Week 2-2018

Combinational circuit I

Binary logic and Gates

- Transistors interconnected each other to form logic gates

- Each gate has inputs and an output. It performs a specific logical operation on its binary inputs and provides a binary value at the output.

- Inputs and output of a logic gate are designated by alphabetical variables .

- These variables can assume only 1 or 0 values and are known as Binary variables.

Basic logical operations

- There are three basic logical operations : **AND, OR,** and **NOT.**

AND

- represented by a dot or an absence of operation

$$Z = X \cdot Y = XY = X \wedge Y$$

- Z is 1 if and only if X=1 and Y=1; otherwise Z = 0

YES = 1

NO = 0

$X \cdot Y$
$0 \cdot 0 = 0$
$0 \cdot 1 = 0$
$1 \cdot 0 = 0$
$1 \cdot 1 = 1$

4

X	Y	$Z = X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table for AND gate

2

2

4 = 2

AND logical operation can be extended to more than two input binary variables.

Basic logical operations

OR

- **represented by a plus symbol or "∨"**

$$Z = X + Y = X \vee Y$$

- **Z is 1 if X=1 or Y=1; Z is 0 if X = 0 and Y= 0**

X + Y
0+0 = 0
0+1 = 1
1+0 = 1
1+1 = 1

X	Y	X + Y
0	0	0
0	1	1
1	0	1
1	1	1

Truth Table for OR gate

OR logical operation can be extended to more than two input binary variables.

Basic logical operations

NOT

- represented by a bar over the variable

  $Z = \overline{X}$

- Z is 1 if X=0 and Z is 0 if X = 1

X
$\overline{0} = 1$
$\overline{1} = 0$

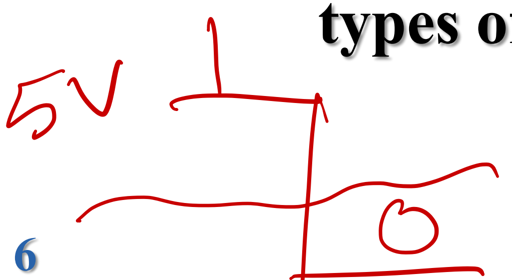
X	$Z = \overline{X}$
0	1
1	0

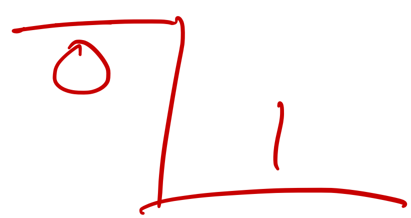
Truth Table for NOT gate

COMPLEMENT

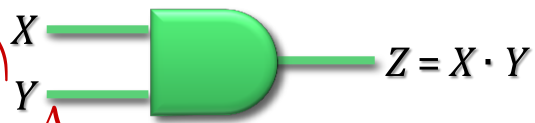
Logic gates

- **Logical Gates -** Electrical circuits that implement logical operations.
- The input terminals accept voltage signals within allowable range (as a binary signals) and gives out at the output terminal a binary signals that also falls with in the allowable range.
- Intermediate values are crossed only during transitions from 0 to 1 or otherwise.
- Graphical (symbolical) representations of the three types of gates: **AND, OR, NOT**

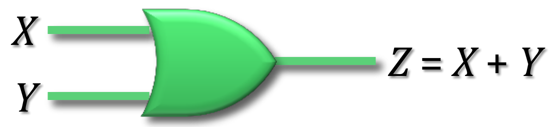




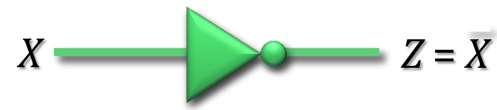
Logic gates



AND gate

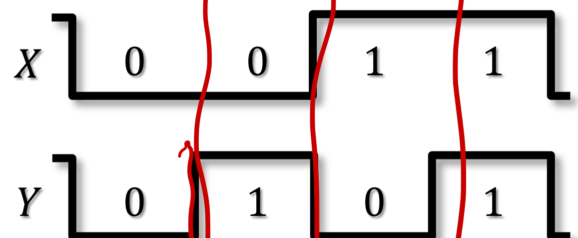


OR gate



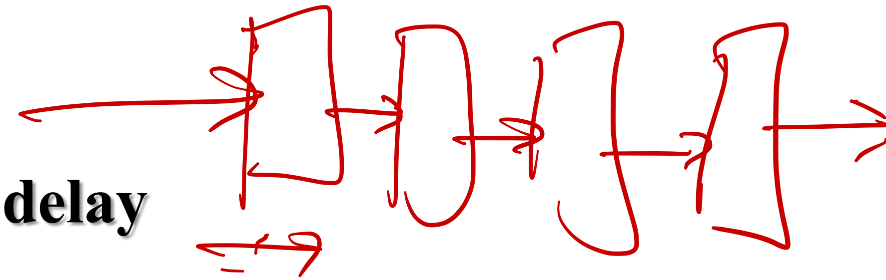
NOT gate (inverter)

- **Time diagram (waveform behavior)**



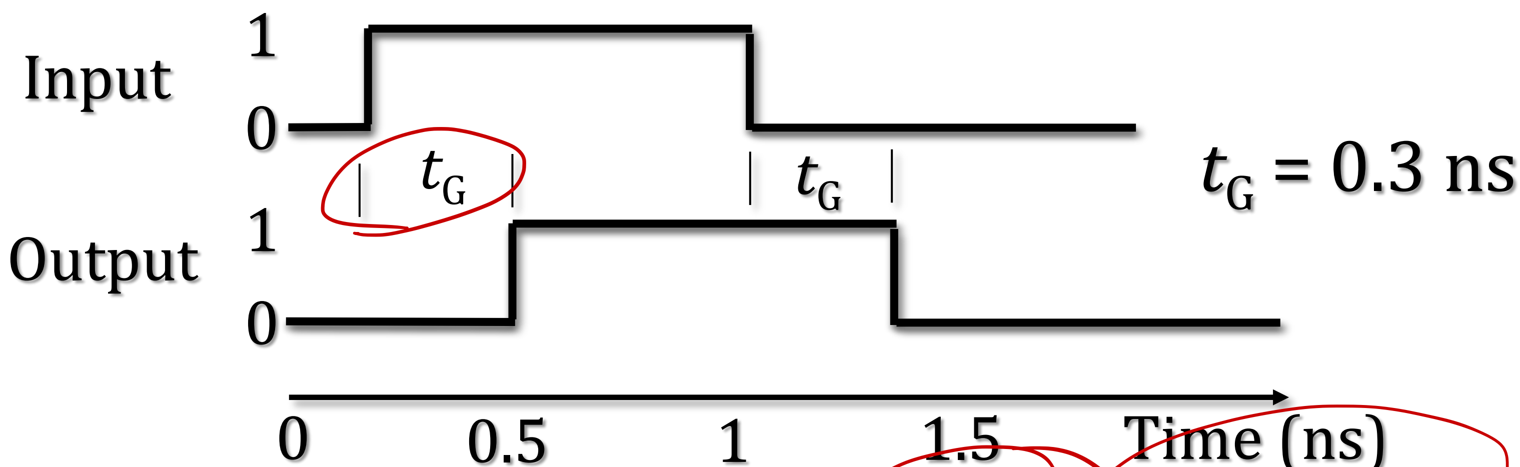
(AND)	$X \cdot Y$	Z	0	0	0	1
(OR)	$X + Y$		0	1	1	1
(NOT)	$\overline{X}$					

Gate delay



- In reality, there is a gate delay

- **Gate delay** – the length of time it takes for an input changes to result in the corresponding output change.



- Gate delay is a function of **gate type**, **number of inputs**, **underlying technology**, and **circuit design of the gate.**

Boolean Algebra

- A *Boolean Algebra* is an algebra dealing with binary variables and logic operations.

- A *Boolean expression* is an algebraic expression formed by using binary variables, the constants 0 and 1, the logic operation symbols, and parentheses

- The order of evaluation in a Boolean expression:

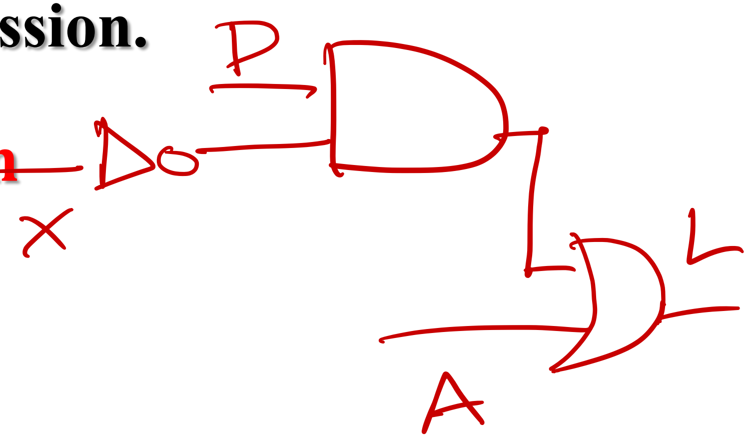
( ), NOT, AND, OR

$Z = AB + CD$

$Z = AB + \overline{C}(D+E)$

Boolean function

- **A *Boolean function* is a boolean equation consisting of a binary variable identifying the function followed by an equal sign and a boolean expression.**

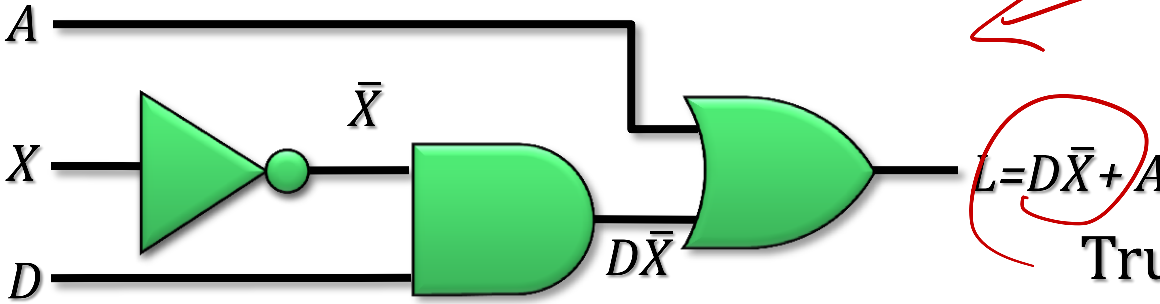


$$\underbrace{L(D,X,A) = \overbrace{D\overline{X} + A}^{\textbf{Boolean expression}}}_{\textbf{Boolean function or Boolean equation}}$$

- **A boolean function can be transformed into a circuit diagram (logic diagram) composed of logic gates and interconnected by wires.**

Boolean function

Logic Diagram



Truth Table

	D	X	A	$L = D\bar{X} + A$	
0	0	0	0	0	
1	0	0	1		
2	0	1	0		
3	0	1	1	1	4
4	1	0	0		
5	1	0	1		
6	1	1	0		
7	1	1	1		

- **A boolean function can be represented by a truth table**

$2^5 - 1$

2^n $2^3 - 1$

$2^n - 1$

Boolean function

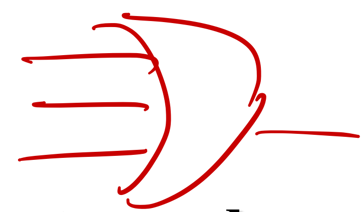
- There is only one way that a boolean function can be represented by a truth table

- The Boolean function, however, can be expressed by various boolean equations, which are not the same but equivalent. Eg. L and F have the same function.

$$L(D,X,A) = D\overline{X} + A$$ 3 x 3 = 9

$$F(D,X,A) = D\overline{X} + AD + A\overline{D}$$ 6 x 3 = 18

- The boolean equation dictates the interconnection of gates in the logic circuit diagram.



- Simpler expression reduces the number of gates and the number of inputs into the gates

Basic identities of boolean algebra

1. $X + 0 = X$	2. $X \cdot 1 = X$	Identity
3. $X + 1 = 1$	4. $X \cdot 0 = 0$	Null
5. $X + X = X$	6. $X \cdot X = X$	Idempotence
7. $X + \overline{X} = 1$	8. $X \cdot \overline{X} = 0$	Complementarity
9. $\overline{\overline{X}} = X$		Involution

Basic identities of boolean algebra

10. $Y+X = X+Y$	11. $Y \cdot X = X \cdot Y$	Commutative
12. $X + (Y + Z) = (X+Y) + Z$	13. $(XY)Z = X(YZ)$	Associative
14. $X (Y + Z) = XY + XZ$	15. $X + Y Z = (X + Y) (X + Z)$	Distributive
16. $\overline{X + Y} = \overline{X}\,\overline{Y}$	17. $\overline{XY} = \overline{X} + \overline{Y}$	De morgan's law

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$$\overline{XY} = \overline{X} + \overline{Y}$$

$$\overline{X+Y} = \overline{X}\,\overline{Y}$$

$$\overline{XYZ} = \overline{X} + \overline{Y} + \overline{Z}$$

$$\overline{X+Y+Z} = \overline{X}\,\overline{Y}\,\overline{Z}$$

$$\overline{\underbrace{XY} + Z} = \overline{W+Z}$$

$$W = XY \qquad = \overline{W}\,\overline{Z}$$

$$= (\overline{XY})\,\overline{Z}$$

$$\overline{W} = \overline{XY} = \overline{X} + \overline{Y}$$

$$= (\overline{X} + \overline{Y})\,\overline{Z}$$

Basic identities of boolean algebra

- Any expression can replace the variable X in all boolean identity.

  Eg. $X + 1 = 1$

  with $X = AB + C$

  $AB + C + 1 = 1$

- Identity 10 -14 are similar to ordinary algebra. However, Identity 15 does not hold in ordinary algebra.
- Identity 16 and 17 are De' morgan's rule and they are very important rules

X	Y	$\overline{X} + \overline{Y}$	$\overline{XY}$	$\overline{X + Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1	1	
1	0	1	1	0	
0	1	1	1	0	
1	1	0	0	0	

Basic identities of boolean algebra

- **De'morgan's law can be extended to three or more variables. The general De'morgan's theorem can be expressed as**

$$\overline{X_1 + X_2 + \cdots + X_n} = \overline{X_1}\,\overline{X_2} \cdots \overline{X_n}$$

$$\overline{X_1 X_2 \ldots X_n} = \overline{X_1} + \overline{X_2} + \cdots + \overline{X_n}$$

Duality

- The *dual* of an algebraic expression is obtained by interchanging + and · and interchanging 0's and 1's
- Any Boolean theorem that can be proven is thus also proven for its dual!
- Note that previous identities appear in dual pairs
- **Example:**

$$XY + \overline{X}Z + YZ = XY + \overline{X}Z$$

Dual $(X+Y)(\overline{X} + Z)(Y+Z) = (X+Y)(\overline{X} + Z)$

Useful theorems of Boolean algebra

$X+XY = X$	$X(X+Y) = X$	**Absorption**
$XY + X\overline{Y} = X$ $X(Y+\overline{Y}) = X(1) = X$	$(X+Y)(\overline{X}+Y) = X$	**Minimization**
$X+\overline{X}Y = X+Y$	$X(\overline{X}+Y) = XY$ $XX + XY = 0 + XY = XY$	**Simplification**
$XY + \overline{X}Z + YZ = XY + \overline{X}Z$	$(X+Y)(\overline{X}+Z)(Y+Z) = (X+Y)(\overline{X}+Z)$	**Consensus**

Proof of consensus theorem

- **Example: Prove the Consensus Theorem**

$$
\begin{aligned}
XY + \bar{X}Z + YZ &= XY + \bar{X}Z + YZ\cdot 1 && \textbf{(2)}\\
&= XY + \bar{X}Z + YZ(X + \bar{X}) && \textbf{(7)}\\
&= XY + \bar{X}Z + XYZ + \bar{X}YZ && \textbf{(14)}\\
&= XY + XYZ + \bar{X}Z + \bar{X}YZ && \textbf{(10)}\\
&= XY(1 + Z) + \bar{X}Z(1 + Y) && \textbf{(14)}\\
&= XY + \bar{X}Z && \textbf{(3)}
\end{aligned}
$$

$W = XY$

$W + WZ = W$

$= \underbrace{XY + (XY)Z}_{XY}$ $\quad \underbrace{\bar{X}Z + \bar{X}YZ}_{\bar{X}Z}$

Expression simplification

$x + xy = x$

- **A *literal* is a complemented or uncomplemented variable in a term**
- **Simplify the following expression to contain the smallest number of literals:**

$$AB + ACD + \bar{A}BD + AC\bar{D} + \bar{A}BCD$$

$$= AB + \bar{A}BCD + ACD + AC\bar{D} + \bar{A}BD$$

$A + \bar{A}D = A + D$

$$= AB + \bar{A}BCD + AC(D + \bar{D}) + \bar{A}BD$$

$= \bar{A}BD(1+C)$

$$= AB + AC + \bar{A}BD = B(A + \bar{A}D) + AC$$

$$= B(A + D) + AC$$

$ACD + AC\bar{D} = AC(D+\bar{D}) = AC$

- **Only 5 literals!**

$\boxed{\bar{A}BD} + \bar{A}BCD = \bar{A}BD$

Complement of a function

$\overline{X+Y} = \bar{X}\bar{Y}$

- The complement of a function F, $\bar{F}$, can be obtained in two ways:
  - By applying DeMorgan's theorem
  - By taking the dual of the function and complementing each literal

$\overline{XY} = \bar{X}+\bar{Y}$

Example: $F = \bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z$

$(X+\bar{Y}+Z)$

$(X+Y+\bar{Z})$

$\overline{\bar{X}Y\bar{Z} + \bar{X}\bar{Y}Z} = (\overline{\bar{X}Y\bar{Z}})(\overline{\bar{X}\bar{Y}Z})$

$\overline{\bar{X}Y\bar{Z}} = \bar{\bar{X}} + \bar{Y} + \bar{\bar{Z}} = X + \bar{Y} + Z$

$\overline{\bar{X}\bar{Y}Z} = X + Y + \bar{Z}$

$F = (X+\bar{Y}+Z)(X+Y+\bar{Z})$

$$F = X(\bar{Y}\bar{Z} + YZ)$$

$$\bar{F} = \overline{X(\underline{\bar{Y}\bar{Z}} + \underline{YZ})}$$

$$W = \bar{Y}\bar{Z} \qquad U = YZ$$

$$\bar{F} = \overline{X(W+U)}$$

DeMorgan's

$$= \bar{X} + \overline{(W+U)}$$

$$= \bar{X} + \bar{W}\bar{U}$$

$$\bar{W} = \overline{\bar{Y}\bar{Z}} = \bar{\bar{Y}} + \bar{\bar{Z}} = Y+Z$$

$$\bar{U} = \overline{YZ} = \bar{Y} + \bar{Z}$$

$$\bar{F} = \bar{X} + (Y+Z)(\bar{Y}+\bar{Z})$$

$$\bar{F} = \bar{X} + (Y+Z)(\bar{Y}+\bar{Z})$$

$$\overline{XY} = \bar{X} + \bar{Y}$$

$$\overline{X+Y} = \bar{X}\bar{Y}$$

Standard Forms

- *Standard forms* are standard ways to express Boolean functions

SOP

- Contain product terms (Minterms) and sum terms (Maxterms)

POS

- Result in more desirable expression for circuit implementation

A	B	F
0	0	1
0	1	1
1	0	1
1	1	1

$\bar{A}\bar{B}$

$\bar{A}B$

$A\bar{B}$

AB

Minterms:

- Minterms are AND terms with every variable present in either true or complement form
- Every variable combination in a truth table has a corresponding minterm

$$\boxed{F = \bar{A}\bar{B} + \bar{A}B + A\bar{B} + AB}$$

$$F = AB + \overline{B}D + C\overline{D}$$

$$G = (\overline{A}+B)(\overline{C}+D+E)$$

$$H = A + (BC + D\overline{E})$$

Minterms

$W = \Sigma(m_0, m_3, m_7)$ SOP $= \Sigma(m_0 - m_7)$

- For a function with *n* variables, there will be 2^n minterms
- The literals are listed in the same order for all minterms (usually alphabetically)

X	Y	Z	Product Term	symbols	m_o	m_1	m_2	m_3	m_4	m_5	m_6	m_7	W
0	0	0	$\bar{X}\bar{Y}\bar{Z}$	m_o	1	0	0	0	0	0	0	0	1
0	0	1	$\bar{X}\bar{Y}Z$	m_1	0	1	0	0	0	0	0	0	0
0	1	0	$\bar{X}Y\bar{Z}$	m_2	0	0	1	0	0	0	0	0	0
0	1	1	$\bar{X}YZ$	m_3	0	0	0	1	0	0	0	0	1
1	0	0	$X\bar{Y}\bar{Z}$	m_4	0	0	0	0	1	0	0	0	0
1	0	1	$X\bar{Y}Z$	m_5	0	0	0	0	0	1	0	0	0
1	1	0	$XY\bar{Z}$	m_6	0	0	0	0	0	0	1	0	0
1	1	1	XYZ	m_7	0	0	0	0	0	0	0	1	1

Minterms

$F = \Sigma(m_1, m_4) = \Sigma m(1,4)$
$= \bar{X}\bar{Y}Z + \bar{X}YZ$

X	Y	Z	Product Term	symbols	m_o	m_1	m_2	m_3	m_4	m_5	m_6	m_7	F
0	0	0	$\bar{X}\bar{Y}\bar{Z}$	m_o	1	0	0	0	0	0	0	0	0
0	0	1	$\bar{X}\bar{Y}Z$	m_1	0	1	0	0	0	0	0	0	1
0	1	0	$\bar{X}Y\bar{Z}$	m_2	0	0	1	0	0	0	0	0	0
0	1	1	$\bar{X}YZ$	m_3	0	0	0	1	0	0	0	0	1
1	0	0	$X\bar{Y}\bar{Z}$	m_4	0	0	0	0	1	0	0	0	0
1	0	1	$X\bar{Y}Z$	m_5	0	0	0	0	0	1	0	0	0
1	1	0	$XY\bar{Z}$	m_6	0	0	0	0	0	0	1	0	0
1	1	1	XYZ	m_7	0	0	0	0	0	0	0	1	0

- **A variable in a minterm *is complemented* for 0 and is *not complemented* for 1**
- **The symbol index corresponds to the binary combination of the variables**

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Sum-of-Minterms Canonical Form

- Express a function as a sum of minterms
- Logical sum (*OR*) of all minterms where the function value is 1

$\overline{F}$

$F = \Sigma(m_1, m_3, m_4, m_6)$

- Example:

	X	Y	Z	F
0	0	0	0	1
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	0
5	1	0	1	1
6	1	1	0	0
7	1	1	1	1

$F = \Sigma(m_0, m_2, m_5, m_7)$

$= \overline{X}\overline{Y}\overline{Z} + \overline{X}Y\overline{Z}$

$+ X\overline{Y}Z$

$+ XYZ$

$= \overline{X}\overline{Z}(\overline{Y}+Y) + XZ(\overline{Y}+Y)$

$= \overline{X}\overline{Z} + XZ$

Maxterms

$F = \Pi(1,5)$
$= (X+Y+\bar{Z})(\bar{X}+Y+\bar{Z})$ POS

- Similarly, maxterms are OR terms with every variable present
- In maxterms, a variable *is complemented* for a 1 and *not complemented* for a 0

X	Y	Z	Sum Term	symbols	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7	F
0	0	0	$\boxed{X+Y+Z}$	M_0	0	1	1	1	1	1	1	1	1
0	0	1	$X+Y+\bar{Z}$	M_1	1	0	1	1	1	1	1	1	0
0	1	0	$X+\bar{Y}+Z$	M_2	1	1	0	1	1	1	1	1	1
0	1	1	$X+\bar{Y}+\bar{Z}$	M_3	1	1	1	0	1	1	1	1	1
1	0	0	$\bar{X}+Y+Z$	M_4	1	1	1	1	0	1	1	1	1
1	0	1	$\bar{X}+Y+\bar{Z}$	M_5	1	1	1	1	1	0	1	1	0
1	1	0	$\bar{X}+\bar{Y}+Z$	M_6	1	1	1	1	1	1	0	1	1
1	1	1	$\bar{X}+\bar{Y}+\bar{Z}$	M_7	1	1	1	1	1	1	1	0	1

$\overline{\bar{X}\bar{Y}\bar{Z}} = X+Y+Z$

Maxterms

- The symbol index corresponds to the binary combination of the variables
- For a function with *n* variables, there will be 2^n maxterms

X	Y	Z	Sum Term	symbols	M_o	M_1	M_2	M_3	M_4	M_5	M_6	M_7
0	0	0	$X + Y + Z$	M_o	0	1	1	1	1	1	1	1
0	0	1	$X + Y + \bar{Z}$	M_1	1	0	1	1	1	1	1	1
0	1	0	$X + \bar{Y} + Z$	M_2	1	1	0	1	1	1	1	1
0	1	1	$X + \bar{Y} + \bar{Z}$	M_3	1	1	1	0	1	1	1	1
1	0	0	$\bar{X} + Y + Z$	M_4	1	1	1	1	0	1	1	1
1	0	1	$\bar{X} + Y + \bar{Z}$	M_5	1	1	1	1	1	0	1	1
1	1	0	$\bar{X} + \bar{Y} + Z$	M_6	1	1	1	1	1	1	0	1
1	1	1	$\bar{X} + \bar{Y} + \bar{Z}$	M_7	1	1	1	1	1	1	1	0

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Maxterms

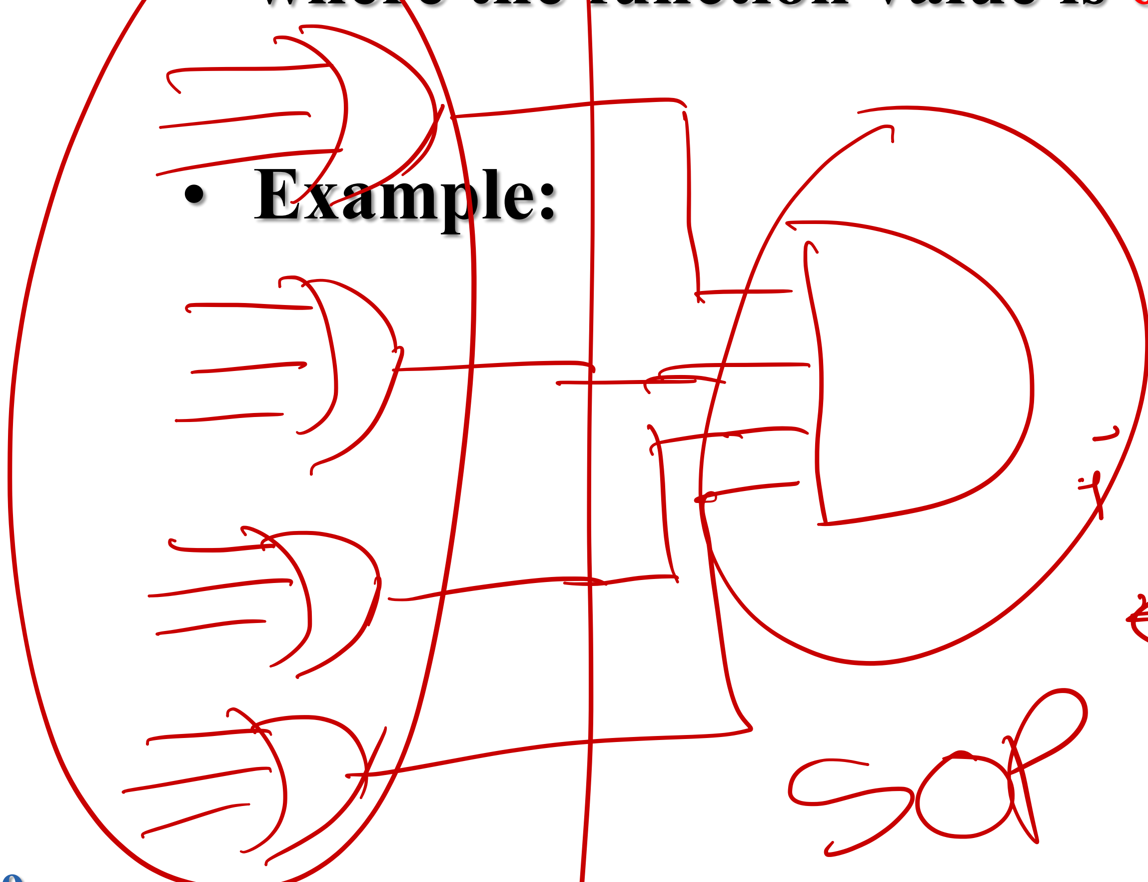
- The literals are listed in the same order for all minterms (usually alphabetically)
- Every variables combination in a truth table has a corresponding minterm

X	Y	Z	Sum Term	symbols	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7
0	0	0	$X+Y+Z$	M_0	0	1	1	1	1	1	1	1
0	0	1	$X+Y+\bar{Z}$	M_1	1	0	1	1	1	1	1	1
0	1	0	$X+\bar{Y}+Z$	M_2	1	1	0	1	1	1	1	1
0	1	1	$X+\bar{Y}+\bar{Z}$	M_3	1	1	1	0	1	1	1	1
1	0	0	$\bar{X}+Y+Z$	M_4	1	1	1	1	0	1	1	1
1	0	1	$\bar{X}+Y+\bar{Z}$	M_5	1	1	1	1	1	0	1	1
1	1	0	$\bar{X}+\bar{Y}+Z$	M_6	1	1	1	1	1	1	0	1
1	1	1	$\bar{X}+\bar{Y}+\bar{Z}$	M_7	1	1	1	1	1	1	1	0

Product-of-Maxterms Canonical Forms

P POS

- Express a function as a product of maxterms
- Logical product (*AND*) of all maxterms where the function value is 0
- Example:



SOP

X	Y	Z	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

M5, M7)

Minterms and Maxterms Relationship

- **Review: DeMorgan's Theorem**

$m_0 = \bar{X}\bar{Y}$ $\quad\quad$ $M_0 = X + Y$

$\bar{m}_0 = \overline{(\bar{X}\bar{Y})} = X + Y = M_0$

$\bar{m}_i = M_i$

X	Y	m	M
0	0	m_0	M_0
0	1	m_1	M_1
1	0	m_2	M_2
1	1	m_3	M_3

$M_i = \bar{m}_i$

- **Two-variable example:**

$F = \Sigma m(1,5,7)$

$\longleftrightarrow F = \Pi M(0,2,3,4,6)$

$\bar{F} = \Sigma m(0,2,3,4,6)$

$\bar{F} = \Pi M(1,5,7)$

	X	Y	Z	F
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	0
5	1	0	1	1
6	1	1	0	0
7	1	1	1	1

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the expression

- Alternatively, the complement of a function expressed as a sum of minterms is simply the product of maxterms with the same indices

Function Complement Example

- Find complement expressions for the function:

$$G(X, Y, Z) = \sum m(1, 3, 5, 7)$$

X	Y	Z	G
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

- As a sum of minterms:

$\overline{G} = \sum m(0, 2, 4, 6)$

- As a product of maxterms:

$\overline{G} = \prod M(1, 3, 5, 7)$

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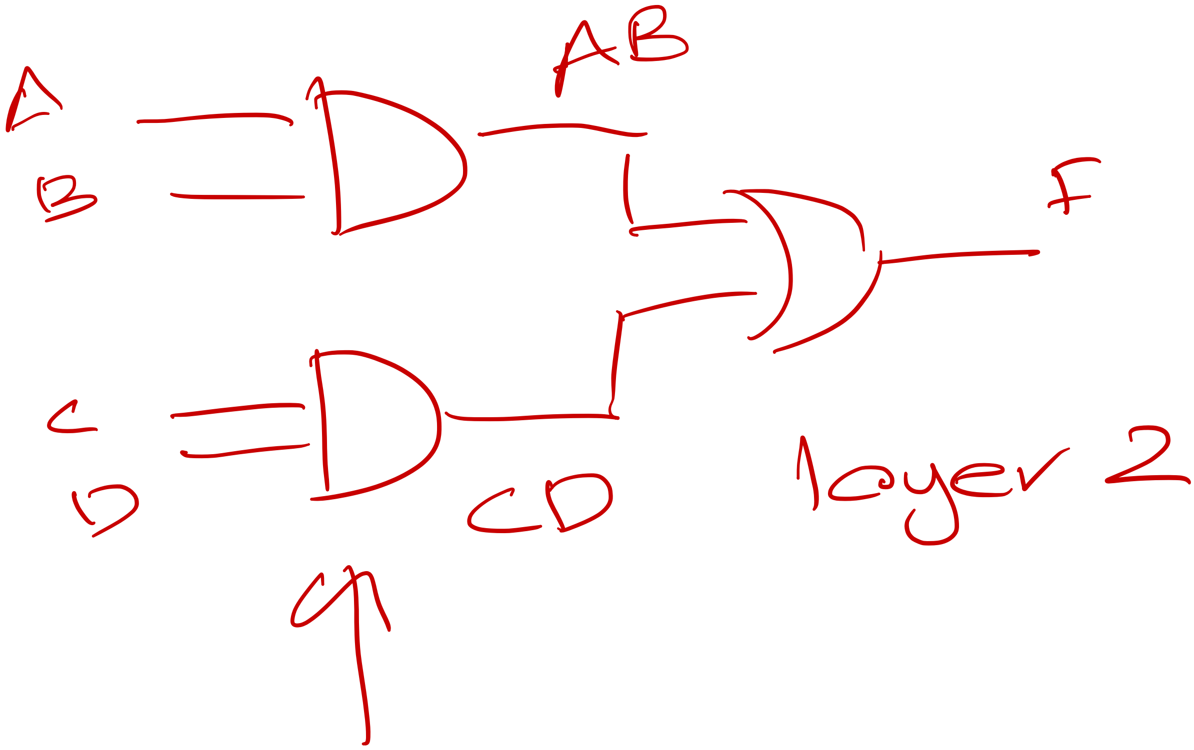
Sum-of-Products

- ***Sum of Products*** **form: equations are written as an OR of AND terms**

- **Similar to sum of minterms but does not need to contain all variables in every term**

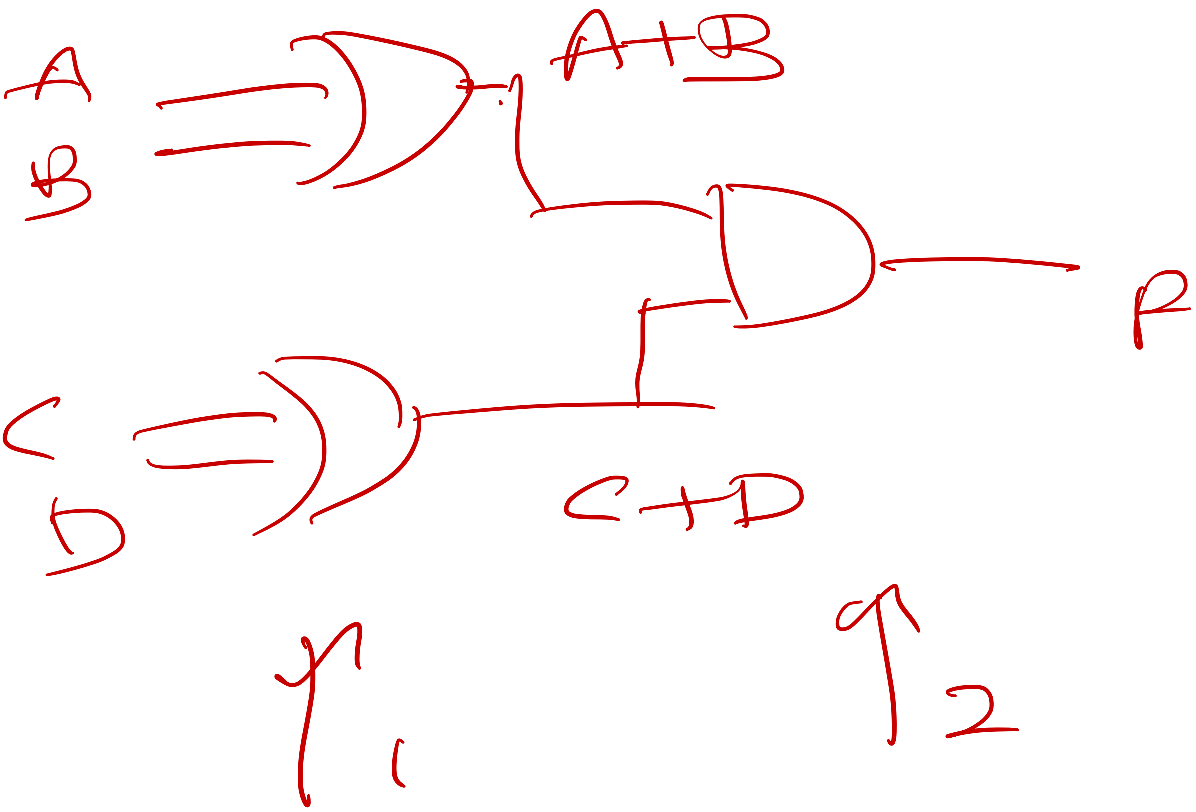
- **Can be directly implemented as a *two-level circuit***

$F = AB + CD$



layer 1

$F = (A+B)(C+D)$



Sum-of-Products Implementation

$F = 1 + 0 + 0 = 1$

- **Example - the function:**

$$F = \overline{Y} + \overline{X}Y\overline{Z} + XY$$

2

1

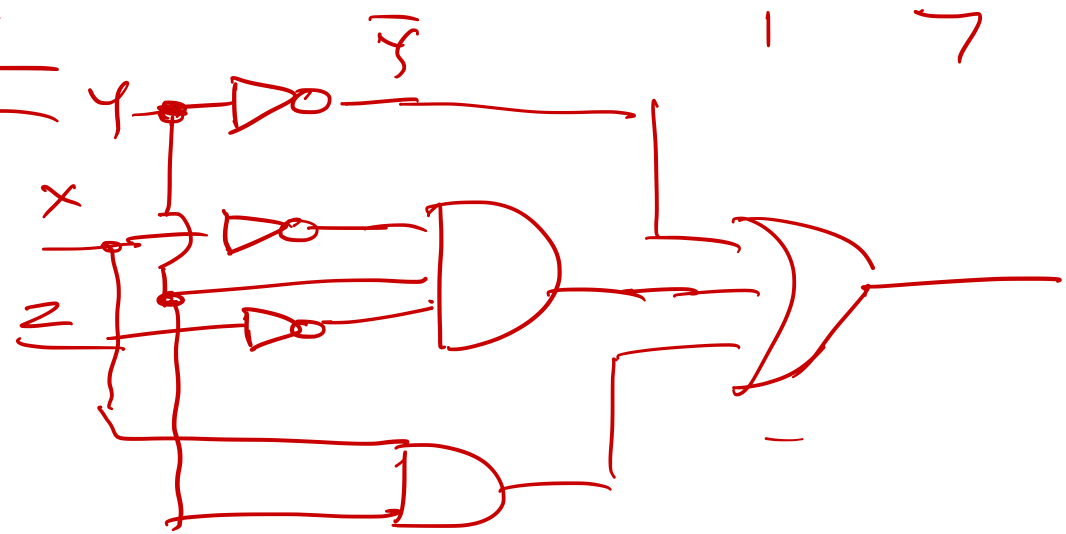
X	Y	Z	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$F = \Sigma(m_0, m_1, m_2, m_4, m_5, m_6, m_7)$

$= \Sigma m(0, 1, 2, 4, 5, 6, 7)$

Draw circuit !

1 7

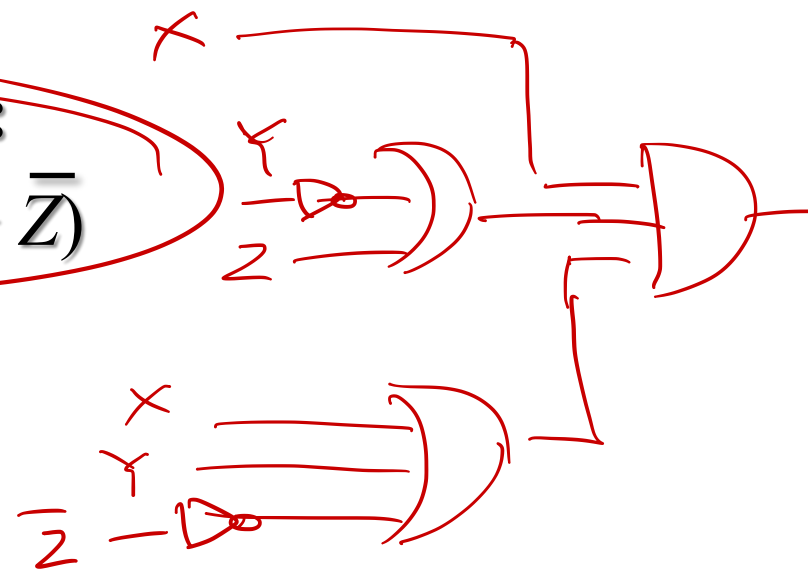


Product-of-Sums

- *Product of Sums* form: equations are written as an AND of OR terms
- Can also be implemented as a two-level circuit
- Example – the function:

$$F = X(\overline{Y} + Z)(X + Y + \overline{Z})$$

is expressed as a POS



Cost Criteria

- Need a method to measure the complexity of a logic circuit

$F = AB + \overline{BC} + DE$ 6 $3+1+2=6$

- Define the *Gate-Input Cost* as the number of inputs to the gates in the implementation

- Can be calculated directly from the equation corresponding to the circuit

- Only count distinct terms and complemented literals

Gate input cost

= sum of

1) all literal appearances

2) number of terms

3) number of complemented literals

$G = ABCD + \bar{A}\bar{B}\bar{C}\bar{D}$ SOP

$H = (\bar{A}+B)(\bar{B}+C)(\bar{C}+D)(\bar{D}+A)$ POS

cost of $G = 8 + 2 + 4$

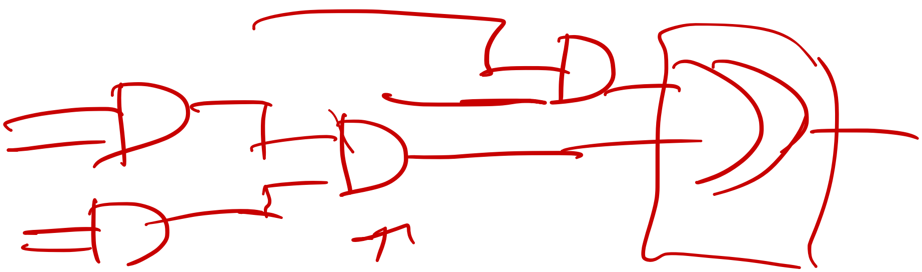
$= 14$

cost of $H = 8 + 4 + 4$

$= 16$

$G = A + (B+C)(\bar{C}D)$ 3

cost $= 5 + 3 + 0 = 8$



Gate-Input Cost

- **Example:**

$$F = AB + \bar{C}(D + E)$$

cost = 5+2+1 = 8

5 — 5

A

1 — 1

B

5 — 10

- **The corresponding logic diagram:**