Adaptivity analysis

Expr.	e	$::=$	$x \mid e_1\ e_2 \mid \lambda x.e$
			$\mathtt{if}(e_1, e_2, e_3) \mid c \mid \delta(e)$
Value	v	$::=$	$c \mid \lambda x.e$
Environment	θ	$::=$	$x_1 \mapsto (v_1, R_1), \ldots, x_n \mapsto (v_n, R_n)$

Index Term	I, Z	$::=$	$i \mid n$
Linear type	τ	$::=$	$A \multimap^{Z} \tau \mid \mathtt{b}$
Nonlinear Type	A	$::=$	$!_I \tau$
Typing context	Γ	$::=$	$x_1 \to A_1, \ldots, x_k \to A_k$

$$\frac{\theta(x) = (v, R)}{\theta, x \Downarrow^{R} v, \theta}\ \mathsf{var} \qquad \frac{}{\theta, c \Downarrow^{0} c, \theta}\ \mathsf{const} \qquad \frac{}{\theta, \lambda x.e \Downarrow^{0} \lambda x.e, \theta}\ \mathsf{lambda}$$

$$\frac{\begin{array}{c}\theta, e_1 \Downarrow^{R_1} \lambda x.e, \theta_1 \\ \theta, e_2 \Downarrow^{R_2} v_2, \theta_2 \qquad \mathsf{fresh}\ x' \qquad (\theta_1 \uplus \theta_2)[x' \to (v_2, R_2)], e[x'/x] \Downarrow^{R_3} v, \theta_3\end{array}}{\theta, e_1\ e_2 \Downarrow^{R_1 + R_3} v, \theta_3}\ \mathsf{app}$$

$$\frac{\theta, e \Downarrow^{R} v', \theta_1 \qquad \delta(v') = v}{\theta, \delta(e) \Downarrow^{R+1} v, \theta_1}\ \mathsf{delta}$$

$$\begin{array}{lcll}
\theta_1 \uplus \emptyset & \triangleq & \theta_1 & \\
\emptyset \uplus \theta_2 & \triangleq & \theta_2 & \\
(\theta_1, [x \to (v, R_1)]) \uplus (\theta_2, [x \to (v, R_2)]) & \triangleq & (\theta_1 \uplus \theta_2), [x \to (v, \max(R_1, R_2))] & \\
\mathtt{adap}(e, \emptyset) & ::= & 0 & \\
\mathtt{adap}(e, [x \to (v, R)] \uplus \theta) & ::= & \max(R, \mathtt{adap}(e[v/x], \theta)) & x \in \mathsf{FV}(e). \\
 & ::= & \mathtt{adap}(e, \theta) & x \notin \mathsf{FV}(e))
\end{array}$$

Figure 1: Big-step semantics

$$\frac{}{\Gamma, x :!_1\tau \vdash_0 x : \tau}\ \textbf{Ax} \qquad \frac{}{\Gamma \vdash_0 c : \mathtt{b}}\ \textbf{const} \qquad \frac{\Gamma, x : A \vdash_Z e : \tau}{\Gamma \vdash_0 \lambda x.e : A \multimap^Z \tau}\ \textbf{lambda}$$

$$\frac{\Gamma_1 \vdash_{Z_1} e_1 :!_I\tau_1 \multimap^Z \tau_2 \qquad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\Gamma_1 + I \times \Gamma_2 \vdash_{Z_1+I\times Z_2+Z} e_1\ e_2 : \tau_2}\ \textbf{app} \qquad \frac{\Gamma \vdash_Z e : \mathtt{b}}{\Gamma \vdash_{1+Z} \delta(e) : \mathtt{b}}\ \textbf{delta}$$

$$\frac{\Gamma' \vdash_{Z'} e : \tau' \qquad \Gamma' \leqslant \Gamma \qquad Z' \leq Z \qquad \tau' <: \tau \qquad \Gamma \vdash_Z e :!_k A}{\Gamma \vdash_Z e : \tau}\ \textbf{subtype}$$

$$\frac{\Gamma, y : \tau', x : \tau, \Gamma' \vdash_Z e : \tau}{\Gamma, x : \tau, y : \tau', \Gamma' \vdash_Z e : \tau}\ \textbf{exchange}$$

$$\begin{array}{lcll}
I \times \Gamma & \triangleq & \Gamma & I = 1 \\
 & \triangleq & \emptyset & I = 0 \\
!_{I_1}\tau + !_{I_2}\tau & \triangleq & !_{\max(I_1,I_2)}\tau & \\
\Gamma + \emptyset & \triangleq & \Gamma & \\
\emptyset + \Gamma & \triangleq & \Gamma & \\
([x : A], \Gamma) + ([x : A'], \Delta) & \triangleq & [x : A + A'], \Gamma + \Delta &
\end{array}$$

Figure 2: Typing rules, first version

$$\frac{k \leq k_1 \qquad A <: A_1}{!_k A <: !_{k_1} A_1}\ \mathsf{bang} \qquad \frac{Z \leq Z' \qquad \tau_1 <: \tau \qquad \tau' <: \tau_1'}{\tau \multimap^Z \tau' <: \tau_1 \multimap^{Z'} A_1'}\ \mathsf{arrow}$$

$$\frac{}{\mathtt{b} <: \mathtt{b}}\ \mathsf{base}$$

Figure 3: subtyping

Theorem 1 (Weaking). 1. If $\Gamma, x : \tau' \vdash_Z e : \tau$ and $x \notin \mathsf{FV}(e)$, then $\Gamma \vdash_Z e : \tau$.

Theorem 2 (Value Adaptivity). 1. for all type $!_k A$, exist value v, then $\vdash_k v :!_k A$.

Theorem 3 (Substitution). 1. If $\Gamma, x :!_k A \vdash_Z e : \tau$ and $\vdash_k v :!_k A$, then $\Gamma \vdash_Z e[v/x] : \tau$.

Theorem 4 (Soundness-original). If $\Gamma \vdash_Z e : \tau$, $\forall \theta$ that $\theta \vDash \Gamma$, exists θ' and v so that $\theta, e \Downarrow^R v, \theta'$, then $R + adap(v, \theta') \leq Z + F(\theta, e)$.

Theorem 5 (Subject Reduction). If $\Gamma \vdash_Z e :!_k A$, $\forall \theta. \theta \vDash \Gamma$, exists θ' and v, $\theta, e \Downarrow^R v, \theta'$, then $\Gamma \vdash_k v :!_k A$.

By induction on the typing derivation.