CSP lecture 25/26 – Problem Set 1

$\mathbb{A} = (A; R_1, R_2, \dots)$ is called *a relational structure* if

- A is a set, called *domain*,
- $R_1, R_2, \dots$ are *relations* on A, i.e. $R_i \subseteq A^{n_i}$ for some finite arity $n_i \geq 1$.

Definition: CSP($\mathbb{A}$)
Given a list of constraints $R_i(x_{i_1}, \dots, x_{i_r})$, $R_j(x_{j_1}, \dots, x_{j_s})$, $R_k(x_{k_1}, \dots, x_{k_t}), \dots$ **Decide** whether they are satisfiable.

Consider the following relations on $\{0,1\}$:

- $C_i := \{i\}$, for $i \in \{0,1\}$
- $R := \{(0,0),(1,1)\}$
- $N := \{(0,1),(1,0)\}$
- $S_{ij} := \{0,1\}^2 \setminus \{(i,j)\}$, for $i,j \in \{0,1\}$
- $H := \{0,1\}^3 \setminus \{(1,1,0)\}$
- $G_1 := \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$, $G_2 := \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$

Problem 1. Find a polynomial–time algorithm for CSP($\mathbb{A}$), where

1. $\mathbb{A} = (\{0,1\}; R)$
2. $\mathbb{A} = (\{0,1\}; R, C_0, C_1)$
3. $\mathbb{A} = (\{0,1\}; S_{10})$
4. $\mathbb{A} = (\{0,1\}; S_{10}, C_0, C_1)$
5. $\mathbb{A} = (\{0,1\}; S_{01}, S_{10}, C_0, C_1)$
6. $\mathbb{A} = (\{0,1\}; N)$
7. $\mathbb{A} = (\{0,1\}; R, N, C_0, C_1)$
8. $\mathbb{A} = (\{0,1\}; R, N, C_0, C_1, S_{00}, S_{01}, S_{10}, S_{11})$
9. $\mathbb{A} = (\{0,1\}; \text{all unary and binary relations})$

Problem 2. Find a polynomial–time algorithm for CSP($\{0,1\}; H, C_0, C_1$).

Problem 3. Find a polynomial–time algorithm for CSP($\{0,1\}; C_0, C_1, G_1, G_2$).

Problem 4. Find a polynomial–time algorithm for CSP($\mathbb{Q}; <$).

Problem 5. Prove that CSP($\mathbb{Q}; <$) $\neq$ CSP($\mathbb{A}$), for every finite relational structure $\mathbb{A} = (A; R)$.

CSP lecture 25/26 – Problem Set 2

The *type* of a relational structure $(A; R_1, \dots, R_s)$ is the tuple $(\mathrm{ar}(R_1), \dots, \mathrm{ar}(R_s))$, where $\mathrm{ar}(R)$ is the arity of the relation R.

Suppose the type of $\mathbb{A} = (A; R_1, \dots, R_t)$ and $\mathbb{B} = (A; S_1, \dots, S_t)$ is $(n_1, \dots, n_t)$. A mapping $\phi : A \to B$ is called *a homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ if $(a_1, \dots, a_{n_i}) \in R_i \Rightarrow (\phi(a_1), \dots, \phi(a_{n_i})) \in S_i$ for every i. If such a homomorphism exists we write $\mathbb{A} \to \mathbb{B}$. A homomorphism $\mathbb{A} \to \mathbb{A}$ is *an endomorphism*, a bijective endomorphism is *an automorphism*.

Hom($\mathbb{A}$)
Given a finite relational structure $\mathbb{X}$ of the same type as $\mathbb{A}$. **Decide** whether $\mathbb{X} \to \mathbb{A}$.

Problem 1. Find a polynomial algorithm for Hom($\mathbb{A}$) where

1. $\mathbb{A} = (\{0,1\}; N)$ (notation is from the 1st problem set)
2. $\mathbb{A} = (\{0,1\}; N, C_0, C_1)$ (notation is from the 1st problem set)
3. $\mathbb{A} = (\{0,1\}; S_{00}, S_{11})$ (notation is from the 1st problem set)

Recall that a decision problem $\mathcal{P}_1$ is *polynomially reducible* to $\mathcal{P}_2$ if there exists a polynomial-time algorithm that transforms an input I of $\mathcal{P}_1$ to an input $r(I)$ of $\mathcal{P}_2$ so that I is a Yes-instance iff $r(I)$ is a Yes-instance. In such a case, we write $\mathcal{P}_1 \leq_P \mathcal{P}_2$. When $\mathcal{P}_1 \leq_P \mathcal{P}_2 \leq_P \mathcal{P}_1$, we write CSP($\mathbb{A}$) $\sim_P$ CSP($\mathbb{B}$) and say that the two problems are *polynomially equivalent*.

Problem 2. $\mathbb{A} = (\{0,1,2\}; N)$, where $N = \{0,1,2\}^2 \setminus \{(0,0),(1,1),(2,2)\}$. Prove that CSP($\mathbb{A}$) is polynomially equivalent to Hom($\mathbb{A}$).

Problem 3. $\mathbb{A}$ is a relational structure. Prove that CSP($\mathbb{A}$) is polynomially equivalent to Hom($\mathbb{A}$).

Observe that if CSP($\mathbb{A}$) $\leq_P$ CSP($\mathbb{B}$) and CSP($\mathbb{B}$) is in P (i.e., solvable in polynomial time), then CSP($\mathbb{A}$) is in P. Similarly, if CSP($\mathbb{A}$) $\leq_P$ CSP($\mathbb{B}$) and CSP($\mathbb{A}$) is NP–complete, then CSP($\mathbb{B}$) is NP–complete.

Problem 4. Prove that CSP($\mathbb{A}$) $\sim_P$ CSP($\mathbb{B}$), where

- $\mathbb{A} = (\{0,1,2\}; C_0, C_1, Q)$, where

$$C_0 = \{0\}, C_1 = \{1\}, Q = \{000, 110, 120, 210, 101, 102, 201, 202, 011, 012, 021\}$$

  (Q is a ternary relation, we omit the commas and parentheses, eg. 110 stands for (1,1,0).)

- $\mathbb{B} = (\{0,1\}; C_0, C_1, G_1)$ (where the notation is from the 1st problem set).

Hint: use homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$.

Problem 5. Prove that for each finite relational structure $\mathbb{A}$ there exists a relational structure $\mathbb{B}$ such that

- there exists a homomorphism $\mathbb{A} \to \mathbb{B}$ and a homomorphism $\mathbb{B} \to \mathbb{A}$, and
- $\mathbb{B}$ is a *core*, that is, each endomorphism of $\mathbb{B}$ is an automorphism.

Problem 5.1. Deduce that we can WLOG concentrate on CSPs over cores.

Problem 5.2. Prove that such a core is unique up to isomorphism.

Problem 5.3. Find a relational structure $\mathbb{A}$ such that every structure $\mathbb{B}$ with homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$ is *not* a core. Hint: $\mathbb{A}$ can be taken to be a directed graph.

Problem 6. Suppose

- $\mathbb{A} = (A; R_1, R_2, R_4)$ is a relational structure, where each R_i is an i-ary relation.
- E is the equality relation, i.e. $E = \{(a, a) : a \in A\}$
- S is the ternary relation on A defined by

$$S(x, y, z) = R_1(x) \wedge R_2(x, z) \wedge R_4(y, z, y, x)$$

- T is the binary relation defined by $T(x, y) = (\exists z \in A)\ S(x, y, z)$

Prove that

1. $\mathrm{CSP}(A; R_1, R_2, R_4, E) \leq_P \mathrm{CSP}(\mathbb{A})$
2. $\mathrm{CSP}(A; R_1, R_2, R_4, E, S) \leq_P \mathrm{CSP}(\mathbb{A})$
3. $\mathrm{CSP}(A; R_1, R_2, R_4, E, S, T) \leq_P \mathrm{CSP}(\mathbb{A})$

Problem 6.1. Try to formulate a general theorem covering these particular cases.

Problem 7. Prove that

1. $\mathrm{CSP}(\{0, 1, 2\}; C_0, C_1, N) \sim_P \mathrm{CSP}(\{0, 1, 2\}; C_0, C_1, C_2, N)$
2. $\mathrm{CSP}(\{0, 1, 2\}; N) \sim_P \mathrm{CSP}(\{0, 1, 2\}; N')$
3. $\mathrm{CSP}(\{0, 1\}; C_0, C_1, R) \sim_P \mathrm{CSP}(\{0, 1\}; R')$

where

$$\begin{aligned} N &= \{0, 1, 2\}^2 \setminus \{(0, 0), (1, 1), (2, 2)\} & N' &= \{0, 1, 2\}^3 \setminus \{(0, 0, 0), (1, 1, 1), (2, 2, 2)\} \\ R &= \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\} & R' &= \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \end{aligned}$$

Hint: try to use the general theorem from Problem 6.1.

Problem 8. Prove that $\mathrm{CSP}(\mathbb{A}), \mathrm{CSP}(\mathbb{B})$ and $\mathrm{CSP}(\mathbb{C})$ are polynomially equivalent, where

$$\begin{aligned} \mathbb{A} &= (\{0, 1, 2\}; C_0, C_1, C_2, N), \quad N = \{0, 1, 2\}^2 \setminus \{(0, 0), (1, 1), (2, 2)\} \\ \mathbb{B} &= (\{0, 1\}; S_{000}, S_{001}, S_{011}, S_{111}), \quad S_{ijk} = \{0, 1\}^3 \setminus \{(i, j, k)\} \\ \mathbb{C} &= (\{0, 1\}; C_0, C_1, R), \quad R = \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\} \end{aligned}$$

Problem 9. Prove that $\mathrm{CSP}(\mathbb{A}) \sim_P \mathrm{CSP}(\{0, 1, 2\}; N)$, where $\mathbb{A}, N$ are from the previous problem.

Problem 10. For each finite relational structure $\mathbb{A}$, find an input of $\mathrm{CSP}(\mathbb{A})$ whose solutions precisely correspond to endomorphisms of $\mathbb{A}$.

Problem 11. Let $\mathbb{A}$ be a finite *core* and let $\mathbb{B}$ be the relational structure formed from $\mathbb{A}$ by adding all the unary relations $C_a = \{a\}$, $a \in A$. Prove that $\mathrm{CSP}(\mathbb{A}) \sim_P \mathrm{CSP}(\mathbb{B})$.

Problem 12. Let $\mathbb{A}$ be a finite relational structure such that $\mathrm{CSP}(\mathbb{A})$ is in P. Prove that there is a polynomial–time algorithm for finding a solution of $\mathrm{CSP}(\mathbb{A})$.

CSP lecture 25/26 – Problem Set 3

An *n-ary operation* on a set A is a mapping $A^n \to A$. *The n-ary projection onto the i-th coordinate* (on a set A) is the operation π_i^n defined by $\pi_i^n(a_1, \dots, a_n) = a_i$ for any $a_1, \dots, a_n \in A$.

An n-ary operation $f : A^n \to A$ *preserves* an m-ary relation $R \subseteq A^m$ if $f(\mathbf{r}_1, \dots, \mathbf{r}_n) \in R$ (operation is applied coordinate-wise) whenever $\mathbf{r}_1, \dots, \mathbf{r}_n \in R$. In other words, for any $m \times n$ matrix whose columns are in R, f applied to the rows of this matrix gives a tuple in R. In such a situation, we also say that R is compatible with f, or R is *invariant under* f, or f is a *polymorphism* of R.

An operation $A^n \to A$ is a *polymorphism* of a relational structure $\mathbb{A} = (A; \dots)$ if it preserves all the relations in $\mathbb{A}$. The set of all polymorphisms of $\mathbb{A}$ is denoted $\mathrm{Pol}(\mathbb{A})$.

Problem 1. Observe that

1. $f : A^n \to A$ is compatible with every singleton unary relation $\{a\}$, $a \in A$, iff $f(a, \dots, a) = a$ for all $a \in A$;
2. the constant unary operation $c_a : A \to A$ (defined by $c_a(x) = a$ for any $x \in A$) is compatible with $R \subseteq A^n$ iff R contains the tuple $(a, a, \dots, a)$.

Problem 2. Let A be a set. Prove that f preserves every relation on A if and only if f is a projection.

Problem 3. Let $\mathbb{A} = (A; \dots)$ be a relational structure, $f \in \mathrm{Pol}(\mathbb{A})$ a binary polymorphism and $g \in \mathrm{Pol}(\mathbb{A})$ a ternary polymorphism. Then the 4-ary operation h defined by

$$h(x_1, x_2, x_3, x_4) = g(x_1, f(x_3, g(x_2, x_2, x_4)), x_3)$$

is a polymorphism of $\mathbb{A}$ as well. Try to formulate a general statement.

Problem 4. Find all unary and binary polymorphisms of the structure $\mathbb{A} = (\{0,1\}; H, C_0, C_1)$ from Problem Set 1 (Problem 2 – HORN-SAT).

Problem 5. Find all unary and binary polymorphisms of the structure

$$\mathbb{A} = (\{0,1\}; \text{all unary and binary relations})$$

from Problem Set 1 (Problem 1 – 2-SAT). Find some nice nontrivial (= not a projection) polymorphism of $\mathbb{A}$.

Problem 6. Find all unary, binary, and ternary polymorphisms of $\mathbb{A} = (\{0,1\}; C_0, C_1, G_1, G_2)$ from Problem Set 1 (Problem 3 – LIN-EQ($\mathbb{Z}_2$)).

A relation $R \subseteq A^m$ is *pp-definable* from $\mathbb{A} = (A; \dots)$ if it can be defined from relations in $\mathbb{A}$ by a pp-formula, that is, a formula which only uses conjunction, equality, and existential quantification. A relational structure $\mathbb{B} = (B; \dots)$ is pp-definable from $\mathbb{A}$ if $A = B$ and each relation in $\mathbb{B}$ is pp-definable from $\mathbb{A}$. We also say that $\mathbb{A}$ pp-defines $\mathbb{B}$.

Problem 7. Prove that any relation pp-definable from $\mathbb{A}$ is invariant under every polymorphism of $\mathbb{A}$.

Problem 8. Find all polymorphisms of the structure $\mathbb{B}$ in Problem Set 2 (Problem 8 – 3-SAT). Hint: only projections; possible approach: (1) pp-define the four-ary relations of the form $R_{a,b,c,d} = \{0,1\}^4 \setminus \{(a,b,c,d)\}$, (2) pp-define all four-ary relations (3) similarly, pp-define every relation, (4) use Problem 2.

Problem 9. Let $\mathbb{A}$ be a finite structure. Prove that a relation invariant under every polymorphism of $\mathbb{A}$ is pp-definable from $\mathbb{A}$. Proof strategy:

(i) Denote $R = \{(c_{11}, \dots, c_{1k}), \dots, (c_{m1}, \dots, c_{mk})\}$

(ii) Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be a complete list of m-tuples of elements of A (i.e. $n = |A|^m$)

(iii) Prove that the relation

$$S = \{(f(\mathbf{a}_1), \dots, f(\mathbf{a}_n)) : f \text{ is an } m\text{-ary polymorphism}\}$$

is pp-definable from $\mathbb{A}$ (no need to use existential quantification)

(iv) Existentially quantify over all coordinates but those corresponding to $(c_{11}, \dots, c_{m1})$, $\dots$, $(c_{1k}, \dots, c_{mk})$

(v) Prove that the obtained relation contains R (because of projections) and is contained in R (because of compatibility)

Problem 10. Let $\mathbb{A} = (\mathbb{Z} \times \mathbb{Z}; R, U)$, where

$$R = \{((x, y), (x', y')) \mid x = x', |y' - y| \in \{1, 2\}\}, \quad U = \{(0, 0)\}.$$

Prove that $\{(0, y) \mid y \in \mathbb{Z}\}$ is invariant under every polymorphism of $\mathbb{A}$, but that this set is not pp-definable from $\mathbb{A}$.

Problem 11. Observe that, for finite structures $\mathbb{A}$ and $\mathbb{B}$,

1. $\mathbb{A}$ pp-defines $\mathbb{B}$ iff $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$ and in such a case $\mathrm{CSP}(\mathbb{B}) \leq_P \mathrm{CSP}(\mathbb{A})$;
2. any CSP over a two–element structure is polynomially reducible to 3–SAT
3. if $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$, then the proof of Problem 9 gives an explicit pp-formulas defining relations in $\mathbb{B}$ from relations in $\mathbb{A}$.
4. In particular, for $\mathbb{B}$ and $\mathbb{C}$ as in Problem Set 2, Problem 4, we get $\mathrm{CSP}(\mathbb{C}) \leq \mathrm{CSP}(\mathbb{B})$. How large are the explicit formulas defining relations in $\mathbb{C}$ from relations in $\mathbb{B}$?

CSP lecture 25/26 – Problem Set 4

A set of operations on a set A is a *(function) clone* on A if it contains all projections and is closed under composition (as in Problem 3, Problem Set 3). A function clone on A is called *idempotent* if for every operation f in it and every $a \in A$, $f(a, a, \dots, a) = a$. For a se

Problem 1. Recall that for any relational structure $\mathbb{A}$, $\mathrm{Pol}(\mathbb{A})$ is a clone.

In this problem set, we focus on function clones on the set $A = \{0, 1\}$. We use the following notation for some special operations on $\{0, 1\}$:

$\wedge$ the binary minimum operation

$\vee$ the binary maximum operation

maj the ternary majority operation defined by $maj(a, a, b) = maj(a, b, a) = maj(b, a, a) := a$ for every $a, b \in \{0, 1\}$

min the ternary minority operation defined by $min(a, a, b) = min(a, b, a) = min(b, a, a) := b$ for every $a, b \in \{0, 1\}$

An operation $f : A^n \to A$ is called *essentially unary* if there exist i and a unary operation $\alpha : A \to A$ such that $f(x_1, \dots, x_n) = \alpha(x_i)$ for every $x_1, \dots, x_n \in A$.

Problem 2. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither $\wedge$ nor $\vee$. Show that the only binary operations in $\mathcal{A}$ are the two projections.

Problem 3. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither of the operations $\wedge, \vee, maj, min$. Show that the only binary and ternary operations in $\mathcal{A}$ are the projections.

Problem 4. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither of the operations $\wedge, \vee, maj, min$. Show that $\mathcal{A}$ contains only projections.

Hint: possible strategy

- Let $f \in \mathcal{A}$ be n-ary with $n \geq 4$.
- Assume first $f(1, 0, 0, \dots, 0) = 1$. Use the binary operation $g(x, y) := f(x, y, \dots, y)$ to show that $f(0, 1, \dots, 1) = 0$. Use ternary operations of the form $g(x, y, z) := f(w_1, w_2, \dots)$ where $w_1, w_2, \dots \in \{x, y, z\}$ to show that f is the projection onto the first coordinate.
- Deduce that if f is not a projection, then $f(x, \dots, x, y, x, \dots, x) = x$ for every x, y and every position of y.
- Assuming this and using appropriate ternary operations (similar as above) show that $f(x, \dots, x, y, y) = x$, …, etc, and derive a contradiction

Problem 5. Let $\mathcal{A}$ be a clone on $A = \{0, 1\}$ with an operation which is not essentially unary. Prove that $\mathcal{A}$ contains a constant unary operation, or at least one of the operations $\wedge, \vee, maj, min$.
Hint: try to reduce to the idempotent case

CSP lecture 25/26 – Problem Set 5

A ternary operation $m : A^3 \to A$ is called a *majority operation* if $m(a,a,b) = m(a,b,a) = m(b,a,a) = a$ for each $a, b \in A$ (note that for $|A| \leq 2$ there is a unique majority operation on A, otherwise there are more of them).

Problem 1. Let $R \subseteq A^n$ be a relation compatible with a majority operation on A. Denote $\pi_{i,j}(R)$ the projection of R onto the coordinates i, j $(1 \leq i, j \leq n)$, that is,

$$\pi_{i,j}(R) = \{(a_i, a_j) : (a_1, \dots, a_n) \in R\} \ .$$

Prove that R is determined by these binary projections, that is,

$$(a_1, \dots, a_n) \in R \text{ if and only if } (\forall i, j,\ 1 \leq i, j \leq n)\ (a_i, a_j) \in \pi_{i,j}(R)$$

Hint: start with $n = 3$

Problem 2. Let $\mathbb{A} = (A; \dots)$ be a relational structure with a majority polymorphism. Show that there exists a relational structure $\mathbb{B} = (A; \dots)$ which contains only binary relations such that $\mathbb{A}$ is pp-definable from $\mathbb{B}$ and $\mathbb{B}$ is pp-definable from $\mathbb{A}$. For $A = \{0, 1\}$, conclude that $\mathrm{CSP}(\mathbb{A}) \leq_P$ 2–SAT (and thus $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time).

Problem 2.1. Let $\mathbb{A} = (\mathbb{Z}; R_1, \dots, R_k)$, where all relations $R_1, \dots, R_k$ admit a quantifier-free definition over the relations $y < x + c$ and $y = x + c$, where $c \in \mathbb{Z}$. E.g. R can be the 4-ary relation that holds on (x, y, z, t) iff $(x > y + 1 \vee x > z - 6) \wedge (x = z \Rightarrow t = y + 1)$ holds. Suppose that the ternary median operation is a polymorphism of $\mathbb{A}$. Show that $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time.

Problem 3. Let $\mathbb{A} = (\{0, 1\}; \dots)$ be a relational structure with polymorphism min (from Problem Set 4). Show that each n-ary relation of $\mathbb{A}$ is an affine subspace of $\mathbb{Z}_2^n$. Conclude that $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time.

Problem 4. Let $\mathbb{A} = (\{0, 1\}; C_0, C_1, H)$ be as in Problem Set 1 (the corresponding CSP is HORN-3-SAT). For every relation $R \subseteq \{0, 1\}^n$ compatible with $\wedge$ find a pp–definition from $\mathbb{A}$.

Problem 5. Prove that for each relational structure $\mathbb{A} = (A; \dots)$ with $A = \{0, 1\}$, either $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time or $\mathrm{CSP}(\mathbb{A})$ is NP–complete (this is *Schaefer's dichotomy theorem* (1978)). Describe the two cases in terms of polymorphisms.