MATH2089
Numerical Methods
Lecture 9

Euler Method

Huen Method

Modified Euler Method

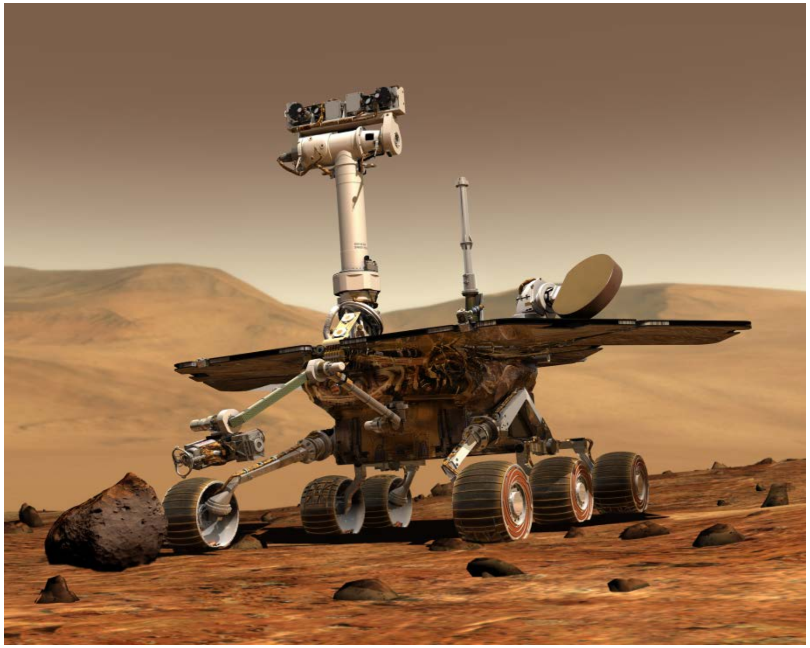
Ordinary Differential Equations

- Importance of ordinary differential equations ➲ most scientific laws are more readily expressed in terms of rates of change
- Solution of a differential equation is sought where the function satisfies the differential equation and also satisfies certain <u>initial conditions</u> on the function
  - For example

$$\frac{dy}{dx} \text{ or } y' = f(x,y)$$

  - Several methods of solving the first-order equations will be explored in this and next couple of lectures

Ordinary Differential Equations - applications



On August 5th 2012, The Mars Science Laboratory entry vehicle successfully entered Mars' atmosphere and landed the Curiosity rover on its surface.

Linear accelerations and angular rates as well as various observed quantities such as radar and air data were modelled with the nonlinear system of differential equations.

Some Engineering Applications

- An equation that defines a relationship between an unknown function and one or more of its derivatives is referred to as a ***differential equation***
- Refer to Newton's second law

$$\sum F = ma = m\frac{dv}{dt}$$

- Balance of forces: $F_D = mg$ and assuming that the opposite force is proportional to velocity $F_v = cv$ where *c* is a drag coefficient

$$\sum F = F_D - F_v = mg - cv$$

Some Engineering Applications (continue)

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

- The above equation is derived based on Newton's second law to compute the velocity *v* of a falling parachutist as a function of time *t*
  - The quantity being differentiated, *v*, is called the dependent variable
  - The quantity with respect to which *v* is differentiated, *t*, is called the independent variable

Some Engineering Applications (continue)

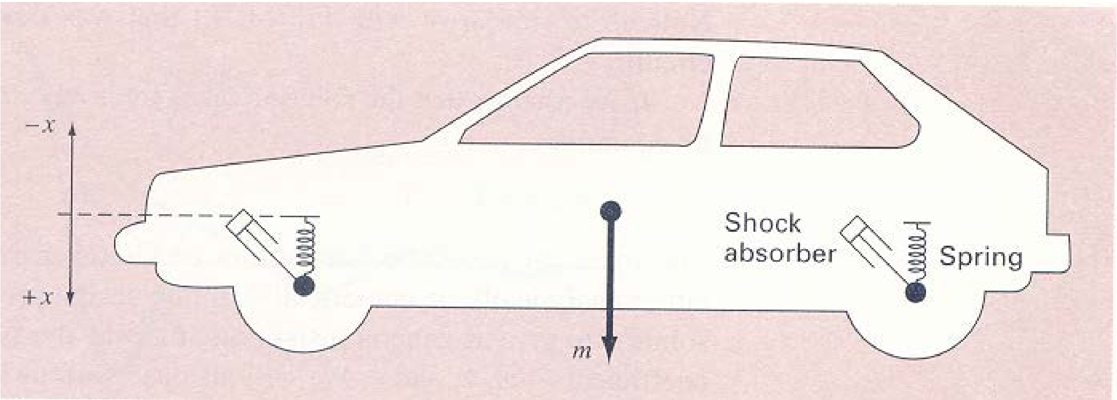
- Classification of differential equations
  - Ordinary Differential Equation (ODE) – one independent variable
  - Partial Differential Equation (PDE) – two or more independent variables
- Newton’s second law is a ***first-order equation*** because the highest derivative is a first order derivative
- For a ***second-order equation*** – Hooke’s law

Some Engineering Applications (continue)

- According to Hooke's law, resistance of the spring is proportional to the spring constant and the distance from the equilibrium position $F_s = -kx$
- Damping force of the shock absorber is given by $F_d = -c\dfrac{dx}{dt}$

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} \quad \text{or} \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

acceleration



Some Engineering Applications (continue)

- An *n*th-order equation would include an *n*th derivative
- For example, a general linear differential equation of *n*th-order is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0 y = g(x)$$

- Homogeneous equation: $g(x) = 0$
- ***Higher order equations*** can be reduced to a system of first-order equations. By defining a new variable *y*, where

$$y = \frac{dx}{dt}$$

Some Engineering Applications (continue)

- it can be differentiated to yield

$$\frac{dy}{dt} = \frac{d^2x}{dt^2}$$

- Recalling $$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

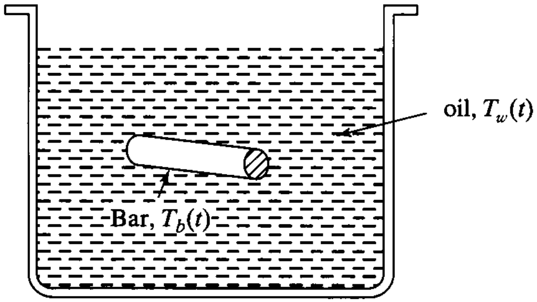
$$y = \frac{dx}{dt}$$

- it can be alternatively expressed as $$m\frac{dy}{dt} + cy + kx = 0$$

A pair of first-order equations that are equivalent to the original second-order equation

Initial Value Problems

- ***Initial value problems (IVP)***: values of the dependent variables or their derivatives are known at the initial value of the independent variable
- Example
  - The falling parachutist with the initial condition $v(t = 0) = 0$
  - The quenching operation used to harden a metal component with initial conditions $T_b(t = 0) = 1200°F$ at $T_w(t = 0) = 65°F$



Consideration of thermal equilibrium of the bar and oil leads to

$$\frac{m_b c_b}{hA}\frac{dT_b}{dt} + T_b = T_w \qquad \frac{m_w c_w}{hA}\frac{dT_w}{dt} + T_w = T_b$$

Initial Value Problems

Example of deriving equation for IVP and finding solution by separation of variables.

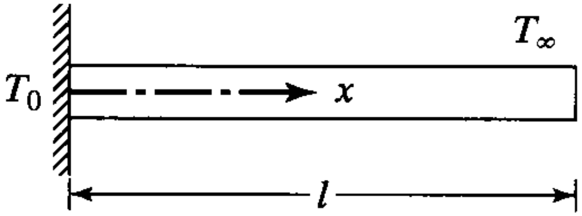
Annual ticket sale for a new professional soccer league are projected to grow at a rate proportional to the difference between sales at time t and an upper bound of $300 million. Assume that annual ticket sales are initially $0 and must be $40 million after 3 years. Based on these assumptions, how long it will take for annual ticket to reach $220 million?

Boundary Value Problems

- ***Boundary Value Problems (BVP)***: value of the dependent variables or their derivatives are known at more than one point of the independent variable (at the boundaries)
- Example
  - A cooling fin extends from a hot surface

$$kA\frac{d^2T}{dx^2} - hP\left(T - T_\infty\right) = 0$$

  - Boundary conditions: $T(x = 0) = 0$ and $-k\frac{dT(x=l)}{dx} = h\left(T(x=l) - T_\infty\right)$



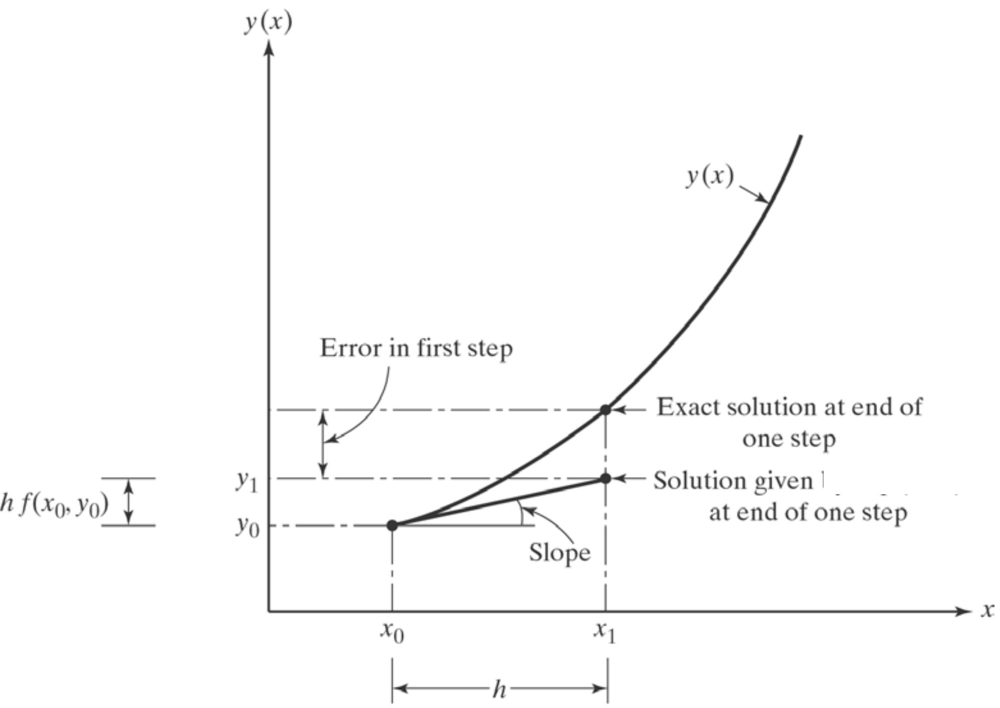
Euler's Method

- A simple method which provides most of the basic features of numerical methods used for the solution of ordinary differential equations
- Recalling Taylor's series expansion

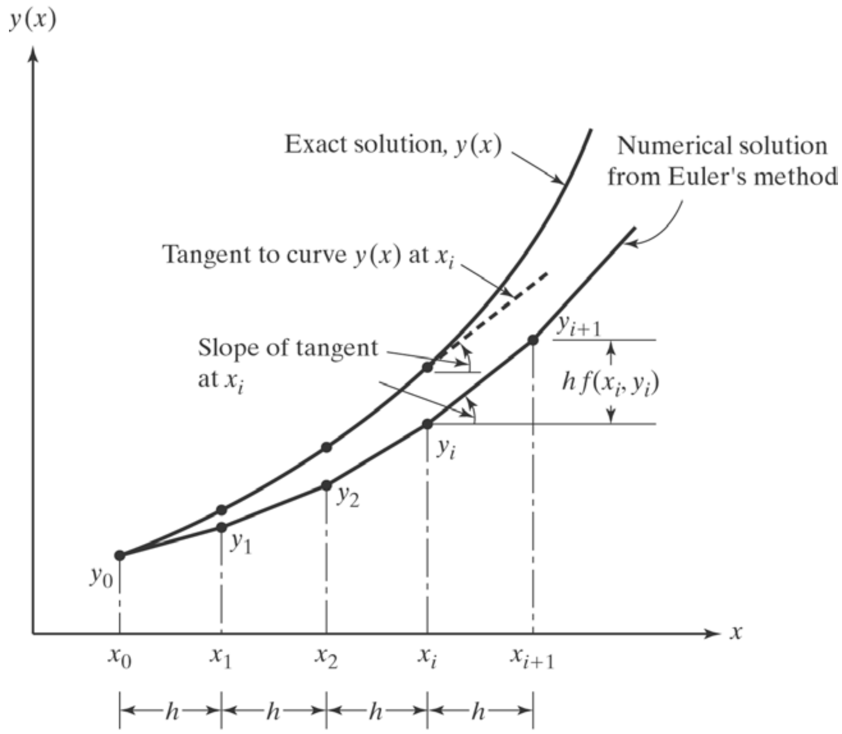
$$y_{i+1} = y_i + h y_i' + \underbrace{\frac{h^2}{2} y_i''(\xi)}_{\text{error or remainder term}} \; ; \; x_i = x_0 + ih, \; x_i < \xi < x_{i+1}$$

- Let $y_i' = f(x_i, y_i)$ $\qquad y_{i+1} = y_i + h\, f(x_i, y_i)$
- Note: if *h* is small, the error term can be neglected
- Sort of a projection method

Euler's Method



Graphical interpretation



Accumulation of error

Example

- Find the solution of the initial value problem $y' = y + 2x - 1$ with $y(x=0) = 1$ over the interval $0 \leq x \leq 1$ using h = 0.1, 0.05 and 0.01

- <u>Solution</u>

Start with $y_0 = y(0) = 1$, calculate y_1, y_2, …, y_9, y_{10}

Example (continue)

- h = 0.1, x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, …, x_{10} = 1, with an initial y_0 = 1

$$y_1 = y(0.1) = y_0 + h\, f(x_0, y_0) = y_0 + h(y_0 + 2x_0 - 1) = 1 + 0.1(1 + 2(0) - 1) = 1$$

$$y_2 = y(0.2) = y_1 + hf(x_1, y_1) = y_1 + h(y_1 + 2x_1 - 1) =$$
$$1 + 0.1(1 + 2(0.1) - 1) = 1 + 0.1(0.2) = 1.02$$

$$y_3 = y(0.3) = y_2 + hf(x_2, y_2) = y_2 + h(y_2 + 2x_2 - 1) =$$
$$1.02 + 0.1(1.02 + 2(0.2) - 1) = 1.02 + 0.1(0.42) = 1.062$$

- Subsequently calculate y_4, y_5, …, y_9, y_{10}

Example (continue)

- Compare with exact solution $y(x) = -1 - 2x + 2e^{x}$

x	y	exact	% error
0	1	1	0
0.1	1	1.010342	1.023598
0.2	1.02	1.042806	2.186939
0.3	1.062	1.099718	3.429755
0.4	1.1282	1.183649	4.684613
0.5	1.22102	1.297443	5.890245
0.6	1.34312	1.444238	7.001313
0.7	1.49743	1.627505	7.99206
0.8	1.68718	1.851082	8.85451
0.9	1.9159	2.119206	9.593726
1	2.18748	2.436564	10.22254

Example (continue)

- h = 0.05, x_0 = 0, x_1 = 0.05, x_2 = 0.1, x_3 = 0.15, …, x_{20} = 1, with an initial y_0 = 1

$$y_1 = y(0.05) = y_0 + hf(x_0, y_0) = y_0 + h(y_0 + 2x_0 - 1) =$$
$$1 + 0.05(1 + 2(0) - 1) = 1 + 0.05(0) = 1$$

$$y_2 = y(0.1) = y_1 + hf(x_1, y_1) = y_1 + h(y_1 + 2x_1 - 1) =$$
$$1 + 0.05(1 + 2(0.05) - 1) = 1 + 0.05(0.1) = 1.005$$

$$y_3 = y(0.15) = y_2 + hf(x_2, y_2) = y_2 + h(y_2 + 2x_2 - 1) =$$
$$1.005 + 0.05(1.005 + 2(0.1) - 1) = 1.005 + 0.05(0.205) = 1.0153$$

- Subsequently calculate y_4, y_5, …, y_{19}, y_{20}

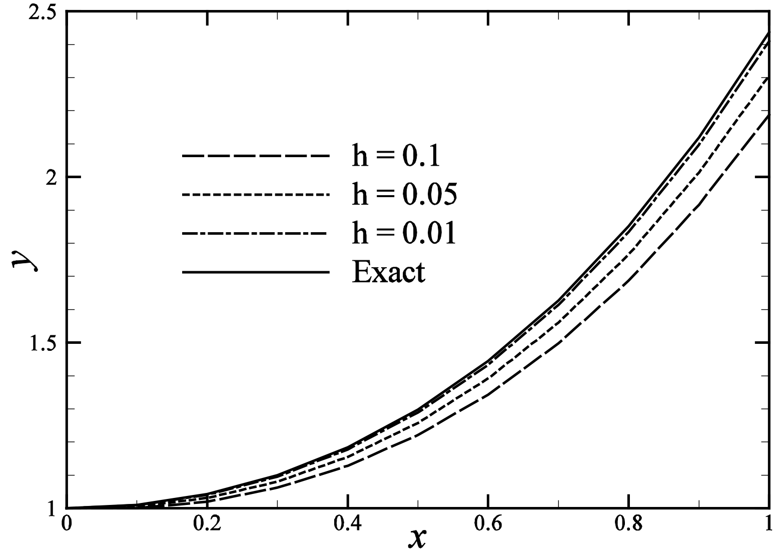
Example (continue)

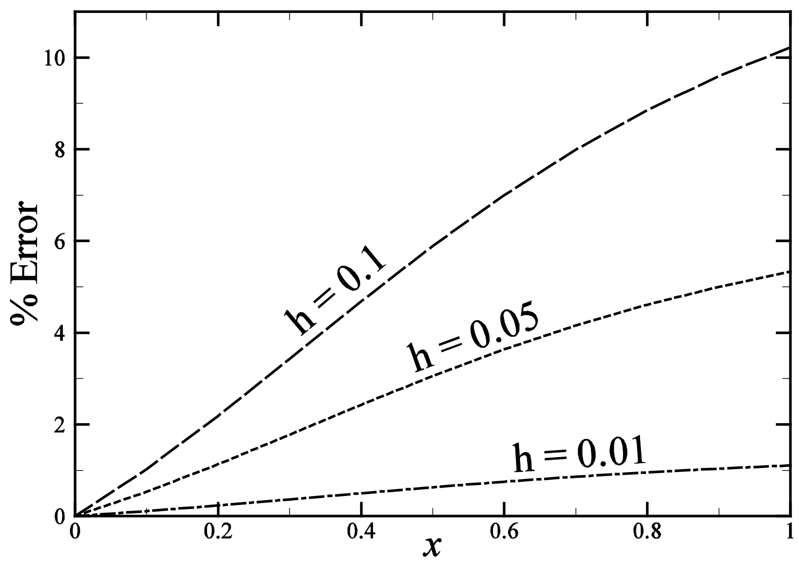
- Compare with exact solution $y(x) = -1 - 2x + 2e^{x}$

x	y	exact	% error
0	1	1	0
0.05	1	1.002542	0.253575
0.1	1.005	1.010342	0.528716
0.15	1.01525	1.023668	0.822384
0.2	1.03101	1.042806	1.130893
0.25	1.05256	1.068051	1.450091
0.3	1.08019	1.099718	1.775577
0.35	1.1142	1.138135	2.102936
0.4	1.15491	1.183649	2.427958
0.45	1.20266	1.236624	2.746828
0.5	1.25779	1.297443	3.056265
0.55	1.32068	1.366506	3.353613
0.6	1.39171	1.444238	3.636863
0.65	1.4713	1.531082	3.90465
0.7	1.55986	1.627505	4.15619
0.75	1.65786	1.734	4.391215
0.8	1.76575	1.851082	4.609882
0.85	1.88404	1.979294	4.81268
0.9	2.01324	2.119206	5.000351
0.95	2.1539	2.271419	5.17381
1	2.3066	2.436564	5.33408

Example (continue)

- Compare with solution of h = 0.01



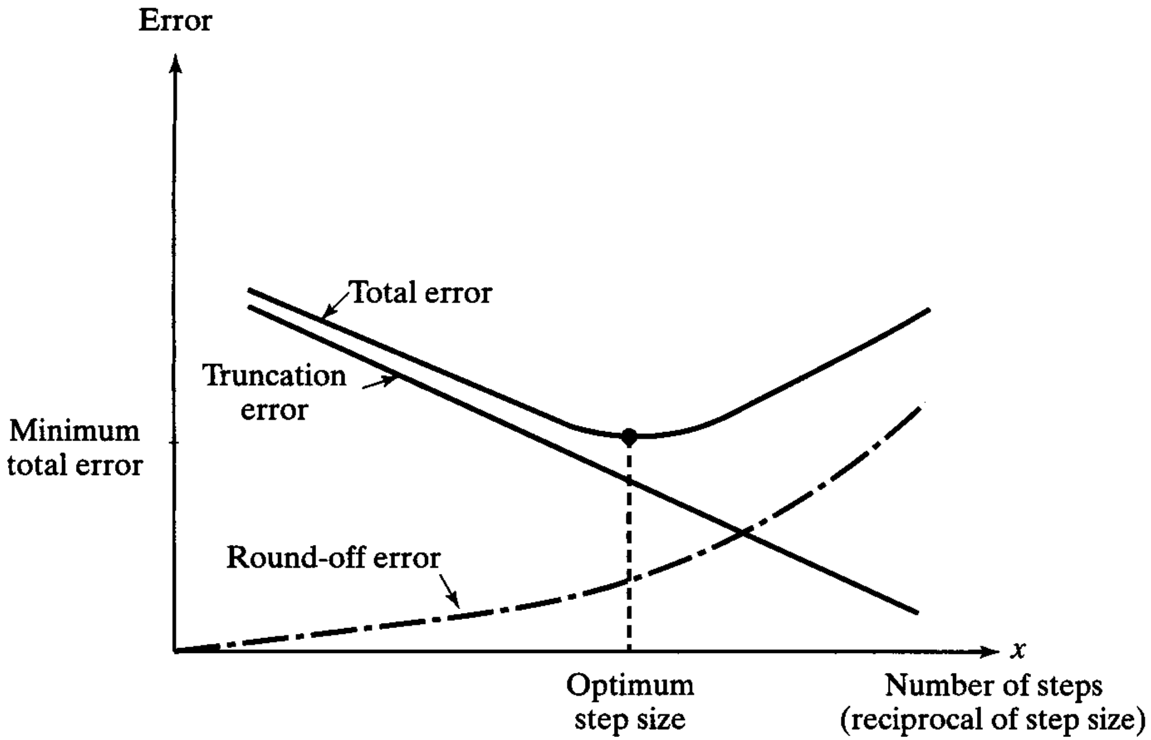


Error Analysis of Euler's Method

- Truncation error per one step (local truncation error) ~ $O(h^2)$
- Sum of local error is the global or total truncation error ~ $O(h)$
- Euler's method is called ***a first-order method***
- ***Truncation error*** is caused by the approximate procedure used in computing the values of y, i.e. the truncation of higher order derivative terms in Taylor's series expansion
- ***Round-off error*** is caused by the limited number of significant digits used in the computation by a computer

Error Analysis of Euler's Method (continue)

- Truncation error can be reduced by using a small step size *h*, but the round-off error increases as the number of steps increases



Higher Order Taylor's Series

- Improvement to first order Euler's method
- Retain more terms in Taylor's series expansion up to *n*th order derivative

$$y(x_{i+1}) = y(x_i) + h\,y'(x_i) + \frac{h^2}{2!}y''(x_i) + \cdots + \frac{h^n}{n!}y^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi), \quad x_i < \xi < x_{i+1}$$

- Using chain rule of differentiation

$$\begin{aligned}
&y' = f(x,y) \\
&y''(x) = f'(x,y) = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f \\
&y'''(x) = f''(x,y) = \frac{\partial f'}{\partial x} + \frac{\partial f'}{\partial y}y' = \frac{\partial f'}{\partial x} + \frac{\partial f'}{\partial y}f \\
&\vdots \\
&y^{(n)}(x) = f^{(n-1)}(x,y)
\end{aligned}$$

Example

- Find the solution of the initial value problem $y' = y + 2x - 1$ with $y(x=0) = 1$ over the interval $0 \le x \le 1$ using Taylor's series methods of order 2, 3 and 4 using $h = 0.1$

- <u>Solution</u>

Start with $y_0 = y(0) = 1$, calculate $y_1, y_2, \ldots, y_9, y_{10}$

Example (continue)

$$f = y + 2x - 1$$

$$f' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f = 2 + (1)(y + 2x - 1) = y + 2x + 1$$

- Second-order method:

$$y_{i+1} = y_i + h f(y_i, x_i) + \frac{h^2}{2!} f'(y_i, x_i) = y_i + h(y_i + 2x_i - 1) + \frac{h^2}{2}(y_i + 2x_i + 1)$$

Example (continue)

$$f = y + 2x - 1$$

$$f' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f = 2 + (1)(y + 2x - 1) = y + 2x + 1$$

$$f'' = \frac{\partial f'}{\partial x} + \frac{\partial f'}{\partial y} f = 2 + (1)(y + 2x - 1) = y + 2x + 1$$

- Third-order method:

$$\begin{aligned} y_{i+1} &= y_i + h f(y_i, x_i) + \frac{h^2}{2!} f'(y_i, x_i) + \frac{h^3}{3!} f''(y_i, x_i) \\ &= y_i + h(y_i + 2x_i - 1) + \frac{h^2}{2}(y_i + 2x_i + 1) + \frac{h^3}{6}(y_i + 2x_i + 1) \end{aligned}$$

Example (continue)

$$f = y + 2x - 1$$

$$f' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f = 2 + (1)(y + 2x - 1) = y + 2x + 1$$

$$f'' = \frac{\partial f'}{\partial x} + \frac{\partial f'}{\partial y} f = 2 + (1)(y + 2x - 1) = y + 2x + 1$$

$$f''' = \frac{\partial f''}{\partial x} + \frac{\partial f''}{\partial y} f = 2 + (1)(y + 2x - 1) = y + 2x + 1$$

- Fourth-order method:

$$y_{i+1} = y_i + h f(y_i, x_i) + \frac{h^2}{2!} f'(y_i, x_i) + \frac{h^3}{3!} f''(y_i, x_i) + \frac{h^4}{4!} f'''(y_i, x_i)$$
$$= y_i + h(y_i + 2x_i - 1) + \frac{h^2}{2}(y_i + 2x_i + 1) + \frac{h^4}{24}(y_i + 2x_i + 1)$$

Example (continue)

- $h = 0.1$, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, …, $x_{10} = 1$, with an initial $y_0 = 1$
- For example, the second-order method yields

$$y_1 = y_0 + h(y_0 + 2x_0 - 1) + \frac{h^2}{2}(y_0 + 2x_0 + 1)$$

$$= 1 + 0.1(1 + 2(0) - 1) + \frac{(0.1)^2}{2}(1 + 2(0) + 1) = 1.01$$

$$y_2 = y_1 + h(y_1 + 2x_1 - 1) + \frac{h^2}{2}(y_1 + 2x_1 + 1)$$

$$= 1.01 + 0.1(1.01 + 2(0.1) - 1) + \frac{(0.1)^2}{2}(1.01 + 2(0.1) + 1) = 1.04205$$

Example (continue)

- Subsequently calculate $y_4, y_5, \ldots, y_9, y_{10}$

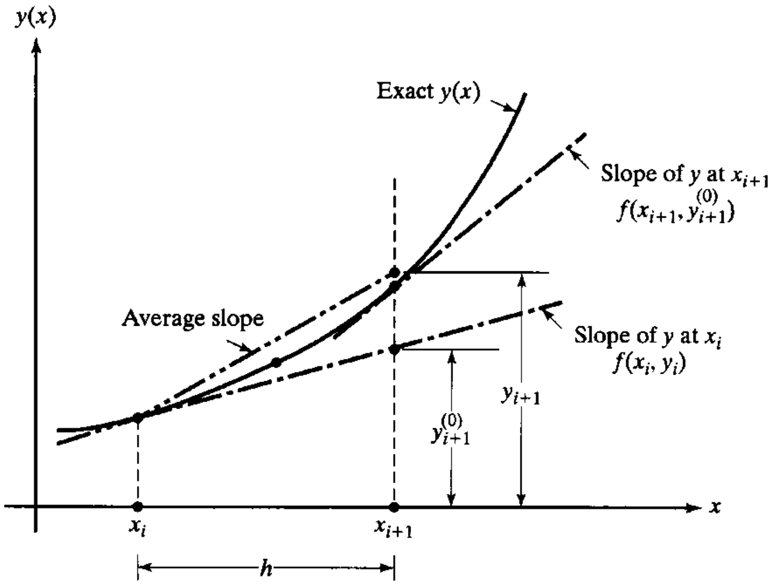
x	Order 1	%Error	Order 2	%Error	Order 3	%Error	Order4	%Error
0	1	0	1	0	1	0	1	0
0.1	1	1.023598	1.01	0.033834	1.010333	0.000842	1.010342	1.68E-05
0.2	1.02	2.186939	1.04205	0.07245	1.042787	0.001802	1.042805	3.59E-05
0.3	1.062	3.429755	1.098465	0.113881	1.099686	0.002833	1.099717	5.65E-05
0.4	1.1282	4.684613	1.181804	0.155899	1.183603	0.003879	1.183648	7.73E-05
0.5	1.22102	5.890245	1.294894	0.196464	1.297379	0.004888	1.297441	9.74E-05
0.6	1.343122	7.001313	1.440857	0.234051	1.444153	0.005824	1.444236	0.000116
0.7	1.497434	7.99206	1.623147	0.267774	1.627397	0.006664	1.627503	0.000133
0.8	1.687178	8.85451	1.845578	0.29734	1.850945	0.0074	1.851079	0.000148
0.9	1.915895	9.593726	2.112364	0.32289	2.119036	0.008036	2.119203	0.00016
1	2.187485	10.22254	2.428162	0.344828	2.436355	0.008583	2.436559	0.000171

Example (continue)

- Note: higher-order Taylor's series approaches have a smaller truncation error, but they require the computation and evaluation of the derivatives $f(x, y)$. This is a very complicated and time-consuming procedure

Heun's Method

- Another Improvement to first order Euler's method
- Regarded as a predictor-corrector method
- In essence, slope is computed at two point (y_i and $y(0)_{i+1}$) and their average value is used to achieve an improvement



Heun's Method (continue)

- Comprise of the following steps
  - Calculate $f(x_i, y_i)$ from the known initial condition $y(x_i) = y_i$
  - Find $y_{i+1}^{(0)} = y_i + h f(x_i, y_i)$ (Predictor step)
  - Calculate $f(x_{i+1}, y_{i+1}^{(0)})$
  - Determine the value of *y* at x_{i+1} (Corrector step)

$$y_{i+1} = y_i + h\left(\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(0)})}{2}\right)$$

- Further improvements by modifying iteratively the corrector equation which can be stopped when $\left|\frac{y_{i+1}^{(k+1)} - y_{i+1}^{(k)}}{y_{i+1}^{(k)}}\right| \le \varepsilon$

$$y_{i+1}^{k+1} = y_i + h\left(\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k)})}{2}\right); k = 0, 1, 2, \ldots$$

Error Analysis of Heun's Method

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y''(x_i) + \frac{h^3}{3!} y'''(\xi), \;\; x_i < \xi < x_{i+1}$$

- Using a forward differentiation

$$y''(x_i) = \frac{y'(x_{i+1}) - y'(x_i)}{h} \qquad\qquad y_{i+1} = y_i + h\left(\frac{y_i' + y_{i+1}'}{2}\right) + O(h^3)$$

- Truncation error per one step (local truncation error) ~ $O(h^3)$
- Sum of local error is the global or total truncation error ~ $O(h^2)$
- Heun's method is called ***a second-order method***

Example

- Find the solution of the initial value problem

  $$y' = -5y + e^{-2x} \; ; y(0) = 1.0$$

  in the interval $0 \leq x \leq 1$ using Heun's method with $h = 0.1$

- Solution

  Divide the interval $0 \leq x \leq 1$ into 10 parts for $h = 0.1$

Example (continue)

- h = 0.1, x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, …, x_{10} = 1, with an initial y_0 = 1
- Evaluation of y_1

$$\text{(i)}\ f(x_0, y_0) = -5y_0 + e^{-2x_0} \quad \Rightarrow \quad f(0,1) = -5(1) + e^{-2(0)} = -4$$

$$\text{(ii)}\ y_1^{(0)} = y_0 + h\, f(x_0, y_0) \quad \Rightarrow \quad \underline{y_1^{(0)} = y^{(0)}(0.1) = 1 + 0.1(-4) = 0.6}$$

$$\text{(iii)}\ f(x_1, y_1^{(0)}) = -5y_1^{(0)} + e^{-2x_1} \quad \Rightarrow \quad f(0.1, 0.6) = -5(0.6) + e^{-2(0.1)} = -2.1813$$

$$\text{(iv)}\ y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] \Rightarrow \underline{\underline{y_1 = y(0.1) = 1 + \frac{0.1}{2}[-4 - 2.1813] = 0.69094}}$$

Step (ii) is the predictor while step (iv) is the corrector

Example (continue)

- Evaluation of y_2

$$\text{(i) } f(x_1,y_1) = -5y_1 + e^{-2x_1} \quad \Rightarrow \quad f(0.1, 0.69094) = -5(0.69094) + e^{-2(0.1)} = -2.636$$

$$\text{(ii) } y_2^{(0)} = y_1 + h\, f(x_1,y_1) \quad \Rightarrow \quad \underline{y_1^{(0)} = y^{(0)}(0.2) = 0.69094 + 0.1(-2.636) = 0.42734}$$

$$\text{(iii) } f(x_2, y_2^{(0)}) = -5y_2^{(0)} + e^{-2x_2} \quad \Rightarrow \quad f(0.2, 0.42734) = -5(0.42734) + e^{-2(0.2)} = -1.4664$$

$$\text{(iv) } y_2 = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)}) \right]$$

$$\Rightarrow \underline{\underline{y_2 = y(0.2) = 0.69094 + \frac{0.1}{2}\left[-2.636 - 1.4664\right] = 0.48582}}$$

Step (ii) is the predictor while step (iv) is the corrector

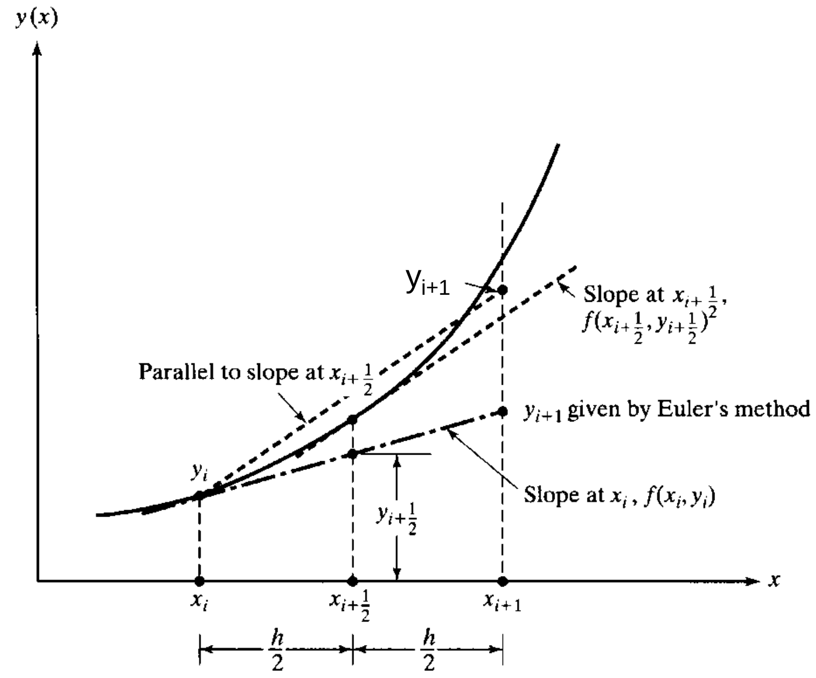
Example (continue)

- Subsequently calculate $y_4, y_5, \ldots, y_9, y_{10}$

x	y	f(x,y)	$y^{(0)}$	$f(x,y^{(0)})$	exact	% error
0	**1**	-4	0.6	-2.1813	1	0
0.1	**0.69094**	-2.6360	0.42734	-1.4664	0.67726	-2.0188
0.2	**0.48582**	-1.7588	0.30994	-1.0008974	0.46869	-3.6541
0.3	**0.34784**	-1.1904	0.22880	-0.6946664	0.33169	-4.8675
0.4	**0.25358**	-0.81859	0.17172	-0.4907454	0.24	-5.6601
0.5	**0.18812**	-0.57271	0.13085	-0.3530387	0.17735	-6.0713
0.6	**0.14183**	-0.40796	0.10103	-0.2585751	0.13359	-6.1686
0.7	**0.10850**	-0.29592	0.078911	-0.1926606	0.10233	-6.0323
0.8	**0.084074**	-0.21848	0.062227	-0.1458354	0.079509	-5.7416
0.9	**0.065859**	-0.164	0.049459	-0.1119613	0.062506	-5.3647
1	**0.052061**	-0.12497	0.039564	0.8021798	0.049604	-4.9539

Modified Euler's Method (Midpoint Method)

- Modification to first order Euler's method
- This involves using the derivative at the midpoint at the interval $(x_{i+\frac{1}{2}} = x_i + \frac{h}{2})$



Modified Euler's Method (continue)

- Comprise of the following steps
  - Calculate $f(x_i, y_i)$ from the known initial condition $y(x_i) = y_i$
  - Find the value of *y* at the midpoint of interval $y_{i+\frac{1}{2}} = y(x_i + \frac{h}{2}) = y_i + \frac{h}{2} f(x_i, y_i)$
  - Find the slope at the midpoint $y'_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$
  - Determine the value of *y* at x_{i+1}

$$y_{i+1} = y_i + h f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

- As in the case of Heun's method, the local truncation error of the modified Euler's method can be shown to be $O(h^3)$
- Modified Euler's method is superior to the original Euler's method

Example

- Find the solution of the initial value problem

$$y' = -5y + e^{-2x}\ ;\ y(0) = 1.0$$

in the interval $0 \le x \le 1$ using modified Euler's method with $h = 0.1$

- Solution

Divide the interval $0 \le x \le 1$ into 10 parts for $h = 0.1$

Example (continue)

- $h = 0.1$, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, …, $x_{10} = 1$, with an initial $y_0 = 1$
- Evaluation of y_1

$$\text{(i)}\ f(x_0, y_0) = -5y_0 + e^{-2x_0} \quad \Rightarrow \quad f(0,1) = -5(1) + e^{-2(0)} = -4$$

$$\text{(ii)}\ y_{0.5} = y_0 + h\, f(x_0, y_0)/2 \quad \Rightarrow \quad \underline{y_{0.5} = y(0.05) = 1 + 0.1(-4)/2 = 0.8}$$

$$\text{(iii)}\ f(x_{0.5}, y_{0.5}) = -5y_{0.5} + e^{-2x_{0.5}} \quad \Rightarrow \quad f(0.05, 0.8) = -5(0.8) + e^{-2(0.05)} = -3.0952$$

$$\text{(iv)}\ y_1 = y_0 + h\, f(x_{0.5}, y_{0.5}) \Rightarrow \underline{\underline{y_1 = y(0.1) = 1 + 0.1(-3.0952) = 0.69048}}$$

Example (continue)

- Evaluation of y_2

(i) $f(x_1, y_1) = -5y_1 + e^{-2x_1} \quad \Rightarrow \quad f(0.1, 0.69048) = -5(0.69048) + e^{-2(0.1)} = -2.6337$

(ii) $y_{1.5} = y_1 + h\, f(x_1, y_1)/2 \quad \Rightarrow \quad \underline{y_{1.5} = y(0.15) = 0.69048 + 0.1(-2.6337)/2 = 0.5588}$

(iii) $f(x_{1.5}, y_{1.5}) = -5y_{1.5} + e^{-2x_{1.5}} \Rightarrow f(0.15, 0.5588) = -5(0.5588) + e^{-2(0.15)} = -2.0532$

(iv) $y_2 = y_1 + h\, f(x_{1.5}, y_{1.5}) \Rightarrow \underline{\underline{y_2 = y(0.2) = 0.69048 + 0.1(-2.0532) = 0.48517}}$

Example (continue)

- Subsequently calculate $y_4, y_5, \ldots, y_9, y_{10}$

x	y	f(x,y)	$y^{(0)}$	$f(x,y^{(0)})$	exact	% error
0	**1**	-4	0.8	-3.0952	1	0
0.1	**0.69048**	-2.6337	0.55880	-2.0532	0.67726	-1.9519
0.2	**0.48517**	-1.7555	0.39739	-1.3804	0.46869	-3.5146
0.3	**0.34712**	-1.1868	0.287783	-0.9423	0.33169	-4.6529
0.4	**0.25289**	-0.8151	0.212134	-0.6541	0.24	-5.3711
0.5	**0.18748**	-0.5695	0.159004	-0.4622	0.17735	-5.7122
0.6	**0.14127**	-0.4051	0.121009	-0.3325	0.13359	-5.7459
0.7	**0.10801**	-0.2935	0.09334	-0.2436	0.10233	-5.5541
0.8	**0.08366**	-0.2164	0.072838	-0.1815	0.079509	-5.2166
0.9	**0.06551**	-0.1622	0.057395	-0.1374	0.062506	-4.8010
1	**0.05177**	-0.1235	0.045591	-0.1055	0.049604	-4.3590

Example (continue)

- Comparison of different methods