# Lecture 8: Inductors and RL Transients

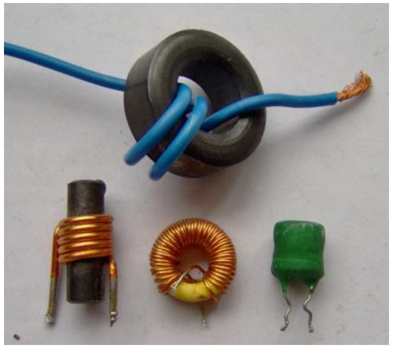
ELEC1111 Electrical and Telecommunications Engineering

UNSW AUSTRALIA

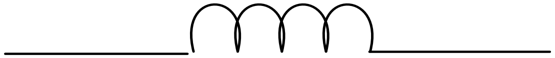
Never Stand Still | Faculty of Engineering | School of Electrical Engineering and Telecommunications

# Inductors

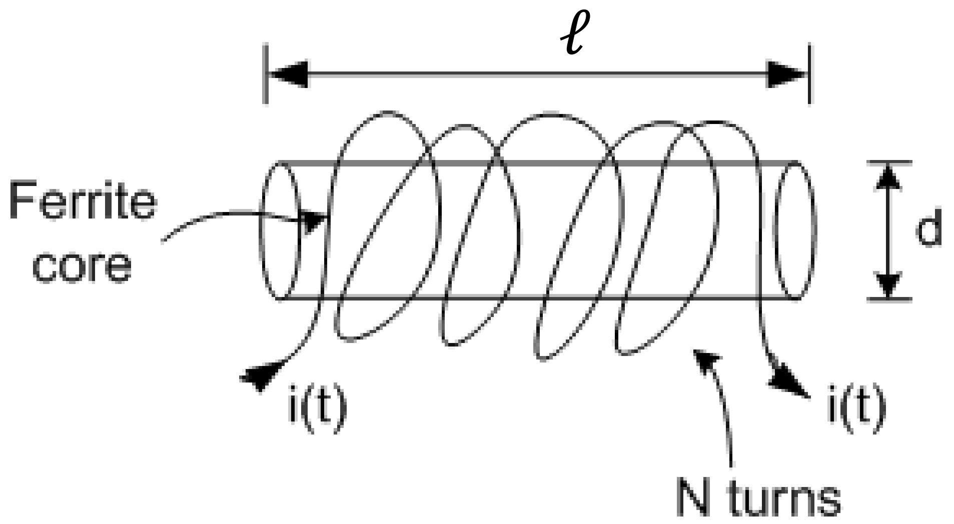
Image



Symbol on circuit



- Inductors are coils of various dimensions designed to introduce specified amounts of inductance (ability to oppose any change in current) into a circuit
- The inductance of a coil varies directly with the magnetic properties of the coil
- Ferromagnetic materials frequently employed to increase the inductance of the coil
- Inductance is measures in henries (H)




- ✓ The inductance of the coil is given by

$$L = \frac{N^2 \mu A}{\ell}$$

where $N$ – number of turns;
$\mu$ – the permeability of the core;
$A$ – the area of the core in m$^2$ and
$\ell$ - the length of the core (m)

# Inductors

$$L = \frac{N^2 \mu_r \mu_o A}{\ell}$$

$$= \mu_r \cdot \frac{N^2 \mu_o A}{\ell}$$

**For an iron core $\mu_r = 2000$**

$$\boxed{L = \mu_r L_o}$$

where $L_o$ is the inductance of the coil with an air core

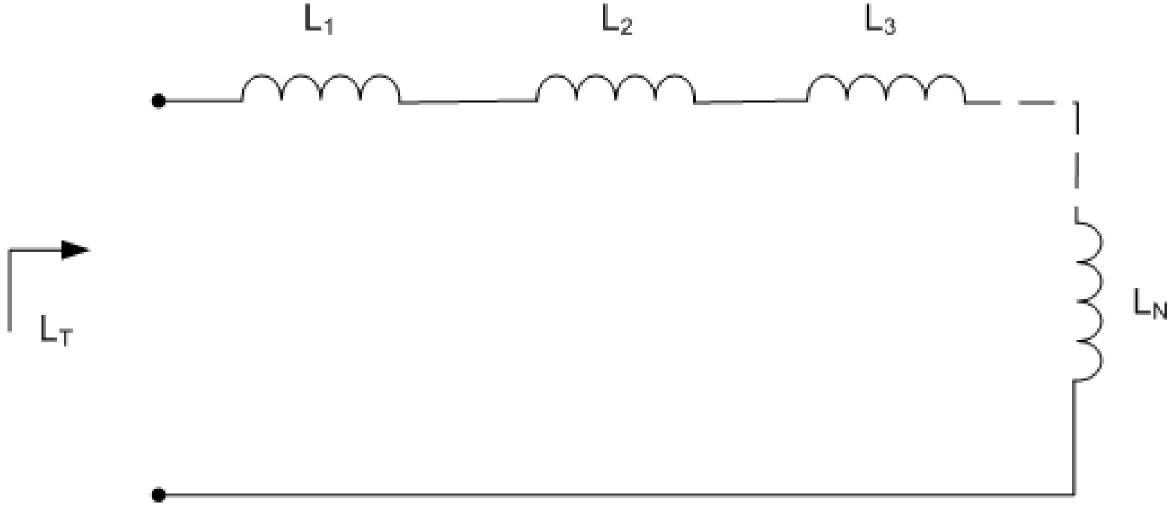
The inductance of a coil with a ferromagnetic core is the relative permeability of the core times the inductance achieved with an air core

# Inductors in Series and Parallel

Inductors, like resistors and capacitors, can be placed in series or parallel.

For inductors in series, the total inductance is formed in the same manner as the total resistance of resistors in series.

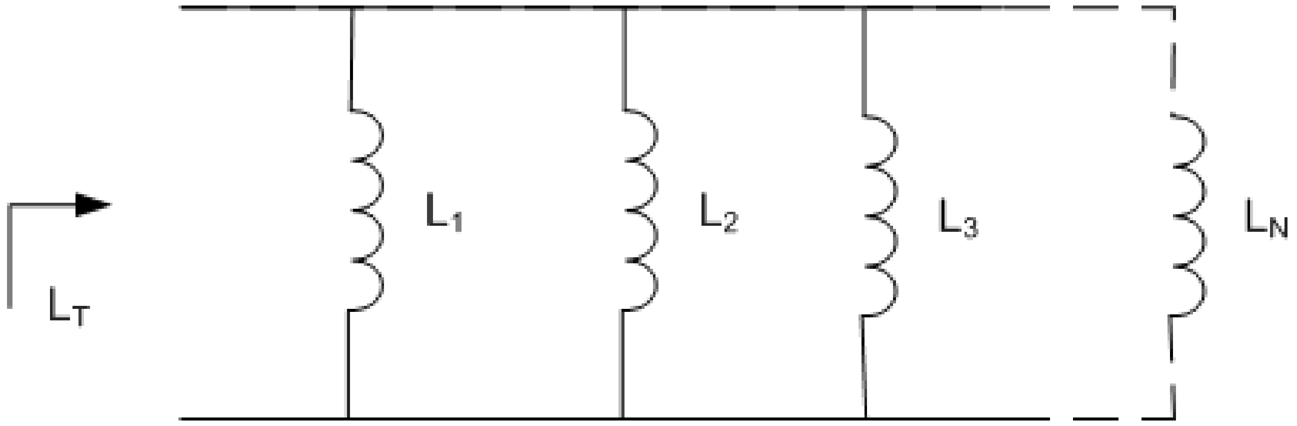
$$L_T = L_1 + L_2 + L_3 + \cdots + L_N$$



# Inductors in Series and Parallel

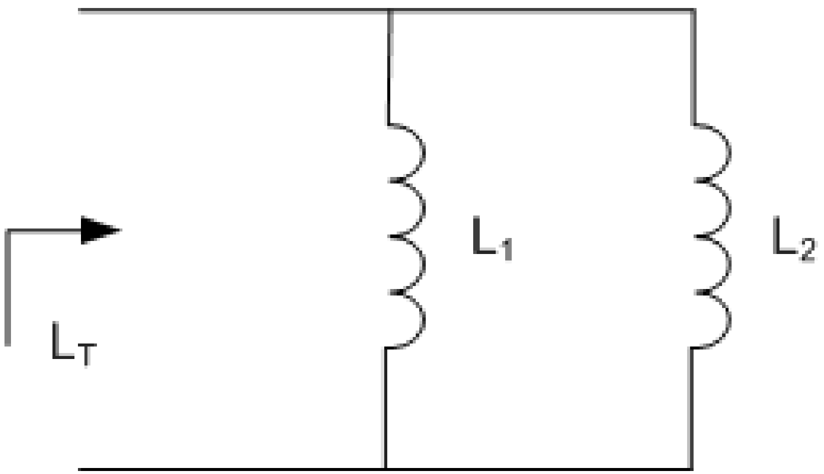
For inductors in parallel, the total inductance is formed in the same manner as the total resistance of resistors in parallel.

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}$$



# Inductors in Series and Parallel

For two inductors in parallel

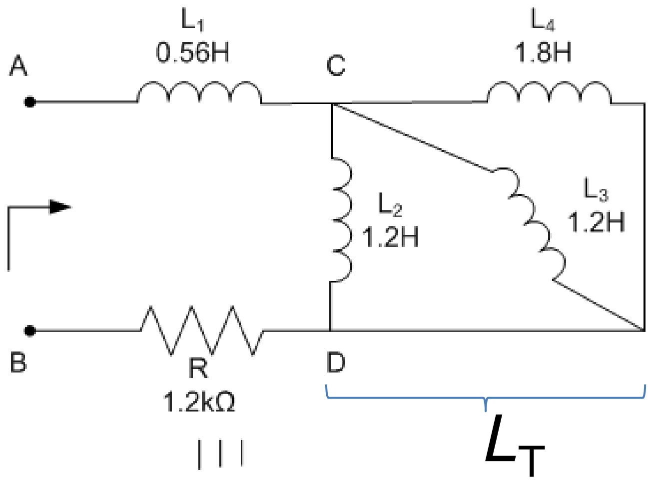


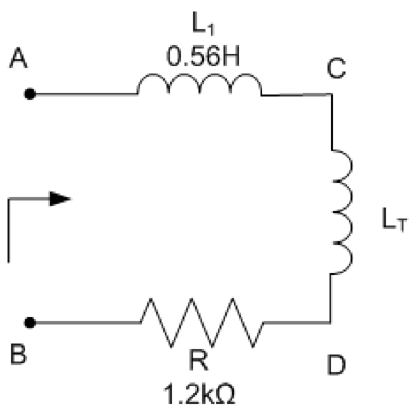

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2}$$

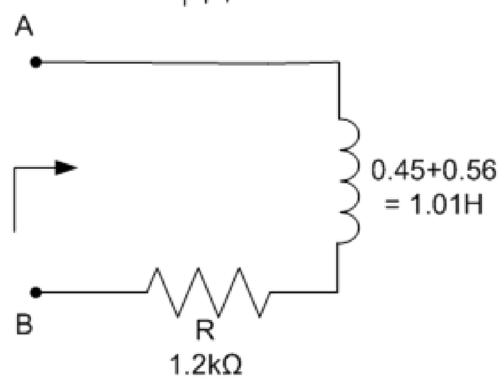
$$\therefore L_T = \frac{L_1 L_2}{L_1 + L_2}$$

# Inductors in Series and Parallel

Reduce the network to its simplest form




<u>Note:</u>

$L_2//L_3//L_4$

$$\therefore \frac{1}{L_T} = \frac{1}{L_2} + \frac{1}{L_3} + \frac{1}{L_4}$$

$$\frac{1}{L_T} = \underbrace{\frac{1}{1.2} + \frac{1}{1.2}}_{} + \frac{1}{1.8}$$

$$= \frac{1}{0.6} + \frac{1}{1.8}$$

$$\therefore L_T = 0.45\text{H}$$

(note: $L_1$ & $L_T$ are in series)

# Induced Voltage

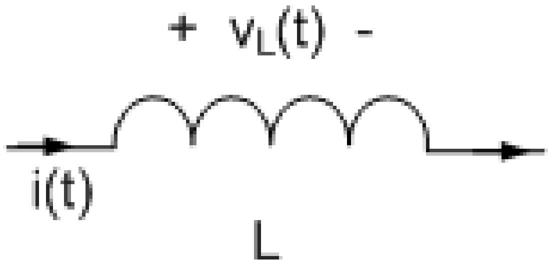
Examples:

$0.1\ \mu\mathrm{H} = 0.1 \times 10^{-6}\mathrm{H}$

$0.27\mu\mathrm{H} = 0.27 \times 10^{-6}\mathrm{H}$

$1\mathrm{mH} = 1 \times 10^{-3}\mathrm{H}$

<u>Induced Voltage</u>




If $i(t)$ is the current through the coil, the voltage across the inductor $v_L(t)$ is given by

**Induced voltage** $\longrightarrow$

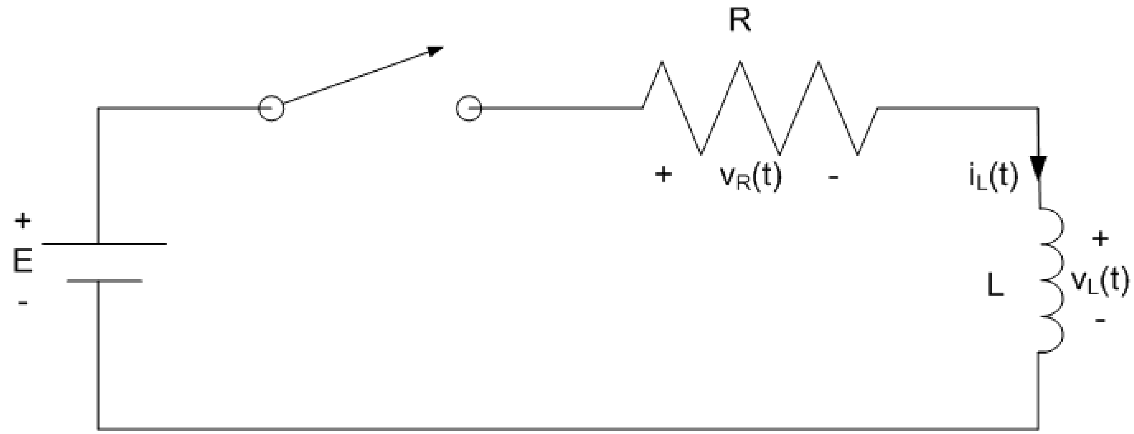
$$\boxed{v_L(t) = L\frac{di(t)}{dt}}$$

The greater the rate of change of current through the coil, the greater will be the induced voltage.

# Transients in Inductive Networks

Reference: Introductory Circuit Analysis – Boylestad, Prentice Hall, 2000

# RL Circuit: Storage Phase

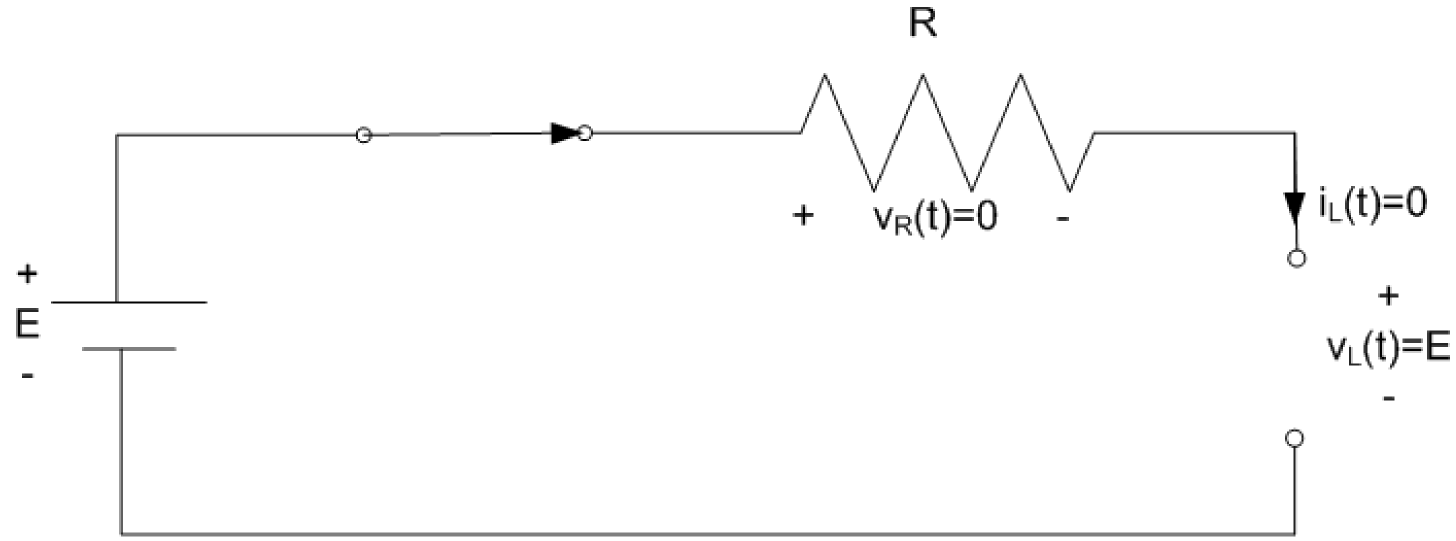



At the instant the switch is closed, the inductance of the coil will prevent an instantaneous change in current through the coil

The potential drop across the coil $v_L(t) = E \; [i.e. \, i(0) = 0]$

# RL circuit: storage phase

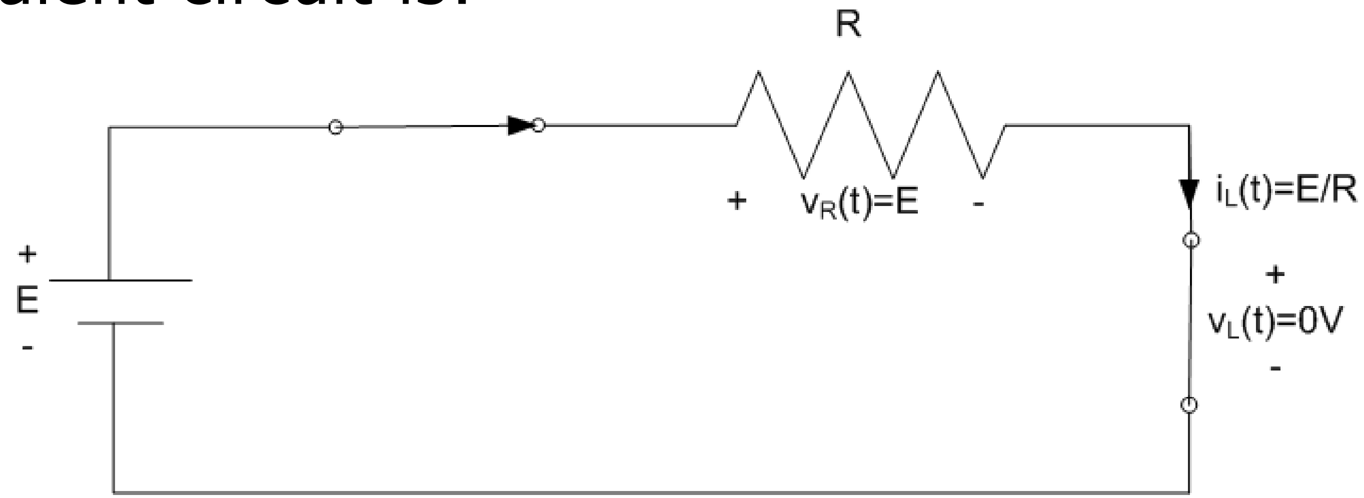
The equivalent circuit **at the instant (*t*=0)** the switch is closed is shown below



# RL circuit: storage phase

The current $i_L(t)$ will then build up from zero, establishing a voltage drop across the resistor. The current will continue to increase until the voltage across the inductor drops to zero volts. At this point $v_R(t) = E$ and $v_L(t) = 0$.

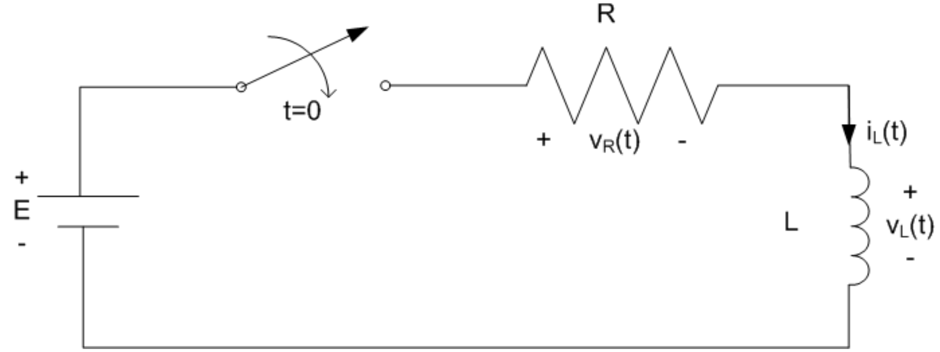
When the **steady-state ($t \to \infty$)** condition has been established and the storage phase is complete, the equivalent circuit is:



# RL circuit transients: Equations for storage phase

$$E = i_L(t)\,R + v_L(t)$$

$$v_L(t) = L\frac{di_L(t)}{dt}$$




$$= i_L(t)\,R + L\frac{di_L(t)}{dt}$$

$$\therefore \frac{di_L(t)}{dt} = \frac{E}{L} - \frac{R}{L} i_L(t)$$

$$\int_0^t \frac{di_L(t)}{E - Ri_L(t)} = \frac{1}{L}\int_0^t dt$$

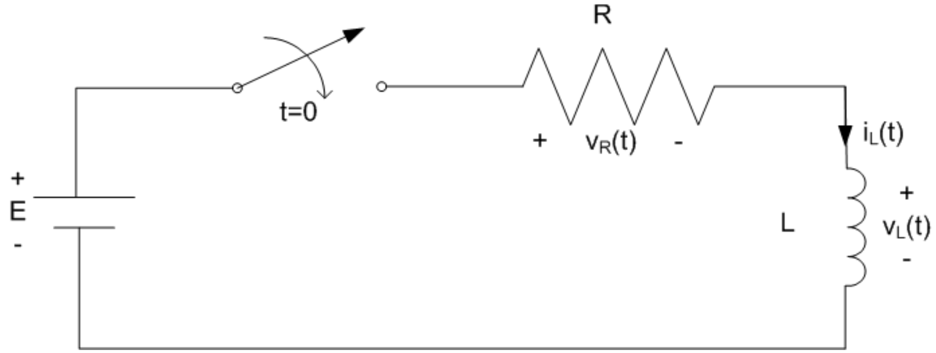
$$-\frac{1}{R}\left[\ln\big(E - Ri_L(t)\big)\right]_0^t = \frac{1}{L}[t]$$

$$i_L(t) = 0 \; at \; t = 0$$

$$-\left[\ln\big(E - Ri_L(t)\big)\right] + \left[\ln\big(E - R\underbrace{i_L(0)}_{=0}\big)\right] = \frac{R}{L}t$$

$$\therefore \ln\frac{E - Ri_L(t)}{E} = -\frac{R}{L}t$$

# RL circuit transients: Equations for storage phase




$$\frac{E - Ri_L(t)}{E} = e^{-\frac{R}{L}t}$$

$$\therefore Ri_L(t) = E - Ee^{-\frac{R}{L}t}$$

$$\boxed{\therefore i_L(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$= \frac{E}{R} \times L \cdot \frac{R}{L} e^{-\frac{R}{L}t}$$

$$\boxed{v_L(t) = Ee^{-\frac{R}{L}t}}$$

$$v_R(t) = i_L(t) \cdot R$$

$$\boxed{v_R(t) = E\left(1 - e^{-\frac{R}{L}t}\right)}$$

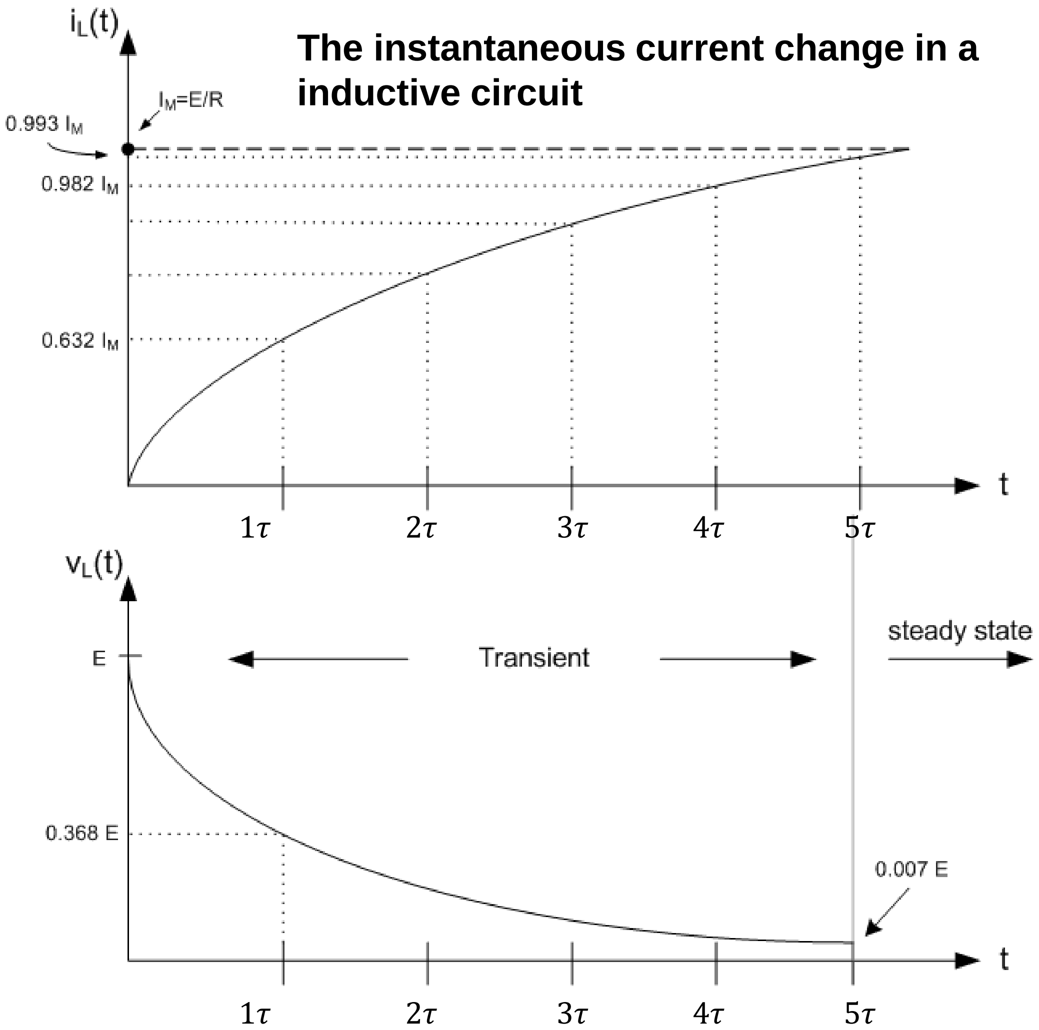
Time constant for inductive circuit is defined as $\tau = \frac{L}{R}$ seconds

$$\therefore\ i_L(t) = \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$v_L(t) = Ee^{-\frac{t}{\tau}}$$

$$v_R(t) = E\left(1 - e^{-\frac{t}{\tau}}\right)$$

# RL circuit transients: Equations for storage phase



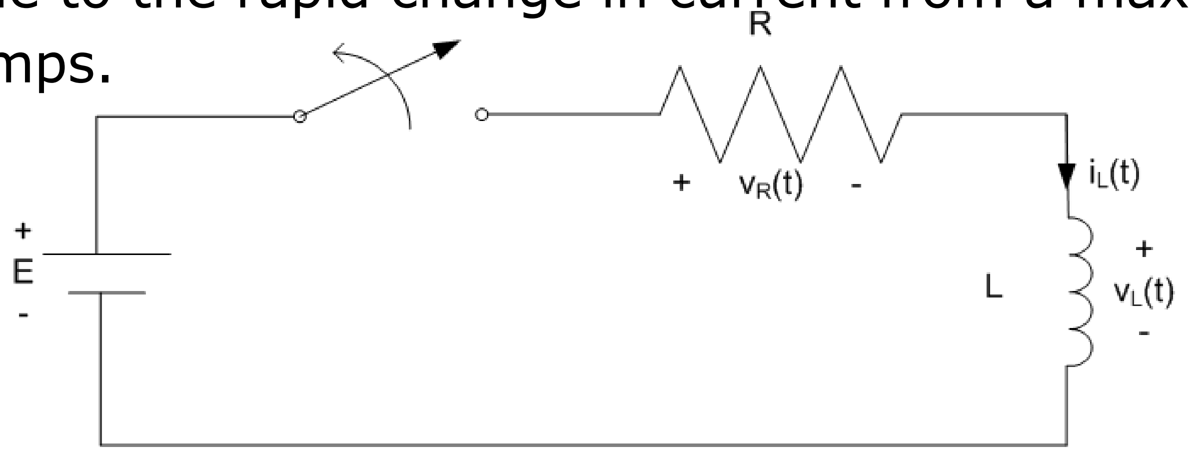

In 5 time constants, we may assume that the storage phase has passed and steady-state conditions have been established.

In five time constants $i_L(t) = \frac{E}{R}$ and $v_L(t) = 0V$ and the inductor can be replaced by its short circuit equivalent.

# RL circuit: storage phase

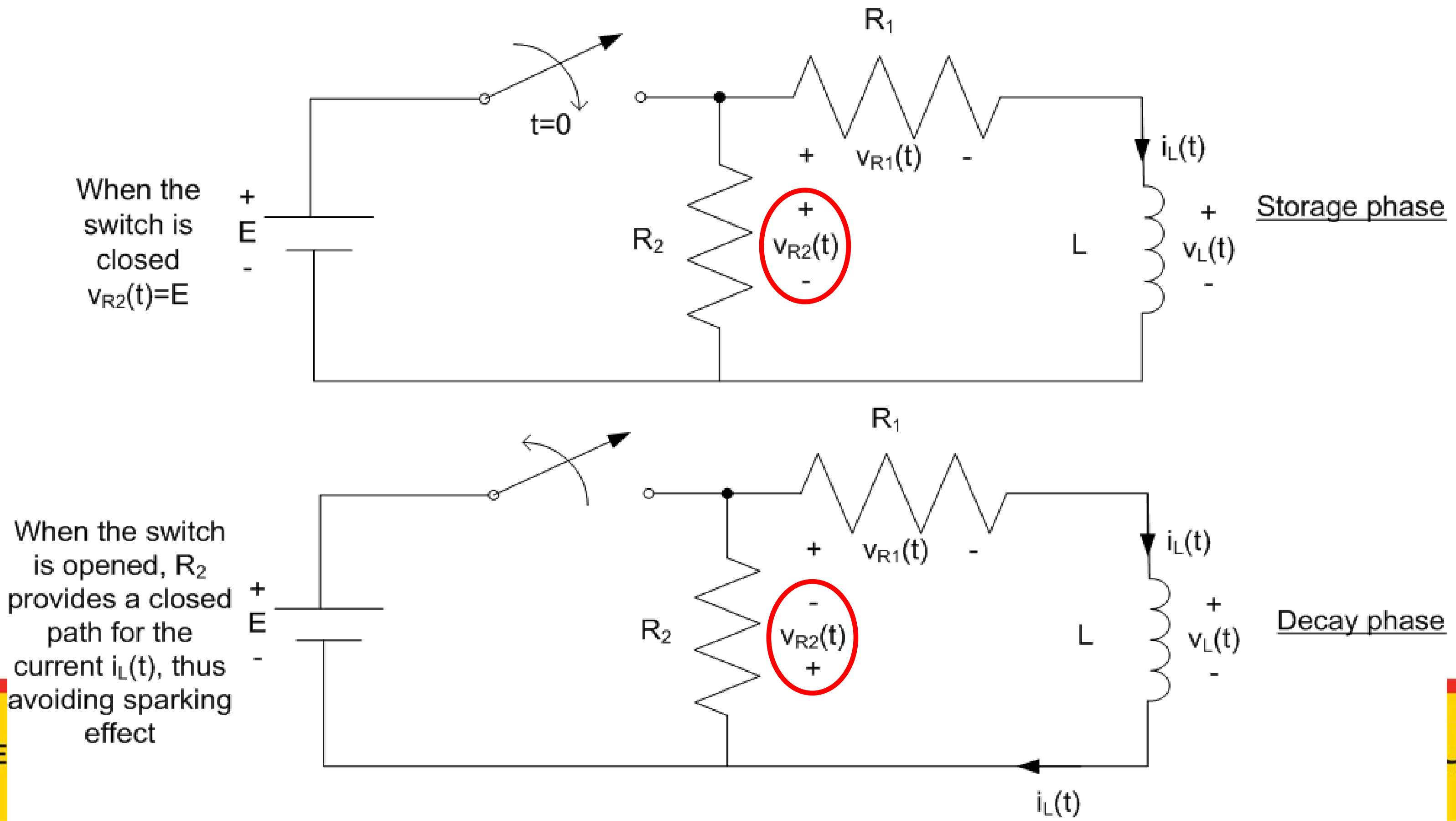
Note:

- In RC circuits, the capacitor holds its charge and stores energy in the form of an electric field for a period of time determined by the leakage factors.
- In RL circuits, the energy is stored in the form of a magnetic field established by the current through the coil. If the switch is opened, the current will drop to zero in the absence of a closed path and the inductor will release the stored energy in the form of a magnetic field. A spark would probably occur across the contacts due to the rapid change in current from a maximum of $\frac{E}{R}$ to zero amps.



# RL circuit: storage and decay phase

- The change in current $\frac{di_L(t)}{dt}$ $\left[v_L(t) = L\frac{di_t(t)}{dt}\right]$ establishes a high voltage $v_L(t)$ across the coil that would spark across the points of the switch.
- To avoid this problem use the following circuit for RL <u>storage</u> and <u>decay</u> phase.



# RL circuit: decay phase

$$v_L(t) + v_{R1}(t) + v_{R2}(t) = 0$$

$$v_L(t) = -[v_{R1}(t) + v_{R2}(t)]$$

$$v_L(t) = -i_L(R_1 + R_2)$$

At the instant the switch is opened ($t$=0)

$$\underbrace{i_L(t)}_{(t=0)} = \frac{E}{R_1}$$

$$\therefore \underbrace{v_L(t)}_{(t=0)} = -\frac{E}{R_1}(R_1 + R_2)$$

$$\boxed{v_L(0) = -\left(1 + \frac{R_2}{R_1}\right)E}$$

Voltage across inductor will reverse polarity and drop from 0 to $-\left(1 + \frac{R_2}{R_1}\right)E$ volts. The minus sign means $v_L(t)$ has a polarity opposite to the defined one.

# RL circuit: decay phase

As inductor releases its stored energy, the voltage across the coil will decay to zero as follows

$$v_L(t) = -\left(1 + \frac{R_2}{R_1}\right) E\, e^{\frac{-t}{\tau\prime}}$$

$$\tau' = \frac{L}{R_1 + R_2}$$

The current will decay from $\frac{E}{R_1}$ to zero as follows

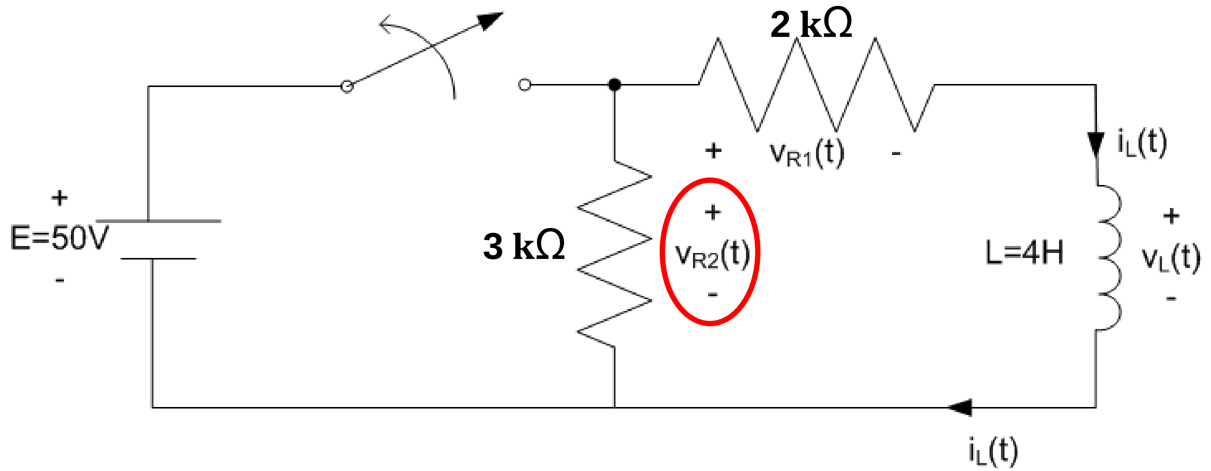
$$i_L(t) = \frac{E}{R_1} e^{\frac{-t}{\tau\prime}}$$

$$v_{R_1}(t) = E e^{\frac{-t}{\tau\prime}}$$

$$v_{R_2}(t) = \frac{R_2}{R_1} E\, e^{\frac{-t}{\tau\prime}}$$

$v_{R_2}(t)$ polarity defined according to **decay phase**

# Example




$v_{R_2}(t)$ polarity defined according to **storage phase**

For the circuit above,

(a) Find the expressions for $i_L(t), v_L(t), v_{R1}(t)$ and $v_{R2}(t)$ after the storage phase has been completed and the switch is opened

(b) Sketch the waveform for each voltage and current for both phases, if five time constants pass between phases

# Example 1

(a)

$$\tau' = \frac{L}{R_1 + R_2} = \frac{4\text{H}}{(2+3)\text{k}\Omega} = 0.8\text{ms}$$

$$v_L(0) = -\left(1 + \frac{R_2}{R_1}\right) E = -\left(1 + \frac{3}{2}\right)(50) = -125\text{V}$$

$$\boxed{v_L(t) = -125 e^{-\frac{t}{0.8\text{ms}}}}$$

$$\begin{cases} i_L(t) = \dfrac{E}{R_1} e^{-\frac{t}{0.8\times 10^{-3}}} = (25 \times 10^{-3}) e^{-\frac{t}{0.8\times 10^{-3}}} \\ v_{R1}(t) = R_1 i_L(t) = E\, e^{\frac{-t}{\tau'}} = 50\, e^{\frac{-t}{0.8\times 10^{-3}}} \\ v_{R2}(t) = -\dfrac{R_2}{R_1} E\, e^{\frac{-t}{\tau'}} = -75\, e^{\frac{-t}{0.8\times 10^{-3}}} \end{cases}$$

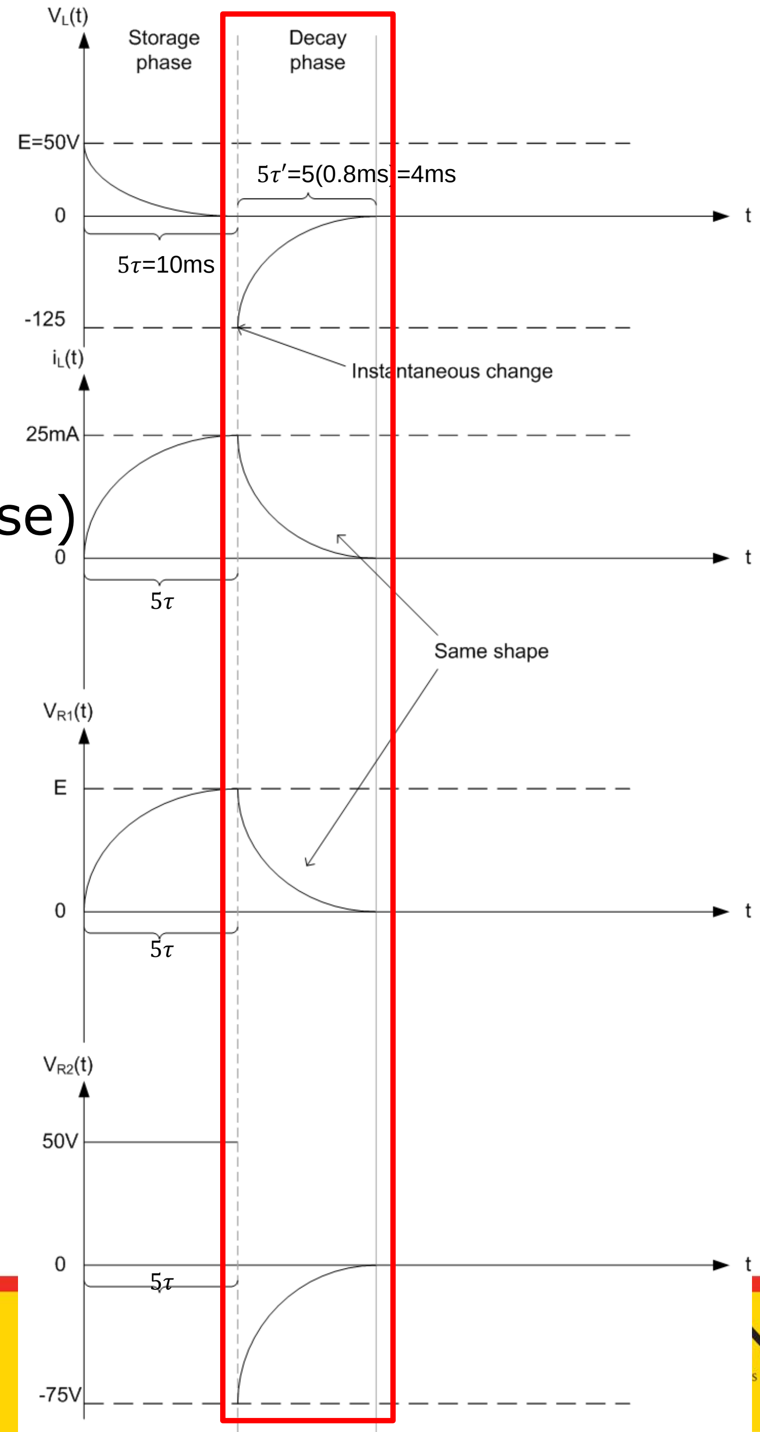
$v_{R_2}(t)$ polarity defined according to **storage phase**

$$v_L(t) = -125e^{-\frac{t}{0.8\text{ms}}} \text{ (Decay Phase)}$$

$$i_L(t) = (25 \times 10^{-3})e^{-\frac{t}{0.8\times10^{-3}}} \text{ (Decay Phase)}$$

$$v_{R1}(t) = 50\, e^{\frac{-t}{0.8\times10^{-3}}} \text{ (Decay Phase)}$$

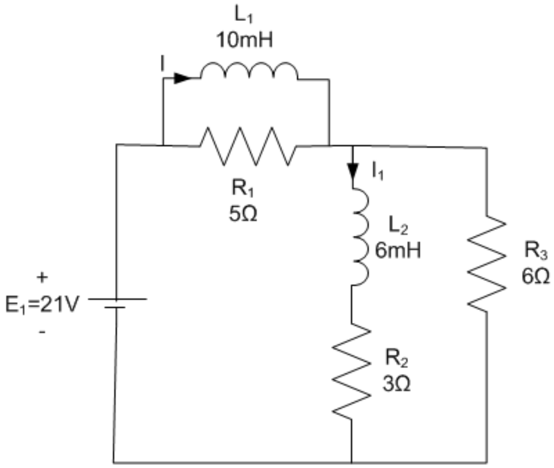
$$v_{R2}(t) = -75\, e^{\frac{-t}{0.8\times10^{-3}}} \text{ (Decay Phase)}$$



# Example 2 (Steady State)

For all practical purposes, an inductor can be replaced by a short circuit in a dc circuit after a period of time greater than five time constants have passed.

For the circuit shown below, find $I_1$ in steady state.

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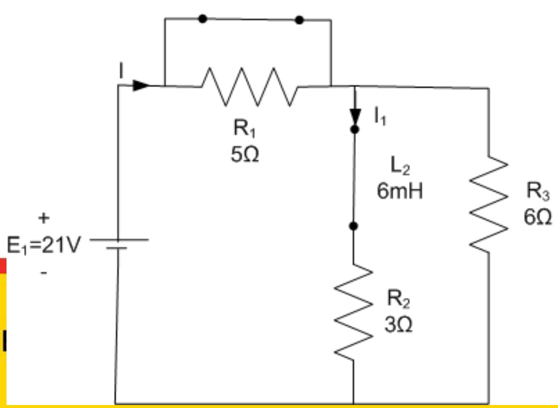



$$I = \frac{E}{R_2 \,//\, R_3} = \frac{21V}{2\Omega} = 10.5A$$

$$\therefore I_1 = R_3 \times \frac{I}{R_3 + R_2} = \frac{6 \times 10.5}{6 + 3}$$

$$= \frac{63}{9} A = 7A$$

# Energy stored by an inductor

An ideal inductor stores energy in the form of magnetic field.

Voltage, current and power to an inductor during the build up of the magnetic field surrounding the inductor:

$$P_L(t) = v_L(t) \cdot i_L(t)$$

$$= L\frac{di_L(t)}{dt} \cdot i_L(t)$$

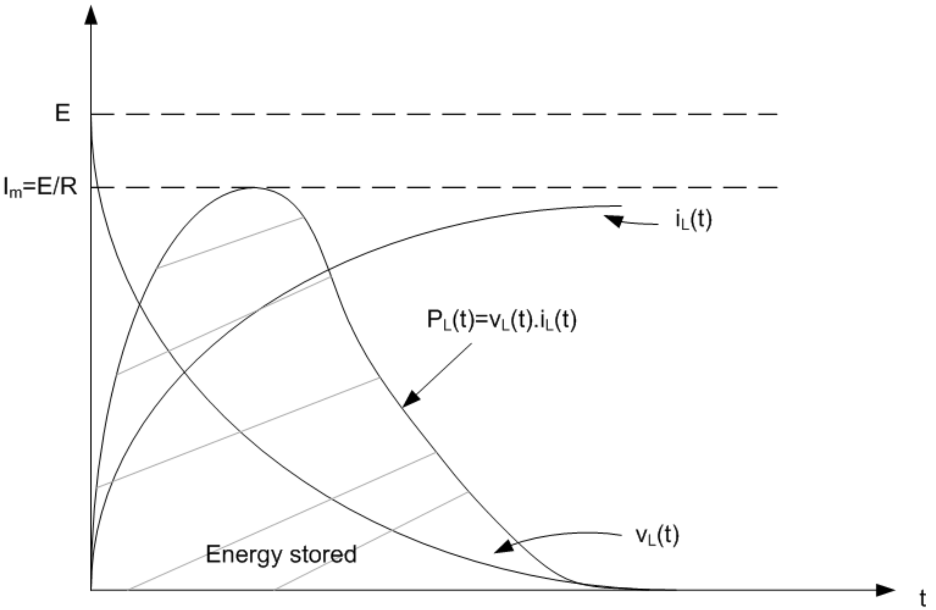
Energy stored = shaded area under the power curve

$$W_L(t) = \int_0^t P_L(t)dt$$

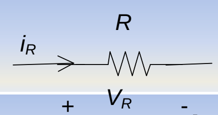
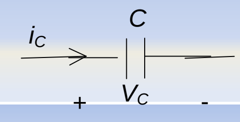
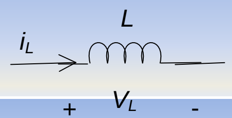
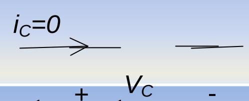
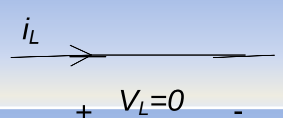
$$= \int_0^t L\, i_L(t)\frac{di_L(t)}{dt}dt$$

$$= L\frac{i^2(t)}{2}$$

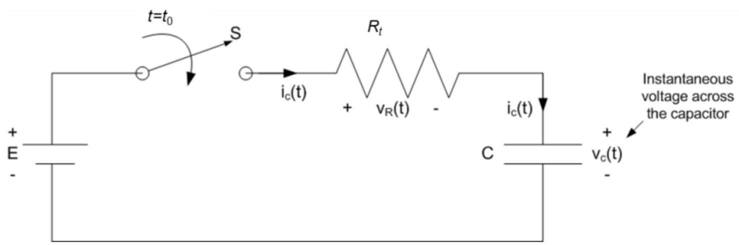
$$W_L(t) = \tfrac{1}{2}Li^2(t) \qquad \text{(Joules, } J\text{)}$$



# R, C, L Revision

| | R | C | L |
|---|---|---|---|
| |  |  |  |
| V-I relation | $v_R=Ri_R$ | $i_c=dv_c/dt$ | $v_L=di_L/dt$ |
| Power | $P_R=V_RI_R=RI_R^2$ | $P_c=v_ci_c$ | $P_L=v_Li_L$ |
| Energy | $E_R = \int_0^t P_R dt$ | $E_c=Cv_c^2/2$ | $E_c=Li_L^2/2$ |
| Steady state | $v_R=Ri_R$ |  |  |
| Series connection | $R_{eq}=R_1+R_2+\ldots R_N$ | $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$ | $L_{eq} = L_1 + L_2 + \cdots L_N$ |
| Parallel connection | $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots \frac{1}{R_N}$ | $C_{eq}=C_1+C_2+\ldots C_N$ | $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots \frac{1}{L_N}$ |

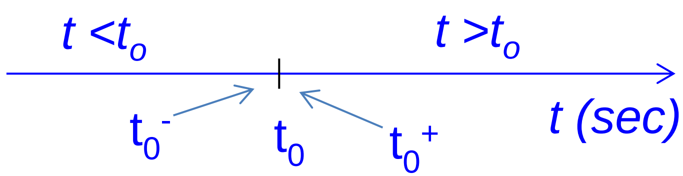
# RC and RL circuits in transient




$v_c(t_0^+)=v_c(t_0^-)$

*$t_o$ is the time when the circuit is interupted*

Voltage across capacitor is continuous




When $t>t_0$

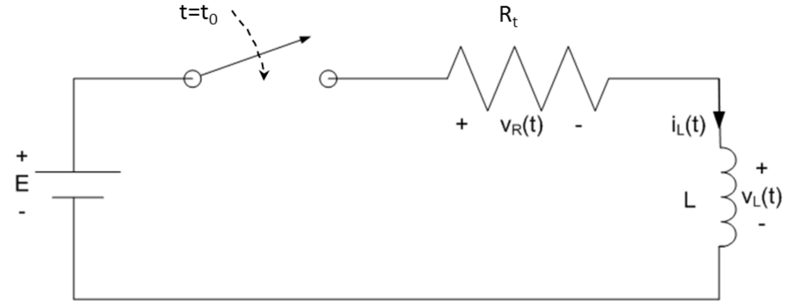
$$\tau = R_t C$$

$$v_C(t) = V_{oc} + \left[ v_C(t_0) - V_{oc} \right] \exp\left( -\frac{t-t_0}{\tau} \right)$$

$V_{oc} = V_c$ when the capacitor C is open (disconnected)

OR: $V_{oc} = V_c\ (+\infty)$

$v_c(t_0) = v_c$ at $t=t_0$




$i_L(t_0^+)=i_L(t_0^-)$

Current through capacitor is continuous

When $t>t_0$

$$\tau = L/R_t$$

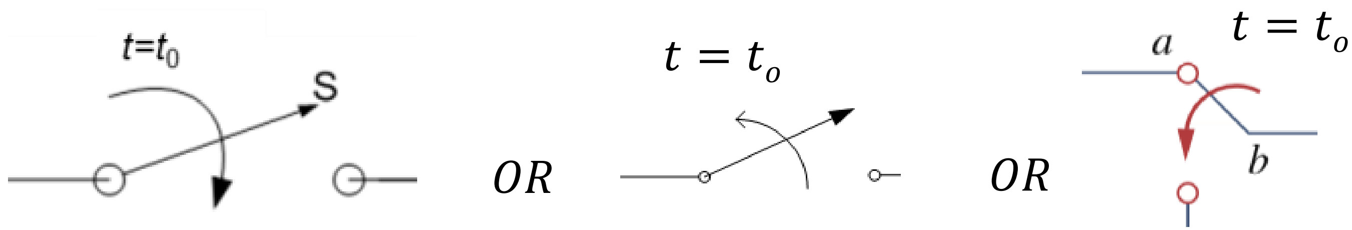
$$i_L(t) = I_{sc} + \left[ i_L(t_0) - I_{sc} \right] \exp\left( -\frac{t-t_0}{\tau} \right)$$

$I_{SC} = i_L$ when inductor is shorted, or $I_{SC}=E/R_t$

OR: $I_{SC} = i_L\ (+\infty)$

$i_L(t_0) = i_L(t)$ at $t=t_0$

# Three Phases in Transient Analysis




$t=t_o$ *After interruption*

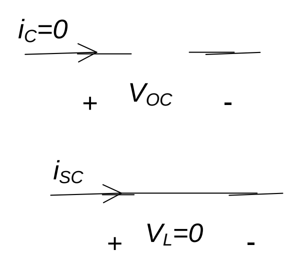
$+\infty$

*Before interruption* $t=5\tau$ $t(sec)$

*Transient* *Steady state*

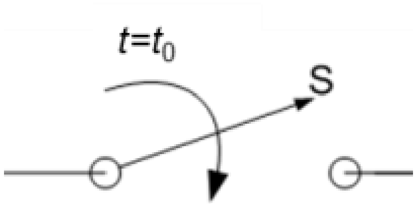
*C is open*

*RC circuit*: $v_c(t_0^+)=v_c(t_0^-)$ $v_C(t)=V_{oc}+\left[v_C(t_0)-V_{oc}\right]\exp\left(-\frac{t-t_0}{\tau}\right)$

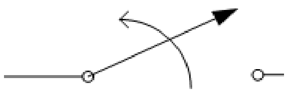



*RL circuit*: $i_L(t_0^+)=i_L(t_0^-)$ $i_L(t)=I_{sc}+\left[i_L(t_0)-I_{sc}\right]\exp\left(-\frac{t-t_0}{\tau}\right)$

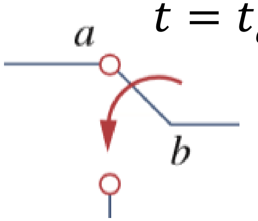
# Three Phases in Transient Analysis




$t = t_o$



$t = t_o$



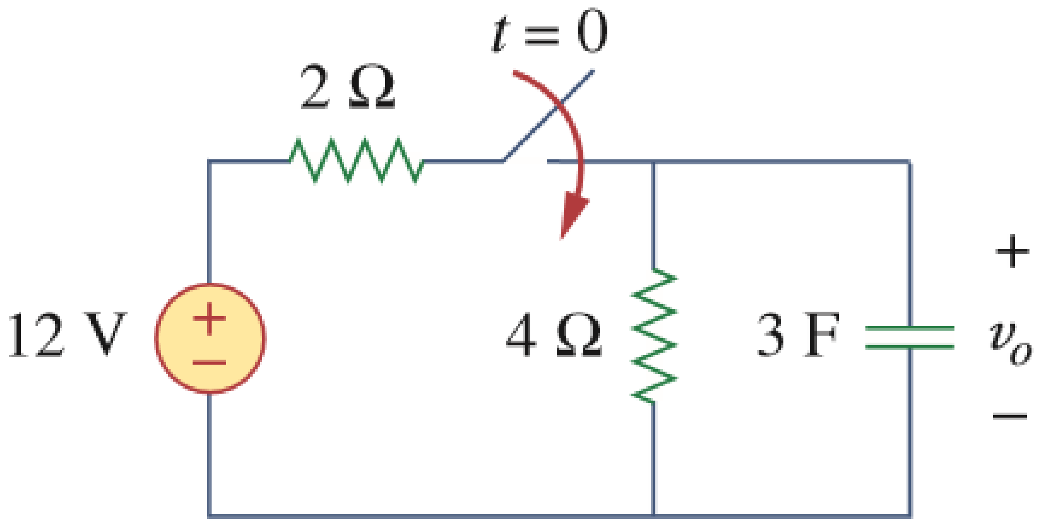

$t < t_o$ $\quad$ $t = t_o$ $\quad$ $t > t_o$

Reference: Introductory Circuit Analysis – Boylestad, Prentice Hall, 2000

# Example 3

For the following circuit, the switch has been open for a long time and is closed at t=0.

- Find $v_o(0^+)$
- Find $v_o(+\infty)$
- Find the equation of $v_o(t)$ when $t \geq 0$

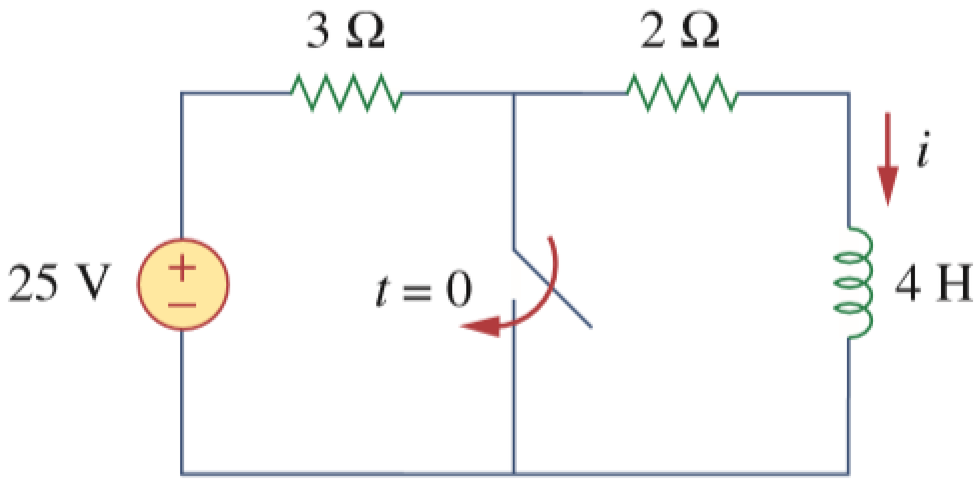



*"Fundamentals of Electric Circuits", Alexander and Sadiku, McGraw-Hill*

# Example 4

For the following circuit, the switch has been open for a long time and is closed at t=0.

- Find $i_o(0^+)$
- Find $i_o(+\infty)$ or $i_{oSC}$
- Find the equation of $i_o(t)$ when $t \geq 0$

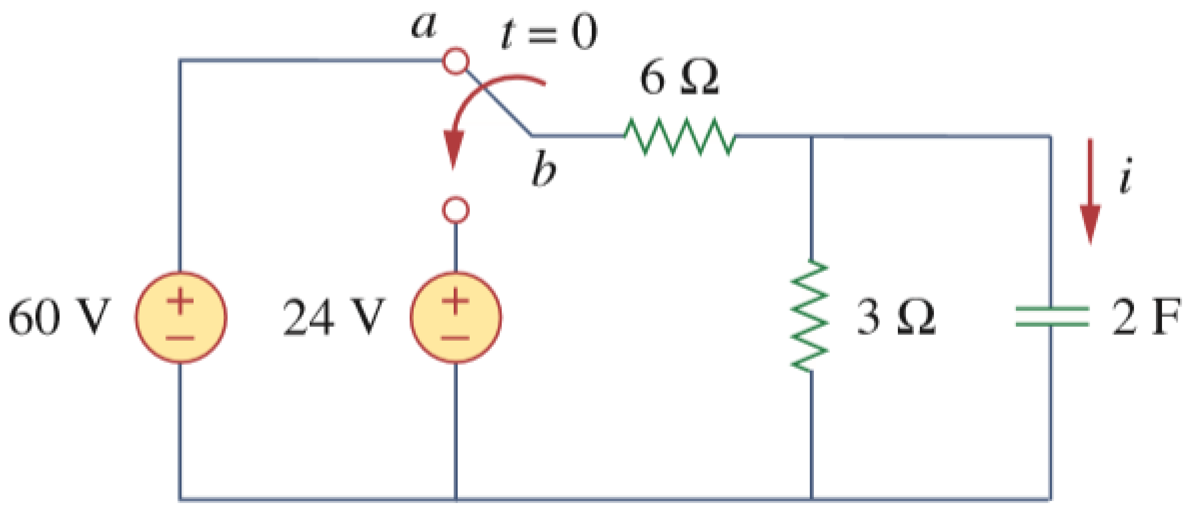



*"Fundamentals of Electric Circuits", Alexander and Sadiku, McGraw-Hill*

# Example 5

For the following circuit, the switch has been in position *a* for a long time and is moved to position b at t=0.

- Find the equation of *i* when t≥0




*"Fundamentals of Electric Circuits", Alexander and Sadiku, McGraw-Hill*

# Example 6

For the following circuit, the switch has been open for a long time and is closed at t=0.

- Find the equation of *i* when t≥0