Adaptivity analysis

Expr.	e	$::=$	$x \mid e_1\ e_2 \mid \lambda x.e$
			$c \mid \delta(e)$
Value	v	$::=$	$c \mid \lambda x.e$
Adaptivity	R	$::=$	n
Environment	θ	$::=$	$x_1 \mapsto (v_1, R_1), \ldots, x_n \mapsto (v_n, R_n)$

Linear type	A	$::=$	$\tau \multimap \tau \mid \mathtt{b}$
Nonlinear Type	τ	$::=$	$!_I A$

$$\frac{\theta(x) = (v, R)}{\theta, x \Downarrow^{R} v, \theta}\ \textsf{var} \qquad \frac{}{\theta, c \Downarrow^{0} c, \theta}\ \textsf{const} \qquad \frac{}{\theta, \lambda x.e \Downarrow^{0} \lambda x.e, \theta}\ \textsf{lambda}$$

$$\frac{\theta, e_1 \Downarrow^{R_1} \lambda x.e, \theta_1 \qquad \theta, e_2 \Downarrow^{R_2} v_2, \theta_2 \qquad (\theta_1 \uplus \theta_2)[x \to (v_2, R_2)], e \Downarrow^{R_3} v, \theta_3}{\theta, e_1\ e_2 \Downarrow^{R_1+R_3} v, \theta_3}\ \textsf{app}$$

$$\frac{\theta, e \Downarrow^{R} v', \theta_1 \qquad \delta(v') = v}{\theta, \delta(e) \Downarrow^{R+1} v, \theta_1}\ \textsf{delta}$$

$$\begin{array}{llll}
\theta_1 \uplus \emptyset & \triangleq & \theta_1 & \\
\emptyset \uplus \theta_2 & \triangleq & \theta_2 & \\
(\theta_1, [x \to (v, R_1)]) \uplus (\theta_2, [x \to (v, R_2)]) & \triangleq & (\theta_1 \uplus \theta_2), [x \to (v, \max(R_1, R_2))] & \\
\texttt{adap}(e, \emptyset) & ::= & 0 & \\
\texttt{adap}(e, [x \to (v, R)] \uplus \theta) & ::= & \max(R, \texttt{adap}(e[v/x], \theta)) & x \in \textsf{FV}(e). \\
 & ::= & \texttt{adap}(e, \theta) & x \notin \textsf{FV}(e))
\end{array}$$

Figure 1: Big-step semantics

$$\frac{}{\Gamma, x :!_Z A, \Gamma' \vdash_Z x :!_Z A}\ \textbf{Ax} \qquad\qquad \frac{}{\Gamma \vdash_Z c :!_Z \mathtt{b}}\ \textbf{b}$$

$$\frac{\Gamma, x : \tau_1 \vdash_Z e : \tau_2}{k + \Gamma \vdash_{k+Z} \lambda x.e :!_k(\tau_1 \multimap \tau_2)}\ \textbf{lambda}$$

$$\frac{\Gamma_1 \vdash_{Z_1} e_1 :!_0(\tau_1 \multimap \tau_2) \qquad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\max(\Gamma_1, \Gamma_2) \vdash_{\max(Z_1,Z_2)} e_1\ e_2 : \tau_2}\ \textbf{app}$$

$$\frac{\Gamma \vdash_Z e :!_k A}{\Gamma', 1 + \Gamma \vdash_{1+Z} \delta(e) :!_k A}\ \textbf{delta}$$

$$\frac{\Gamma' \vdash_{Z'} e : \tau' \qquad \Gamma' \leqslant \Gamma \qquad Z' \leq Z \qquad \tau' <: \tau \qquad \Gamma \vdash_Z e :!_k A}{\Gamma \vdash_Z e : \tau}\ \textbf{subtype}$$

$$\frac{\Gamma, y : \tau', x : \tau, \Gamma' \vdash_Z e : \tau}{\Gamma, x : \tau, y : \tau', \Gamma' \vdash_Z e : \tau}\ \textbf{exchange}$$

$$\begin{array}{lcl}
k+!_r A & \triangleq & !_{k+r} A \\
k + \emptyset & \triangleq & \emptyset \\
k + ([x : \tau], \Gamma) & \triangleq & [x : k + \tau], k + \Gamma \\
\max(!_{k_1} A, !_{k_2} A) & \triangleq & !_{\max(k_1,k_2)} A \\
\max(\Gamma, \emptyset) & \triangleq & \Gamma \\
\max(\emptyset, \Gamma) & \triangleq & \Gamma \\
\max\Big(([x : \tau], \Gamma), ([x : \tau'], \Delta)\Big) & \triangleq & [x : \max(\tau, \tau')], \max(\Gamma, \Delta) \\
\Gamma <: \Delta & \triangleq & \mathtt{dom}(\Gamma) = \mathtt{dom}(\Delta) \land \forall x \in \mathtt{dom}(\Gamma), \Delta(x) <: \Gamma(x)
\end{array}$$

Figure 2: Typing rules, first version

$$\frac{k_1 \leq k \qquad A <: A_1}{!_k A <:!_{k_1} A_1}\ \textsf{bang} \qquad \frac{\tau_1 <: \tau \qquad \tau' <: \tau_1'}{\tau \multimap \tau' <: \tau_1 \multimap A_1'}\ \textsf{arrow} \qquad \frac{}{\mathtt{b} <: \mathtt{b}}\ \textsf{base}$$

Figure 3: subtyping

Theorem 1 (Substitution). 1. If $\Gamma, x : \tau' \vdash_Z e : \tau$ and $\vdash_{Z'} v : \tau'$, then $\Gamma \vdash_{\max(Z,Z')} e[v/x] : \tau$.

Proof. By induction on the typing derivation. □

Lemma 2 (Parameter Decreasing). if $k + \Gamma \vdash_Z v : k + \tau$, then exists Z' so that $\Gamma \vdash_{Z'} v : \tau$ and $Z' \le Z - k$.

Proof. if v is a constant, then it is trivial, assume $\tau =!_r \mathtt{b}$, choose $Z' = r, k' = k$, from the rule b.
If $v = \lambda x.e$. Assume $\tau =!_r\tau_1 \multimap A_2$, then $k + \tau =!_{k+r}\tau_1 \multimap A_2$. From its typing derivation, we know: $\Gamma - r, x : \tau_1, \vdash_{Z-(k+r)} e : \tau_2$ (1). Choose $Z' = Z - r$, we know that $\Gamma \vdash_{Z'} v :!_r\tau_1 \multimap A_2$ from the rule lambda. □

$$
\begin{array}{lll}
\theta \vDash \Gamma & \triangleq & \forall x_i \in \mathtt{dom}(\Gamma).\theta(x_i) = (v_i, R_i) \wedge \vdash_{R_i} v_i : \Gamma(x_i) \\
F(\theta, e) & ::= & \max(R_i) \\
 & \textit{where} & \forall x_i \in \mathsf{FV}(e).\theta(x_i) = (v_i, R_i).
\end{array}
$$

Theorem 3 (Soundness). If $\Gamma \vdash_Z e : \tau$, $\forall \theta$ that $\theta \vDash \Gamma$, exists θ' and v so that $\theta, e \Downarrow^R v, \theta'$, then $R + \mathit{adap}(v, \theta') \le Z + F(\theta, e)$.

Proof. By Induction on the typing derivation.

$$\frac{}{\Gamma, x :!_Z A \vdash_Z x :!_Z A}\ \textbf{Ax}$$

Assume $\theta = \Big(\theta_1, [x \to (v, R)], \Big) \vDash (\Gamma, x :!_Z A)$ where $\theta_1 \vDash \Gamma$. We know that $\vdash_R v :!_Z A$. From the evaluation rule var, we know $\theta, x \Downarrow^R v, \theta$. TS: $R + \mathit{adap}(v, \theta) \le Z + F(\theta) \implies R + 0 \le Z + \max(R, F(\theta_1))$.It is trivially true.

$$\frac{\Gamma, x : \tau_1 \vdash_Z e : \tau_2}{k + \Gamma \vdash_{k+Z} \lambda x.e :!_k(\tau_1 \multimap \tau_2)}\ \textbf{lambda}$$

Choose $\theta \vDash (k + \Gamma)$ so that $\forall x_i \in (\Gamma).\theta(x_1) = (v_i, R_i) \wedge \vdash_{R_i} v_i : k + \Gamma(x_i)$. By the evaluation rule we know $\theta, \lambda x.e \Downarrow^0 \lambda x.e, \theta$, TS: $0 + \mathtt{adap}(\lambda x.e, \theta) \le k + Z + F(\theta)$, which is trivially true because $\mathtt{adap}(\lambda x.e, \theta) \le F(\theta)$.

$$\frac{\Gamma_1 \vdash_{Z_1} e_1 :!_0(\tau_1 \multimap \tau_2) \qquad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\max(\Gamma_1, \Gamma_2) \vdash_{\max(Z_1,Z_2)} e_1\ e_2 : \tau_2}\ \textbf{app}$$

Choose $\theta = [x_i \to (v_i, 0)]$ for all x_i in $\mathtt{dom}(\max(\Gamma_1, \Gamma_2))$ so that $\vdash_{Z_i} v_i : (\max(\Gamma_1, \Gamma_2)(x_i)$. From the definition, we know that $\theta \vDash \Gamma_1$ and $\theta \vDash \Gamma_2$. Because e_1 has the arrow type and will be evaluated to a function, assume exists θ_1 so that $\theta, e_1 \Downarrow^{R_1} \lambda x.e, \theta_1$. By induction hypothesis on the first premise, we know that: $R_1 + \mathtt{adap}(\lambda x.e, \theta_1) \le Z_1 + F(\theta, \Gamma_1)$ (1).Assume exists θ_2 so that e_2 is evaluated to an arbitrary value $v_2 : \theta, e_2 \Downarrow^{R_2} v_2, \theta_2$, by induction hypothesis, we conclude that : $R_2 + \mathtt{adap}(v, \theta_2) \le Z_2 + F(\theta, \Gamma_2)$ (2).

$$\frac{\theta, e_1 \Downarrow^{R_1} \lambda x.e, \theta_1 \qquad \theta, e_2 \Downarrow^{R_2} v_2, \theta_2 \qquad (\theta_1 \uplus \theta_2)[x \to (v_2, R_2)], e \Downarrow^{R_3} v, \theta_3}{\theta, e_1\ e_2 \Downarrow^{R_1+R_3} v, \theta_3}\ \mathsf{app}$$

$\square$

Theorem 4 (Subject Reduction)**.** If $\Gamma \vdash_Z e : \tau$, $\forall \theta. \theta \vDash \Gamma$, exists θ' and v, $\theta, e \Downarrow^R v, \theta'$, then $\Gamma \vdash_? v : \tau$.

By induction on the typing derivation.