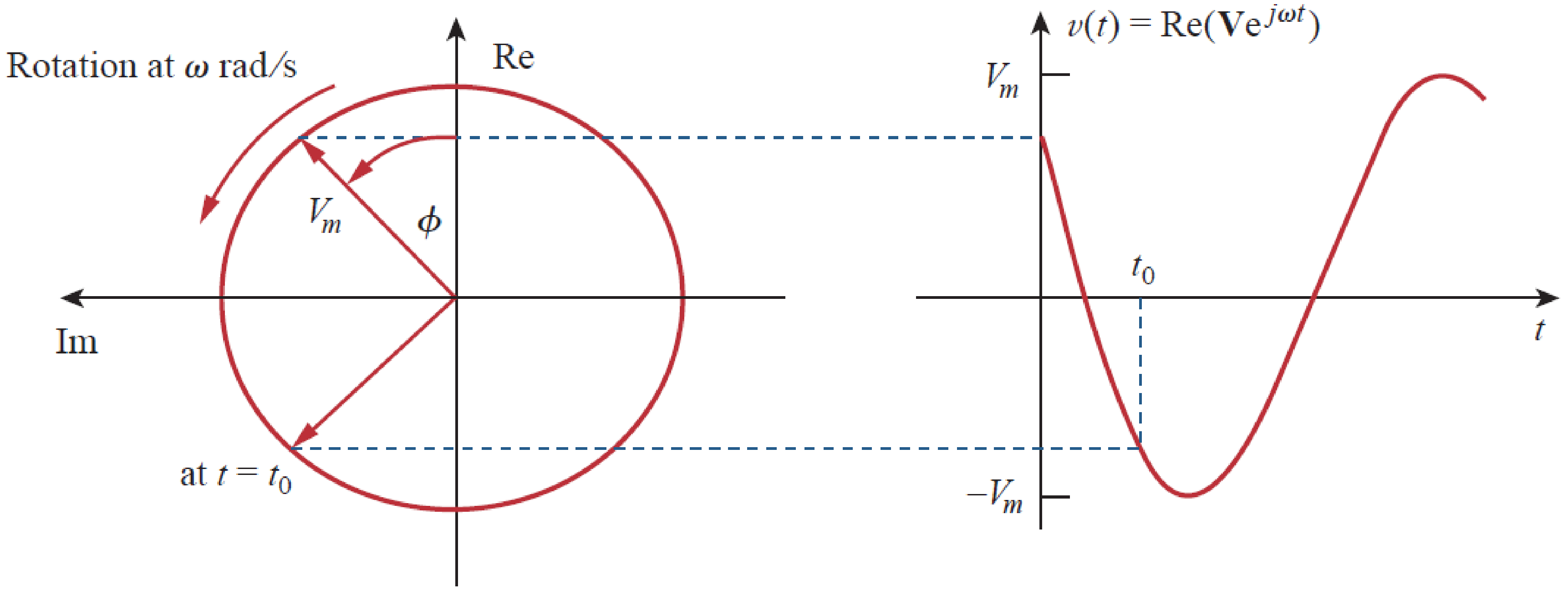
$$\mathbf{V}_5 = -3 - j2$$

Phasors

$$V_m e^{j(\omega t+\phi)} = \underbrace{V_m \cos(\omega t+\phi)}_{\text{Real}} + j\underbrace{V_m \sin(\omega t+\phi)}_{\text{Imaginary}}$$

$$v(t) = V_m \cos(\omega t+\phi) = \text{Re}(V_m e^{j(\omega t+\phi)}) = \text{Re}(\boxed{V_m e^{j\phi}} e^{j\omega t})$$

$$v(t) = \text{Re}(\boxed{\mathbf{V}} e^{j\omega t})$$

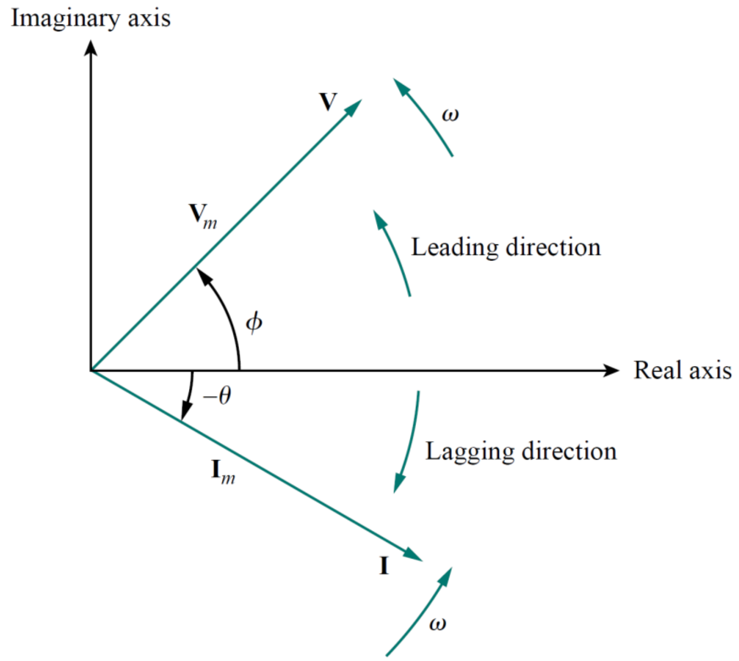


Phasor diagram

- A phasor has a magnitude and phase, behaves as a vector and can by represented graphically in a phasor diagram

$$v(t) = V_m \cos(\omega t + \phi) \iff \mathbf{V} = V_m \angle \phi$$

time domain phasor domain



Phasor diagram

$$v(t) = V_m \cos(\omega t + \phi)$$

- Differentiating a sinusoid is equivalent to multiplying its corresponding phasor by *jω*

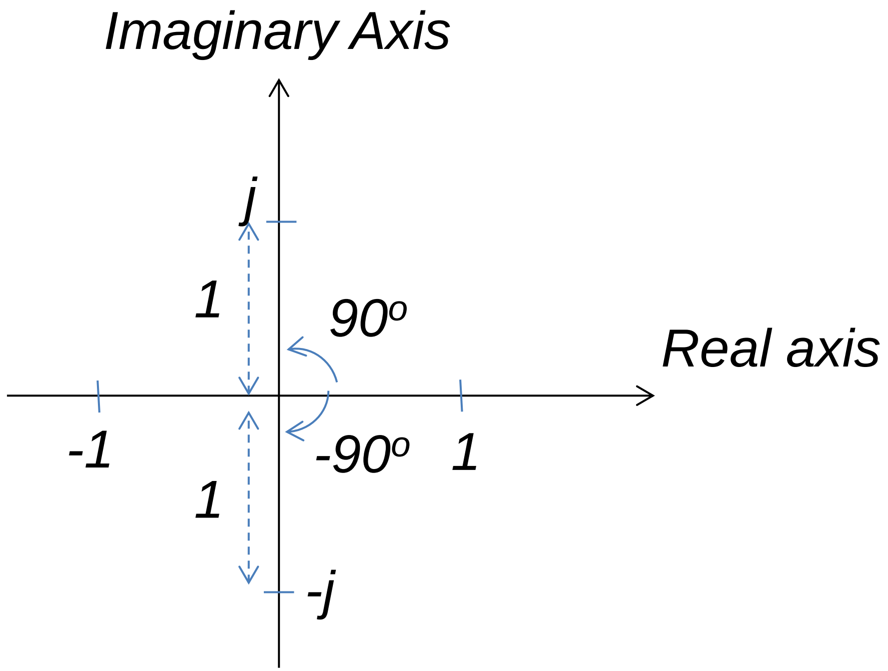
$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ)$$

$$= \mathrm{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \mathrm{Re}(j\omega \mathbf{V} e^{j\omega t})$$

- Integrating a sinusoid is equivalent to dividing its corresponding phasor by *jω.*
- Adding sinusoids of the same frequency is equivalent to adding their corresponding phasors.

Phasor diagram

- *Note:* $-\sin(\omega t+\phi)=\cos(\omega t+\phi+90^\circ)= j\cos(\omega t+\phi)$



$$e^{j90^\circ} = \cos(90^\circ) + j\sin(90^\circ) = j = 1\angle 90^\circ$$

$$e^{-j90^\circ} = \cos(-90^\circ) + j\sin(-90^\circ) = -j = 1\angle -90^\circ$$

$$-1 = \cos(180^\circ) + j\sin(180^\circ) = e^{j180} = 1\angle 180^\circ = 1\angle -180^\circ$$

Phasor relationships for circuit elements

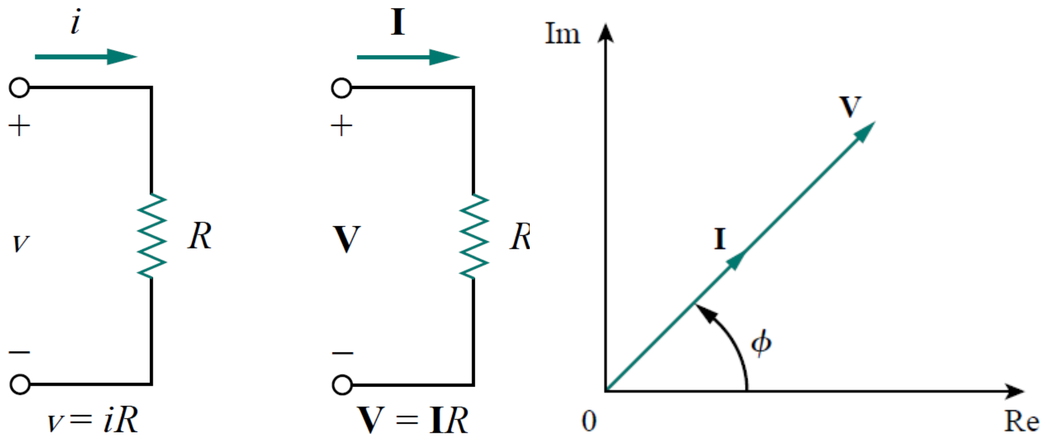
- Transform the voltage-current relationship of circuit elements from the time domain to the frequency domain.
- Resistor

$$i = I_m \cos(\omega t + \phi) \rightarrow I = I_m \angle \phi$$

$$v = iR = RI_m \cos(\omega t + \phi)$$

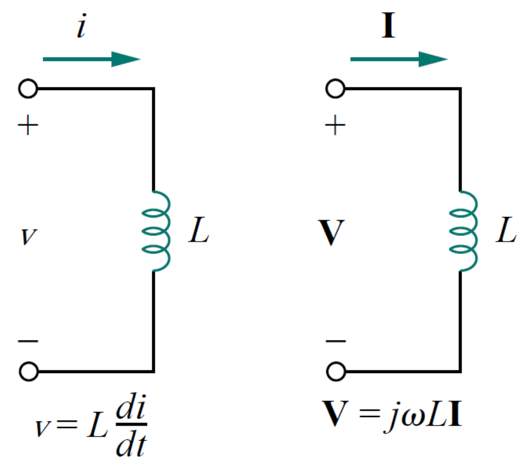
$$\mathbf{V} = RI_m \angle \phi$$

$$\mathbf{V} = R\mathbf{I}$$



Phasor relationships for circuit elements

- Inductor



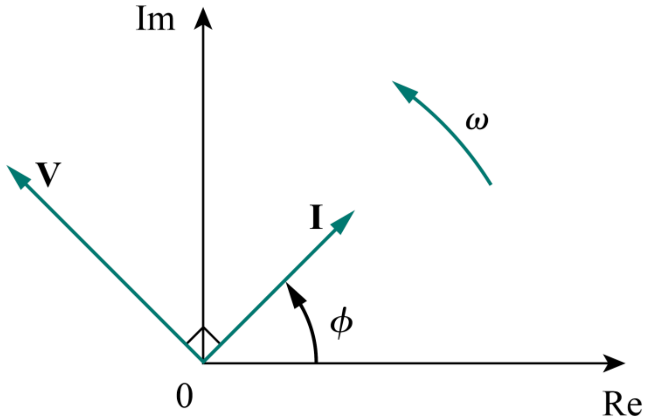
$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

$$v = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

$$\mathbf{V} = \omega L I_m e^{j(\phi + 90^\circ)} = \omega L I_m e^{j\phi} e^{j90^\circ} = \omega L I_m \angle(\phi + 90^\circ)$$

$$\mathbf{V} = j\omega L\mathbf{I}$$

- Current lags (behind) the voltage by 90 degrees

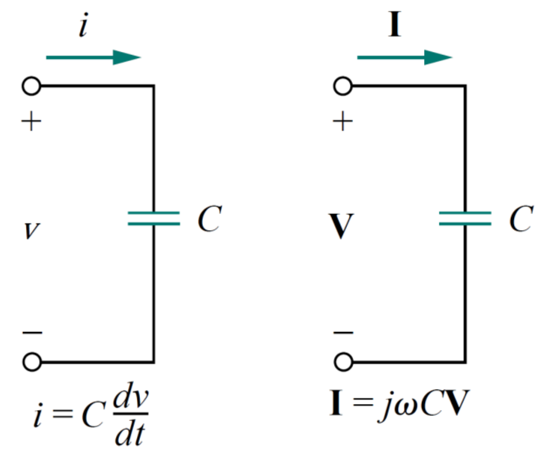


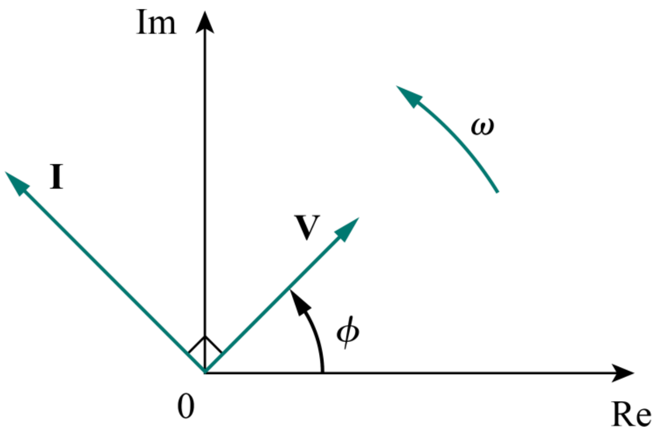
Phasor relationships for circuit elements

- Capacitor

$$i = C\frac{dv}{dt}$$

$$\mathbf{I} = j\omega C\mathbf{V} \Rightarrow \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$





- The current leads (advances) the voltage 90 degrees.
- Convention to give the current relative to the voltage.

Impedance and Admittance

- Impedance ***Z*** of a circuit is defined as the ratio of the phasor voltage ***V*** to the phasor current ***I***, measured in Ohms (Ω).

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

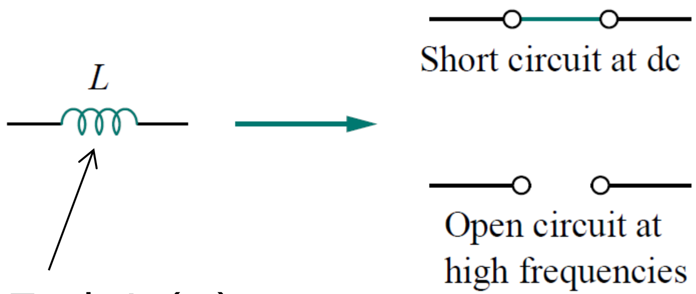
- The impedance represents the opposition of the circuit to the flow of sinusoidal current.

- Admittance is the reciprocal of impedance, measured in Siemens (S).

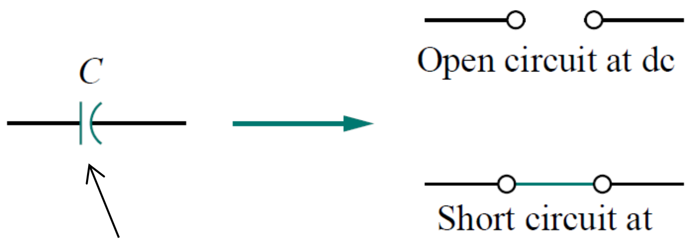
$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}$$

Impedance and Admittance

Resistor	Inductor	Capacitor
$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$
$\dfrac{\mathbf{V}}{\mathbf{I}} = R$	$\dfrac{\mathbf{V}}{\mathbf{I}} = j\omega L$	$\dfrac{\mathbf{V}}{\mathbf{I}} = \dfrac{1}{j\omega C}$



$Z_L = j\omega L\ (\Omega)$



$$Z_c = \frac{1}{j\omega C}\ (\Omega)$$

Impedance and Admittance

- Impedance

$$\mathbf{Z} = R + jX$$

Where R is the resistance and X is the reactance

Impedance and Admittance

- Admittance

$$\mathbf{Y} = G + jB$$

Where *G* is the conductance and *B* the susceptance.

$$G + jB = \frac{1}{R + jX}$$

$$G = \frac{R}{R^2 + X^2} \qquad B = -\frac{X}{R^2 + X^2}$$

Kirchhoff's laws in the frequency domain

- Kirchhoff's Voltage Law
  - Considering the voltages across a closed loop

$$v_1 + v_2 + \ldots + v_n = 0$$

  - In the sinusoidal state

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \ldots + V_{mn}\cos(\omega t + \theta_n) = 0$$

which can be written as

$$\mathrm{Re}[(\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_n)e^{j\omega t}] = 0$$

And since the exponential term cannot be equal to zero

$$\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_n = 0$$

Kirchhoff's laws in the frequency domain

- Kirchhoff's Current Law
  - Similarly to KVL, considering the currents on a node

$$i_1 + i_2 + \ldots + i_n = 0$$

  and in the phasor form for frequency domain representation

$$\mathbf{I}_1 + \mathbf{I}_2 + \ldots + \mathbf{I}_n = 0$$

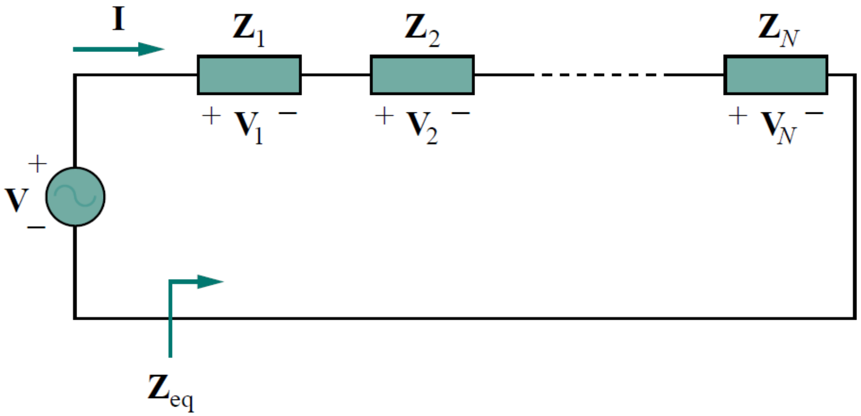
  - Based on the frequency domain forms of the KVL and KCL, impedance combination, nodal and mesh analysis, superposition and source transformation can be easily performed

Impedance combinations

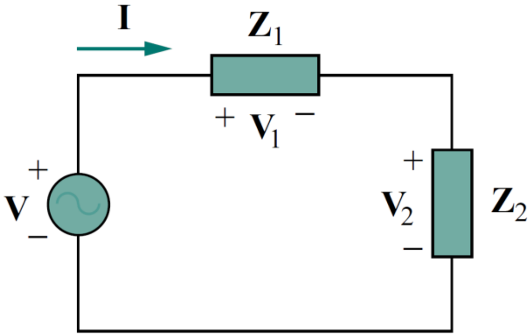
- Impedances connected in series

$$\mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \ldots + \mathbf{Z}_N)$$

$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \ldots + \mathbf{Z}_N$$



- Voltage divider



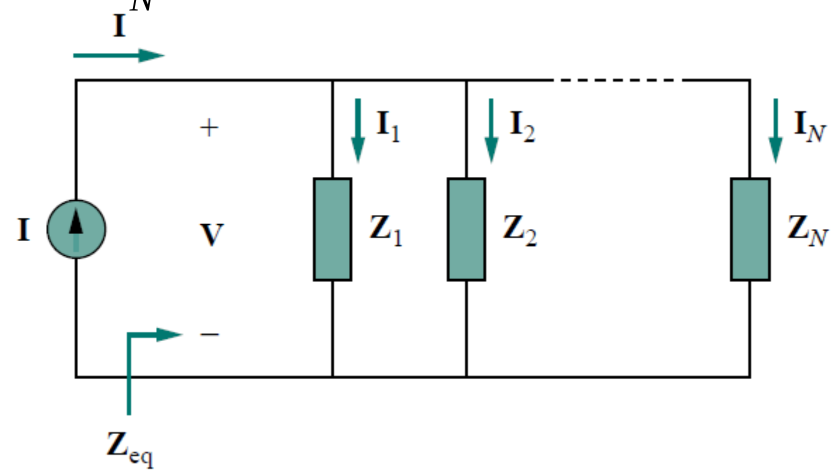
$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V} \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

Impedance combinations

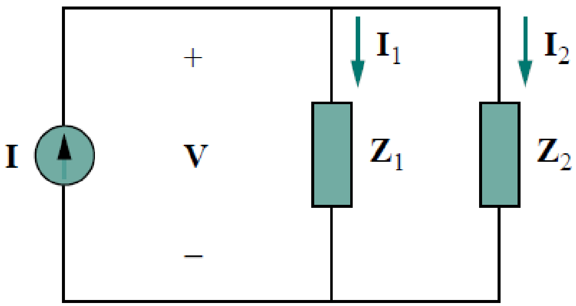
- Impedances connected in parallel

$$\mathbf{I}_1 + \mathbf{I}_2 + \ldots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \ldots + \frac{1}{\mathbf{Z}_N}\right)$$

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \ldots + \frac{1}{\mathbf{Z}_N}$$



- Current divider

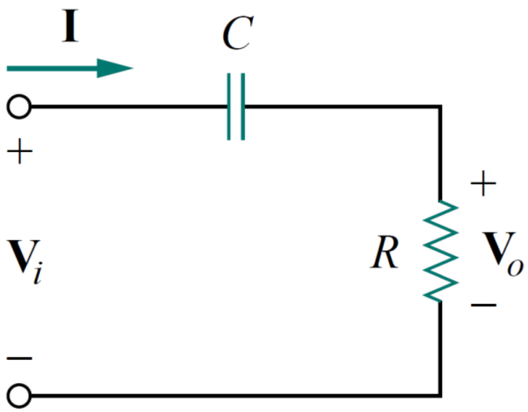


$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

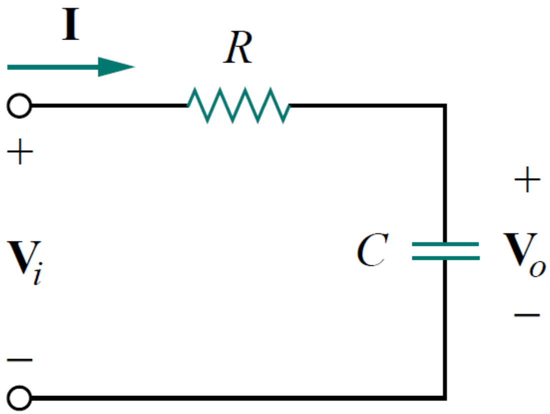
Applications

- Phase shifters
  - Used to correct undesirable phase-shift or to produce a required phase shift in signal.
  - Simplest configuration of an RC circuit

$$\theta = tan^{-1}\frac{X}{R}$$



Leading arrangement



Lagging arrangement

Applications

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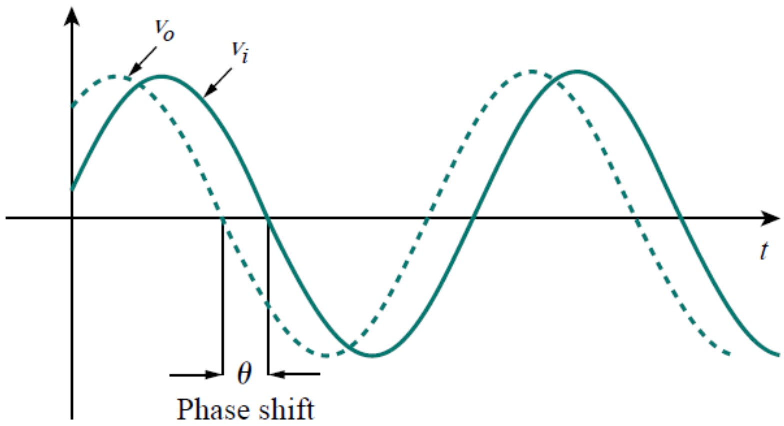
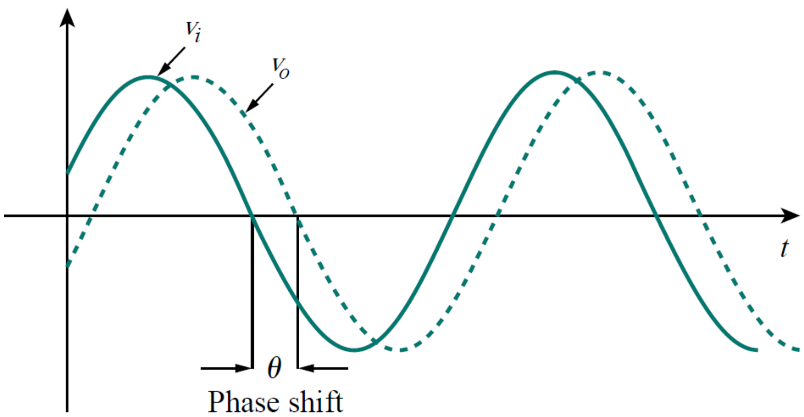


Leading arrangement



Lagging arrangement

Applications

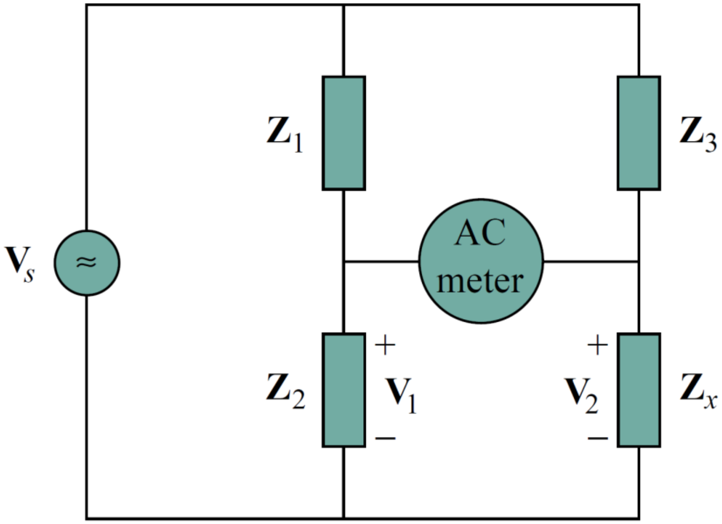
- AC Bridges
  - Used in measurements of inductances and capacitances of elements.
  - In a general configuration, the bridge is balanced when no current flows through the meter.
  - In a balanced bridge

$$\mathbf{V}_1 = \mathbf{V}_2$$

$$\mathbf{V}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_x}{\mathbf{Z}_3 + \mathbf{Z}_x}\mathbf{V}_s$$

hence

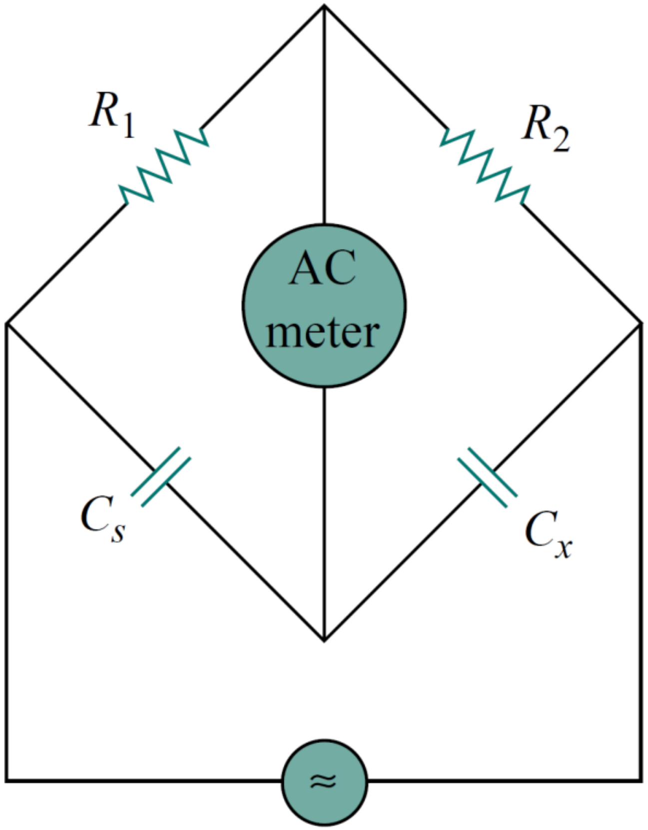
$$\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{\mathbf{Z}_x}{\mathbf{Z}_3 + \mathbf{Z}_x} \Rightarrow \mathbf{Z}_x = \frac{\mathbf{Z}_3}{\mathbf{Z}_1}\mathbf{Z}_2$$



Applications

- AC bridge configuration for measuring capacitance (C)

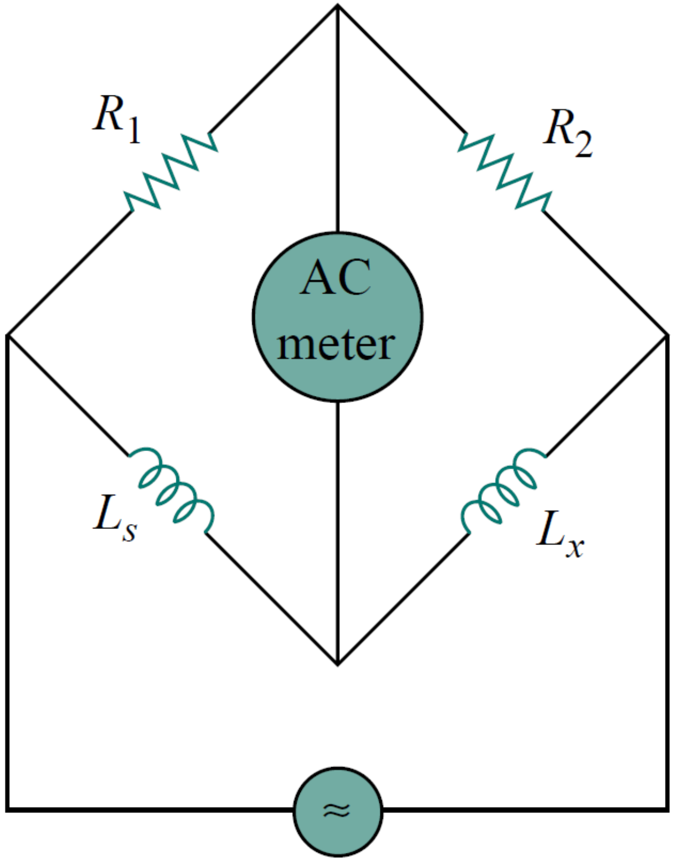
$$C_x = \frac{R_2}{R_1} C_s$$



Applications

- AC bridge configuration for measuring inductance (L)

$$L_x = \frac{R_2}{R_1} L_s$$



- Balancing of the bridges is not affected by the frequency of the AC source.

Summary

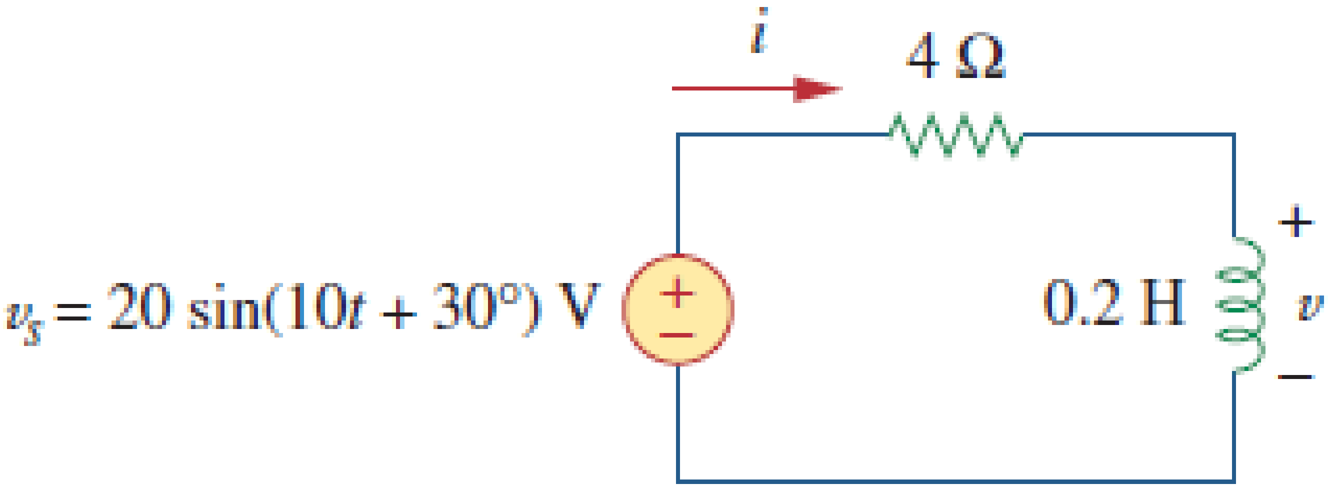
- Sinusoids and phasors
  - Application in the analysis of AC circuits
- Phasor relations for circuit elements
  - Resistor
  - Inductor
  - Capacitor
- Impedance and Admittance
- KCL and KVL in the frequency domain
- Applications of RLC circuits with AC signals

Sinusoidal steady-state analysis

- Steps to analyze AC circuits
  1. Transform the circuit to the frequency domain.
  2. Solve the problem using circuit techniques learnt previously.
     - Analysis is similar to the DC circuit analysis but uses complex numbers.
  3. Transform the resulting phasor to the time domain.

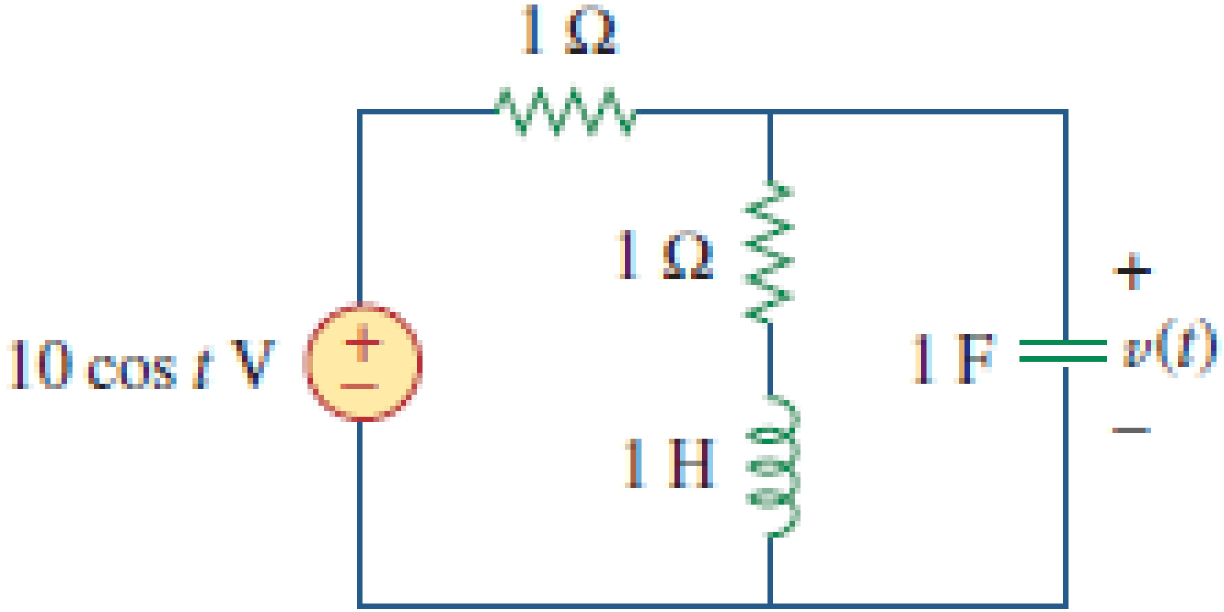
Example 1

Determine v(t) and i(t). Show V and I in phasor diagram



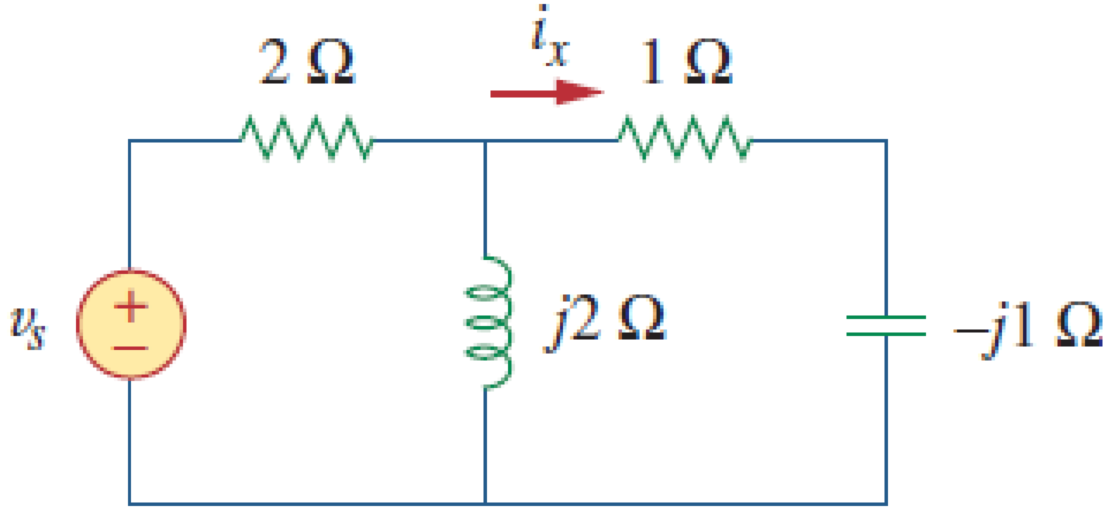
Example 2

Determine v(t)



Example 3

Find $v_s(t)$ in the following circuit if it is known that the current $i_x(t)=0.5\sin 200t$ A



Ans: $v_s(t)=1.41\sin(200t-15^o)V$

Example 4

$v_1(t)=30\sin(10t+60^o)$V and
$v_2(t)=40\cos(10t-45^o)$V.
Which voltage is leading and in how many degree?

Ans: v_2 leads v_1 by 15^o