Adaptivity analysis

Expr.	e	$::=$	$x \mid e_1\ e_2 \mid \lambda x.e$
			$c \mid \delta(e)$
Value	v	$::=$	$c \mid \lambda x.e$
Adaptivity	R	$::=$	n
Environment	θ	$::=$	$x_1 \mapsto (v_1, R_1), \ldots, x_n \mapsto (v_n, R_n)$

Linear type	A	$::=$	$\tau \multimap \tau \mid \mathsf{b}$
Nonlinear Type	τ	$::=$	$!_I A$

$$\frac{\theta(x) = (v, R)}{\theta, x \Downarrow^R v, \theta}\ \mathsf{var} \qquad \frac{}{\theta, c \Downarrow^0 c, \theta}\ \mathsf{const} \qquad \frac{}{\theta, \lambda x.e \Downarrow^R \lambda x.e, \theta}\ \mathsf{lambda}$$

$$\frac{\theta, e_1 \Downarrow^{R_1} \lambda x.e, \theta_1 \qquad \theta, e_2 \Downarrow^{R_2} v_2, \theta_2 \qquad (\theta_1 \uplus \theta_2)[x \to (v_2, R_2)], e \Downarrow^{R_3} v, \theta_3}{\theta, e_1\ e_2 \Downarrow^{R_1 + R_3} v, \theta_3}\ \mathsf{app}$$

$$\frac{\theta, e \Downarrow^R v', \theta_1 \qquad \delta(v') = v}{\theta, \delta(e) \Downarrow^{R+1} v, \theta_1}\ \mathsf{delta}$$

Figure 1: Big-step semantics

$$\frac{}{\Gamma, x :!_Z A, \Gamma' \vdash_Z x :!_Z A}\ \mathbf{Ax} \qquad\qquad \frac{}{\Gamma \vdash_Z c :!_Z \mathtt{b}}\ \mathbf{b}$$

$$\frac{\Gamma, x : \tau_1 \vdash_Z e : \tau_2}{k + \Gamma \vdash_{k+Z} \lambda x.e :!_k(\tau_1 \multimap \tau_2)}\ \mathbf{lambda}$$

$$\frac{\Gamma_1 \vdash_{Z_1} e_1 :!_0(\tau_1 \multimap \tau_2) \qquad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\max(\Gamma_1, \Gamma_2) \vdash_{\max(Z_1,Z_2)} e_1\ e_2 : \tau_2}\ \mathbf{app}$$

$$\frac{\Gamma \vdash_Z e :!_k A}{\Gamma', 1 + \Gamma \vdash_{1+Z} \delta(e) :!_k A}\ \mathbf{delta}$$

$$\frac{\Gamma' \vdash_{Z'} e : \tau' \qquad \Gamma' \leqslant \Gamma \qquad Z' \leq Z \qquad \tau' <: \tau \qquad \Gamma \vdash_Z e :!_k A}{\Gamma \vdash_Z e : \tau}\ \mathbf{subtype}$$

$$\frac{\Gamma, y : \tau', x : \tau, \Gamma' \vdash_Z e : \tau}{\Gamma, x : \tau, y : \tau', \Gamma' \vdash_Z e : \tau}\ \mathbf{exchange}$$

Figure 2: Typing rules, first version

$$\frac{k_1 \leq k \qquad A <: A_1}{!_k A <: !_{k_1} A_1}\ \mathsf{bang} \qquad \frac{\tau_1 <: \tau \qquad \tau' <: \tau_1'}{\tau \multimap \tau' <: \tau_1 \multimap A_1'}\ \mathsf{arrow} \qquad \frac{}{\mathtt{b} <: \mathtt{b}}\ \mathsf{base}$$

Figure 3: subtyping

Theorem 1 (Substitution)**.** 1. If $\Gamma, x : \tau' \vdash_Z e : \tau$ and $\vdash_{Z'} v : \tau'$, then $\Gamma \vdash_{\max(Z,Z')} e[v/x] : \tau$.

Proof. By induction on the typing derivation.

$$\frac{}{\Gamma, x :!_Z A \vdash_Z x :!_Z A}\ \textbf{Ax}$$

Assume $\vdash_{Z'} v :!_Z A$, TS: $\Gamma \vdash_{\max(Z,Z')} x[v/x] : \tau$. proved by subtype rule on the assumption.

$$\frac{}{\Gamma, y : \tau', x :!_Z A \vdash_Z x :!_Z A}\ \textbf{Ax2}$$

Assume $\vdash_{Z'} v :!_Z A$, TS: $\Gamma, x :!_Z A \vdash_{\max(Z,Z')} x[v/y] : \tau$. proved by rule AX and then subtype.

$$\frac{\Gamma, x : \tau_1, y : \tau' \vdash_Z e : \tau_2}{k + \Gamma, y : k + \tau' \vdash_{k+Z} \lambda x.e :!_k(\tau_1 \multimap \tau_2)}\ \textbf{lambda}$$

Assume $\vdash_{k+Z'} v : k + \tau'$, TS: $k + \Gamma \vdash_{\max(k+Z,k+Z')} (\lambda x.e)[v/y] : \tau$. From the Lemma 2 on the assumption, we know: $\vdash_{Z'} v : \tau'$ (1).
By Induction hypothesis on the premise, we get: $\Gamma, x : \tau_1 \vdash_{\max(Z,Z')} e[v/y] : \tau_2$ (2). By rule lambda, we conclude that $k + \Gamma \vdash_{k+(\max(Z,Z)} \lambda x.e[v/y] : \tau_2$.

$$\frac{\Gamma_1, x : \tau' \vdash_{Z_1} e_1 :!_0(\tau_1 \multimap \tau_2) \qquad \Gamma_2, x : \tau'', \vdash_{Z_2} e_2 : \tau_1}{\max(\Gamma_1, \Gamma_2), x : \max(\tau', \tau'') \vdash_{\max(Z_1,Z_2)} e_1\ e_2 : \tau_2}\ \textbf{app}$$

Assume $\vdash_{Z'} v : \max(\tau', \tau'')$, TS: $\max(\Gamma_1, \Gamma_2) \vdash_{\max(Z_1,Z_2,Z')} (e_1\ e_2)[v/x] : \tau_2$.
From the definition of $\max(\tau', \tau'')$, we know that τ' and τ'' have similar form.
Let us assume $\tau' =!_{k_1} A$ and $\tau'' =!_{k_2} A$ so that $\max(\tau', \tau'') =!_{\max(k_1,k_2)} A$.
From the Lemma 2 on the assumption, we have $\vdash_{Z'-(\max(k_1,k_2)-k_1)} v :!_{k_1} A$ (1) and $\vdash_{Z'-(\max(k_1,k_2)-k_2)} v :!_{k_2} A$ (2).
By induction hypothesis on (1) and (2) respctively, we know that: $\Gamma_1 \vdash_{\max(Z_1,Z'-(\max(k_1,k_2)-k_1))} e_1[v/x] :!_0(\tau_1 \multimap \tau_2)$ (3) and $\Gamma_2 \vdash_{\max(Z_2,Z'-(\max(k_1,k_2)-k_2))} e_2[v/x] : \tau_1$ (4).
By the rule app and (3), (4), we conclude that

$$\max(\Gamma_1, \Gamma_2) \vdash_{\max(\max(Z_1,Z'-(\max(k_1,k_2)-k_1)),\max(Z_2,Z'-(\max(k_1,k_2)-k_2)))} e_1[v/x]\ e_2[v/x] : \tau_2 \ (5).$$

Because $\max(Z' - (\max(k_1, k_2) - k_1)), Z' - (\max(k_1, k_2) - k_2)) \le Z'$, by subtype, we raise the adaptivity to $\max(Z_1, Z_2, Z')$ from (5).

$$\frac{\Gamma_1, x : \tau' \vdash_{Z_1} e_1 :!_0(\tau_1 \multimap \tau_2) \qquad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\max(\Gamma_1, \Gamma_2), x : \tau' \vdash_{\max(Z_1,Z_2)} e_1\ e_2 : \tau_2}\ \textbf{app2}$$

It is another case for application when x only appear in the first premise. In this case, $e_2[v/x] = e_2$. Another case when variable x only appears in the second premise can be proved in a similar way.
Assume $\vdash_{Z'} v : \tau'$. TS:$\max(\Gamma_1, \Gamma_2) \vdash_{\max(Z_1,Z_2,Z')} (e_1\ e_2)[v/x] : \tau_2$. By Induction Hypothesis on the first premise using the assumption, we get: $\Gamma_1 \vdash_{\max(Z_1,Z')} e_1[v/x] :!_0(\tau_1 \multimap \tau_2)$ (1). By the rule app using (1) and the second premise, we conclude that

$$\max(\Gamma_1, \Gamma_2) \vdash_{\max(\max(Z_1,Z'),Z_2)} e_1[v/x]\ e_2 : \tau_2$$

$$\frac{\Gamma, x : \tau' \vdash_Z e :!_k A}{\Gamma', 1 + \Gamma, x : 1 + \tau' \vdash_{1+Z} \delta(e) :!_k A}\ \textbf{delta}$$

Assume $\vdash_{Z'+1} v : 1 + \tau'$, TS: $\Gamma', 1 + \Gamma \vdash_{\max(1+Z, 1+Z')} \delta(e)[v/x] :!_k A$. By Lemma 2 on the assumption, we have $\vdash_{Z'} v : \tau'$ (1). By IH on the first premise along with (1), we have: $\Gamma \vdash_{\max(Z,Z')} e[v/x] :!_k A$ (2). By the rule delta using (2), we conclude that $\Gamma', 1 + \Gamma \vdash_{1+(Z,Z')} \delta(e[v/x]) :!_k A$. □

Lemma 2 (Parameter Decreasing)**.** if $k + \Gamma \vdash_Z v : k + \tau$, then exists Z' so that $\Gamma \vdash_{Z'} v : \tau$ and $Z' \leq Z - k$.

Proof. if v is a constant, then it is trivial, assume $\tau =!_r \mathtt{b}$, choose $Z' = r, k' = k$, from the rule b.
If $v = \lambda x.e$. Assume $\tau =!_r \tau_1 \multimap A_2$, then $k + \tau =!_{k+r} \tau_1 \multimap A_2$. From its typing derivation, we know: $\Gamma - r, x : \tau_1, \vdash_{Z-(k+r)} e : \tau_2$ (1). Choose $Z' = Z - r$, we know that $\Gamma \vdash_{Z'} v :!_r \tau_1 \multimap A_2$ from the rule lambda. □