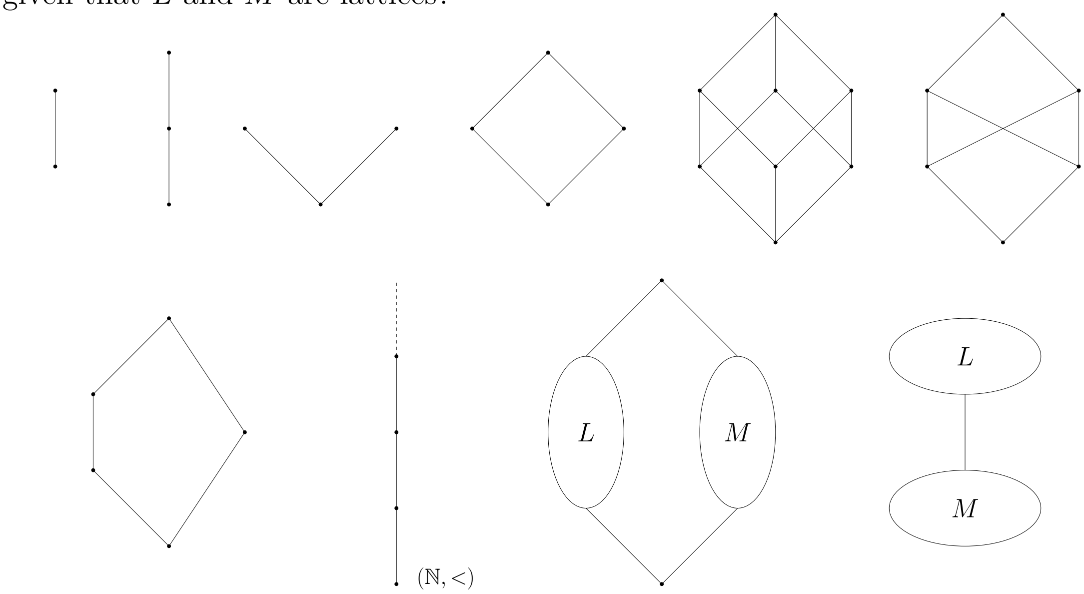
Universal Algebra Exercises - Sheet 1

Recall the definitions of (semi)lattices and (semi)lattice ordered sets.

Exercise 1. Which of the following Hasse Diagrams represent (semi)lattices, given that L and M are lattices?



Exercise 2. Find the correspondence between lattices and lattice ordered sets:

(i) For a given semilattice $(S, \wedge)$ find a natural definition of a semilattice order on S, and vice-versa.

(ii) Conclude that there is a 1-to-1 correspondence between semilattices and semilattice ordered sets.

(iii) Prove that there is a 1-to-1 correspondence between lattices and lattice ordered sets.

Exercise 3. Show that every homomorphism of lattices is order preserving. What about the converse?

Exercise 4. Choose your favorite (finite) group and draw the poset of its normal subgroups, ordered by inclusion of their underlying sets.

Exercise 5. Given a group G, prove that the poset of its normal subgroups is a lattice (ordered set). What are the operations meet and join?

Exercise 6. Let $(P, \leq)$ be a poset. Show that there is a linear order $\leq'$ on P such that $p \leq q \implies p \leq' q$ for all $p, q \in P$.

Exercise 7 (*)**.** Let $(P, \leq)$ be a poset, $(\mathcal{U}_P, \subseteq)$ the poset of its upsets and $(\mathcal{D}_P, \subseteq)$ the poset of its downsets.

(i) Find an injective order preserving map $(P, \leq) \to (\mathcal{D}_P, \subseteq)$

(ii) Show that there is no surjective order preserving map $(P, \leq) \to (\mathcal{D}_P, \subseteq)$

(iii) Conclude that for any set X, there is no surjective map $X \to 2^X$

Exercise 8. Prove or disprove that for every poset $(P, \leq)$ we have

$$(\mathcal{D}_P, \subseteq) \cong (\mathcal{U}_P, \supseteq)$$

Universal Algebra Exercises - Sheet 2

Definition. A lattice L is called *distributive*, if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{2.1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{2.2}$$

Exercise 9. Show that every lattice with less than four elements is distributive. Find examples of distributive lattices with a large number of elements.

Exercise 10. Show that in the definition of distributive lattices (3.1) and (3.2) are equivalent.

Definition. A lattice L is called *modular*, if for all $x, y, z \in L$

$$x \leq z \implies x \vee (y \wedge z) = (x \vee y) \wedge z$$

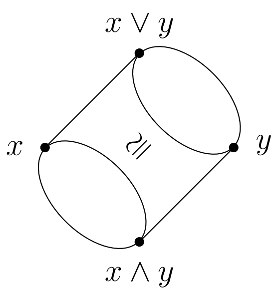
Exercise 11. Show that every distributive lattice is modular and disprove the converse, i.e. find a modular lattice that is not distributive.

Exercise 12. Show that the following two statements hold for all lattices L and all $x, y, z \in L$

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

$$x \leq z \implies x \vee (y \wedge z) \leq (x \vee y) \wedge z$$

Exercise 13 (Diamond isomorphism theorem)**.** Let L be a modular lattice and $x, y \in L$. Show that the intervals $I[x \wedge y, x]$ and $I[y, x \vee y]$ are isomorphic lattices.

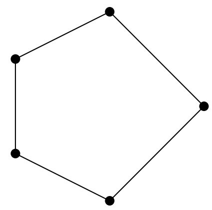


Exercise 14. A term $m(x, y, z)$ of an algebra A is called *majority* if it satisfies the identities

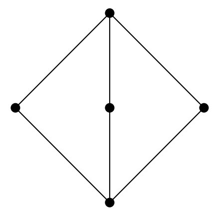
$$x \approx m(x, x, y) \approx m(x, y, x) \approx m(y, x, x)$$

Show that every lattice has a majority term.

Theorem (Dedekind)**.** Prove that a lattice is modular if and only if it does not contain the following lattice as a sublattice.



Theorem (Birkhoff)**.** A modular lattice is distributive if and only if it does not contain the following lattice as a sublattice.



Universal Algebra Exercises - Sheet 3

Exercise 15. Show that every complete lattice is bounded.

Exercise 16. Find examples of lattices L that contain a sublattice S such that

(i) L is complete but S is not complete

(ii) L is not complete but S is complete

(iii) both L and S are complete lattices but S is not a complete sublattice

Exercise 17. Let L be a complete lattice and $a, b \in L$ two compact elements.

(i) Is $a \vee b$ compact?

(ii) Is $a \wedge b$ compact?

Exercise 18. Let C be a closure operator on a set X. Prove that L_C is closed under finite unions if and only if for all subsets $U, V \in 2^X$

$$C(U \cup V) = C(U) \cup C(V)$$

Exercise 19. Let X be a set and let ϕ be the binary relation on 2^X defined by

$$(U, V) \in \phi \iff U \cap V \neq \emptyset$$

Consider the Galois correspondence on the sets 2^{2^X} and 2^{2^X} induced by this relation

(i) Let $X = \{1, 2, 3, 4\}$. Compute $A^{\leftarrow\rightarrow}$ and $A^{\rightarrow\leftarrow}$ for both $A = \Big\{\{1, 2\}, \{2, 3\}\Big\}$ and $A = \Big\{\{1, 2\}, \{2\}\Big\}$. Compare the results.

(ii) Prove that if a Galois correspondence is defined by a symmetric relation on a set, then the closure operators induced by it coincide.

(iii) Prove that for every $A \subseteq 2^X$ we have

$$A^{\rightarrow\leftarrow} = \{U \in 2^X \mid \exists V \in A, V \subseteq U\}$$

Exercise 20. Let C be a closure operator on a set X. Find a relation $\phi \subseteq X \times 2^X$ whose induced Galois correspondence gives

$$C(U) = U^{\rightarrow\leftarrow}$$

for all subsets $U \subseteq X$.