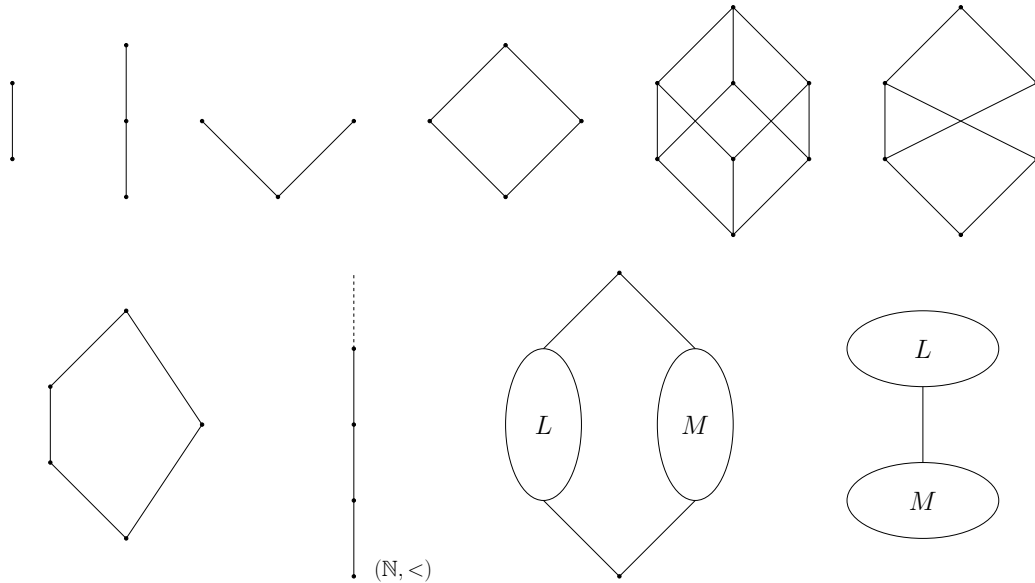# Exercises for Universal Algebra 1

Exercise 1. Which of the following Hasse Diagrams represent (semi)lattices, given that L and M are lattices?

Exercise 2. Find the correspondence between lattices and lattice ordered sets:

(i) For a given semilattice $(S, \wedge)$ find a natural definition of a semilattice order on S, and vice-versa.

(ii) Conclude that there is a 1-to-1 correspondence between semilattices and semilattice ordered sets.

(iii) Prove that there is a 1-to-1 correspondence between lattices and lattice ordered sets.

Exercise 3. Show that every homomorphism of lattices is order preserving. What about the converse?

Exercise 4. Choose your favorite (finite) group and draw the poset of its normal subgroups, ordered by inclusion of their underlying sets.

Exercise 5. Given a group G, prove that the poset of its normal subgroups is a lattice (ordered set). What are the operations meet and join?

Exercise 6. Let $(P, \leq)$ be a poset. Show that there is a linear order $\leq'$ on P such that $p \leq q \implies p \leq' q$ for all $p, q \in P$.

Exercise 7 (*). Let $(P, \leq)$ be a poset, $(\mathcal{U}_P, \subseteq)$ the poset of its upsets and $(\mathcal{D}_P, \subseteq)$ the poset of its downsets.

(i) Find an injective order preserving map $(P, \leq) \to (\mathcal{D}_P, \subseteq)$

(ii) Show that there is no surjective order preserving map $(P, \leq) \to (\mathcal{D}_P, \subseteq)$

(iii) Conclude that for any set X, there is no surjective map $X \to 2^X$

Exercise 8. Prove or disprove that for every poset $(P, \leq)$ we have

$$(\mathcal{D}_P, \subseteq) \cong (\mathcal{U}_P, \supseteq)$$

Definition. A lattice L is called *distributive*, if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1.1}$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{1.2}$$

Exercise 9. Show that every lattice with less than four elements is distributive. Find examples of distributive lattices with a large number of elements.

Exercise 10. Show that in the definition of distributive lattices (3.1) and (3.2) are equivalent.

Definition. A lattice L is called *modular*, if for all $x, y, z \in L$

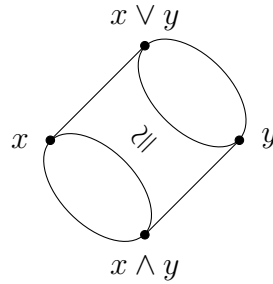$$x \leq z \implies x \vee (y \wedge z) = (x \vee y) \wedge z$$

Exercise 11. Show that every distributive lattice is modular and disprove the converse, i.e. find a modular lattice that is not distributive.

Exercise 12. Show that the following two statements hold for all lattices L and all $x, y, z \in L$

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

$$x \leq z \implies x \vee (y \wedge z) \leq (x \vee y) \wedge z$$

Exercise 13 (Diamond isomorphism theorem). Let L be a modular lattice and $x, y \in L$. Show that the intervals $I[x \wedge y, x]$ and $I[y, x \vee y]$ are isomorphic lattices.
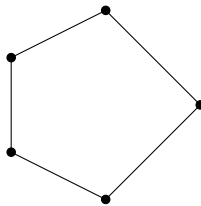

Exercise 14. A term $m(x, y, z)$ of an algebra A is called *majority* if it satisfies the identities
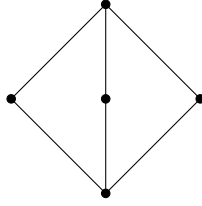
$$x \approx m(x, x, y) \approx m(x, y, x) \approx m(y, x, x)$$

Show that every lattice has a majority term.

Theorem (Dedekind). Prove that a lattice is modular if and only if it does not contain the following lattice as a sublattice.

Theorem (Birkhoff). A modular lattice is distributive if and only if it does not contain the following lattice as a sublattice.

Exercise 15. Show that every complete lattice is bounded.

Exercise 16. Find examples of lattices L that contain a sublattice S such that

(i) L is complete but S is not complete

(ii) L is not complete but S is complete

(iii) both L and S are complete lattices but S is not a complete sublattice

Exercise 17. Let L be a complete lattice and $a, b \in L$ two compact elements.

(i) Is $a \vee b$ compact?

(ii) Is $a \wedge b$ compact?

Exercise 18. Let C be a closure operator on a set X. Prove that L_C is closed under finite unions if and only if for all subsets $U, V \in 2^X$

$$C(U \cup V) = C(U) \cup C(V)$$

Exercise 19. Let X be a set and let ϕ be the binary relation on 2^X defined by

$$(U, V) \in \phi \iff U \cap V \neq \emptyset$$

Consider the Galois correspondence on the sets 2^{2^X} and 2^{2^X} induced by this relation

(i) Let $X = \{1, 2, 3, 4\}$. Compute $A^{\leftarrow\rightarrow}$ and $A^{\rightarrow\leftarrow}$ for both $A = \big\{\{1, 2\}, \{2, 3\}\big\}$ and $A = \big\{\{1, 2\}, \{2\}\big\}$. Compare the results.

(ii) Prove that if a Galois correspondence is defined by a symmetric relation on a set, then the closure operators induced by it coincide.

(iii) Prove that for every $A \subseteq 2^X$ we have

$$A^{\rightarrow\leftarrow} = \{U \in 2^X \mid \exists V \in A, V \subseteq U\}$$

Exercise 20. Let C be a closure operator on a set X. Find a relation $\phi \subseteq X \times 2^X$ whose induced Galois correspondence gives

$$C(U) = U^{\rightarrow\leftarrow}$$

for all subsets $U \subseteq X$.

Exercise 21. Let $\mathbb{A} = (A, *)$ be a binary algebra and θ an equivalence relation on A. Show that θ is a congruence relation if and only if for all $a, b, c \in A$ we have

$$(a, b) \in \theta \implies \begin{cases} (a * c, b * c) \in \theta \quad \text{and} \\ (c * a, c * b) \in \theta \end{cases}$$

Exercise 22. Let $\mathbb{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

$*$	0	1	2	3
0	0	2	1	1
1	2	1	0	2
2	1	0	2	0
3	1	2	0	3

Draw the lattice of subalgebras and the lattice of congruences of $\mathbb{A}$.

Exercise 23. Consider the algebra $(\mathbb{Z}, +, \cdot) \times (\mathbb{Z}, \cdot, +)$. What is the subalgebra generated by the pairs $(0, 1)$ and $(1, 0)$?

Exercise 24. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras in the same signature and let $f : \mathbb{A} \to \mathbb{B}$ be a homomorphism.

- Given two subalgebras $U \leq \mathbb{A}$ and $V \leq B$, are $f(U) \subseteq \mathbb{B}$ and $f^{-1}(V) \subseteq \mathbb{A}$ subalgebras?

- Given two congruences $\theta \in \mathrm{Con}(\mathbb{A})$ and $\psi \in \mathrm{Con}(\mathbb{B})$, is $f(\theta) \in \mathrm{Con}(\mathbb{B})$ and $f^{-1}(\psi) \in \mathrm{Con}(\mathbb{A})$?

- Given a subset $X \subseteq A$ is $f(\mathrm{Sg}_{\mathbb{A}}(X)) = \mathrm{Sg}_{\mathbb{B}}(f(X))$?

Exercise 25. Given a binary algebra $\mathbb{A} = (A, *)$ define its *nucleus* as

$$B := \{a \in A \mid \forall x, y \in A, (x * a) * y = x * (a * y)\}$$

Show that B is a subalgebra of $\mathbb{A}$ and find and an example of an algebra $\mathbb{A}$ whose nucleus is empty.

Exercise 24. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras in the same signature and let $f : \mathbb{A} \to \mathbb{B}$ be a homomorphism.

- Given two subalgebras $U \leq \mathbb{A}$ and $V \leq B$, are $f(U) \subseteq \mathbb{B}$ and $f^{-1}(V) \subseteq \mathbb{A}$ subalgebras?

- Given two congruences $\theta \in \mathrm{Con}(\mathbb{A})$ and $\psi \in \mathrm{Con}(\mathbb{B})$, is $f(\theta) \in \mathrm{Con}(\mathbb{B})$ and $f^{-1}(\psi) \in \mathrm{Con}(\mathbb{A})$?

- Given a subset $X \subseteq A$ is $f(\mathrm{Sg}_{\mathbb{A}}(X)) = \mathrm{Sg}_{\mathbb{B}}(f(X))$?

Exercise 26. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras of the same type and let $f : A \to B$ be a map. Show that f is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ if and only if its graph is a subalgebra of $\mathbb{A} \times \mathbb{B}$.

$$\{(a, f(a)) \mid a \in A\} \leq \mathbb{A} \times \mathbb{B}$$

Exercise 27. Let $f, g : \mathbb{A} \to \mathbb{B}$ be two homomorphisms and let $X \subseteq A$ with $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}(X)$. Show that

$$f|_X = g|_X \implies f = g.$$

Remark. The converse is also true: Given $\mathbb{A}$ and $X \subseteq A$, then $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}(X)$ if and only if $f|_X = g|_X$ implies $f = g$ for all algebras $\mathbb{B}$ and all homomorphisms $f, g : \mathbb{A} \to \mathbb{B}$.

Exercise 28. Let $X \subseteq A$ be a subset that generates the algebra $\mathbb{A}$ such that no proper subset of X generates $\mathbb{A}$. Is it true that every map $f : X \to B$ to any algebra $\mathbb{B}$ can be extended to an homomorphism $\mathbb{A} \to \mathbb{B}$?

Exercise 29. Find all homomorphisms $(\mathbb{N}, +)^2 \to (\mathbb{Z}_2, +)$.

Exercise 30. Show that a map $f : A \to B$ is injective if and only if its kernel is the equality relation.

Exercise 31 (Second isomorphism theorem)**.** Let $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ be two homomorphisms and let $\alpha \leq \beta$ be two congruences on $\mathbb{A}$ and let ϕ be a congruence on $\mathbb{B}$. Prove that

(i) if f is surjective and $\ker(f) \subseteq \ker(g)$, then there exists a homomorphism $h : \mathbb{B} \to \mathbb{C}$ such that $g = h \circ f$.

$$
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\ f\ } & \mathbb{B} \\
 & {\scriptstyle g}\searrow & \downarrow {\scriptstyle \exists h} \\
 & & \mathbb{C}
\end{array}
$$

(ii) there is an embedding $\mathbb{A}/f^{-1}(\phi) \to \mathbb{B}/\phi$.

(iii) there is a congruence β/α on $\mathbb{A}/\alpha$ such that

$$\mathbb{A}/\beta \cong (\mathbb{A}/\alpha)\big/(\beta/\alpha).$$

Exercise 32. Find classes of algebras witnessing that

$$\mathrm{PS} \lneq \mathrm{SP} \qquad \mathrm{PH} \lneq \mathrm{HP} \qquad \mathrm{SH} \lneq \mathrm{HS}$$

Exercise 31 (Second isomorphism theorem)**.** Let $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ be two homomorphisms and let $\alpha \leq \beta$ be two congruences on $\mathbb{A}$ and let ϕ be a congruence on $\mathbb{B}$. Prove that

(i) if f is surjective and $\ker(f) \subseteq \ker(g)$, then there exists a homomorphism

$h : \mathbb{B} \to \mathbb{C}$ such that $g = h \circ f$.

$$\mathbb{A} \xrightarrow{\ f\ } \mathbb{B}$$

with g going diagonally to $\mathbb{C}$ and $\exists h$ going down from $\mathbb{B}$ to $\mathbb{C}$.
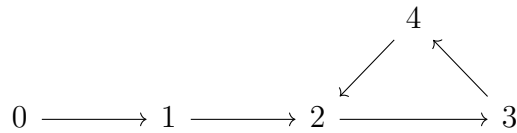
(ii) there is an embedding $\mathbb{A}/f^{-1}(\phi) \to \mathbb{B}/\phi$.

(iii) there is a congruence β/α on $\mathbb{A}/\alpha$ such that

$$\mathbb{A}/\beta = (\mathbb{A}/\alpha)\big/(\beta/\alpha).$$

Exercise 33. Consider the algebra $C_n = (\{0, 1, \ldots, n - 1\}, f)$, where f is the unary function $x \mapsto x + 1 \mod n$. Decide for each $n \in \{2, 3, 4, 5, 6\}$ if C_n is simple, subdirectly irreducible or directly indecomposable.

Exercise 34. Consider the algebra $\mathbb{A} = (\{0, 1, 2, 3, 4\}, g)$, where g is the unary function given by the following diagram.
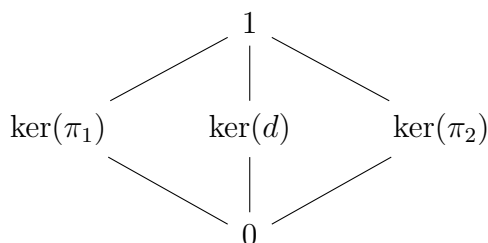

Draw the congruence lattice of $\mathbb{A}$ and then decide whether $\mathbb{A}$ is subdirectly irreducible and or directly indecomposable.

Exercise 35. Consider the to algebras $\mathbb{A} := (\{0, 1\}, +, \mathrm{id})$ and $\mathbb{B} = (\{0, 1\}, +, f)$, where $+$ is addition modulo 2 and f is the unary function given by $x \mapsto x + 1 \mod 2$.

(i) Show that $d : \mathbb{B}^2 \to \mathbb{A}, (x, y) \mapsto x + y$ is a surjective homomorphism.

(ii) Show that the congruence lattice of $\mathbb{B}^2$ is

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & \diagup & | & \diagdown & \\
\ker(\pi_1) & & \ker(d) & & \ker(\pi_2) \\
 & \diagdown & | & \diagup & \\
 & & 0 & &
\end{array}
$$

(iii) Show that $\mathbb{B}^2 \cong \mathbb{B} \times \mathbb{A} \not\cong \mathbb{A}^2$ and conclude that direct decompositions are in general not unique.

Exercise 36. Find algebras $\mathbb{A}$ and $\mathbb{B}$ such that there are no homomorphisms

$$\mathbb{A} \to \mathbb{A} \times \mathbb{B} \quad \text{and} \quad \mathbb{B} \to \mathbb{A} \times \mathbb{B}$$

Exercise 37. Find a proper subdirect composition of the three element lattice into subdirectly irreducible lattices.

Exercise 38. Recall from the homework that, given a group G, its lattice of congruences is isomorphic to its lattice of normal subgroups. Prove an analogous statement for rings.

Exercise 39. Let p be a prime number. Show that the group $(\mathbb{Z}, +)$ is a subdirect product of the groups $\mathbb{Z}_{p^k}$.

Exercise 40. Let $\mathbb{Q}$ be the additive group of rational numbers and let p be a prime number.

(i) Let $\mathbb{Q}_p := \{a/p^k \mid a \in \mathbb{Z}, k \geq 0\}$. Show that $\mathbb{Q}_p$ is a subgroup of $\mathbb{Q}$ and that $\mathbb{Z}$ is a normal subgroup of $\mathbb{Q}_p$.

(ii) Let $\mathbb{Z}_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}$. Prove that every element of $\mathbb{Z}_{p^\infty}$ has finite order.

(iii) Let H be a subgroup of $\mathbb{Z}_{p^\infty}$ such that the order of elements in H if bounded. Show that H is a cyclic group of order p^k.

(iv) Show that $\mathbb{Z}_{p^\infty}$ is subdirectly irreducible by showing that the lattice of (normal) subgroups is a chain.

$$0 < H_1 < H_2 < \cdots < \mathbb{Z}_{p^\infty}$$

(v) Show that $\mathbb{Z}_{p^\infty}/H_k = \mathbb{Z}_{p^\infty}$ for all k.

Definition. A variety $\mathcal{V}$ is called *finitely generated* if it contains some finite algebras $A_1, \ldots, A_n$ such that $\mathcal{V} = \mathrm{HSP}(A_1, \ldots, A_n)$.

Definition. A variety $\mathcal{V}$ is called *locally finite* if every finitely generated algebra in $\mathcal{V}$ is finite.

Exercise 41. Show that every finitely generated variety is locally finite.
  Hint: let $B \in \mathrm{HSP}(A_1, \ldots, A_n)$ be finitely generated ...

If $\mathcal{V}$ is a variety and X a set, let $F_\mathcal{V}(X)$ the free algebra in $\mathcal{V}$ on the set X.

Exercise 42. Let $\mathcal{V}$ be the variety of all algebras (A, f) where f is unary and $f^6 = f^2$. Determine and draw $F_\mathcal{V}(\{x\})$ and $F_\mathcal{V}(\{x, y\})$

Exercise 43. Let $\mathcal{S}$ be the variety of semigroups. Show that

$$F_\mathcal{S}(X) = (\{\text{nonempty words over } X\}, \circ)$$

where $\circ$ is concatenation of words (for example $(xyz) \circ (zy) = (xyzzy)$).

Exercise 44. Let $\mathcal{R}$ be the variety of semigroups satisfying

$$(x \cdot y) \cdot z \approx x \cdot z \quad \text{and} \quad x \cdot x \approx x$$

(i) Describe $F_\mathcal{R}(X)$ for any set X.

(ii) Find a natural homomorphism $F_\mathcal{S}(X) \to F_\mathcal{R}(X)$.

(iii) Generalize (ii) to free algebras of any varieties $\mathcal{W} \subseteq \mathcal{V}$.

Exercise 45. Let $\mathcal{V}$ be the variety of distributive lattices.

(i) Describe $F_{\mathcal{V}}(\{x\})$ and $F_{\mathcal{V}}(\{x, y\})$.

(ii) Find an upper bound on the size of $F_{\mathcal{V}}(\{x_1, \ldots, x_n\})$

Remark. The question of whether $F_{\mathcal{V}}(X)$ is always finite for finite X is in general undecidable. It is even unknown if $F_{\mathcal{V}}(\{x, y\})$ is finite when $\mathcal{V}$ is the variety of groups with $(a^5 \approx 1)$. This is part of "Burnsides Problem".

Exercise 46. Let $\mathcal{V}$ be the variety of semilattices. Show that

$$F_{\mathcal{V}}(X) = (2^X \setminus \{\emptyset\}, \cup)$$

Exercise 47. Given two varieties of groups $\mathcal{V}$ and $\mathcal{W}$, define

$$\mathcal{V} \cdot \mathcal{W} := \Big\{ G \text{ group} \mid \exists N \trianglelefteq G, \ N \in \mathcal{V}, \ G/N \in \mathcal{W} \Big\}.$$

Show that this is a variety of groups.

Exercise 48. Let G be a group and let $\mathcal{A}$ be the variety of abelian groups. Let $\lambda_{\mathcal{A}}^G$ be the smallest congruence of G with an abelian quotient.

- Show that the congruence class of the identity element is the subgroup of G generated by elements of the form $[x, y] := xyx^{-1}y^{-1}$.

$$1/\lambda_{\mathcal{A}}^G = \mathrm{Sg}_G([x, y] \mid x, y \in G)$$

- Show that the variety $\mathcal{A} \cdot \mathcal{A}$ is axiomatized by the group laws and $[[x, y], [z, w]] = 1$.

Exercise 49. Let $\mathcal{A}_n$ be the variety of abelian groups satisfying $x^n \approx 1$. Show that

- $\mathcal{A}_3 \cdot \mathcal{A}_2 = \mathrm{Mod}\Big(\text{group axioms} \cup \{x^6 \approx 1, \quad [x^2, y^2] \approx 1, \quad [x, y]^3 \approx 1\}\Big)$

- $\mathcal{A}_2 \cdot \mathcal{A}_2 = \mathrm{Mod}\Big(\text{group axioms} \cup \{(x^2y^2)^2 \approx 1\}\Big)$

Exercise 50. Let cRing_n be the variety of commutative rings satisfying $x^n \approx x$, and let $\mathbb{F}_9$ be the field of order 9. Show that $\text{HSP}(\mathbb{F}_9)$ is axiomatized by the axioms of cRing_9 together with

$$x + x + x \approx 0.$$

Exercise 48. Let G be a group and let $\mathcal{A}$ be the variety of abelian groups. Let $\lambda_{\mathcal{A}}^G$ be the smallest congruence of G with an abelian quotient.

- Show that the congruence class of the identity element is the subgroup of G generated by elements of the form $[x, y] := xyx^{-1}y^{-1}$.

$$1/\lambda_{\mathcal{A}}^G = \text{Sg}_G([x, y] \mid x, y \in G)$$

- Show that the variety $\mathcal{A} \cdot \mathcal{A}$ is axiomatized by the group laws and $[[x, y], [z, w]] = 1$.

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- $\mathcal{A}_3 \cdot \mathcal{A}_2 = \text{Mod}\Big(\text{ group axioms } \cup \{x^6 \approx 1, \quad [x^2, y^2] \approx 1, \quad [x, y]^3 \approx 1\}\Big)$

- $\mathcal{A}_2 \cdot \mathcal{A}_2 = \text{Mod}\Big(\text{ group axioms } \cup \{(x^2y^2)^2 \approx 1\}\Big)$

Exercise 50. Let cRing_n be the variety of commutative rings satisfying $x^n \approx x$, and let $\mathbb{F}_9$ be the field of order 9. Show that $\text{HSP}(\mathbb{F}_9)$ is axiomatized by the axioms of cRing_9 together with

$$x + x + x \approx 0.$$

Exercise 51. Let $\mathbb{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

$*$	0	1	2	3
0	1	2	1	0
1	0	3	2	3
2	1	0	1	0
3	2	3	2	1

Show that there is no function $f \in \mathrm{Clo}(\mathbb{A})$ satisfying the following.

(i) $f(3, 1, 3, 3, 3) = 0$

(ii) $f(1, 0, 2, 3, 2) = 0$ and $f(1, 0, 0, 3, 2) = 1$