Adaptivity analysis

$$\begin{array}{llll}
\text{Expr.} & e & ::= & x \mid e_1\ e_2 \mid \lambda x.e \\
 & & & \texttt{true} \mid \texttt{false} \mid \texttt{if}\ e\ \texttt{then}\ e_2\ \texttt{else}\ e_3 \mid c \mid \delta(e) \\
\text{Value} & v & ::= & c \mid \lambda x.e \\
\text{Environment} & \theta & ::= & x_1 \mapsto (v_1, R_1), \ldots, x_n \mapsto (v_n, R_n)
\end{array}$$

[MG: I renamed var2 into var.]

$$\begin{array}{llll}
\text{Index Term} & k, Z & ::= & i \mid n \\
\text{Linear type} & \tau & ::= & A \multimap^{Z} \tau \mid \texttt{b} \mid \texttt{bool} \\
\text{Nonlinear Type} & A & ::= & !_k \tau \\
\text{Typing context} & \Gamma & ::= & x_1 : A_1, \ldots, x_n : A_n
\end{array}$$

$$\frac{}{v, \theta \Downarrow^{0} v, \theta}\ \textsf{val} \qquad \frac{\theta(x) = (v, \theta_1, R)}{x, \theta \Downarrow^{R} v, \theta_1}\ \textsf{var} \qquad \frac{}{c, \theta \Downarrow^{0} c, \theta}\ \textsf{const}$$

$$\frac{}{\lambda x.e, \theta \Downarrow^{0} \lambda x.e, \theta}\ \textsf{lambda}$$

$$\frac{e_2, \theta_2 \Downarrow^{R_2} v_2, \theta'_2 \qquad \textsf{fresh } x' \qquad \begin{array}{c} e_1, \theta_1 \Downarrow^{R_1} \lambda x.e, \theta'_1 \\ e[x'/x], \theta'_1[x' \to (v_2, \theta'_2, R_2)] \Downarrow^{R_3} v, \theta_3 \end{array}}{e_1\ e_2, (\theta_1 \uplus \theta_2) \Downarrow^{R_1 + R_3} v, \theta_3}\ \textsf{app}$$

$$\frac{e, \theta \Downarrow^{R} v, \theta_1 \qquad \delta(v\theta) = v' \qquad FV(v') = \emptyset}{\delta(e), \theta \Downarrow^{R+1} v, \theta_1}\ \textsf{delta}$$

$$\frac{e, \theta \Downarrow^{R} \texttt{false}, \theta' \qquad e_2, \theta \Downarrow^{R_2} v_2, \theta_2}{\texttt{if } e \texttt{ then } e_1 \texttt{ else } e_2, \theta \Downarrow^{R+R_2} v_2, \theta_2}\ \textsf{if-f}$$

$$\frac{e, \theta \Downarrow^{R} \texttt{true}, \theta' \qquad e_1, \theta \Downarrow^{R_1} v_1, \theta_1}{\texttt{if } e \texttt{ then } e_1 \texttt{ else } e_2, \theta \Downarrow^{R+R_1} v_1, \theta_1}\ \textsf{if-t}$$

$$\begin{aligned} \theta_1 \uplus \emptyset &\triangleq \theta_1 \\ \emptyset \uplus \theta_2 &\triangleq \theta_2 \end{aligned}$$

Figure 1: Big-step semantics

$$\frac{}{v, \theta \Downarrow^0 v, \theta}\ \textsf{val} \qquad \frac{\theta(x) = (v, \theta_1, R)}{x, \theta \Downarrow^R v, \theta_1}\ \textsf{var} \qquad \frac{}{c, \theta \Downarrow^0 c, \theta}\ \textsf{const}$$

$$\frac{}{\lambda x.e, \theta \Downarrow^0 \lambda x.e, \theta}\ \textsf{lambda}$$

$$\frac{\begin{array}{c} e_1, \theta_1 \Downarrow^{R_1} \lambda x.e, \theta_1' \\ e_2, \theta_2 \Downarrow^{R_2} v_2, \theta_2' \qquad \textsf{fresh } x' \qquad e[x'/x], \theta_1'[x' \to (v_2, \theta_2', R_2)] \Downarrow^{R_3} v, \theta_3 \end{array}}{e_1\ e_2, (\theta_1 \uplus \theta_2) \Downarrow^{[\max(R_1, R_3)]} v, \theta_3}\ \textsf{app}$$

$$[\frac{e, \theta \Downarrow^R v, \theta_1}{\delta(e), \theta \Downarrow^{R+1} \delta(v), \theta_1}\ \textsf{delta}] \qquad \frac{e, \theta \Downarrow^R \texttt{false}, \theta' \qquad e_2, \theta \Downarrow^{R_2} v_2, \theta_2}{\texttt{if } e \texttt{ then } e_1 \texttt{ else } e_2, \theta \Downarrow^{R+R_2} v_2, \theta_2}\ \textsf{if-f}$$

$$\frac{e, \theta \Downarrow^R \texttt{true}, \theta' \qquad e_1, \theta \Downarrow^{R_1} v_1, \theta_1}{\texttt{if } e \texttt{ then } e_1 \texttt{ else } e_2, \theta \Downarrow^{R+R_1} v_1, \theta_1}\ \textsf{if-t}$$

$$[\frac{\begin{array}{c} e, \theta \Downarrow^R (e_1, e_2), \theta_1 \qquad e_1, \theta_1 \Downarrow^{R_1} v_1, \theta_1' \qquad e_2, \theta_1 \Downarrow^{R_2} v_2, \theta_2' \\ \textsf{fresh } x', y' \qquad e'[x'/x][y'/y], \theta_1[x' \to (v_1, \theta_1', R_1), y' \to (v_2, \theta_2', R_2)] \Downarrow^{R'} v, \theta' \end{array}}{\texttt{let } (x, y) = e \texttt{ in } e', \theta \Downarrow^{\max(R, R')} v, \theta'}\ \textsf{bind}]$$

$$\begin{array}{rcl} \theta_1 \uplus \emptyset & \triangleq & \theta_1 \\ \emptyset \uplus \theta_2 & \triangleq & \theta_2 \end{array}$$

Figure 2: Big-step semantics - Jan.28

$$\frac{}{\Gamma, x :!_1\tau \vdash_0 x : \tau}\ \textbf{Ax} \qquad \frac{}{\Gamma \vdash_0 c : \mathtt{b}}\ \textbf{const} \qquad \frac{\Gamma, x : A \vdash_Z e : \tau}{\Gamma \vdash_0 \lambda x.e : A \multimap^Z \tau}\ \textbf{lambda}$$

$$\frac{\Gamma_1 \vdash_{Z_1} e_1 :!_k\tau_1 \multimap^Z \tau_2 \qquad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\Gamma_1 + k \times \Gamma_2 \vdash_{Z_1+k\times Z_2+Z} e_1\ e_2 : \tau_2}\ \textbf{app} \qquad \frac{\Gamma \vdash_Z e : \mathtt{b}}{\Gamma \vdash_{1+Z} \delta(e) : \mathtt{b}}\ \textbf{delta}$$

$$\frac{\Gamma' \vdash_{Z'} e : \tau' \qquad \Gamma' \leqslant \Gamma \qquad Z' \leq Z \qquad \tau' <: \tau}{\Gamma \vdash_Z e : \tau}\ \textbf{subtype}$$

$$\frac{\Gamma, y : \tau', x : \tau, \Gamma' \vdash_Z e : \tau}{\Gamma, x : \tau, y : \tau', \Gamma' \vdash_Z e : \tau}\ \textbf{exchange}$$

$$\frac{\begin{array}{c}\Gamma \vdash_Z e : \mathtt{bool} \qquad \Gamma_1 \vdash_{Z_1} e_1 : \tau \\ \Gamma_2 \vdash_{Z_2} e_2 : \tau \qquad \Gamma' = \Gamma + \Gamma_1 + \Gamma_2 \qquad Z' = Z + \max(Z_1, Z_2)\end{array}}{\Gamma' \vdash_{Z'} \mathtt{if}\ e\ \mathtt{then}\ e_1\ \mathtt{else}\ e_2 : \tau}\ \textbf{if}$$

$$\begin{array}{lcll}
k \times \Gamma & \triangleq & \Gamma & k = 1 \\
 & \triangleq & !_0\Gamma & k = 0 \\
!_{k_1}\tau + !_{k_2}\tau & \triangleq & !_{\max(k_1,k_2)}\tau & \\
\Gamma + \emptyset & \triangleq & \Gamma & \\
\emptyset + \Gamma & \triangleq & \Gamma & \\
([x : A], \Gamma) + ([x : A'], \Delta) & \triangleq & [x : A + A'], \Gamma + \Delta & \\
\Gamma <: \Delta & \triangleq & \mathtt{dom}(\Gamma) = \mathtt{dom}(\Delta) & \\
 & & \wedge \forall x \in \mathtt{dom}(\Gamma), \Delta(x) <: \Gamma(x) &
\end{array}$$

Figure 3: Typing rules, first version

$$\frac{k_1 \leq k \qquad A <: A_1}{!_k A <: !_{k_1} A_1}\ \mathsf{bang} \qquad \frac{Z \leq Z' \qquad \tau_1 <: \tau \qquad \tau' <: \tau_1'}{\tau \multimap^Z \tau' <: \tau_1 \multimap^{Z'} A_1'}\ \mathsf{arrow}$$

$$\frac{}{\mathtt{b} <: \mathtt{b}}\ \mathsf{base}$$

Figure 4: subtyping

$$\frac{\theta(x) = (v, \theta', R) \qquad \vdash_Z (v, \theta') : \tau}{\vdash_{R+Z} (x, \theta) : \tau}\ \textbf{C-Ax} \qquad \frac{}{\vdash_0 (c, \theta) : \mathsf{b}}\ \textbf{C-const}$$

$$\frac{\mathsf{fresh}\ x' \quad \forall R' \qquad \begin{array}{c} \vdash_{Z'} (v', \theta') : \tau_1 \\ \vdash_{S+k\times(R'+Z')+Z} (e[x'/x], \theta[x' \to (v', \theta', R')]) : \tau_2 \end{array}}{\vdash_S (\lambda x.e, \theta) :!_k \tau_1 \multimap^Z \tau_2}\ \textbf{C-lambda}$$

$$\frac{\vdash_{Z_1} (e_1, \theta_1) :!_k \tau_1 \multimap^Z \tau_2 \qquad \vdash_{Z_2} (e_2, \theta_2) : \tau_1}{\vdash_{Z_1+k\times Z_2+Z} (e_1\ e_2, \theta_1 \uplus \theta_2) : \tau_2}\ \textbf{C-app}$$

$$\frac{\vdash_Z (e, \theta) : \mathsf{b}}{\vdash_{1+Z} (\delta(e), \theta) : \mathsf{b}}\ \textbf{C-delta}$$

$$\begin{array}{lll} \theta & \triangleq (x_i \to (v_i, \theta_i, R_i)) & i \in \mathbb{N} \\ (x_i :!_k \tau_i), \Gamma \vDash (x_i \to (v_i, \theta_i, R_i)) \uplus \theta & \triangleq \quad \vdash_{\_} (v_i, \theta_i) : \tau_i & \wedge\ \Gamma \vDash \theta \end{array}$$

Figure 5: Typing rules, configure

$$\begin{array}{lll} [\![\mathsf{b}]\!]_{\mathrm{V}} & = & \{(c, \theta, Z)\} \\ [\![!_k \tau]\!]_{\mathrm{V}} & = & \{(v, \theta, Z) | (v, \theta, Z) \in [\![\tau]\!]_{\mathrm{V}}\} \\ [\![!_k \tau_1 \multimap^Z \tau_2]\!]_{\mathrm{V}} & = & \{(\lambda x.e, \theta, Z_1) \mid \forall v', \theta', Z'.(v', \theta', Z') \in [\![!_k \tau_1]\!]_{\mathrm{V}}. \\ & & \Longrightarrow \mathsf{fresh}\ x' \wedge \\ & & \forall R.(e[x'/x], \theta[x' \mapsto (v', \theta', R)]) \in [\![\tau_2]\!]_{\mathrm{E}}^{Z_1+Z+k\times(R+Z')}\} \\ \\ [\![\tau]\!]_{\mathrm{E}}^{Z} & = & \{(e, \theta) \mid (e, \theta \Downarrow^R v, \theta') \\ & & \qquad \Longrightarrow R \leq Z \wedge (v, \theta', Z - R) \in [\![\tau]\!]_{\mathrm{V}})\} \end{array}$$

Figure 6: Logical relation without step-indexing

Theorem 1 (Monotonicity).

1. If $(e, \theta) \in [\![\tau]\!]_{\mathrm{E}}^{Z}$ and $Z' \geq Z$, then $(e, \theta) \in [\![\tau]\!]_{\mathrm{E}}^{Z'}$.
2. If $(v, \theta, Z) \in [\![\tau]\!]_{\mathrm{V}}$ and $Z' \geq Z$, then $(v, \theta, Z') \in [\![\tau]\!]_{\mathrm{V}}$.
3. If $\vdash_Z (e, \theta) : \tau$ and $Z \leq Z'$, then $\vdash_{Z'} (e, \theta) : \tau$.
4. If $\Gamma \vdash_Z e : \tau$ and $Z \leq Z'$, then $\Gamma \vdash_{Z'} e : \tau$.

$$F_{c2t}(e, x_i) = \begin{array}{ll} 1 & x_i \in \mathsf{FV}(e) \\ 0 & x_i \notin \mathsf{FV}(e) \end{array}$$

Theorem 2 (ConfigurationToTyping)**.** 1. If $\vdash_Z (e, \theta) : \tau$ and $\forall x_i \in \texttt{dom}(\theta).\theta(x_i) = (v_i, \theta_i, R_i) \wedge \exists Z_i. \vdash_{Z_i} (v_i, \theta_i) : \tau_i$, then $x_i :!_{F_{c2t}(e,x_i)}\tau_i \vdash_{Z - F_{c2t}(e,x_i) \times (R_i + Z_i)} e : \tau$.

$$
\begin{array}{ll}
F(e,\phi) & \textit{where} \quad \phi(x_i) = (k_i, R_i, Z_i) \\
F(x,\phi) & = \sum_{x_i \in \mathsf{FV}(x)} k_i \times (R_i + Z_i) \\
F(\lambda x.e,\phi) & = \sum_{x_i \in \mathsf{FV}(\lambda x.e)} k_i \times (R_i + Z_i) \\
F(\delta(e),\phi) & = \sum_{x_i \in \mathsf{FV}(\delta(e))} k_i \times (R_i + Z_i) \\
F(c,\phi) & = 0 \\
F(e_1\, e_2,\phi) & = F(e_1,\phi) + F(e_2,\phi)
\end{array}
$$

Theorem 3 (TypingtoConfiguration)**.** 1. If $x_1 :!_{k_1}\tau_1, \ldots, x_i :!_{k_i}\tau_i \vdash_Z e : \tau$, and $\vdash_{Z_I} (v_i, \theta_i) : \tau_i$, and $\theta = [x_1 \to (v_1, \theta_1, R_1), \ldots, x_i \to (v_i, \theta_i, R_i)]$, $\phi = [x_1 \to (k_1, R_1, Z_1), \ldots, x_i \to (k_i, R_i, Z_i)]$, then $\vdash_{Z+F(e,\phi)} (e, \theta) : \tau$.

Proof. By induction on the typing derivation.

Case $\dfrac{}{\Gamma, x :!_1\tau \vdash_0 x : \tau}$ **Ax**

assume $\vdash_Z (v, \theta') : \tau$ and $\vdash_{Z_i} (v_i, \theta_i) : \tau_i$,
assume $\theta = [x \to (v, \theta', R)] \uplus [x_1 \to (v_1, \theta_1, R_1), \ldots, x_i \to (v_i, \theta_i, R_i)]$.
So $\phi = [x \to (1, R, Z)] \uplus [x_1 \to (k_1, R_1, Z_1), \ldots, x_i \to (k_i, R_i, Z_i)]$
$F\big(x, \phi\big) = 1 \times (R + Z)$.
TS:$\vdash_{0+1\times(R+Z)} (x, \theta) : \tau$.
We conclude from the configuratio rule C-Ax.

$$
\dfrac{\theta(x) = (v, \theta', R) \qquad \vdash_Z (v, \theta') : \tau}{\vdash_{R+Z} (x, \theta) : \tau} \ \textbf{C-Ax}
$$

Case $\dfrac{\Gamma, x : A \vdash_Z e : \tau}{\Gamma \vdash_0 \lambda x.e : A \multimap^Z \tau}$ **lambda**

let $\Gamma = x_1 :!_{k_1}\tau_1, \ldots, x_i :!_{k_i}\tau_i$ and $A = !_k\tau_1$.
Assume $\vdash_{Z'} (v', \theta') : \tau_1$ (1) and $\vdash_{Z_i} (v_i, \theta_i) : \tau_i$.
Assume $\theta = [x_1 \to (v_1, \theta_1, R_1), \ldots, x_i \to (v_i, \theta_i, R_i)]$.
So $\phi = [x_1 \to (k_1, R_1, Z_1), \ldots, x_i \to (k_i, R_i, Z_i)]$
TS: $\vdash_{0+F(\lambda x.e,\phi)} (\lambda x.e, \theta) :!_k\tau_1 \multimap^Z \tau_2$.
Let $S = \sum_{x_i \in \mathsf{FV}(\lambda x.e)} k_i \times (R_i + Z_i)$.
From assumption (1), we know : $\vdash_{Z'} (v', \theta) : \tau_1$ $(\star)$.

Take a fresh variable x', doing alpha renaming on the premise, pick R' so that $\theta' = [x' \to (v', \theta', R')] \uplus \theta$ and $\phi' = [x' \to (k, R', Z')] \uplus \phi$.

By induction hypothesis on the premise, we know: $\vdash_{Z+F(e[x'/x],\phi')} (e[x'/x], [x' \to (v', \theta', R')] \uplus \theta) : \tau$ $(\diamond)$.

$F(e[x'/x], \phi') = \sum_{x_i \in \mathsf{FV}(\lambda x.e)} k_i \times (R_i + Z_i) + k \times (R' + Z') = S + k \times (R' + Z')$.

We can conclude the following by the configuration rule.

$$\frac{\text{fresh } x' \qquad \begin{array}{c}\vdash_{Z'} (v',\theta') : \tau_1 \ (\star) \\ \vdash_{S+k\times(R'+Z')+Z} (e[x'/x], \theta[x' \to (v',\theta',R')]) : \tau \ (\diamond)\end{array}}{\vdash_S (\lambda x.e, \theta) :!_k \tau_1 \multimap^Z \tau} \textbf{ C-lambda}$$

Case $\dfrac{\Gamma_1 \vdash_{Z_1} e_1 :!_k \tau_1 \multimap^Z \tau_2 \qquad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\Gamma_1 + k \times \Gamma_2 \vdash_{Z_1+k\times Z_2+Z} e_1\ e_2 : \tau_2}$ **app**

Let us assume $\Gamma_1 = x_i :!_{k_i}\tau_i$ and $\Gamma_2 = x'_i :!_{k'_i}\tau'_i$,($\Gamma_1$ and Γ_2 may overleap.).

Forall the variables x''_i in $\texttt{dom}(\Gamma_1 + k \times \Gamma_2)$, we assume $(\Gamma_1 + k \times \Gamma_2)(x''_i) = !_{k''_i}\tau''_i$ so that $\vdash_{Z''_i} (v''_i, \theta''_i) : \tau''_i$

assume $\theta = [x''_1 \to (v''_1, \theta''_1, R''_1), \ldots, x''_i \to (v''_i, \theta''_i, R''_i)]$.

So $\phi = [x''_1 \to (k''_1, R''_1, Z''_1), \ldots, x''_i \to (k''_i, R''_i, Z''_i)]$

TS: $\vdash_{Z_1+k\times Z_2+Z+F(e_1\ e_2,\phi)} (e_1\ e_2, \theta) : \tau_2$.

let $\theta_1 = \{[x''_i \to (v''_i, \theta''_i, R''_i)] | x''_i \in \texttt{dom}(\Gamma_1)\}$.

let $\theta_2 = \{[x''_i \to (v''_i, \theta''_i, R''_i)] | x''_i \in \texttt{dom}(\Gamma_2)\}$.

We know $\texttt{dom}(\theta) = \texttt{dom}(\theta_1) \uplus \texttt{dom}(\theta_2) \implies \theta = \theta_1 \uplus \theta_2$

We set $\phi_1 = \{[x_i \to (k''_i, R''_i, Z''_i)] | x''_i \in \texttt{dom}(\Gamma_1)\}$. and $\phi_2 = \{[x_i \to (k''_i, R''_i, Z''_i)] | x''_i \in \texttt{dom}(\Gamma_2)\}$

By induction hypothesis on the first premise, we have: $\vdash_{Z_1+F(e_1,\phi_1)} (e_1, \theta_1) :!_k \tau_1 \multimap^Z \tau_2 \ (\star)$.

By induction hypothesis on the second premise, we have: $\vdash_{Z_2+F(e_2,\phi_2)} (e_2, \theta_2) : \tau_1$.

From the definition, we know: $F(e_1, \phi_1) = F(e_1, \phi)$ and $F(e_2, \phi_2) = F(e_2, \phi)$.

From the configuration rule C-app, we get:

$$\frac{\vdash_{Z_1+F(e_1,\theta_1,\Gamma_1)} (e_1, \theta_1) :!_k \tau_1 \multimap^Z \tau_2 \ (\star) \qquad \vdash_{Z_2+F(e_2,\theta_2,\Gamma_2)} (e_2, \theta_2) : \tau_1 \ (\diamond)}{\vdash_{Z_1+F(e_1,\theta_1,\Gamma_1)+k\times(Z_2+F(e_2,\theta_2,\Gamma_2))+Z} (e_1\ e_2, \theta_1 \uplus \theta_2) : \tau_2 \ (\clubsuit)} \textbf{ C-app}$$

Because $F(e_1, \phi_1) + k \times F(e_2, \phi_2) \le F(e_1\ e_2, \phi)$, so we conclude :$Z_1 + F(e_1, \phi_1) + k \times (Z_2 + F(e_2, \phi_2)) + Z \le Z_1 + k \times Z_2 + Z + F(e_1\ e_2, \phi)$.

The TS can be shown using Lemma 1 on the $\clubsuit$.

$\square$

Theorem 4 (ConfigurationSoundness)**.** 1. $\vdash_Z (e, \theta) : \tau$, then $(e, \theta) \in [\![\tau]\!]_{\mathrm{E}}^{Z}$

2. $\vdash_Z (v, \theta) : \tau$, then $(v, \theta, Z) \in [\![\tau]\!]_{\mathrm{V}}$.

Proof. Statement (1) is proved by induction on the configuration derivation.

$$\textbf{Case}\ \frac{\theta(x) = (v, \theta', R) \qquad \vdash_Z (v, \theta') : \tau\ (\star)}{\vdash_{R+Z} (x, \theta) : \tau}\ \textbf{C-Ax}$$

TS: $(x, \theta) \in [\![\tau]\!]_{\mathrm{E}}^{R+Z}$.
Let us first assume:

$$\frac{\theta(x) = (v, \theta', R)}{x, \theta \Downarrow^{R} v, \theta'}\ \mathsf{var2}$$

By unfolding the definition, STS: $R \leq R + Z$ and $(v, \theta', Z) \in [\![\tau]\!]_{\mathrm{V}}$.
By induction hypothesis on $(\star)$, we know $(v, \theta') \in [\![\tau]\!]_{\mathrm{E}}^{Z}$, by unfolding its definition and $v, \theta \Downarrow^{0} v, \theta$, we know that : $(v, \theta', Z) \in [\![\tau]\!]_{\mathrm{V}}$.

$$\textbf{Case}\ \frac{\mathsf{fresh}\ x' \qquad \begin{array}{c}\vdash_{Z'} (v', \theta') : \tau_1\ (\star) \\ \vdash_{S+k\times(R'+Z')+Z} (e[x'/x], \theta[x' \to (v', \theta', R')]) : \tau_2\ (\diamond)\end{array}}{\vdash_S (\lambda x.e, \theta) :!_k\tau_1 \multimap^{Z} \tau_2}\ \textbf{C-lambda}$$

TS: $(\lambda x.e, \theta) \in [\![!_k\tau_1 \multimap^{Z} \tau_2]\!]_{\mathrm{E}}^{S}$.
By unfoling the definition, as well as $\lambda x.e$ is value, we know :$v, \theta \Downarrow^{0} v, \theta$.
STS: $0 \leq S$ and $(\lambda x.e, \theta, S - 0) \in [\![!_k\tau_1 \multimap^{Z} \tau_2]\!]_{\mathrm{V}}$.
By induction hypothesis on $(\star)$, we know : $(v', \theta', Z' - 0) \in [\![!_k\tau_1]\!]_{\mathrm{V}}$ (1).
Unfolding the definition of $[\![!_k\tau_1 \multimap^{Z} \tau_2]\!]_{\mathrm{V}}$, pick $(v', \theta', Z') \in [\![!_k\tau_1]\!]_{\mathrm{V}}$.
STS: $\mathsf{fresh}\ x' \wedge \forall R.(e[x'/x], \theta[x' \to (v', \theta', R)]) \in [\![\tau_2]\!]_{\mathrm{E}}^{S+Z+k\times(R+Z')}$.
Pick $R = R'$. STS: $(e[x'/x], \theta[x' \to (v', \theta', R')]) \in [\![\tau_2]\!]_{\mathrm{E}}^{S+Z+k\times(R'+Z')}$
It is proved by Induction hypothesis on $(\diamond)$.

$$\textbf{Case}\ \frac{\vdash_{Z_1} (e_1, \theta_1) :!_k\tau_1 \multimap^{Z} \tau_2\ (\star) \qquad \vdash_{Z_2} (e_2, \theta_2) : \tau_1\ (\diamond)}{\vdash_{Z_1+k\times Z_2+Z} (e_1\ e_2, \theta_1 \uplus \theta_2) : \tau_2}\ \textbf{C-app}$$

TS: $(e_1\ e_2, \theta_1 \uplus \theta_2) \in [\![\tau_2]\!]_{\mathrm{E}}^{Z_1+k\times Z_2+Z}$.
Let us first assume:

$$\frac{\mathsf{fresh}\ x' \qquad \begin{array}{c} e_1, \theta_1 \Downarrow^{R_1} \lambda x.e, \theta_1'\ (a) \qquad e_2, \theta_2 \Downarrow^{R_2} v_2, \theta_2'\ (b) \\ e[x'/x], \theta_1'[x' \to (v_2, \theta_2', R_2)] \Downarrow^{R_3} v, \theta_3\ (c)\end{array}}{e_1\ e_2, (\theta_1 \uplus \theta_2) \Downarrow^{R_1+R_3} v, \theta_3}\ \mathsf{app.}$$

By unfolding the definition:
STS1: $R_1 + R_3 \leq Z_1 + k \times Z_2 + Z$.
STS2: $(v, \theta_3, Z_1 + k \times Z_2 + Z - (R_1 + R_3)) \in [\![\tau_2]\!]_{\mathrm{V}}$.
By Induction hypothesis on $(\star)$, we get: $(e_1, \theta_1) \in [\![!_k\tau_1 \multimap^{Z} \tau_2]\!]_{\mathrm{E}}^{Z_1}$ (1).

Unfolding (1), from the assumption (a), we know: $R_1 \leq Z_1 \wedge (\lambda x.e, \theta_1', Z_1 - R_1) \in [\![!_k \tau_1 \multimap^Z \tau_2]\!]_{\mathrm{V}}$ (2).

By Induction hypothesis on $(\diamond)$, we get: $(e_2, \theta_2) \in [\![\tau_1]\!]_{\mathrm{E}}^{Z_2}$ (3).

Unfolding (3), from the assumption (b), we know: $R_2 \leq Z_2 \wedge (v_2, \theta_2', Z_2 - R_2) \in [\![\tau_1]\!]_{\mathrm{V}}$ (4).

Unfolding (2), pick $(v_2, \theta_2', Z_2 - R_2) \in [\![\tau_1]\!]_{\mathrm{V}}$ from (4).

We know: $\mathsf{fresh}\ x' \wedge \forall R.(e[x'/x], \theta_1'[x' \to (v_2, \theta_2', R)]) \in [\![\tau_2]\!]_{\mathrm{E}}^{(Z_1 - R_1) + Z + k \times (R + Z_2 - R_2)}$.

Pick $R = R_2$. We have: $\mathsf{fresh}\ x' \wedge (e[x'/x], \theta_1'[x' \to (v_2, \theta_2', R_2)]) \in [\![\tau_2]\!]_{\mathrm{E}}^{(Z_1 - R_1) + Z + k \times (R_2 + Z_2 - R_2)}$ (5).

Unfolding (5),we conclude that: $R_3 \leq (Z_1 - R_1) + Z + k \times (Z_2 - R_2 + R_2)$ (6). and $(v, \theta_3, (Z_1 - R_1) + Z + k \times R_2 - R_3) \in [\![\tau_2]\!]_{\mathrm{V}}$ (7) .

STS1 is proved by using both (6) . STS2 is proved by (7).

Case $\dfrac{\vdash_Z (e, \theta) : \mathtt{b}\ (\star)}{\vdash_{1+Z} (\delta(e), \theta) : \mathtt{b}}$ **C-delta**

TS: $(\delta(e), \theta) \in [\![\mathtt{b}]\!]_{\mathrm{E}}^{1+Z}$. We first assume:

$$\dfrac{e, \theta \Downarrow^R v', \theta_1\ (a) \qquad \delta(v') = v}{\delta(e), \theta \Downarrow^{R+1} v, \theta_1}\ \mathsf{delta}.$$

By unfolding the definition:

STS1: $R + 1 \leq Z + 1$.

STS2: $(v, \theta_1, Z - R) \in [\![\mathtt{b}]\!]_{\mathrm{V}}$.

By induction hypothesis on $(\star)$, we get: $(e, \theta) \in [\![\mathtt{b}]\!]_{\mathrm{E}}^{Z}$. Unfold this statement, from the assumption (a), we get: $R \leq Z$ (1) and $(v', \theta_1, Z - R) \in [\![\mathtt{b}]\!]_{\mathrm{V}}$ (2).

STS1 is proved by (1), STS2 is proved by (2) and the interpretation of $\delta(v')$.

Statement (2) is proved by induction on the value v.

Case $\dfrac{\mathsf{fresh}\ x' \qquad \begin{array}{c}\vdash_{Z'} (v', \theta') : \tau_1\ (\star) \\ \vdash_{S + k \times R' + Z} (e[x'/x], \theta[x' \to (v', \theta', R')]) : \tau_2\ (\diamond)\end{array}}{\vdash_S (\lambda x.e, \theta) :!_k \tau_1 \multimap^Z \tau_2}$ **C-lambda**

TS: $(\lambda x.e, \theta, S) \in [\![!_k \tau_1 \multimap^Z \tau_2]\!]_{\mathrm{V}}$.

By induction hypothesis on $(\star)$, we know : $(v', \theta', Z' - 0) \in [\![!_k \tau_1]\!]_{\mathrm{V}}$ (1).

Unfolding the definition of $[\![!_k \tau_1 \multimap^Z \tau_2]\!]_{\mathrm{V}}$, pick $(v', \theta', Z') \in [\![!_k \tau_1]\!]_{\mathrm{V}}$.

STS: $\mathsf{fresh}\ x' \wedge \forall R.(e[x'/x], \theta[x' \to (v', \theta', R)]) \in [\![\tau_2]\!]_{\mathrm{E}}^{S + Z + k \times R}$.

Pick $R = R'$. STS: $(e[x'/x], \theta[x' \to (v', \theta', R')]) \in [\![\tau_2]\!]_{\mathrm{E}}^{S + Z + k \times R'}$

It is proved by Induction hypothesis on $(\diamond)$.

$\square$

1 Typable Approach

$$
\begin{array}{ll}
F(e, \phi) & \textit{where} \quad \phi(x_i) = (k_i, R_i, Z_i) \\
F(x, \phi) & = \sum_{x_i \in \mathsf{FV}(x)} k_i \times (R_i + Z_i) \\
F(\lambda x.e, \phi) & = \sum_{x_i \in \mathsf{FV}(\lambda x.e)} k_i \times (R_i + Z_i) \\
F(\delta(e), \phi) & = \sum_{x_i \in \mathsf{FV}(\delta(e))} k_i \times (R_i + Z_i) \\
F(c, \phi) & = 0 \\
F(e_1\ e_2, \phi) & = F(e_1, \phi) + F(e_2, \phi) \\
F(\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2, \phi) & = F(e, \phi) + \max(F(e_1, \phi), F(e_2, \phi))
\end{array}
$$

Definition 5 (Typable)**.** A closure $(e, [x_1 \to (v_1, \theta_1, R_1), \ldots, x_i \to (v_i, \theta_i, R_i)])$ is typable with type τ and adaptivity J if exists k_i

$$x_1 :!_{k_1}\tau_1, \ldots, !_{k_i}\tau_i \vdash_Z e : \tau$$

and each closure (v_i, θ_i) is also typable with type $!_{k_i}\tau_i$ and adaptivity Z_i, $\phi = [x_1 \to (k_1, R_1, Z_1), \ldots, x_i \to (k_i, R_i, Z_i)]$, $J = Z + F(e, \phi)$.

Definition 6 (ClosedClosure)**.** A closure (e, θ) is closed if $\mathsf{FV}(e) \subseteq \mathtt{dom}(\theta)$.

Lemma 7 (programTypable)**.** If $\vdash_Z e : \tau$, then $(e, \emptyset)$ is typable with τ and adaptivity Z.

Lemma 8 (TypableMono)**.** If a closure is D is typable with τ and Z, and $Z \leq Z'$, then D is typable with τ and Z'.

Lemma 9 (TypableSoundness)**.** If a closure D is typable with τ and J, and $D \Downarrow^R E$, then closure E is typable with τ and adaptivity $J - R$.

Proof. By induction on the evaluation semantics.

Case

$$\frac{\theta(x) = (v, \theta', R)}{x, \theta \Downarrow^R v, \theta'}\ \textsf{var2}$$

Suppose (x, θ) is typable with τ and J and $\theta = [x \to (v, \theta', R), x_1 \to (v_1, \theta_1, R_1), \ldots, x_i \to (v_i, \theta_i, R_i)]$

We know that: exists k_i so that $x_1 :!_{k_1}\tau_1, \ldots, !_{k_i}\tau_i, x :!_1\tau \vdash_0 x : \tau$, and the each closure (v_i, θ_i) is typable with τ_i and Z_i as well as (v, θ') is typable with τ and Z. and $J = 0 + (R + Z)$.

TS: (v, θ') is typable with τ and $J - R$, it is proved by Lemma 8 on the assumption.

Case

$$\frac{e_2, \theta_2 \Downarrow^{R_2} v_2, \theta_2' \qquad \textsf{fresh } x' \qquad \begin{array}{c} e_1, \theta_1 \Downarrow^{R_1} \lambda x.e, \theta_1' \\ e[x'/x], \theta_1'[x' \to (v_2, \theta_2', R_2)] \Downarrow^{R_3} v, \theta_3 \ (\clubsuit) \end{array}}{e_1\ e_2, (\theta_1 \uplus \theta_2) \Downarrow^{R_1+R_3} v, \theta_3} \textsf{ app}$$

Suppose for each variable $x_i \in \texttt{dom}(\theta_1)$, (v_i, θ_i) is tyable with type τ_i and I_i. For each variable $x_j \in \texttt{dom}(\theta_2)$, (v_j, θ_j) is tyable with type τ_j and I_j.

We assume exists k_i and $\Gamma_1 = x_1 :!_{k_1}\tau_1, \ldots, x_i :!_{k_i}\tau_i$ for $x_i \in \texttt{dom}(\theta_1)$, so that $\Gamma_1 \vdash_{Z_1} e_1 :!_k\tau_1 \multimap^Z \tau_2$.

Similarly, we assume exists k_j and $\Gamma_2 = x_1 :!_{k_1}\tau_1, \ldots, x_j :!_{k_j}\tau_j$ for $x_j \in \texttt{dom}(\theta_j)$, so that $\Gamma_2 \vdash_{Z_2} e_2 : \tau_1$.

By the typing rule app, we have:

$$\frac{\Gamma_1 \vdash_{Z_1} e_1 :!_k\tau_1 \multimap^Z \tau_2 \qquad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\Gamma_1 + k \times \Gamma_2 \vdash_{Z_1+k\times Z_2+Z} e_1\ e_2 : \tau_2} \textbf{ app}$$

We know that $\texttt{dom}(\Gamma_1 + k \times \Gamma_2) = \texttt{dom}(\Gamma_1) \cup \texttt{dom}(\Gamma_2) = \texttt{dom}(\theta_1 \uplus \theta_2)$. So forall all the closure (v_i, θ_i) assigned by variable $x_i \in \texttt{dom}(\Gamma_1 + k \times \Gamma_2)$, (v_i, θ_i) is typable with τ_i and I_i from our assumption.

$\phi = [x_1 \to (k_1, R_1, I_1), \ldots, x_i \to (k_i, R_i, I_i)]$ where $x_i \in \texttt{dom}(\theta_1 \uplus \theta_2)$.

So, we know that : $(e_1\ e_2, \theta_1 \uplus \theta_2)$ is typable with τ_2 and $J = Z_1 + k \times Z_2 + Z + F(e_1\ e_2, \phi)$.

TS: (v, θ_3) is typable with τ_2 and $J - (R_1 + R_3)$.

Set $\phi_1 = [x_1 \to (k_1, R_1, I_1), \ldots, x_i \to (k_i, R_i, I_i)]$ where $x_i \in \texttt{dom}(\theta_1)$.

Set $\phi_2 = [x_1 \to (k_1, R_1, I_1), \ldots, x_i \to (k_i, R_i, I_i)]$ where $x_i \in \texttt{dom}(\theta_2)$.

From our assumption, we also know that (e_1, θ_1) is typable with $!_k\tau_1 \multimap^Z \tau_2$ and $J_1 = Z_1 + F(e_1, \phi_1)$.

Similarly, we have: (e_2, θ_2) is typable with τ_1 and $J_2 = Z_2 + F(e_2, \phi_2)$ for $x_j \in \texttt{dom}(\theta_2)$ (2) .

By induction hypothesis on (1), (2) respectively, we know that:

$(\lambda x.e, \theta_1')$ is typable with $!_k\tau_1 \multimap^Z \tau_2$ and $J_1 - R_1$ (3).

(v_2, θ_2') typable with τ_1 and $J_2 - R_2$ (4).

From (3), we know the following if we assume exists k_i'' and $\Gamma_1'' = x_1'' :!_{k_1''}\tau_1'', \ldots, x_i'' :!_{k_i''}\tau_i''$, and $\theta_1'(x_i'') = (v_i'', \theta_i'', R_i'')$ and each closure (v_i'', θ_i'') is typable with τ_i'' and I_i''.

Set $\phi_1'' = [x_1'' \to (k_1'', R_1'', I_1''), \ldots, x_i'' \to (k_i'', R_i'', I_i'')]$ where $x_i'' \in \texttt{dom}(\theta_1')$.

$$\frac{\Gamma_1'', x :!_k\tau_1 \vdash_Z e : \tau_2 \ (\star)}{\Gamma_1'' \vdash_0 \lambda x.e :!_k\tau_1 \multimap^Z \tau_2} \textbf{ lambda}$$

and we know $J_1 - R_1 = 0 + F(\lambda x.e, \phi_1'')$.

Take a fresh variable x', from $(\star)$, we know: $\Gamma_1, x' :!_k\tau_1 \vdash_Z e[x/x'] : \tau_2$ $(\star\star)$.

Set $\phi_1''' = \phi_1''[x' \to (k, R_2, J_2 - R_2)]$.

From $(\star\star)$ and (4), we conclude that $(e[x'/x], \theta_1'[x' \to (v_2, \theta_2', R_2)])$ is typable with τ_2 and $Z + F(e[x'/x], \phi_1''')$.

Because $F(e[x'/x], \phi_1''') = F(\lambda x.e, \phi_1'') + k \times (R_2 + J_2 - R_2) = J_1 - R_1 + k \times J_2$.

By induction hypothesis using $(\clubsuit)$ and the above statement, we know: v, θ_3 is typable with τ_2 and $Z + J_1 - R_1 + k \times J_2 - R_3$.

Unfold J_1, J_2, we know $Z + J_1 - R_1 + k \times J_2 - R_3 = Z + Z_1 + F(e_1, \phi_1) + k \times Z_2 + k \times F(e_2, \phi_2) - (R_1 + R_3) \leq J - (R_1 + R_3)$.

The TS is proved by lemma 8 on the above statement.

Case

$$\frac{e, \theta \Downarrow^R \texttt{false}, \theta' \qquad e_2, \theta \Downarrow^{R_2} v_2, \theta_2}{\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2, \theta \Downarrow^{R+R_2} v_2, \theta_2}\ \textsf{if-f}$$

Assume for all the variables $x_i \in \texttt{dom}(\theta)$, $\theta(x_i) = (v_i, \theta_i, R_i)$ and (v_i, θ_i) is typable with τ_i and I_i.

We assume exists $\Gamma' = x_1 :!_{k_1}\tau_1, \ldots, x_i :!_{k_i}\tau_i$ and $\texttt{dom}(\Gamma') = \texttt{dom}(\theta)$, and assume the follwoing using the typing rule if.

$$\frac{\begin{array}{c}\Gamma \vdash_Z e : \texttt{bool}\ (\star) \qquad \Gamma_1 \vdash_{Z_1} e_1 : \tau \\ \Gamma_2 \vdash_{Z_2} e_2 : \tau\ (\diamond) \qquad \Gamma' = \Gamma + \Gamma_1 + \Gamma_2 \qquad Z' = Z + \max(Z_1, Z_2)\end{array}}{\Gamma' \vdash_{Z'} \texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 : \tau}\ \textsf{if}$$

Set $\phi = [x_1 \to (k_1, R_1, I_1), \ldots, x_i \to (k_i, R_i, I_i)]$ where $x_i \in \texttt{dom}(\theta)$.

We have : $(\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2, \theta)$ is typable with τ and J, where $J = Z' + F(\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2, \phi)$

TS: (v_2, θ_2) is typable with τ and $J - (R + R_2)$.

From $(\star)$, we know from weaking rule that $\Gamma, x_j :!_0\tau_j \vdash_Z e : \texttt{bool}$ (1) where $x_j \in \texttt{dom}(\theta) \setminus \texttt{dom}(\Gamma)$.

Set $\phi_1 = [x_1 \to (k_1, R_1, I_1), \ldots, x_i \to (k_i, R_i, I_i)]$ where $k_j = 0$ for $x_j \in \texttt{dom}(\theta) \setminus \texttt{dom}(\Gamma)$.

Similaly, from $(\diamond)$, we have $\Gamma_2, x_j :!_0\tau_j \vdash_{Z_2} e_2 : \tau$ (2) where $x_j \in \texttt{dom}(\theta) \setminus \texttt{dom}(\Gamma_2)$.

Set $\phi_2 = [x_1 \to (k_1, R_1, I_1), \ldots, x_i \to (k_i, R_i, I_i)]$ where $k_j = 0$ for $x_j \in \texttt{dom}(\theta) \setminus \texttt{dom}(\Gamma_2)$.

From (1), we know that (e, θ) is typable with $\texttt{bool}$ and J_1, where $J_1 = Z + F(e, \phi_1)$ (3).

From (2), we know that (e_2, θ) is typable with τ and J_2, where $J_2 = Z_2 + F(e_2, \phi_2)$ (4).

By IH on (3), (4), we have ($\texttt{false}, \theta_1'$) is typable with $\texttt{bool}$ and $J_1 - R$ (5) and (v_2, θ_2) is typable with τ and $J_2 - R_2$(6).

we know that $J = Z' + F(\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2, \phi) = Z + F(e, \phi) + \max(Z_1, Z_1) + \max(F(e_1, \phi), F(e_2, \phi)) \geq (J_1 + J_2)$.

We also know $J_1 - R \geq 0$ from (5),

So we know $J - (R + R_2) \geq (J_1 + J_2) - (R + R_2) \geq J_2 - R_2$. Our TS can be proved using Lemma 8 on (6).

□

2 Typable Approach by Marco

Definition 10 (Typable Closures)**.** Let $\theta = [x_1 \to (v_1, \theta_1, R_1), \ldots, x_n \to (v_n, \theta_n, R_n)]$. The closure (e, θ) is typable with type τ and adaptivity J if:

1. $x_1 :!_{k_1}\tau_1, \ldots, !_{k_i}\tau_i \vdash_Z e : \tau$, for some types $!_{k_i}\tau_i$ for $(1 \leq i \leq n)$,
2. each closure (v_i, θ_i) for $(1 \leq i \leq n)$ is typable with type $!_{k_i}\tau_i$ and adaptivity Z_i,
3. $J = Z + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i \times (R_i + Z_i)$.

To justify why we chose $\sum$ in the third clause above it is worth to consider the following configuration:

$$[x \mapsto (\lambda u.\lambda w.\delta(u) + \delta(w), [\,], 0), y :\mapsto (v, [\,], 2)], x\, y\, y$$

[MG: To elaborate the example above]

Lemma 11 (Soundness)**.** If a closure D is typable with type τ and adaptivity J, and $D \Downarrow^R E$, then the closure E is typable with type τ and adaptivity I, where $I + R \leq J$.

Proof. By induction on the derivation showing $D \Downarrow^R E$. The rules lambda,val, and const dollow from the hypothesis. We will discuss the other cases.

Case

$$\frac{\theta(x) = (v, \theta', R)}{x, \theta \Downarrow^R v, \theta'} \;\textsf{var}$$

By assumption, the closure (x, θ) is typable with type τ and adaptivity J. Let us assume without loss of generality that $\theta = [x \to (v, \theta', R), x_1 \to (v_1, \theta_1, R_1), \ldots, x_n \to (v_n, \theta_n, R_n)]$. By Definition 10 we have that there exist k and $!_{k_i}\tau_i$ for $(1 \leq i \leq n)$ such that:

$$x :!_k\tau, x_1 :!_{k_1}\tau_1, \ldots, x_n :!_{k_n}\tau_n \vdash_Z x : \tau$$

Moreover, we have that the closure (v, θ') is typable with type τ and adaptivity I, and that each closure (v_i, θ_i) is typable with type τ_i and adaptivity Z_i, and that

$$J = Z + k \times (R + I) + \sum_{1\leq i\leq n, i\neq j} k_i \times (R_i + Z_i).$$

By inversion on the typing rules, we have that $k = 1$, hence we can rewrite J as

$$J = Z + R + I + \sum_{1\leq i\leq n, i\neq j} k_i \times (R_i + Z_i)).$$

Hence, it is easy to see that $I + R \leq J$ which is what we need to show.

Case

$$\frac{e_2, \theta \Downarrow^{R_2} v_2, \theta^2 \qquad \textsf{fresh } x' \qquad \begin{array}{c} e_1, \theta \Downarrow^{R_1} \lambda x.e, \theta^1 \\ e[x'/x], \theta^1[x' \to (v_2, \theta^2, R_2)] \Downarrow^{R_3} v, \theta^3 \ (\clubsuit) \end{array}}{e_1\ e_2, \theta \Downarrow^{R_1+R_3} v, \theta^3}\ \textsf{app}$$

By assumption, the closure $(e_1\ e_2, \theta)$ is typable with type τ and adaptivity J. Let us assume without loss of generality that $\theta = [x_1 \to (v_1, \theta_1, S_1), \ldots, x_n \to (v_n, \theta_n, S_n)]$. By Definition 10 we have that there exist $!_{k_i}\tau_i$ for $(1 \le i \le n)$ such that:

$$x_1 :!_{k_1}\tau_1, \ldots, x_n :!_{k_n}\tau_n \vdash_Z e_1 e_2 : \tau \tag{1}$$

Moreover, we have that each closure $(v_i, \theta_i) \in \theta$ is typable with type τ_i and adaptivity Z_i, and that

$$J = Z + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i \times (S_i + Z_i).$$

By inversion on Fact 1 we have that there exist σ, Z', Z^1, k and k_i^1 for $(1 \le i \le n)$ such that:

$$x_1 :!_{k_1^1}\tau_1, \ldots, x_n :!_{k_n^1}\tau_n \vdash_{Z^1} e_1 :!_k\sigma \multimap^{Z'} \tau \tag{2}$$

Similarly, we have that there exist Z^2 and k_i^2 for $(1 \le i \le n)$ such that:

$$x_1 :!_{k_1^2}\tau_1, \ldots, x_n :!_{k_n^2}\tau_n \vdash_{Z^2} e_2 : \sigma \tag{3}$$

Moreover, we have that $Z = Z^1 + k \times Z^2 + Z'$, and that $k_i = k_i^1 + k \times k_i^2$.

Using Fact 2 and the assumption that each closure (v_i, θ_i) in θ is typable with type τ_i and adaptivity Z_i we have that the closure (e_1, θ) is typable with type $!_k\sigma \multimap^{Z'} \tau$ and adaptivity:

$$Z^1 + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^1 \times (S_i + Z_i)$$

By induction hypothesis we then have that the closure $(\lambda x.e, \theta^1)$ is typable with type $!_k\sigma \multimap^{Z'} \tau$ and adaptivity I_1 where

$$I_1 + R_1 \le Z^1 + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^1 \times (S_i + Z_i)$$

It is worth to stress here that by definition, using the fact that

$$x_1 :!_{k_1^1}\tau_1, \ldots, x_n :!_{k_n^1}\tau_n \vdash_0 \lambda x.e :!_k\sigma \multimap^{Z'} \tau$$

which follows from Fact 2 and inversion on the rule **lambda** [MG: here we actually want a lemma that says that values are typable with $\vdash_0$, and that takes care of the free variables that we may have added] we have:

$$I_1 = \sum_{(v_i,\theta_i,S_i)\in\theta^1} k_i^1 \times (S_i + Z_i)$$

Using Fact 3 and again the assumption that each closure (v_i, θ_i) in θ is typable with type τ_i and adaptivity Z_i we have that the closure (e_2, θ) is typable with type τ and adaptivity:

$$Z^2 + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^2 \times (S_i + Z_i)$$

By induction hypothesis we than have that the closure (v_2, θ_2) is typable with type τ and adaptivity I_2 where

$$I_2 + R_2 \leq Z^2 + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^2 \times (S_i + Z_i)$$

It is again worth to stress here that by definition, using the fact that v_2 is a value we have:

$$I_2 = \sum_{(v_i,\theta_i,S_i)\in\theta^2} k_i^2 \times (S_i + Z_i)$$

By Fact 2, and type preservation and some other Lemma [MG: TODO: we need to be more clear and handle the fact that the adaptivity can be 0 in the lambda rule] we have

$$x_1 :!_{k_1^1}\tau_1, \ldots, x_n :!_{k_n^1}\tau_n, x :!_k\sigma \vdash_{Z'} e : \tau \tag{4}$$

By composing this with the previous assumptions, we have that the closure $(e[x'/x], \theta^1[x' \mapsto (v_2, \theta_2, R_2)])$ is typable with type τ and adaptivity I_3 where

$$I_3 = Z' + k \times (R_2 + I_2) + \sum_{(v_i,\theta_i,S_i)\in\theta^1} k_i^1 \times (S_i + Z_i)$$

By induction hypothesis we then get that the closure (v, θ^3) is typable with type τ and adaptivity I where $I + R_3 \leq I_3$. Notice also that we have a θ^4 such that $\theta^3 = \theta^4[x' \mapsto (v_2, \theta_2, R_2)]$. Moreover, once again, note that by Lemma [MG: Lemma showing that we can type values with adaptivity 0] we have:

$$I = \sum_{(v_i,\theta_i,S_i)\in\theta^3} k_i \times (S_i + Z_i) = k \times (R_2 + I_2) + \sum_{(v_i,\theta_i,S_i)\in\theta^4} k_i \times (S_i + Z_i)$$

We want to prove $I + R_1 + R_3 \leq J$.

Starting from the left hand side we have

$$\begin{array}{rcl}
I + R_1 + R_3 & = & \sum_{(v_i,\theta_i,S_i)\in\theta^3}(k_i \times (S_i + Z_i)) + R_1 + R_3 \\
& = & k \times (R_2 + I_2) + \sum_{(v_i,\theta_i,S_i)\in\theta^4} k_i \times (S_i + Z_i) + R_3 + R_1 \\
& \leq & k \times (R_2 + I_2) + Z' + \sum_{(v_i,\theta_i,S_i)\in\theta^1} k_i^1 \times (S_i + Z_i) + R_1 \\
& \leq & k \times (R_2 + I_2) + Z' + Z^1 + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^1 \times (S_i + Z_i) \\
& = & k \times (R_2 + \sum_{(v_i,\theta_i,S_i)\in\theta^2} k_i^2 \times (S_i + Z_i)) \\
& & +Z' + Z^1 + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^1 \times (S_i + Z_i) \\
& \leq & k \times (Z^2 + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^2 \times (S_i + Z_i)) \\
& & +Z' + Z^1 + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^1 \times (S_i + Z_i) \\
& = & k \times Z^2 + Z' + Z^1 + \\
& & \sum_{(v_i,\theta_i,S_i)\in\theta} k \times k_i^2 \times (S_i + Z_i)) + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i^1 \times (S_i + Z_i) \\
& = & k \times Z^2 + Z' + Z^1 + \sum_{(v_i,\theta_i,S_i)\in\theta}(k_i^1 + k \times k_i^2) \times (S_i + Z_i)) \\
& = & J
\end{array}$$

This concludes this case.

Case

$$\frac{e, \theta \Downarrow^R v, \theta_1 \qquad \delta(v\theta_1) = v' \qquad FV(v') = \emptyset}{\delta(e), \theta \Downarrow^{R+1} v', \theta_1} \textbf{ delta}$$

By assumption, the closure $(\delta(e), \theta)$ is typable with type τ and adaptivity J. Let us assume without loss of generality that $\theta = [x_1 \to (v_1, \theta_1, S_1), \ldots, x_n \to (v_n, \theta_n, S_n)]$. By Definition 10 we have that there exist Z and $!_{k_i}\tau_i$ for $(1 \leq i \leq n)$ such that:

$$x_1 :!_{k_1}\tau_1, \ldots, x_n :!_{k_n}\tau_n \vdash_Z \delta(e) : \tau \tag{5}$$

Moreover, we have that each closure $(v_i, \theta_i) \in \theta$ is typable with type τ_i and adaptivity Z_i, and that

$$J = Z + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i \times (S_i + Z_i).$$

By inversion on Fact 6 we have that $Z = Z' + 1$ for some Z', that $\tau = \mathtt{b}$, that $k_i = k_i' + 1$ for $(1 \leq i \leq n)$, and that:

$$x_1 :!_{k_1'}\tau_1, \ldots, x_n :!_{k_n'}\tau_n \vdash_{Z'} e : \mathtt{b}. \tag{6}$$

Hence, it is easy to see that the closure (e, θ) is typable with type $\mathtt{b}$ and adaptivity

$$Z' + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i' \times (S_i + Z_i)$$

By applying the induction hypothesis we have that also the closure (v, θ') is typable with type $\mathtt{b}$ and adaptivity I' such that:

$$I' + R \leq Z' + \sum_{(v_i,\theta_i,S_i)\in\theta} k_i' \times (S_i + Z_i)$$

From this and the fact that $\vdash_0 v' : \mathtt{b}$, we have:

$$I' + R + 1 \leq Z' + 1 + \sum_{(v_i, \theta_i, S_i) \in \theta} (k'_i + 1) \times (S_i + Z_i)$$

This councludes this case. $\square$