GROUP SCHEDULING AND ASSIGNMENT: COMPLEXITY AND ALGORITHMS

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Abstract

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Complexity of Group Scheduling Problem I

Scheduling an event for a group of invitees is a frustrating task; it tends to be tedious and time consuming. A typical scheduling process can be described as iterative approval voting: An event organizer selects a candidate set of date/time options, and asks invitees to respond with their availability. Given the responses, the organizer chooses an agreeable option and announces it, or she may repeat the process by proposing another set of options if no feasible option is found. Naturally the organizer and her invitees wish to reach an agreement within a small number of iterations and proposed options – the more iterations and proposed options there are, the more laborious a scheduling process becomes.

There exist several software tools that are designed to help an event organizer handle a scheduling process more efficiently – one of the most well-known tools is Doodle [1]. In Doodle, an organizer can simply list as many date/time options as she likes, and each invitee is asked to respond with her availability. Essentially, invitees participate in approval voting upon all proposed options. Hence if too many options are proposed, then invitees are given the burden of answering them all. This often leads to undesired behaviors of invitees such as herding or procrastination, instead of honest, quick responses [46]. On the other hand, if the organizer proposes only few options, there may not exist an agreeable outcome after

[1] http://www.doodle.com

all, which may result in another iteration of proposals and responses. In fact, surveys find that the most challenging part of group scheduling is due to "chasing people who do not answer" and "finding a suitable time." [2]

Doodle has many practical advantages. One of them is its simplicity – invitees simply need to approve a subset of options based on their availability. Another is a short duration of scheduling process – the duration it takes until the last invitee responds and the meeting time is settled. Each invitee needs to respond only once, which limits the degree to which the process is hijacked by invitees' delayed responses. But Doodle has many drawbacks, even in this idealized setting. In particular, Doodle forces the invitees to examine many potential date/time options. This can be inconvenient not so much due to the effort involved (though that is a factor), but mostly because invitees need to block their available slots off until a option is announced. We use the expected number of time slots floated by the organizer as a proxy for this inconvenience.

To avoid incurring much inconvenience, the organizer can select only a few options, and poll the invitees about those. If a feasible option is not found among them, then repeat with another batch of date/time options. We call this broad class of mechanisms *B-Doodle* (for "Batched Doodle"). Clearly, Doodle is a special case with one batch consisting of all options. Another extreme case is the OAAT (one-at-a-time) mechanism, in which the organizer tests a single option at each iteration. Doodle minimizes the total number of iterations, whereas OAAT minimizes (expected) inconvenience caused by the scheduling process. In-between lie many other mechanisms, with different batching schemes, that trade off time against inconvenience differently.

Figure 2.1 illustrates this via a simple example. In this scenario there are six options, four invitees, each of whom is available at each option independently with probability .8. The figure depicts a scatter plot of all 32 different B-Doodle mechanisms, including Doodle and OAAT. The vertical axis depicts the expected number of rounds to determine the option, and the horizontal axis the expected inconvenience.

We can clearly observe a time-inconvenience Pareto frontier in this plot. If in addition there is an overall cost function combining time and inconvenience, one can identify an

[2] http://en.blog.doodle.com/2012/07/26/new-findings-a-small-number-of-initiators-organize-most-of-the-meetings/

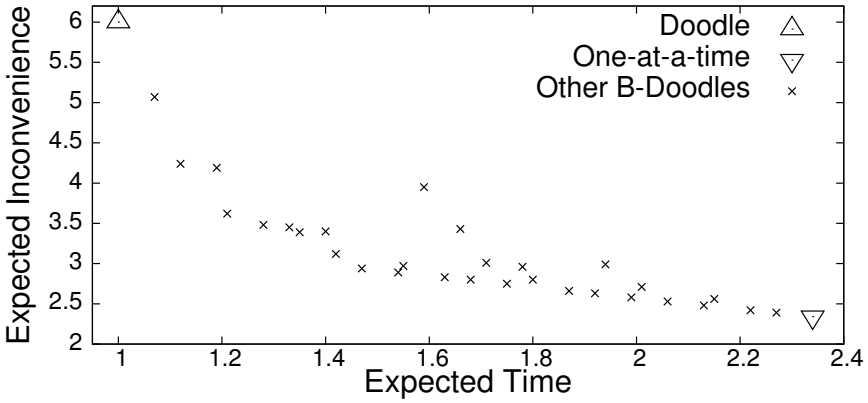

Figure 2.1: A scatter plot of B-Doodle Mechanisms.

optimal B-Doodle mechanism along this frontier as illustrated in Figure 2.2. In this example, if the overall cost is the linear function $3 \cdot \text{Time} + \text{Inconvenience}$, the optimal mechanism happens to be the "Half-n-half" mechanism; this mechanism sends out 3 options in the first batch, and if no feasible time slot is found then sends out the remaining 3 options in the next batch.

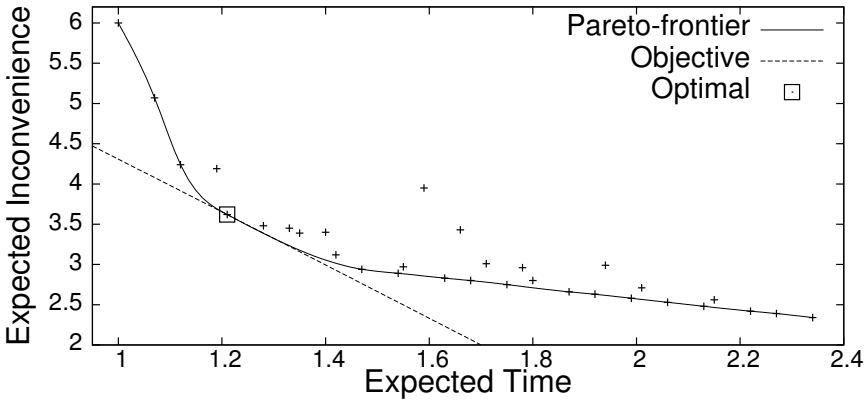

Figure 2.2: Pareto-frontier and objective function.

In this chapter we will investigate the difficulty of determining the optimal B-Doodle mechanism, and to what degree it improves on the simple Doodle in realistic scenarios. Under the assumption that the set of events in which an invitee is available for a particular option is mutually independent, we provide an efficient recursive algorithm for computing

the optimal batching scheme, for a broad class of objective functions of time and inconvenience, including the linear combination as in the above example.[3]

Lastly, we assume that the event organizer is given probability estimates on availability of agents, but it is not clear how one can obtain such probabilities. Probability estimation is an interesting and challenging research question on its own, and we do not attempt to solve the question in this work. However, we provide several plausible methods for estimating the probabilities in the context of group scheduling, which can enable our model and algorithm to be deployed as a real-world application in the future. In the psychology literature, Mann et al. found that cultural differences between the Western, individualistic countries (such as the United States) and the Eastern, collectivistic countries (such as China and Japan) lead to different behaviors of respondents when it comes to a group-decision making process [35]. More recently, Reinecke et al. analyzed more than 1.5 million Doodle date/time polls from 211 countries, and confirmed similar findings regarding time perception and group's behavior [39]. Among others, they found that "in comparison to predominantly individualist societies, poll participants from collectivist countries respond earlier, agree to fewer options but find more consensus," which agrees with the findings of Mann et al. Besides the cultural differences, Doodle's own surveys on event scheduling found that people tend to respond to the scheduling surveys on Mondays, while Monday is the least popular day for having a meeting. [4] We believe that these studies and findings can be used to design a reasonable estimator for availability of agents, by utilizing the features that are known to be crucial – such as demographics of the group and purpose of the event being scheduled. Recent work by Zou et al. analyzed over 340,000 Doodle polls data to study behavioral patterns of the users, and they were able to identify response functions that match the response patterns observed in the real data [46]. We believe that a similar approach can be taken to tackle the problem of probability estimation in the context of group scheduling.

[3] As another, more minor generalization of the problem, we will allow that the organizer is content with only a fraction of the invitees attending, not necessarily requiring everyone be available. This fraction will be a part of input to the problem; of course, as a special case, 1 implies full attendance.

[4] http://en.blog.doodle.com/2012/05/23/mondays-for-planning-busy-weekends/

2.1 Notation and Definitions

We mentioned two dimensions of optimality in the scheduling process: Time and Inconvenience. Time captures the duration of the scheduling process and Inconvenience measures how much inconvenience is caused for each invitee during the scheduling process. In this section we formally define the class of B-Doodle mechanisms and the Batched Doodle Problem (BDP).

Consider a setting where there are n invitees and s date/time options from which the organizer can choose to propose. The organizer wishes to find a *feasible* date/time option that works for at least $\lceil f \cdot n \rceil$ invitees (we call $f \in [0,1]$ the *feasibility threshold*) or determines that there is no feasible option. We model the uncertainty of availability of invitees as a random matrix as follows. Consider a matrix of mutually independent Bernoulli random variables where rows represent invitees and columns represent a set of date/time options. Given probability of success for each Bernoulli random variable, the organizer can "inspect" a batch of columns at once to know of the realization of those entries. This corresponds to polling invitees' availability for a batch of date/time options. The organizer wishes to determine with certainty whether the matrix contains a feasible column or it does not. Using the random matrix model, we first investigate interesting properties of an optimal solution, and then discuss how it performs better than the classic Doodle under various settings.

Definition 1 (Feasibility)**.** Let A be a matrix whose entries are from $\{0,1\}$, and refer to the entry of A at row r and column c as $a_{r,c}$. We say that a column c of A is *feasible* if it consists only of 1's (otherwise it is *infeasible*). We say that A is *feasible* if it contains at least one feasible column (otherwise it is *infeasible*).

The organizer iteratively sends out a batch of options until a feasible option is found. In our setting Time spent by the scheduling process is measured by the number of iterations and Inconvenience caused is measured by the number of options that have been sent out by the organizer. Let us define a class of B-Doodle mechanisms that describe how the organizer sends out options in each iteration.

Definition 2 (B-Doodle Mechanism)**.** Let $S = \{1,2,\ldots,s\}$ be a set of s options. We define a B-Doodle mechanism for S as an ordered partition of S. Let $B = \langle S_1, S_2, \ldots, S_m \rangle$ be a

partition of S into m subsets such that $S_j \neq \emptyset$, $S_j \cap S_k = \emptyset$ for $j \neq k$, and $\cup_{l=1}^{m} S_l = S$ where $1 \leq j,k \leq m$. We define $b_j = |S_j|$ for all $j \leq m$ and call b_j the size of the j-th batch. We write B_m to emphasize that B has m batches.

We interpret a B-Doodle mechanism B as follows: the organizer sends out options in S_1 during the first iteration. If a feasible option is found, the process ends. Otherwise she sends out the next batch, S_2, and so on. Note that there exist exponentially many B-Doodle mechanisms for any a given S (exponential in cardinality of S).

The goal is to find a B-Doodle mechanism that minimizes the expected cost of scheduling process, given an objective function of Time and Inconvenience. Earlier we considered a simple cost function that is a linear combination of the two. While this cost function fits in many realistic situations, we want to explore a larger class of cost functions. We define a class of cost functions that depend only on the number of iterations and batch sizes.

Definition 3 (Cost function)**.** A cost function c takes two integers j and b as arguments, and $c(j,b) > 0$ describes the aggregate cost of Time and Inconvenience that is incurred by sending out a batch of b options during the j-th iteration. We assume that cost is additive so that the overall cost of executing the first j iterations of $B_m = \langle S_1, S_2, \ldots, S_m \rangle$ is simply the sum, $\sum_{k=1}^{j} c(k, b_k)$, which we denote by $C(j, B_m)$; recall that $b_k = |S_k|$. A cost function c is said to be θ-*simple* if there exists some constant $\theta > 0$ such that $c(j,b) = \theta \cdot c(j-1,b)$ for all $j > 1$ and for all $b \geq 1$. We often drop θ and just state that a cost function is *simple*, for brevity.

It is reasonable to assume that cost is additive (with respect to iterations) because an additional iteration adds the time spent and inconvenience caused for the invitees.

Note that $C(j, B_m)$ is strictly increasing in j for any fixed B_m because $c(j,b) > 0$ for any j and b. While the class of simple cost functions may seem too restrictive, we present a few natural choices of cost functions that belong to the class of simple cost functions. In this work we assume that the underlying cost function is simple; further we assume that $c(j,b)$ for all $1 \leq j,b \leq s$ can be computed in polynomial time with respect to s.

Example 1 (Examples of simple cost functions)**.** The simplest choice of a cost function is a linear combination of Time and Inconvenience, parameterized by some constant $\alpha > 0$;

we call this a *linear* cost function. This cost function is 1-simple. Notice that the first term captures Time and the second captures Inconvenience in both expressions.

$$c_\alpha(j, b_j) = \alpha + b_j$$
$$C_\alpha(j, B_m) = (\alpha \cdot j) + \left(\sum_{k=1}^{j} b_k \right)$$

In some cases the organizer may want to penalize the mechanism for executing too many iterations by making later iterations to cost more than earlier iterations. The following describes such cost function, parameterized by some constant $\beta > 1$; we call this a *time-averse* cost function. Notice that the term β^j is strictly increasing as j increases (recall $\beta > 1$). This cost function is β-simple.

$$c_\beta(j, b_j) = \beta^j \cdot b_j$$
$$C_\beta(j, B_m) = \left(\sum_{k=1}^{j} \beta^k \cdot b_k \right)$$

In contrast if the organizer is interested in reducing the size of each batch because it may cause too much inconvenience, then the following cost function could be used, which is parameterized by some constant $\gamma > 1$; we call this an *inconvenience-averse* cost function which is 1-simple.

$$c_\gamma(j, b_j) = \gamma^{b_j}$$
$$C_\gamma(j, B_m) = \left(\sum_{k=1}^{j} \gamma^{b_k} \right)$$

We now formally define the Batched Doodle Problem.

Definition 4 (Batched Doodle Problem)**.** An instance of the Batched Doodle Problem (BDP) is a tuple (A, P_A, f, c) where A is a matrix of Bernoulli random variables, P_A is the probability matrix associated with it, f is the feasibility threshold with $f \in [0, 1]$, and c is a cost function.

The objective in BDP is to find an optimal B-Doodle mechanism B^* such that B^* minimizes the expected cost of the scheduling process (expectation with respect to P_A), given (A, P_A, f, c).

Since there are exponentially many B-Doodle mechanisms, it is impractical to enumerate each B-Doodle and compute its expected cost in order to find an optimal one. In what follows we make some simplifying assumptions that are assumed throughout this paper, and we present an efficient algorithm to solve BDP in next section.

2.2 Efficient Algorithm for Finding Optimal B-Doodle

2.2.1 Preliminaries

Intuitively, if the organizer knows the probability distribution of availability of invitees, then he should inspect the columns that are more likely to be feasible first. We will prove this claim using the following lemma.

Lemma 1. *Given (A, P_A, f, c), let q_t for each column t be the probability that column t is f-feasible. We can compute q_t in polynomial time.*

Proof. For any fixed t, let us define $V_t(i,z)$ to be the probability that among the column t of A, exactly z entries out of the first i entries (in their row indices) are 1's. $V_t(i,z)$ is well-defined where $1 \leq i \leq n$ and $0 \leq z \leq i$. We further define $V_t(i,z)$ for the following degenerate cases:

$$V_t(i,z) = \begin{cases} 1 & \text{if } i = z = 0 \\ 0 & \text{if } z = -1 \text{ or } (i = 0 \wedge z > 0) \end{cases}$$

Then we can compute $V_t(i,z)$ using a simple dynamic programming algorithm according to the following recurrence relation where $1 \leq i \leq n$ and $0 \leq z \leq i$:

$$V_t(i,z) = V_t(i-1, z-1) \cdot p_{i,t} + V_t(i-1,z) \cdot (1 - p_{i,t}) \tag{2.1}$$

It is easy to verify that the recurrence relation is correct. For the degenerate cases (or base cases) when $i = z = 0$, no entries (out of 0 entries) in column t are equal to 1, and

thus $V_t(i,z) = 1$ in this case. When $(z = -1)$ or $(i = 0 \wedge z > 0)$, $V_t(i,z) = 0$ because it is impossible to have z entries out of i entries.

For the main recurrence relation, the entry $a_{r,t}$ is either 1 or 0, and we compute $V_t(i,z)$ by considering both cases. For each fixed t, there are $O(n^2)$ entries of V_t to be computed, and computing each entry takes $O(1)$; therefore it takes $O(sn^2)$ time to compute $V_t(i,z)$ for all t,i,z.

Finally we can compute q_t after we compute $V_t(i,z)$ for all i,z, as follows:

$$q_t = \sum_{z=\lceil f \cdot n \rceil}^{n} V_t(n,z).$$

□

Lemma 1 allows us to describe a B-Doodle mechanism in a simpler form, along with the following theorem because an optimal B-Doodle mechanism must inspect the columns in non-increasing order of q_t.

Theorem 1. *Consider a B-Doodle mechanism B described by $\langle S_1, S_2, \ldots, S_m \rangle$. Suppose that there is some option $t \in S_l$ and $t' \in S_{l'}$ with $q_t < q_{t'}$ and $l < l'$. Then B is not an optimal in the sense that we can find another mechanism B^* whose expected cost is strictly less than the expected cost of B.*

Proof. Given B, let us construct $B^* = \langle S_1^*, S_2^*, \ldots, S_m^* \rangle$ that is the same as B except that we swap t and t'. That is, $S_i^* = S_i$ for all $i \neq l$ and $i \neq l'$, and $S_l^* = (S_l \cup t') \setminus t$ and $S_{l'}^* = (S_{l'} \cup t) \setminus t'$. Note that the two mechanisms have the same batch sizes ($b_j = b_j^*$ for all j).

Let w_j be the probability that the scheduling process ends after the j-th iteration if we use B, and w_j^* if we use B^*. Then $w_j = w_j^*$ for all $j < l$ because $S_j = S_j^*$ for all j with $1 \leq j < l$. Clearly it holds that $w_l < w_l^*$ because of the swap of t and t'. For j with $l < j \leq l'$, it holds that $w_j > w_j^*$; this is not obvious, but the intuition is that if w_l^* increases then the subsequent w_j^*'s with $l < j \leq l'$ must decrease as they depend on $1 - w_l^*$, until this effect is canceled out in the l'-th batch. Since the probability of finding no feasible options during l' iterations is the same in both cases (because $\cup_{i=1}^{l'} S_i = \cup_{i=1}^{l'} S_i^*$) it holds that $w_j = w_j^*$ for

all $j > l'$.

Let E_B be the expected cost of B and E_{B^*} of B^*. Since $b_j = b_j^*$ for all j, it holds that $C(j,B) = C(j,B^*)$ for all j, and we have:

$$\begin{aligned} E_B - E_{B^*} &= (w_l - w_l^*)C(l,B) + \sum_{j=l+1}^{l'} (w_j - w_j^*)C(j,B) \\ &= \sum_{j=l+1}^{l'} (w_j - w_j^*)(C(j,B) - C(l,B)) \end{aligned}$$

The first inequality holds by definition of the expected cost and canceling out some terms; the second inequality holds because $(w_l - w_l^*) + \sum_{j=l+1}^{l'}(w_j - w_j^*) = 0$ (as probabilities must add up to one). Since $w_j > w_j^*$ and $C(j,B) > C(l,B)$ for all j (because $C(j,B)$ is an increasing function in j for fixed B), we conclude that $E_B > E_{B^*}$. □

Theorem 1 implies that an optimal B-Doodle that minimizes the expected cost must have the following property: for any two batches S_j and S_k with $j < k$, it must hold that $q_t \geq q_{t'}$ for all $t \in S_j$ and $t' \in S_k$ (otherwise we can apply the theorem and swap the two options to obtain a mechanism with smaller expected cost). Therefore we can limit our attention to the sub-class of B-Doodle mechanisms that only describe batch sizes, but not explicitly which options in each batch. Due to Lemma 1 we can compute q_t for each t in polynomial time, and thus we can simply sort options by q_t in non-increasing order as a pre-processing step.

Therefore we can focus on the following sub-class of *simplified* B-Doodle mechanisms when we seek an optimal B-Doodle mechanism.

Definition 5 (Simplified B-Doodle Mechanism)**.** Let $S = \{1,2,\ldots,s\}$ be a set of s options. A simplified B-Doodle mechanism B for S is a vector of integers, described as $B = \langle b_1, b_2, \ldots, b_m \rangle$ where ($b_j \geq 1$ for all $j \leq m$) and ($\sum_{k=1}^{m} b_k = s$). We write B_m to emphasize that B has m batches. It is assumed that B_m partitions S into m batches according to b_j's where options (or columns) are sorted by q_t in non-increasing order.

From now on we use the definition of a simplified B-Doodle, and simply write a B-Doodle mechanism to mean a simplified B-Doodle mechanism but this should cause no

confusion.

2.2.2 Optimal Algorithm

In this section we present an algorithm that finds an optimal B-Doodle mechanism that minimizes the expected cost, given an instance of BDP. Let us first describe how one can express the expected cost of a B-Doodle mechanism.

Given an instance (A, P_A, f, c), consider some B-Doodle mechanism $B_m = \langle b_1, b_2, \ldots, b_m \rangle$ with $\sum_{k=1}^{m} b_k = |S|$. Let $\Pr(j)$ be the probability that the scheduling process ends after j-th iteration. Let us denote the expected cost of B_m by $\mathbb{E}_c[B_m]$, which can be expressed in terms of $\Pr(\cdot)$ and the cost function c:

$$\mathbb{E}_c[B_m] = \sum_{j=1}^{m} \Pr(j) \cdot C(j, B_m) = \sum_{j=1}^{m} \Pr(j) \cdot \left(\sum_{k=1}^{j} c(k, b_k) \right) \tag{2.2}$$

While one can compute $\Pr(j)$ in polynomial time, it is not necessary for our algorithm. Instead we present an important lemma that leads us to an efficient algorithm. The following lemma states that if c is simple, then we can compute $\mathbb{E}_c[B_m]$ in a recursive manner.

Lemma 2. *Consider any mechanism $B_m = \langle b_1, b_2, \ldots, b_m \rangle$. Let us denote another mechanism that is obtained after removing the first batch (b_1) from B_m, as $\hat{B}_{m-1}$ (i.e. $\hat{B}_{m-1} = \langle b_2, b_3, \ldots, b_m \rangle$). For each option $t \in S$, let q_t be the probability that t is f-feasible.*

If c is a θ-simple cost function with with $\theta > 0$, the following equality holds:

$$\mathbb{E}_c[B_m] = c(1, b_1) + \left(\prod_{t=1}^{b_1} (1 - q_t) \right) \cdot \theta \cdot \mathbb{E}_c[\hat{B}_{m-1}] \tag{2.3}$$

Proof. Let us first provide an intuitive way to understand the recurrence relation given by Equation 2.3. The first term $c(1, b_1)$ captures the cost of sending out the first batch; regardless of when the scheduling process ends, this cost incurs with probability of 1. If it turns out that the first batch does not contain a feasible option with probability of $(1 - \Pr(1))$, then the organizer must send out the remaining batches, which is precisely $\hat{B}_{m-1}$. It is easy to verify that the product term in the equation is equal to $(1 - \Pr(1))$. The expected cost of

$\hat{B}_{m-1}$ is adjusted by a factor of θ in the equation because the cost function is θ-simple.

We now formally prove the claim. Given B_m as described above, for each batch $j \in \{1,2,\ldots,m\}$, let r_j be the probability that the j-th batch has at least one f-feasible option. Let us define $v_0 = 0$ and $v_j = \sum_{k=1}^{j} b_k$ (i.e. v_j is the number of options in batches 1 through j). We can express r_j as:

$$r_j = 1 - \prod_{t=v_{j-1}+1}^{v_j} (1 - q_t) \tag{2.4}$$

$\Pr(j)$ is the probability that the scheduling process ends after j-th iteration (with $\Pr(m) = 1 - \sum_{k=1}^{m-1} \Pr(k)$). For $1 \leq j < m$, $\Pr(j)$ is given by:

$$\Pr(j) = r_j \prod_{k=1}^{j-1} (1 - r_k) \tag{2.5}$$

It is understood that $\prod_{k=x}^{y}(\cdot)$ is equal to 1 when $x > y$.

Recall that $\hat{B}_{m-1} = \langle b_2, b_3, \ldots, b_m \rangle$. Not to be confused, let us express it as $\hat{B}_{m-1} = \langle \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_{m-1} \rangle$ with $\hat{b}_j = b_{j+1}$ for all $j \geq 1$. Let $\hat{r}_j$ be the probability that a feasible option exists in j-th batch of $\hat{B}_{m-1}$, which is equal to r_{j+1} for all $j \geq 1$. Let $\hat{Pr}(j)$ be the probability that scheduling ends after the j-th iteration of $\hat{B}_{m-1}$:

$$\hat{Pr}(j) = \hat{r}_j \prod_{k=1}^{j-1} (1 - \hat{r}_k) = r_{j+1} \prod_{k=2}^{j} (1 - r_k) \tag{2.6}$$

We can express $\mathbb{E}_c[\hat{B}_{m-1}]$ as follows:

$$\begin{aligned}
\mathbb{E}_c[\hat{B}_{m-1}] &= \sum_{j=1}^{m-1} \hat{Pr}(j) \left(\sum_{k=1}^{j} c(k, \hat{b}_k) \right) \\
&= \sum_{j=1}^{m-1} r_{j+1} \left(\prod_{k=2}^{j} (1 - r_k) \right) \left(\sum_{k=2}^{j+1} c(k-1, b_{k+1}) \right) \\
&= \sum_{j=2}^{m} r_j \left(\prod_{k=2}^{j-1} (1 - r_k) \right) \left(\sum_{k=2}^{j} c(k-1, b_{k+1}) \right)
\end{aligned}$$

The first equality holds by definition. The second equality is due to Equation 2.6 and because $\hat{b}_k = b_{k+1}$. The third equality is by changing the range of j in the summation.

If we multiply $\mathbb{E}_c[\hat{B}_{m-1}]$ by $(1-r_1)\theta$, we get the following:

$$(1-r_1)\theta\mathbb{E}_c[\hat{B}_{m-1}] = \sum_{j=2}^{m} r_j \left(\prod_{k=1}^{j-1}(1-r_k)\right)\left(\sum_{k=2}^{j} c(k,b_{k+1})\right) \tag{2.7}$$

Notice that the product term now runs from $k=1$ to $j-1$ as we multiply by $(1-r_1)$ and the inner-most summation has $c(k,b_{k+1})$ as we multiply by θ (recall that c is θ-simple).

Finally we can express $\mathbb{E}_c[B_m]$ in terms of $\mathbb{E}_c[\hat{B}_{m-1}]$ as follows (where $c_1 = c(1,b_1)$ for brevity):

$$\begin{aligned}
\mathbb{E}_c[B_m] &= \sum_{j=1}^{m} \Pr(j)\left(\sum_{k=1}^{j} c(k,b_k)\right) \\
&= c_1 + \left(\sum_{j=2}^{m} r_j \left(\prod_{k=1}^{j-1}(1-r_k)\right)\left(\sum_{k=2}^{j} c(k,b_k)\right)\right) \\
&= c_1 + (1-r_1)\theta\mathbb{E}_c[\hat{B}_{m-1}] \\
&= c(1,b_1) + \left(\prod_{t=1}^{b_1}(1-q_t)\right)\cdot\theta\cdot\mathbb{E}_c[\hat{B}_{m-1}]
\end{aligned}$$

The first equality holds by definition of $\mathbb{E}_c[B_m]$. The second equality is obtained by taking $c(1,b_1)$ out from the summation (and note that $\Pr(j)$ adds up to 1) first and then applying Equation 2.5. The last two inequalities hold due to Equation 2.7 and 2.4, respectively. The last expression exactly matches Equation 2.3 in the lemma. □

Using the recurrence relation in Lemma 2 and the computing method in Lemma 1, we can now design an efficient recursive algorithm finds the optimal B-Doodle mechanism. Our algorithm is presented in Algorithm 1 as a recursive method $Rec(x)$. We assume that the values of q_t have been computed as a pre-processing step prior to running our algorithm, using the method in Lemma 1. We further assume that options are sorted in decreasing order of q_t (i.e. $q_1 \geq q_2 \geq \cdots \geq q_s$); therefore we simply refer to the option by its index (i.e. $t=3$ refers to the third option in the sorted list of options).

Given $1 \leq x \leq s$, $Rec(x)$ returns optimal B-Doodle (B^*) that consists of options $\{x, x+1, \ldots, s\}$ and its expected cost (EC^*). To solve for a given instance of the problem, we simply call the method $Rec(x)$ with $x = 1$.

Algorithm 1 Recursive Algorithm: $Rec(x)$

1: $B^* \leftarrow \langle s-x+1 \rangle$, $EC^* \leftarrow c(1, s-x+1)$
2: **for** $b := 1, 2, \ldots, s-x$ **do**
3: $\quad (B, EC) \leftarrow Rec(x+b)$
4: $\quad EC_b \leftarrow c(1,b) + \left(\prod_{t=x}^{x+b-1}(1-q_t)\right) \cdot \theta \cdot EC$
5: $\quad$ **if** $EC^* > EC_b$ **then**
6: $\quad\quad EC^* \leftarrow EC_b$
7: $\quad\quad B^* \leftarrow \langle b, B \rangle$
8: $\quad$ **end if**
9: **end for**
10: **return** (B^*, EC^*)

Theorem 2. *Algorithm 1 runs in polynomial time, and returns an optimal B-Doodle mechanism that minimizes the expected cost, given an instance* (N, S, f, c, P) *of* BDP *when c is* θ*-simple.*

Proof. Our recursive method is very simple: given options $\{x, x+1, \ldots, s\}$, it iteratively considers the case of sending out b options in the first iteration, and computes the expected cost EC_b of doing so, for each b with $1 \leq b \leq s-x+1$. In line 1, our method checks for the trivial case when $b = s-x+1$ (i.e. all options are sent in a single batch); we store this mechanism in B^* and its expected cost in EC^*. Then we iterate b from 1 to $s-x$ and compute the expected cost EC_b, and compare with the optimal expected cost found so far (lines 2-9). For each b, we first recursively compute the expected cost for the remaining options (namely, $\{x+b, x+b+1, \ldots, s\}$) by calling our method $Rec(x+b)$, and store the expected cost in EC (line 3). Using Lemma 2 we can compute EC_b as in line 4 (note that the first batch contains b options from x to $x+b-1$). In lines 5-8, we simply compare EC_b with the current optimal, EC^*, and update if necessary; $\langle b, B \rangle$ should be interpreted as the concatenation of b and B into a single vector of integers. Finally in line 10, our method returns the optimal B-Doodle found, B^* and its expected cost EC^*.

We prove correctness of the algorithm by induction on x (from $x = s$ to $x = 1$). The base case is trivial: if $x = s$ the only B-Doodle mechanism is $\langle 1 \rangle$ and our method finds it in line 1 and returns it in line 10. Suppose our method correctly returns the optimal B-Doodle mechanism (and its expected cost) for all $x > k$ for some k (the inductive hypothesis), and we prove for the case of $x = k$. When $x = k$, there are precisely $(s-k+1)$ options that are to be sent, and the first batch can contain any number of options between 1 and $(s-k+1)$, inclusive. In lines 1-9 our method considers all such cases, and for each case it computes the expected cost correctly (in line 4) due to our inductive hypothesis and Lemma 2. Therefore our method finds the optimal B-Doodle for all x with $1 \leq x \leq s$; in particular when $x = 1$, it returns the desired optimal B-Doodle mechanism for the given problem instance.

Our recursive method runs in polynomial time when we use memoization on $Rec(x)$ with respect to x; that is, for each x we cache what $Rec(x)$ returns for the first time, and for any subsequence calls to $Rec(x)$ we simply return the cached values. Therefore lines 1-9 are executed at most once for each x with $1 \leq x \leq s$. Also note that $Rec(x)$ makes a call to $Rec(x+b)$ with $b \geq 1$ and $x+b \leq s$, which means there are no infinite loops. Once $Rec(x)$ is computed for all $x > k$, it takes $O(|S|^2)$ time to compute $Rec(k)$, and thus the overall running time of our algorithm is $O(|S|^3)$ when we start with $Rec(1)$. Pre-processing steps (of computing q_t) also run in polynomial time due to Lemma 1. □

2.3 Experimental Results with Synthetic Data

We showed that we can find an optimal B-Doodle mechanism that minimizes the expected cost. But a very important question remains: Is our optimal B-Doodle substantially better than Doodle in realistic settings? If the answer is no, we do not have much incentive to discard the simplest mechanism, Doodle, over using our sophisticated algorithm. Intuitively we expect Doodle to be inefficient if there are many options (i.e., s is large) because it causes much inconvenience for invitees to examine the options. Reasoning further, we can also see that inefficiency of Doodle depends on the probability of each option being feasible (i.e. q_t values), which in turn depends on $p_{i,t}$ values. If invitees are relatively free, then a small number of options would be sufficient for finding a feasible option, which makes

Doodle inefficient. Last but not least, the underlying cost function plays an important role as well. These all sound plausible, and we validate our intuition with simulation results.

In our setting there are many experimental choices one can choose from. In what follows we look at a representative example in which each invitee is available for each option with the same probability of p; that is, $p_{i,t} = p$ for some constant p. (While in realistic situations the $p_{i,t}$'s may all differ, we note that in our simulations our results seem to carry over to these more general settings as well.) We present experimental results for each of the three cost functions discussed in Example 1: The linear cost function, the time-averse cost function, and the inconvenience-averse function.

2.3.1 Case of Linear Cost Function

Recall that the linear cost function is parameterized by some constant $\alpha > 0$ and that $c_\alpha(j,b) = \alpha + b$. We used $\alpha = 2$ as an experimental choice, and we later discuss what we observed for different values of α.

We reasoned that Doodle becomes suboptimal if the number of options, s, is large. Therefore it is interesting to know for what ranges of s, Doodle is suboptimal. For some fixed (n,s,p), let $C^D(n,s,p)$ be the cost of Doodle and $C^*(n,s,p)$ be the expected cost of B^*. We want to find the smallest critical point $S^*(n,p)$ such that for any $s \geq S^*(n,p)$ we have $C^D(n,s,p) > C^*(n,s,p)$.

In Table 2.1, we show S^* for various (n,p) when $f = 1$ (i.e. the organizer requires everyone be available). For instance we find $S^*(6,.8) = 5$, which implies that Doodle is suboptimal for all $s \geq 5$ given that $n = 6$ and $p = .8$. The smaller S^* is, the less practical Doodle is for the corresponding (n,p). Across the table we can observe that S^* is small when n is small and/or when p is high – this agrees with our intuition. We highlighted 8 entries in boldface to emphasize that $S^* \leq 15$; for the corresponding (n,p) values, Doodle is suboptimal if there are 15 or more options being considered.

Not only do we want to know when Doodle is suboptimal, but we also want to know how inefficient Doodle is in realistic situations. For some fixed (n,s,p), we define the efficiency of Doodle, $e_D(n,s,p)$, as the ratio of the optimal expected cost to the cost of Doodle: $e_D(n,s,p) = C^*(n,s,p)/C^D(n,s,p)$.

S^*	$n=2$	$n=4$	$n=6$	$n=10$	$n=15$
$p=.8$	**3**	**4**	**5**	**8**	**15**
$p=.5$	**5**	**11**	22	90	>300
$p=.2$	**14**	70	>300	>300	>300

Table 2.1: Critical point S^* for various (n,p) when $f=1,\alpha=2$. Entries in boldface emphasize that $S^* \leq 15$.

In Table 2.2 we show e_D for the same settings of (n,p) we used in Table 2.1. For this experiment we used $s=15$ and $f=1$. If $e_D=1$ Doodle is optimal, and if e_D is close to zero then Doodle is very inefficient. We find that $e_D(2,.8)=.270$, which implies that Doodle is very inefficient in this case; on the other hand $e_D(10,.5)=1$, in which case Doodle is optimal. Across the table we observe the same pattern we observed before – for small n and high p, Doodle is substantially inefficient.

e_D	$n=2$	$n=4$	$n=6$	$n=10$	$n=15$
$p=.8$	**.270**	**.361**	**.486**	.777	.986
$p=.5$	**.502**	.904	1	1	1
$p=.2$	.970	1	1	1	1

Table 2.2: Efficiency of Doodle e_D for various (n,p) when $f=1,\alpha=2,s=15$. Entries in boldface emphasize that $e_D<.750$.

Inefficiency of Doodle is more pronounced when the organizer has a lower feasibility threshold such as $f=.7$; in such case, only a fraction of invitees need to be available. We can clearly observe the worsened inefficiency of Doodle in Table 2.3. Here we show e_D values for the same set of (n,p) as in Table 2.2 but with $f=.7$. Previously we highlighted four entries with $e_D<.750$ when $f=1$ in Table 2.2; when $f=.7$ the number of entires with $e_D<.750$ doubled to eight. Note that the first column is identical between the two tables; this is because $f=.7$ still requires both invitees to be available when $n=2$. Also notice that e_D is surprisingly low in the first row of Table 2.3 across all columns. This shows that regardless of the number of invitees, Doodle is significantly inefficient when the invitees are highly available ($p \geq .8$) and the organizer has a relaxed feasibility threshold.

We ran experiments with different values of (n,s,p,f,α), and observed the same trends that are presented here.

e_D	$n=2$	$n=4$	$n=6$	$n=10$	$n=15$
$p=.8$	**.270**	**.215**	**.267**	**.201**	**.211**
$p=.5$	**.502**	**.434**	.772	**.628**	.913
$p=.2$	.970	1	1	1	1

Table 2.3: Efficiency of Doodle e_D for various (n,p) when $f=.7, \alpha=2, s=15$. Entries in boldface emphasize that $e_D < .750$.

2.3.2 Case of Other Cost Functions

We present some of the experimental results we obtained with different cost functions, namely the time-averse cost function and the inconvenience-averse cost function. As the names suggest, the former places more weight on optimizing Time, while the latter on optimizing Inconvenience.

For the time-averse cost function (recall $c_\beta(j,b)=\beta^j \cdot b$), we chose $\beta=2$ as an experimental choice. Notice that the cost increases exponentially for later iterations, which forces the organizer to send out a small number of batches (as in Doodle). We ran the same set of experiments as before to measure efficient of Doodle, e_D.

The result is summarized in Table 2.4, which shows the same trends as we observed in previous experiments. Notice that in the first row ($p=.8$), as n increases we expect e_D to decrease, but e_D fluctuates while decreasing in general. The fluctuation is due to the rounding of attendance requirement ($\lceil a \cdot n \rceil$). For instance when $n=4$, $\lceil a \cdot n \rceil$ = 3 which effectively requires 75% of attendees be available.

e_D	$n=2$	$n=4$	$n=6$	$n=10$	$n=15$
$p=.8$	**.180**	**.104**	**.175**	**.088**	**.099**
$p=.5$	**.561**	**.457**	.876	**.726**	.987
$p=.2$	1	1	1	1	1

Table 2.4: Efficiency of Doodle e_D for various (n,p) when $f=.7, \beta=2, s=15$. Entries in boldface emphasize that $e_D < .750$.

For the inconvenience-averse cost function (recall $c_\gamma(j,b)=\gamma^b$), we chose $\gamma=1.1$ as an experimental choice. While γ is fairly small, because the cost function is an exponential function of b, an optimal B-Doodle must send out small-size batches. This cost function

optimizes Inconvenience primarily. We ran the same set of experiments as before to measure efficiency of Doodle, e_D.

The result is summarized in Table 2.5. Notice that e_D is not equal to 1 even when $p = .2$ in this setting, which we did not observe with the other cost functions previously. Due to integer-rounding, we again observe some fluctuations in e_D across columns within the same row, but the general trend is that e_D decreases as n increases.

e_D	$n = 2$	$n = 4$	$n = 6$	$n = 10$	$n = 15$
$p = .8$	**.333**	**.299**	**.329**	**.294**	**.298**
$p = .5$	**.499**	**.450**	**.695**	**.592**	.799
$p = .2$	.850	.887	.974	.976	.980

Table 2.5: Efficiency of Doodle e_D for various (n, p) when $f = .7, \gamma = 1.1, s = 15$. Entries in boldface emphasize that $e_D < .750$.

2.3.3 Summary of Experiments

While we only presented experimental results with specific values of (n, s, p, f) and parameters (α, β, γ), we observed that Doodle is in general substantially inefficient, including (but not limited to) when one or more of the following conditions hold:

- There is a relatively small number of invitees ($n \leq 10$).
- There is a large number of options ($s \geq 15$).
- invitees are relatively free ($p > .5$).
- Feasibility threshold is relaxed ($f < .8$).
- The cost function places more weight on Inconvenience than Time ($\alpha < 20$, $\beta < 5$, or $\gamma > 1.05$).

Intuitively the first four conditions affect q_t (the probability that option t is feasible) in the same way, and if q_t's are high then Doodle is more likely to be inefficient because a few options may be sufficient for finding a feasible one. While the last condition is independent of q_t, the cost function determines what is being optimized, and Doodle becomes more inefficient if c favors reducing Inconvenience over reducing Time.

2.4 Discussion

In this chapter we identified two important dimensions of optimality in group scheduling: Time and Inconvenience. We generalized the popular Doodle mechanism to a class of B-Doodle mechanisms that partition date/time options into batches. We showed an example of the Pareto-frontier of B-Doodle mechanisms on the Time-Inconvenience dimensions. We then described an efficient algorithm for finding an optimal B-Doodle mechanisms, given a simple cost function that aggregates Time and Inconvenience, assuming probabilistic independence among availability of invitees. We showed in simulations that optimal B-Doodle mechanism is superior to Doodle in realistic situations, sometimes greatly so.

Chapter 3

Complexity of Group Scheduling Problem II

In Chapter 2, we assumed that an event organizer is only allowed to inspect a group of columns (as a batch) by polling all invitees at once. In this chapter, we relax this restriction, and assume that the organizer can inspect each entry of a random matrix at unit cost (by querying a certain invitee about a certain date/time option). Consequently, Timeand Inconvenienceare always equal in this setting, and the goal is to find an optimal inspection sequence which is a permutation of entries of a given random matrix. As in BDP, we would like to optimize the expected cost, which is the expected number of inspections until a feasible column is found or all columns are deemed infeasible – we call this problem the Probabilistic Matrix Inspection Problem (PMIP).

The main difference between the Batched Doodle Problem (BDP) and the Probabilistic Matrix Inspection Problem (PMIP) lies in the solution domain (a unit operation on a group of columns versus entries) and the cost model (a function of Time and Inconvenience versus the number of inspections). The Batched Doodle generalizes Doodle by allowing columns-by-columns inspections, and we further generalize it by allowing entry-by-entry inspections.

3.1 Notation and Definitions

We use the same definition of feasibility as in previous chapter, which is copied below.

Definition 6 (Feasibility)**.** Let A be a matrix whose entries are from $\{0,1\}$, and refer to the entry of A at row r and column c as $a_{r,c}$. We say that a column c of A is *feasible* if it consists only of 1's (otherwise it is *infeasible*). We say that A is *feasible* if it contains at least one feasible column (otherwise it is *infeasible*).

In the Probabilistic Matrix Inspection Problem, we do not know of the values of the entries of A, as they are Bernoulli random variables, but we know of probability distribution of each entry. An input to the problem is this probability distribution.

Definition 7 (Input instance)**.** An input instance of the Probabilistic Matrix Inspection problem is a pair of matrices (A, P_A) of size n by m where A is a matrix of Bernoulli random variables and P_A is the probability matrix associated with it. We denote an entry of A as $a_{r,c}$ and of P_A as $p_{r,c}$. Each entry $a_{r,c}$ of A is a Bernoulli random variable, and $p_{r,c}$ is the probability of success for $a_{r,c}$ (i.e., $p_{r,c} = \mathbf{P}[a_{r,c} = 1]$). We define $s_c = \prod_{r=1}^{n} p_{r,c}$ to denote the probability that column c is feasible (probability of success for column c).

Throughout this work we will assume that the set of random variables $\{a_{r,c}\}$ are mutually independent. This assumption is crucial to our technical results because it allows to compute (in polynomial time) the probability of a specific realization of A conditioning on the event in which some entries of A have already been realized. Without this assumption, it is unclear how one can compute such probabilities without having an access to the joint probability distribution over all realizations of A (whose size is exponential in the size of A).

We define an "inspection" as an operation that can be performed on A. One can inspect an arbitrary entry $a_{r,c}$ of A at unit cost, so as to know of the realization of the random variable. In group scheduling, an inspection corresponds to querying an agent about her availability for a certain outcome. The objective of the problem is to determine whether A is feasible or not, with minimum (expected) number of inspections possible. The expectation is with respect to the probability distribution specified by P_A.

Because we are interested in determining feasibility of A with minimum number of inspections, there are certain "unnecessary inspections" that an optimal strategy must avoid. For instance, if a certain entry $a_{r,c}$ is found to be unsuccessful (i.e., $a_{r,c} = 0$ is realized), then there is no need to inspect any other entry from the same column because the column is already known to be infeasible. Similarly, if a certain column is found to be feasible (which implies A is feasible) or if all columns are found to be infeasible (which implies A is infeasible), then there is no need to inspect any other entries of the matrix. Lastly, if $p_{r,c} = 0$ or $p_{r,c} = 1$, then there is no need to inspect the entry $a_{r,c}$ because we already know its realization with probability 1. Therefore, without loss of generality, we will assume that $p_{r,c} \in (0,1)$ (i.e., $p_{r,c} \neq 0,1$) in this work.

Let us define what constitutes a solution to the problem.

Definition 8 (Inspection policy)**.** Given an input instance (A, P_A), a solution is any permutation of the entries of A, and we call it an "inspection policy" (or simply, a "policy").

The interpretation of a permutation is as follows. The entries of A will be inspected in order specified by the permutation. After each inspection, if A is found to be feasible or infeasible, the inspection process ends. Otherwise, it continues inspecting the entries as specified, but it will not make any unnecessary inspections as mentioned earlier.

If π is a permutation of the entries of A, we write $C(\pi)$ to denote the number of inspections performed by π. $C(\pi)$ is a random variable whose probability distribution is determined by P_A. We are interested in finding an optimal policy which minimizes the expected number of inspections, $\mathbf{E}[C(\pi)]$. Note that there are $(nm)!$ permutations of the entries of A, and therefore exhaustive search for an optimal permutation will not produce an efficient algorithm.

3.1.1 Example

Consider a 2-by-2 matrix A of Bernoulli random variables whose probability of success is given by P_A as follows. In group scheduling, this corresponds to two agents and a set of

two date/time options being considered.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, P_A = \begin{bmatrix} 0.6 & 0.7 \\ 0.9 & 0.8 \end{bmatrix}$$

Let us consider an inspection policy π which inspects the entries of A column-by-column while inspecting them from top to bottom within a column:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a_{1,1} & a_{2,1} & a_{1,2} & a_{2,2} \end{pmatrix}.$$

Suppose that the realization of A happens to be the identity matrix of size 2 (i.e., $a_{1,1} = a_{2,2} = 1$ and $a_{2,1} = a_{1,2} = 0$). If we use π, it will first inspect $a_{1,1}$ and learn its realization. Since $a_{1,1} = 1$, it will inspect $a_{2,1}$ next only to find that column 1 is infeasible after all. It will then inspect $a_{1,2}$ and learn that column 2 is also infeasible, which implies that A is infeasible. At this point, the inspection process terminates without inspecting $a_{2,2}$. This specific realization of A happens with probability $p_{1,1}(1-p_{2,1})(1-p_{1,2})p_{2,2}$, and yields $C(\pi) = 3$ because 3 inspections would occur. If the realization of A happens to be the null matrix (i.e., all entries are 0's), then π would only inspect $a_{1,1}$ and $a_{1,2}$ but skip $a_{2,1}$ and $a_{2,2}$. In this manner one can consider all $2^{2\cdot 2} = 16$ possible realizations of A, and compute $\mathbf{E}[C(\pi)]$ in this example.

Another way to compute $\mathbf{E}[C(\pi)]$ is by de-coupling $C(\pi)$ into two random variables $N(\pi, c)$ with $c \in \{1,2\}$ where $N(\pi, c)$ denotes the number of inspections (on column c) performed by π conditioning on the event that (at least one element of) column c is inspected. We can efficiently compute these: $\mathbf{E}[N(\pi,c)] = 1 \cdot \mathbf{P}[a_{1,c} = 0] + 2 \cdot \mathbf{P}[a_{1,c} = 1]$ for $c \in \{1,2\}$. To express $\mathbf{E}[C(\pi)]$ in $N(\pi,c)$'s, we need to take conditional probability into account, as column 2 is inspected only if column 1 is infeasible: $\mathbf{E}[C(\pi)] = \mathbf{E}[N(\pi,1)] + \mathbf{E}[N(\pi,2)](1-s_1) = 2.382$. Recall that s_c is the probability of success for column c.

3.2 Technical Results and Algorithm

We first consider two special cases (1-row or 1-column matrices) of the Probabilistic Matrix Inspection problem, which admit intuitive, greedy algorithms. We then discuss a couple of interesting properties of an optimal inspection policy, which leads to our main result and algorithm.

3.2.1 1-Row Matrix and 1-Column Matrix

Let us first consider the case where an input matrix A has only one row (i.e., $n = 1$). In this case it is natural to inspect entries with largest probability first because we can stop as soon as we find an entry whose value is 1 – which makes its column and A feasible. This intuition is exactly what an optimal policy should do in the single row case.

Lemma 3 (1-Row Matrix)**.** *When $n = 1$, an inspection policy π is optimal if and only if it inspects the entries in non-increasing order of their associated probabilities.*

Proof. Without loss of generality let us assume that a policy π inspects the entries in increasing order of their column index; that is, $\pi(i) = a_{1,i}$. Recall that $C(\pi)$ is a random variable that denotes the number of inspections π incurs. We can express the expectation of $C(\pi)$ in terms of $p_{1,c}$'s as follows:

$$\mathbf{E}[C(\pi)] = m\left(\prod_{k=1}^{m-1}(1-p_{1,k})\right) + \sum_{j=1}^{m-1} j\left(p_j \prod_{k=1}^{j-1}(1-p_{1,k})\right). \tag{3.1}$$

Suppose that there exists some c^* such that $p_{1,c^*} < p_{1,c^*+1}$ (if no such c^* exists, then π is an inspection policy that inspects the entries in non-increasing order of probabilities). Let π' be the same policy as π except we swap the order of a_{1,c^*} and a_{1,c^*+1}. That is, π' is defined as follows.

$$\pi'(j) = \begin{cases} \pi'(j) = \pi(c^*+1) & \text{if } j = c^* \\ \pi'(j) = \pi(c^*) & \text{if } j = c^*+1 \\ \pi'(j) = \pi(j) & \text{if } j \neq c^* \wedge j \neq c^*+1 \end{cases}$$

After expressing $\mathbf{E}[C(\pi)]$ and $\mathbf{E}[C(\pi')]$ as in Equation 3.1, one can re-arrange the terms to obtain the following:

$$\mathbf{E}\left[C(\pi)\right] - \mathbf{E}\left[C(\pi')\right] = \left(\prod_{j=1}^{c^*-1}(1-p_{1,j})\right)(p_{1,c^*+1} - p_{1,c^*}). \tag{3.2}$$

This quantity is positive if $p_{1,c^*+1} > p_{1,c^*}$ (recall that $p_{1,j} \in (0,1)$ for all j as mentioned in Section 3.1).

This proves the lemma because any policy that inspects an entry with smaller probability before another entry with higher probability is suboptimal, and therefore an optimal policy must inspect entries in non-increasing order of their associated probabilities. □

Although the proof of Lemma 3 is simple, it confirms correctness of our intuition. Equation 3.2 illustrates this intuition; conditioning on the event that the first $c^* - 1$ inspections fail (whose probability is the product term in Equation 3.2), the difference $\mathbf{E}[C(\pi)] - \mathbf{E}[C(\pi')]$ depends on the difference in the probabilities of success between the next-entry-to-be-inspected by π and π'.

We can also consider the case where an input matrix A has only one column. Intuitively, if we wish to minimize the expected number of inspections, we must inspect entries with smallest probability first because we can stop as soon as we determine that A is infeasible. Lemma 4 formally states this intuition about optimal policy, and we omit a proof of it as it can be easily done by following the proof of Lemma 3.

Lemma 4 (1-Column Matrix)**.** *When $m = 1$, an inspection policy π is optimal if and only if it inspects the entries in non-decreasing order of their associated probabilities.*

3.2.2 Inspection of Entire Column

Another interesting property of an optimal inspection policy is that once it inspects the first entry of a column, then it must commit to it and continue inspecting the remaining entries of the column until feasibility of the column is determined. Otherwise, if the policy switches to another column too soon, then it is not optimal.

Theorem 3 (Optimality of inspecting entire column)**.** *Consider any inspection policy π. Without loss of generality, let us assume that for each column c, π inspects $a_{n,c}$ the last among n entries of the column. Let b_c be the index of π such that $\pi(b_c) = a_{n,c}$. Without loss of generality, assume $b_1 < b_2 < \cdots < b_m$ (we can do this by re-labeling the columns of A). If there is some column c^* such that $b_{c^*} > n \cdot c^*$, then π is not optimal.*

Proof. First, note that $b_c \geq n \cdot c$ for all c because we assumed $b_1 < b_2 < \cdots < b_m$, and therefore the entries of previous columns must appear before the last entry of each column.

Let π be an inspection policy being considered in the theorem for which there exists some c with $b_c > n \cdot c$. Let us construct a different inspection policy π'. First, π' inspects all entries of column 1 in the same order π does. Then, π' inspects all entries of column 2 in the same order π does, and so on. In particular, π' inspects all entries of a column before inspecting another column, while preserving the original ordering of the entries within each column that is given by π. We will show that $\mathbf{E}[C(\pi')] < \mathbf{E}[C(\pi)]$.

Let us define a set of new random variables which can be used to express $C(\cdot)$, as we did in Section 3.1.1 when analyzing an example. Recall that $s_c = \prod_{r=1}^{n} a_{r,c}$ is the probability of success for column c. Let $N(\pi, c)$ ($N(\pi', c)$, respectively) be a random variable that denotes the number of entries of column c that is inspected by π (by π', respectively), conditioning on the event that column c is inspected (i.e., when the previous $c-1$ columns are infeasible). We can then express $\mathbf{E}[C(\pi)]$ and $\mathbf{E}[C(\pi')]$ as follows:

$$\mathbf{E}[C(\pi)] = \sum_{c=1}^{m} \mathbf{E}[N(\pi, c)] \left(\prod_{k=1}^{c-1} (1 - s_c) \right) \tag{3.3}$$

and

$$\mathbf{E}[C(\pi')] = \sum_{c=1}^{m} \mathbf{E}[N(\pi', c)] \left(\prod_{k=1}^{c-1} (1 - s_c) \right). \tag{3.4}$$

To prove the theorem we will first show that for any realization of A, $N(\pi, c) \geq N(\pi', c)$ holds for all c; this immediately implies $\mathbf{E}[C(\pi)] \geq \mathbf{E}[C(\pi')]$. We will then show that there exists at least one realization of A such that for some column c' the strict inequality $N(\pi, c') > N(\pi', c')$ holds. These two statements together imply that $\mathbf{E}[C(\pi)] > \mathbf{E}[C(\pi')]$.

Consider any realization of A with the condition that the first $m-1$ columns are infeasible (recall that m is the number of columns of A). Then $N(\pi,c) = N(\pi',c)$ for all c regardless of feasibility of column m. To see why, both π and π' would inspect the same set of entries in each of the first $m-1$ columns in the same order until the column is determined to be infeasible, and therefore $N(\pi,c) = N(\pi',c)$ if $c < m$. If column m is feasible, then both π and π' would inspect all n entries of it, and thus we have $N(\pi,m) = N(\pi',m) = n$. Otherwise, if column m is also infeasible (in which case A is infeasible), then π and π' would inspect the same set of entries of column m in the same order until the first infeasible entry of the column is found. Therefore if the first $m-1$ columns are infeasible we have $N(\pi,c) \geq N(\pi',c)$ for all c.

Now consider any realization of A with the condition that at least one of the first $m-1$ columns is feasible. Let c' be the smallest index of feasible columns of A. Because the columns from 1 to $c'-1$ are infeasible, $N(\pi,c) = N(\pi',c)$ for all $c < c'$ for the same reason we stated earlier for the other case. Since c' is feasible, $N(\pi,c') = N(\pi',c') = n$ as both policies would inspect all n entries of c'. By our construction of π' it is clear that $N(\pi',c) = 0$ for all $c > c'$; therefore we have $N(\pi,c) \geq N(\pi',c)$ for all $c > c'$. In summary $N(\pi,c) \geq N(\pi',c)$ holds for all c in this case as well.

So far we proved the first claim we stated earlier: for all realizations of A, we have $N(\pi,c) \geq N(\pi',c)$ for all c. Let us now prove the second claim. Let c^* be the smallest index c of columns such that $b_c > nc$ (note that $c^* < m$ because $b_m = nm$ by definition). Consider any realization of A with the condition that the first c^*-1 columns are infeasible and column c^* is feasible (feasibility of other columns do not matter). Using the same arguments we used earlier, we can show that $N(\pi,c) = N(\pi',c)$ for all $c < c^*$, that $N(\pi,c^*) = N(\pi',c^*) = n$, and that $N(\pi',c) = 0$ for all $c > c^*$. However, because $b_{c^*} > n \cdot c^*$, there is at least one entry $a_{r',c'}$ with $c' > c^*$ which appears before b_{n,c^*} in π (otherwise, if no such entry exists, then b_{c^*} would be equal to $n \cdot c^*$). This implies that there exists some c' with $c' > c^*$ such that $N(\pi,c') > 0$. This proves the second claim that for some realization of A, there is some column c' for which $N(\pi,c') > N(\pi',c')$, and together with the first claim we proved earlier, this implies that $\mathbf{E}[C(\pi)] > \mathbf{E}[C(\pi')]$.

This proves the theorem: Any policy that does not inspect all entries of a column consecutively is suboptimal. □

By Theorem 3, when seeking an optimal policy, it is sufficient to consider the set of policies that inspect an entire column before committing to another column. Lemma 4 hints that one should inspect the entries of each column in increasing order of probabilities, and this is what we prove next.

3.2.3 Optimal Ordering within Column

Lemma 4 states that an optimal policy must inspect the entries in increasing order of their probability of success, if A is a 1-column matrix. This argument can be generalized to the case where there is more than one column: If an optimal policy is to inspect an entry of some column c, it must inspect the entry with smallest probability of success first.

Theorem 4 (Optimal ordering within column)**.** *Consider any inspection policy π. If there exist two entries $a_{r_1,c}$ and $a_{r_2,c}$ from the same column such that $a_{r_1,c}$ appears before $a_{r_2,c}$ in π and $p_{r_1,c} > p_{r_2,c}$, then π is not optimal. In other words, when restricted to each column, an optimal policy must inspect the entries of the column in non-decreasing order of probabilities.*

Proof. Let π be an inspection policy being considered in the theorem. Because of Theorem 3 we can assume, without loss of generality, that π inspects all entries of column 1, followed by column 2, and so on. Further let us assume that π inspects the entries of each column in increasing order of their row index (we can do so by re-labeling the indices of entries). Precisely, $\pi(r+n(c-1)) = a_{r,c}$ defines π. Let p_{r,c^*} and p_{r+1,c^*} be the entries with $p_{r,c^*} > p_{r+1,c^*}$. Let us consider a different inspection policy π' that is the same as π except that π' inspects p_{r+1,c^*} before p_{r,c^*}, by swapping the ordering of them.

$$\pi'(j) = \begin{cases} \pi'(j) = a_{r+1,c^*} & \text{if } \pi(j) = a_{r,c^*} \\ \pi'(j) = a_{r,c^*} & \text{if } \pi(j) = a_{r+1,c^*} \\ \pi'(j) = \pi(j) & \text{otherwise} \end{cases}$$

We claim that $\mathbf{E}[C(\pi')] < \mathbf{E}[C(\pi)]$, which implies that π is not optimal.

Let us define new random variables $N(\pi,c)$ and $N(\pi',c)$ as we did in our proof of Theorem 3 (i.e., the number of inspections performed by the respective policy on column

c, conditioning on the event that the column is inspected). Then we can express $\mathbf{E}[C(\pi)]$ and $\mathbf{E}[C(\pi')]$ in terms of the new random variables and s_c's as we did in Equations 3.3 and 3.4.

Observe that $N(\pi,c) = N(\pi',c)$ for any realization of A if $c \neq c^*$. To see this, first note that column c would not be inspected by π or by π' if any of the previous columns (that is, columns 1 through $c-1$) is found to be feasible, in which case $N(\pi,c) = N(\pi',c) = 0$. Otherwise, if column c is inspected, both policies would inspect the entries of c in the very same order, so $N(\pi,c) = N(\pi',c)$ must hold. Therefore we conclude that $\mathbf{E}[N(\pi,c)] = \mathbf{E}[N(\pi',c)]$ when $c \neq c^*$.

We will now show that $\mathbf{E}[N(\pi,c^*)] > \mathbf{E}[N(\pi',c^*)]$ holds. This immediately implies $\mathbf{E}[C(\pi)] > \mathbf{E}[C(\pi')]$ due to Equations 3.3 and 3.4. Let us express $\mathbf{E}[N(\pi,c^*)]$ in terms of p_{r,c^*}'s.

$$\mathbf{E}[N(\pi,c^*)] = n\left(\prod_{k=1}^{n-1} p_{k,c^*}\right) + \sum_{j=1}^{n-1} j(1-p_{j,c^*})\left(\prod_{k=1}^{j-1} p_{k,c^*}\right)$$

Note that the event $N(\pi,c^*) = j$ occurs if the first $j-1$ entries are feasible while the j-th entry is not feasible when $j < n$, and $N(\pi,c^*) = n$ occurs if the first $n-1$ entries are feasible (but the n-th entry's feasibility does not matter).

We can express $\mathbf{E}[N(\pi',c^*)]$ in a similar manner, and simplify $\mathbf{E}[N(\pi,c^*)] - \mathbf{E}[N(\pi',c^*)]$ as follows:

$$\mathbf{E}[N(\pi,c^*)] - \mathbf{E}[N(\pi',c^*)] = r\left(\prod_{k=1}^{r-1} p_{k,c^*}\right)(p_{r,c^*} - p_{r+1,c^*}).$$

The quantity above is positive if $p_{r,c^*} > p_{r+1,c^*}$, which is the assumption we began with. This proves the theorem. □

Theorems 3 and 4 together tell us that in order to find an optimal policy we only need to decide the ordering of the columns. There are still $m!$ orderings of columns, and an exhaustive search algorithm would not be efficient. As we were able to generalize Lemma 4 to Theorem 4 by generalizing the optimal solution for 1-column case, it would be natural to consider generalizing Lemma 3 in a similar manner.

This idea leads to the following greedy algorithm: First we sort columns by their probability of success ($s_c = \prod_{r=1}^{n} p_{r,c}$) in decreasing order, and inspect the entries of each column in increasing order of their associated probabilities. However, as the following example

shows, this algorithm is suboptimal.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, P_A = \begin{bmatrix} 0.4459 & 0.2262 \\ 0.4459 & 0.8114 \end{bmatrix}$$

Here we have $s_1 = 0.199$ and $s_2 = 0.184$, and the greedy algorithm would produce $\pi = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{1,2} & a_{2,2} \end{pmatrix}$. Its expected cost, $\mathbf{E}[C(\pi)]$, is 2.428, but if we inspect the second column first, then the expected cost is 2.407 which is optimal in this example. One can consider another greedy algorithm which inspects the columns in increasing order of their expected number of inspections (within column), but this algorithm turns out to be suboptimal as well.

3.2.4 Main Result and Algorithm

Let us present the main result that leads to an efficient algorithm for finding an optimal inspection policy.

Theorem 5. *Let s_c be the probability of success for column c as before. Let μ_c be the expected number of inspections that column c incurs if its entries are inspected in increasing order of their probability of success, conditioning on the event that column c is inspected and infeasible. An optimal policy must be a column-by-column policy (due to Theorem 3), must inspect the entries of each column in non-decreasing order of probabilities (due to Theorem 4), and must inspect the columns in non-decreasing order of $\mu_c(1-s_c)/s_c$.*

Proof. Consider a column-by-column inspection policy π which inspects the column 1 through m in increasing order of their index (we can assume this without loss of generality by re-labeling columns).

As before, let $N(\pi, c)$ be a random variable that denotes the number of inspections performed by π on column c, conditioning on the event that column c is inspected. Then we can express $\mathbf{E}[N(\pi, c)]$ in terms of s_c and μ_c as follows.

$$\mathbf{E}[N(\pi, c)] = s_c \cdot n + (1 - s_c) \cdot \mu_c \tag{3.5}$$

This equation holds because if the column is feasible (with probability s_c), it would require

n inspections, but if it is not (with probability $1 - s_c$), it would require μ_c inspections in expectation. The equation above simply considers these two events, and calculates the expected value of $N(\pi, c)$.

Suppose that there is some column c^* such that $\mu_{c^*}(1-s_{c^*})/s_{c^*} > \mu_{c^*+1}(1-s_{c^*+1})/s_{c^*+1}$. Because π inspects column c^* before column c^*+1, it would not be inspecting the columns in increasing order of $\mu_c(1-s_c)/s_c$. Consider a different inspection policy π' which inspects the columns in the same order as π except that π' inspects column c^*+1 before c^* by swapping the inspection ordering of the two. We can relate $N(\pi,\cdot)$ to $N(\pi',\cdot)$ as follows.

$$N(\pi', c) = \begin{cases} N(\pi, c^*+1) & \text{if } c = c^* \\ N(\pi, c^*) & \text{if } c = c^*+1 \\ N(\pi, c) & \text{otherwise} \end{cases}$$

As we did in proofs of Theorems 3 and 4, we can use Equations 3.3 and 3.4, and simplify $\mathbf{E}[C(\pi)] - \mathbf{E}[C(\pi')]$ as follows.

$$\begin{aligned} \mathbf{E}[C(\pi)] - \mathbf{E}[C(\pi')] = &\left(\frac{\mathbf{E}[N(\pi, c^*)]}{s_{c^*}} - \frac{\mathbf{E}[N(\pi, c^*+1)]}{s_{c^*+1}} \right) \\ &\cdot \left(\prod_{j=1}^{c^*-1} (1-s_j) \right) s_{c^*} s_{c^*+1} \end{aligned} \tag{3.6}$$

The quantity in Equation 3.6 is positive if the difference of the weighted expected values (in the first parentheses) are positive. Using Equation 3.5 we obtain the following inequality.

$$\begin{aligned} &\frac{\mathbf{E}[N(\pi, c^*)]}{s_{c^*}} > \frac{\mathbf{E}[N(\pi, c^*+1)]}{s_{c^*+1}} \\ \Leftrightarrow &\mu_{c^*}(1-s_{c^*})/s_{c^*} > \mu_{c^*+1}(1-s_{c^*+1})/s_{c^*+1} \end{aligned}$$

By definition of c^*, the second inequality above holds, which implies $\mathbf{E}[C(\pi)] > \mathbf{E}[C(\pi')]$. Therefore, an optimal inspection policy must inspect the columns in non-decreasing order of $\mu_c(1-s_c)/s_c$. □

Because there is a unique ordering of columns if we sort them by $\mu_c(1-s_c)/s_c$ (up to ties), Theorem 5 leads to the following algorithm: We inspect the columns in increasing

order of $\mu_c(1-s_c)/s_c$, and in each column, we inspect the entries of it in increasing order of probabilities. This algorithm can easily be implemented to run in polynomial time.

3.3 Discussion and Future Work

In this work we defined the Probabilistic Matrix Inspection problem motivated by group scheduling and Doodle. We first considered two special cases, and discovered interesting properties of an optimal inspection policy which agree with our intuition. We then generalized our findings to design an efficient algorithm to solve the general case, and along the way we showed that two natural greedy algorithms fail to find an optimal solution. While we believe that our technical results make a great starting point for studying and optimizing a group scheduling process, there remain several open problems and future work to be done.

As we discussed in previous chapter, our model and algorithm rely on the assumption that probability estimates on availability of agents are available. We suggested several ideas motivated by previous work in the literature, but it will be important to deploy such ideas into a system, and integrate it with our algorithm. From a theoretical perspective, there remain several open problems. While we assumed that an inspection can be performed on a single entry at unit cost, one can generalize the cost model by allowing an inspection of any subset of entries whose cost depends on, for example, the number of entries being inspected. In the context of group scheduling, an inspection on many entries means querying multiple agents at the same time for one or many outcomes (but an inspection is not limited to the entries from the same column or row). This generalization is particularly useful when scheduling takes place in a hierarchical setting such as corporates. For instance the event organizer may feel that the cost of querying a supervisor is significantly different from that of querying a colleague.

Lastly, although we focused on relating our model to group scheduling, the Probabilistic Matrix Inspection problem has other applications. Finding the right childcare facility, for example, involves extensive inquiries as parents wish to gather more information about how they would handle certain situations, what benefits and environments they provide, and so on. Through advertisements or brochures parents may even be able to gauge the

likelihood of a certain facility satisfying their needs. Yet they still need to inquire facilities for precise information, which can be modeled by our probabilistic matrix model where columns correspond to facilities and rows correspond to the needs of parents.

Chapter 4

Complexity of Group Activity Selection Problem

4.1 Introduction and Related Work

Imagine an event in which several activities are to take place concurrently. A group of agents are willing to participate and announce their activity preferences to the event organizer who is to assign agents to activities. In many settings, the agent preferences include not only which activities the agent is willing to participate in, but also the number(s) of participants in each activity that are acceptable to the agent. For example, agents may wish to have enough participants in certain activities (such as group bus tour) to split the cost associated with it, whereas they may wish to have just few participants in activities with limited resources (such as a demo booth with a limited number of devices). We assume that an agent can be assigned to at most one activity (which naturally occurs when activities are run concurrently).

Given the preferences of agents, the organizer wishes to find a "good" assignment subject to certain rationality and/or stability conditions. The first condition is *individual rationality*: everyone who is assigned to some activity is willing to participate. In addition to individual rationality, the organizer may want to ensure that agents who are not assigned

to any activity do not prefer to deviate from their assignment by joining an activity (*Nash stability*). Other concepts include *envy-freeness* that asserts that no unassigned agent would prefer to take the place of an assigned agent, and *perfection* in which all agents must be assigned.

To model this setting, in WINE'12, Darmann et al. [14] proposed the Group Activity Selection Problem (GASP) and defined three solution concepts (individual rationality, stability, and perfection). In addition, they also provided many computational complexity results, including NP-hardness results for GASP, even under various restrictions on agent preferences.

These hardness results essentially argue that it is hard to find an assignment that maximizes the number of participants in activities. Suppose, however, that we are satisfied if we can assign a small number k out of the n participants to some (up to k) of the p activities while satisfying our rationality criteria. There is a brute-force solution: try all possible $O(p^k)$ ordered choices of up to k activities, all $O(k^k)$ choices for the number of participants in each activity, and all $O(n^k)$ ordered choices of k participants; then check whether the desired criterion is satisfied by the induced assignment. This brute-force solution runs in $O((pnk)^k)$ time. While this running time is polynomial for any fixed k, it is not very desirable. A much better running time would be one of the form $O(f(k) \cdot (p+n))$ - such a runtime would be *linear*, regardless of the constant k, and the function f. More generally, on input size N, one would like an algorithm with runtime $f(k) \cdot N^c$, where c is independent of k. Such an algorithm is known as fixed parameter tractable (FPT), and the problems that admit such algorithms are said to be in the class FPT. Developing FPT algorithms, especially linear time ones, greatly mitigates the NP-hardness of problems as it shows that these problems are actually quite tractable for many instances.

The field of parameterized complexity strives to classify NP-hard problems on a finer scale by analyzing time complexity in terms of both the input size and an additional parameter. The FPT problems are viewed as the most tractable problems in NP (of course, all polytime problems are naturally in FPT). There is a hierarchy of complexity classes under parameterization (called the W-hierarchy), including FPT, W[1], W[2], etc. Hierarchy theorems show that FPT is contained in W[1] which is contained in W[2], and so on (see [17] for more details). It is believed that FPT $\neq$ W[1] and W[2] $\neq$ W[1], so that

the NP-hard parameterized problems in W[2] are believed to be harder than those in W[1] that are themselves believed to be harder than the FPT problems. Furthermore, popular hardness assumptions such as the Exponential Time Hypothesis (ETH) of Impagliazzo and Paturi [28] can often be used to show that particular W[1]-hard problems cannot be solved in $N^{o(k)}$ time, giving concrete runtime lower bounds.

The goal of this work is to investigate how different solution concepts and different restrictions on agent preferences influence the difficulty of GASP. We focus on GASP with the size of the solution as our parameter, i.e. the number of assigned agents, or in the case of perfection, the number of used activities. We place different NP-hard versions of GASP under this parameterization into different parts of the W-hierarchy. Our classification is nearly complete, as seen in Table 4.1 (in Section 4.3). We show that GASP for individual rationality is in FPT, whereas for the other solution concepts the problem is W[1]-complete or W[2]-hard even for restricted types of agent preferences.

The restrictions on preferences that we consider are quite natural – we consider the cases where agents have thresholds for each activity such that they are willing to participate if the number of participants is above or below the thresholds. In the case of *increasing* preferences, every agent i approves an activity a_j as long as at least $l_i(a_j)$ participants are assigned to it (hence the threshold is a lower-bound on the number of participants). In the case of *decreasing* preferences, agent i approves an activity a_j if at most $u_i(a_j)$ agents are assigned to it. Surprisingly to us, the case of decreasing preferences is actually easier than that of increasing preferences as it is FPT even when a stable assignment is to be found, whereas GASP for increasing preferences is W[1]-complete for all concepts except for individual rationality.

We also consider the special case of GASP in which all activities are equivalent. Here the preferences of the agents can vary but for each particular agent the preferences are the same for all copies of the activity. This special case is natural and captures many applications. We show that even though the problem is still NP-hard in this case [14], all parameterized versions of it (except possibly perfection) are FPT.

Related Work. Darmann et al. [14] defined GASP, and provided a number of computational complexity results – mostly, NP-completeness results. In this work, we adopt their

definitions of the problem and solution concepts, but we also consider the new solution concept of envy-freeness, while we analyze the parameterized complexity of the problem. As Darmann et al. point out, it is worth noting that GASP is closely related to Hedonic Games (see Section 2.2 of [14] and Section 2 of [31] for more details); in fact, GASP can be viewed as a class of hedonic coalition games with concise representation of preferences of agents (linear in the number of activities and players as opposed to all subsets of players in the general setting). Much work has been devoted to analyzing solution concepts in hedonic coalition games such as stability and Pareto-optimality [8, 18, 4]. Ballester [5] provides a number of computational complexity results (in fact, hardness results) for finding a core-stable, Nash-stable, or individually rational outcome in hedonic games and anonymous hedonic games. These hardness results do not imply similar hardness results for GASP, however, because of the concise representation of an input to GASP. Lastly, Darmann [13] considered a different setting of GASP where agents are assumed to have strict, ordinal preferences over the outcomes, whereas both our work and [14] assumes that agents have a dichotomic preference over outcomes in the sense that each agent is indifferent among all outcomes that she approves of. It is an interesting future problem to consider how our results in this work can be extended to the ordinal setting that Darmann [13] considered.

The rest of this chapter is structured as follows. In Section 4.2 we formally define GASP, and summarize the known complexity results. In Section 4.3 we provide our main technical results. In Section 4.4 we summarize our findings, and describe several open problems.

4.2 Definitions and Known Results

To make this work self-contained, we begin by introducing the formal definitions proposed by Darmann et al. in their work [14], yet we make slight modifications to notation for readability and consistency in this chapter.

Definition 9. In the Group Activity Selection Problem (GASP), we are given a set of agents $N = \{1, 2, \ldots, n\}$, a set of non-void activities $A^* = \{a_1, a_2, \ldots, a_p\}$, and the *void activity* ($a_\emptyset$) which refers to the case when an agent does not participate in any of the activities in A^*. An *outcome* is a pair $(a_j, x) \in A^* \times [n]$ which is interpreted as x agents participating in non-void activity a_j. For each agent i we are given a set S_i of outcomes (called *approval set*) such

that the outcomes in S_i are equally liked and strictly preferred to $a_\emptyset$, where $S_i \subseteq A^* \times [1,n]$. We write $S_i(a_j) = \{x : (a_j,x) \in S_i\}$ to refer to the set of sizes which agent i approves for activity a_j.

Similarly to the work of [14], in this work we assume that each agent is indifferent among the outcomes in S_i; that is, the void-activity ($a_\emptyset$) draws the line between which outcomes are approved and which ones are not by the agent. While this is a simplifying assumption, note that hardness results immediately imply the same hardness for the general case without this assumption. By abusing notation, we also use $a_\emptyset$ to refer to the outcome in which an agent is not assigned to any non-void activity.

Example 2. Consider $N = \{1,2,3\}$ and $A^* = \{a_1,a_2\}$. There are six outcomes (besides $a_\emptyset$) in $A^* \times [3]$. Suppose $S_1 = \{(a_1,1),(a_1,2),(a_1,3)\}$, $S_2 = \{(a_1,2),(a_2,2),(a_2,3)\}$, and $S_3 = \{(a_1,1),(a_2,1),(a_2,2)\}$. In particular, agent 1 approves a_1 for any size (i.e., unconditional approval) while does not approve a_2 for any size (i.e., unconditional refusal). Using our notation, $S_1(a_1) = \{1,2,3\}$ and $S_1(a_1) = \emptyset$. If we assign all agents to a_1, then $(a_1,3)$ is the outcome realized by all agents – notice that only agent 1 approves it (and thus is willing to participate) while agents 2 and 3 do not (and thus are unwilling to participate). Naturally, this assignment induces instability.

Let us define what constitutes a solution to GASP, and define four solution concepts.

Definition 10. Let $A = A^* \cup \{a_\emptyset\}$. An assignment in GASP is a mapping $\pi : N \to A$ where $\pi(i) = a_\emptyset$ means that agent i is not assigned to any non-void activity. Each assignment naturally partitions the agents into at most $|A|$ groups. We define $\pi^0 = \{i : \pi(i) = a_\emptyset\}$ and $\pi^j = \{i : \pi(i) = a_j\}$ for $j = 1,\dots p$, so that $|\pi^j|$ refers to the number of agents assigned by π to a specific activity (including the void activity). Let us define the size of an assignment, denoted by $|\pi|$, as the number of agents that are assigned to non-void activities; that is, $|\pi| = \sum_{j=1}^{p} |\pi^j|$.

Note that π induces an outcome for each agent: If $\pi(i) = a_\emptyset$, then agent i does not participate in any non-void activity (and thus outcome $a_\emptyset$ is induced), and if $\pi(i) = a_j \in A^*$, then $(a_j, |\pi^j|)$ is the induced outcome for agent i.

The objective in GASP is to find a "good" assignment of maximum size thereby assigning as many agents to activities as possible. We define four solution concepts which require different levels of rationality/stability in an assignment.

Definition 11. An assignment $\pi : N \to A$ is said to be *individually rational* (IR) if for every $j \in [p]$ and every agent $i \in \pi^j$ it holds that $(a_j, |\pi^j|) \in S_i$.

An assignment $\pi : N \to A$ is said to be *(Nash) stable* if it is individually rational and for every agent $i \in N$ such that $\pi(i) = a_\emptyset$ and every $a_j \in A^*$ it holds that $(a_j, |\pi^j| + 1) \notin S_i$.

An assignment $\pi : N \to A$ is said to be *envy-free* (EF) if it is individually rational and for every agent $i \in N$ such that $\pi(i) = a_\emptyset$ and every agent agent $i' \in N$ such that $\pi(i') = a_j \in A^*$, it holds that $(\pi(i'), |\pi^j|) \notin S_i$.

An assignment $\pi : N \to A$ is said to be *perfect* if it is individually rational and $\pi(i) \neq a_\emptyset$ for all $i \in N$.

Individual rationality requires that every agent who's assigned to a non-void activity is willing to participate because participation is preferred to non-participation. Therefore everyone who is assigned has no incentive to deviate from the assignment. Stability further requires that every agent who is assigned to the void activity is also unwilling to deviate (unilaterally, without permission of other agents) from the given assignment. Envy-freeness requires that every agent who is assigned to the void activity is not envious of someone else who is assigned to a non-void activity.[1] Lastly, a perfect assignment is the strongest solution concept which implies individual rationality, stability and envy-free by definition.

As Darmann et al. showed in their work [14], finding a solution in GASP is NP-hard even under various restrictions on inputs. Yet we shall see that some of such restrictions make the problem less complex under parameterization. First we define two natural restricted domains of preferences of agents, called *increasing* and *decreasing* preferences.

Definition 12. We say that agent i has an *increasing* preference for activity a_j if there exists a threshold $l_i(a_j) \in \{1, 2, \ldots, n+1\}$ such that $S_i(a_j) = [l_i(a_j), n]$ (where $[n+1, n] = \emptyset$). Similarly, we say that agent i has a *decreasing* preference for activity a_j if there exists a threshold $u_i(a_j) \in \{0, 1, \ldots, n\}$ such that $S_i(a_j) = [1, u_i(a_j)]$ (where $[1, 0] = \emptyset$).

[1] While one can naturally define another solution concept which requires both stability and envy-freeness, it is not considered in this work.

A natural example that accounts for an increasing preference is when an activity is associated with a cost that is to be split by the participants (e.g., group bus tourism in the city). For decreasing preferences, participants in some activity may need to share limited resources (e.g., a trial-demo of new wearable devices).

Next, we consider a restriction on the activities when there may exist multiple "copies" of the same activity (e.g., chess matches with multiple chessboards). We define equivalence of activities, and consider a special case of the problem in which all activities are equivalent.

Definition 13. We say that two activities a_j and $a_{j'}$ are equivalent if for every agent $i \in N$, $S_i(a_j) = S_i(a_{j'})$.

Let us re-visit the problem instance from Example 2, and relate it to various definitions and concepts we have defined in this section.

Example 3. In Example 2, note that agent 2 has an increasing preference for a_2 with $l_2(a_2) = 2$ while agent 3 has a decreasing preference for a_2 with $u_3(a_2) = 2$. Agent 1 has (degenerate) increasing/decreasing preferences for both a_1 and a_2 with $l_1(a_1) = 1, u_1(a_2) = 3$ and $l_1(a_2) = 4, u_1(a_2) = 0$.

Consider an assignment π with $\pi(1) = \pi(2) = a_1$ and $\pi(3) = a_2$; under the assignment π, agents $1,2$ realize the outcome $(a_1,2)$ and agent 3 realizes $(a_2,1)$. It is easy to check that π is prefect (and thus IR, stable, and EF). Consider another assignment π' with $\pi'(1) = a_1$ and $\pi'(2) = \pi'(3) = a_\emptyset$; under π', agent 1 realizes the outcome $(a_1,1)$ and agents $2,3$ realize $a_\emptyset$. π' is individually rational (as $(a_1,1) \in S_1$), but it is not stable (as $(a_1,2) \in S_2$) or envy-free (as $(a_1,1) \in S_3$).

Darmann et al. proved many hardness results of the Group Activity Selection Problem even under restrictions on inputs of `GASP` [14]. Here we mention the most relevant results of theirs to this work.

Theorem 6. *Finding a perfect assignment is NP-hard. It remains to be NP-hard even if all agents have increasing preferences for all activities, even if all agents have decreasing preferences for all activities, or even if all activities are equivalent (Theorems 4.1, 4.2, 4.3, and 4.4 of [14]).*

As corollaries, the following four problems are also NP-hard even under any of the three restrictions mentioned in Theorem 6. Note that the first three problems parameterize the size of an assignment while the last problem parameterizes the number of used activities in a perfect assignment.

- k-IR-GASP: Does there exist an individually rational assignment of size k?
- k-Stable-GASP: Does there exist a stable assignment of size k?
- k-EF-GASP: Does there exist an envy-free assignment of size k?
- k-Perfect-GASP: Does there exist a perfect assignment using k non-void activities?

In what follows, we show that the parameterized complexity of k-GASP varies with different solution concepts and under different restrictions on inputs to GASP.

4.3 Parameterized Complexity

In this section, we provide parameterized complexity of the four problems mentioned in previous section: k-IR-GASP, k-Stable-GASP, k-EF-GASP, and k-Perfect-GASP. Our main contributions are summarized in Table 4.1. Note that all problems considered in this work are known to be NP-hard (and NP-complete) due to Darmann et al. [14]. The "general case" refers to the case where approval set of each agent can be any set of outcomes (i.e., no restrictions on preferences of agents). "Increasing (decreasing) preferences" refer to the case where all agents have increasing (decreasing) preferences for all activities. "Equivalent activities" refer to the case where all activities are (pairwise) equivalent (i.e., they are copies of one kind of an activity).

	k-IR-GASP	k-Stable-GASP	k-EF-GASP	k-Perfect-GASP
General case	FPT	$W[1]$-hard	$W[1]$-complete	$W[2]$-hard
Increasing preferences	FPT	$W[1]$-complete	$W[1]$-complete	$W[2]$-hard
Decreasing preferences	FPT	FPT	$W[1]$-complete	$W[2]$-hard
Equivalent activities	FPT	FPT	FPT	Unknown

Table 4.1: Summary of results on parameterized complexity of GASP.

4.3.1 k-IR-GASP: Finding Individually Rational Assignments in GASP

Recall that k-IR-GASP is the problem of finding an IR assignment of size k. We show that k-IR-GASP is in FPT. As seen below, our algorithm runs in $\exp knp \log n$ time, where n is the number of agents and p the number of activities. The input size to GASP is $\Theta(n^2 p)$ as the number of possible outcomes is np and each of the n agents needs to specify a subset of approved ones. As we only care about solutions of size k, we are only interested in those preferences of the agents for numbers of activity participants that are at most k. We can hence prune the input to size $\Theta(nkp)$ (in about that much time, assuming random access to preferences). In general, we cannot prune the input more, and for any constant k, our FPT algorithm below runs in near linear time in the input size!

Theorem 7. *k-IR-GASP is in FPT, and can be solved in time $2^{O(k)}(np \log n)$ where $n = |N|$ and $p = |A^*|$.*

Proof. We use "Color Coding" to design a randomized (Monte Carlo) algorithm, which can easily be de-randomized using a family of k-perfect hash functions as shown in the work [2].

Recall that $N = \{1, 2, \ldots, n\}$ is the set of agents, $A^* = \{a_1, a_2, \ldots, a_p\}$ is the set of non-void activities, and S_i is the set of approved outcomes for agent i. We first color each agent using k colors independently and uniformly at random. We seek to assign exactly one agent of each color to some activity such that the resulting assignment is IR and of size k.

Let $c(i)$ denote the color of agent i where $c(i) \in [k]$. For each activity $a_j \in A^*$ and every subset C of colors (i.e., $C \subseteq [k]$), we will first determine whether it is possible to assign to activity a_j exactly $|C|$ agents with distinct colors specified by C while satisfying the IR constraint for each agent; we refer to this subproblem by $T(C, j)$ for every set of colors C and activity a_j. For any fixed a_j and C, we can check for every color $d \in C$ whether there exists an agent i with $c(i) = d$ and $(a_j, |C|) \in S_i$ in time $O(n)$ by iterating over the set of agents and look up her approval set, S_i. If the test is affirmative, we conclude that we can assign exactly $|C|$ agents with distinct colors specified by C to activity a_j. We perform this sub-routine for every activity $a_j \in A^*$ and every subset of colors, which can be done in time $O(n \cdot p \cdot 2^k)$ overall.

Next, we solve another type of sub-problems (which we call $R(C, j)$) to check if it is

possible to assign $|C|$ agents of distinct colors in C to activities in $A_j = \{a_1, a_2, \ldots, a_j\}$ for every $j \leq p$ and $C \subseteq [k]$. When $j = 1$, the sub-problems $R(C, j)$ and $T(R, j)$ are equivalent, and thus we simply use the result of $T(R, j)$ to solve $R(C, j)$. To solve $R(C, j)$ when $j > 1$, we enumerate over every subset $C' \subseteq C$, and solve $R(C', j-1)$ and lookup the result of $T(C \setminus C', j)$. If both subproblems $R(C', j-1)$ and $T(C \setminus C', j)$ are affirmative for some $C' \subset C$, then we conclude that $R(C, j)$ is also affirmative.

If $R([k], p)$ is affirmative, then we can find an IR assignment of size k where those k agents are distinctly colored. There are at most $O(2^k \cdot p)$ subproblems in the form of $R(C, j)$ to be solved, and each subproblem can be solved in time $O(2^k)$ (as we enumerate over all subsets of C).

Therefore the overall running time of this algorithm is bounded by $O(4^k \cdot p + 2^k \cdot (np)) = 2^{O(k)}(np)$; in particular, it is exponential only in k. It is easy to confirm that this algorithm is a Monte Carlo algorithm (i.e., if it finds a solution, it is guaranteed to be IR). On the other hand, if there exists an IR assignment of size k in the original instance, there is a chance that this algorithm does not find it when the k agents are not colored properly. The probability of a proper coloring (i.e., the k agents in the assumed assignment are colored distinctly) is at least $k!/k^k > 1/e^k$, and this is only exponentially small in k. Therefore one can repeat this randomized algorithm $e^k \ln n$ times to increase the probability of success to $1 - 1/n$ (with overall runtime $2^{O(k)}(np \log n)$).

To de-randomize this algorithm, one can use a k-perfect family of hash functions from N to $[k]$. Specifically, if we have a list of colorings of agents N such that for every subset $N' \subseteq N$ of size $|N'| = k$ there exists a coloring in the list that gives each agent in N' a distinct color, then we can simply enumerate over this list of colorings. This is precisely what a k-perfect family of hash functions from N to $[k]$ is, and it is known that the size of the family can be specified using $2^{O(k)} \log n$ bits (for details please see [2]). Therefore we can obtain a deterministic algorithm whose runtime is bounded by $2^{O(k)}(np \log n)$, and conclude that k-IR-GASP is in FPT. □

Individual rationality is the weakest solution concept among the four we consider, and naturally k-IR-GASP is the least complex problem. On the other hand, we will show that other problems are $W[1]$- or $W[2]$-hard unless additional restrictions are assumed. Note that Theorem 7 proves an easiness result, and therefore it is implied that k-IR-GASP is in

FPT under any of the three restrictions on inputs we consider (see the k-IR-GASP column in Table 4.1).

4.3.2 k-Stable-GASP: Finding Stable Assignments

Recall that k-Stable-GASP is the problem of finding a stable assignment of size k. Stability is a stronger solution concept than individual rationality, and the problem of finding a stable assignment is harder than that of finding an IR assignment. This relationship is not apparent under the classic complexity hierarchy as both problems are NP-complete. However, under parameterization, k-IR-GASP is FPT whereas k-Stable-GASP is $W[1]$-hard.

Theorem 8. *k-Stable-GASP is $W[1]$-hard. The problem remains to be $W[1]$-hard even if each agent approves at most one size per activity (i.e., $|S_i(a_j)| \leq 1$ for all $i \in N$ and $a_j \in A^*$).*

Proof. The k-Clique problem is known to be $W[1]$-hard, and we reduce the k-clique problem to k-Stable-GASP.

Construction of `GASP` instance. Consider an instance of the k-Clique problem, $G = (V,E)$ and a parameter k where $V = \{v_1, v_2, \ldots, v_n\}$. Let us create an instance of `GASP` as follows: Let $N = V \cup \{w_{i,x} : (1 \leq i \leq n) \wedge (1 \leq x \leq k-1)\}$; that is, we create n node-agents v_i's (by abusing notation) and $(k-1)$ copies of neighbor-agents ($w_{i,x}$'s) for each v_i. The neighbor-agents will be used to "select" the $k-1$ edges incident to each node if the node is to be included in a clique we are seeking. Let $A^* = \{a_1, \ldots, a_k\} \cup \{e_{i,j} : 1 \leq i < j \leq n\}$; we create k clique-activities (which are used to determine membership of a node in a clique) and $\binom{n}{2}$ edge-activities $e_{i,j}$ (where $i < j$). For each node-agent v_i, we set its approval set $S_{v_i} = \{(a_j, 1) : 1 \leq j \leq k\} \cup \{(e_{i,j}, 3) : i \neq j\}$. For each neighbor-agent $w_{i,x}$, we set its approval set $S_{w_{i,x}} = \{(e_{i,j}, 2) : (v_i, v_j) \in E\}$. Finally we set the parameter k' of `GASP` (to distinguish from k in the Clique problem) to $k' = k + 2\binom{k}{2}$. This is a valid fpt-reduction as k' depends only on k but not on n, and the size of our instance of `GASP` is bounded by $O(nk^3)$ as there are $O(nk)$ agents and $O(k^2)$ activities in the instance.

Let us first describe how a clique in the original instance and a stable assiggnment in the `GASP` instance we created are related. A node-agent is assigned to a clique-activity if and only if its corresponding node belongs to a (corresponding) clique. For each node-agent,

there exists $k-1$ neighbor-agents, and these neighbor-agents must be assigned properly to edge-activities in order to ensure that the resulting set of nodes is indeed a clique. We claim that there exists a clique of size k in the original instance if and only if there exists a stable assignment of size k' in the GASP instance we constructed.

Proof of equivalence between instances. First suppose there exists a clique C of size k in G, and without loss of generality assume $C = \{v_1, v_2, \ldots, v_k\}$. Consider the following assignment π:

$$\pi(v_i) = \begin{cases} a_i & i \leq k \\ a_\emptyset & i > k \end{cases} \quad \text{and} \quad \pi(w_{i,x}) = \begin{cases} e_{i,x+1} & i \leq k \wedge i \leq x \\ e_{x,i} & i \leq k \wedge i > x \\ a_\emptyset & i > k \end{cases}.$$

That is, node-agents are assigned to the clique-activities and their associated neighbor-agents are assigned to the edge-activities; all other agents are assigned to the void activity. Clearly π assigns exactly $k + 2\binom{k}{2} = k'$ agents to non-avoid activities. While the details are omitted, it is easy to verify that π is indeed a stable assignment.

Conversely, suppose there is a stable assignment π of size $k' = k + 2\binom{k}{2}$, and we want to show that a clique of size k exists in G. First notice that for each edge-activity $e_{i,j}$ there are precisely two agents who approve the outcome $(e_{i,j}, 3)$ – namely, v_i and v_j. Therefore if π is stable, it cannot assign any node-agents (of the form v_i) to any edge-activity (of the form $e_{i,j}$). In other words, for each v_i, $\pi(v_i) \in \{a_\emptyset\} \cup \{a_1, \ldots, a_k\}$. Let $C = \{v_i : \pi(v_i) \neq a_\emptyset\}$; since there are k clique-activities, $|C| \leq k$. We claim that $|C| = k$ if π is stable; if $|C| < k$, then there exists some a_j such that no agent is assigned to it; since $k \leq n$, there must be some v_i such that $\pi(v_i) = a_\emptyset$. This implies that π is not stable because $(a_j, 1) \in S_{v_i}$ while $\pi(v_i) = a_\emptyset$; hence $|C| = k$ must hold. Without loss of generality, we now assume that $\pi(v_i) = a_i$ if $i \leq k$ and $\pi(v_i) = a_\emptyset$ if $i > k$ (by re-labeling), and we claim that $C = \{v_1, v_2, \ldots, v_k\}$ is a k-clique in the original instance.

We argued earlier that π never assigns node-agents to any edge-activities if it is stable. This implies that, if π assigns any agent to an edge-activity, it must be the case that π assigns exactly two neighbor-agents (of the form $w_{i,x}$) to it (due to the construction of

$S_{w_{i,x}}$'s). If π is stable, then π must assign no neighbor-agents to $e_{i,j}$ if $i > k$ or $j > k$ and exactly two neighbor-agents to $e_{i,j}$ if $i \leq k$ and $j \leq k$. To prove the first claim, suppose that π assigns two neighbor-agents to $e_{i,j}$ where $i > k$ (and recall that $\pi(v_i) = a_\emptyset$ when $i > k$). Then π is not stable because $(e_{i,j}, 3) \in S_{v_i}$, and thus v_i wishes to join $e_{i,j}$, and this is a contradiction. Similarly one can prove the claim in the case where $j > k$. To prove the second part, recall that $|\pi| = k' = k + 2\binom{k}{2}$. Since π assigns exactly k node-agents to non-void activities, it must assign $k(k-1) = 2\binom{k}{2}$ neighbor-agents to $\binom{k}{2}$ edge-activities from $\{e_{i,j} : i, j \leq k\}$. By the Pigeon Hole principle, π must assign two agents to each of the edge-activities in $\{e_{i,j} : i, j \leq k\}$. This implies that there is an edge between v_i and v_j in the original instance if $i, j \leq k$. Otherwise, if $(v_i, v_j) \notin E$ where $i, j \leq k$, then there is no neighbor-agents who can be assigned to $e_{i,j}$, which contradicts the assumption that π is of size k'. This completes the proof of the claim that if a stable assignment of size k' exists, then a clique of size k exists, and one can construct a clique by choosing the corresponding nodes to the node-agents that are assigned to one of the clique-activities.

Note that in our reduction each agent approves at most one size per activity, proving the second statement in the theorem. □

Because our reduction increases the parameter quadratically (i.e., $k' = k + 2\binom{k}{2} = k^2$), the following corollary follows immediately.

Corollary 1. *Unless the Exponential Time Hypothesis (ETH) fails, k-Stable-GASP cannot be solved in time* $(np)^{o(\sqrt{k})}$.

We now consider k-Stable-GASP with the restriction that all agents have increasing preferences for all activities.

Theorem 9. *k-Stable-GASP is* $W[1]$*-complete when all agents have increasing preferences for all activities.*

Proof. We show $W[1]$-hardness of the problem by reducing from the k-Clique problem, but we omit the proof of completeness which can be shown by reducing k-Stable-GASP to the k-Clique problem.

Construction of GASP instance. Let $G = (V, E)$ be a graph instance of the k-Clique problem. For each vertex $v_i \in V$, we create k^2 copies of v_i as agents (call them copies of v_i) and create an activity a_i; this creates $k^2|V|$ agents and $|V|$ activities. For each edge $e_{i,j} = (v_i, v_j) \in E$, we create two copies of $e_{i,j}$ as agents (call them copies of $e_{i,j}$) and create an activity $w_{i,j}$; this creates $2|E|$ agents and $|E|$ activities. Let $k' = k^3 + k^2 - k$, and we create $k' + 1$ copies of dummy agents (call them copies of z). For each of the k^2 copies of v_i agents, we set its approval set such that $l_{v_i}(a_i) = k^2$ (i.e., approves any outcome with a_i and size k^2 or larger) and $l_{v_i}(w_{i,j}) = 3$ if $(v_i, v_j) \in E$ and $l_{v_i}(\cdot) = n + 1$ for all other activities (where $n = k^2|V| + 2|E| + k' + 1$ is the total number of agents we create). For each of the two copies of $e_{i,j}$ agents, we set its approval set such that $l_{e_{i,j}}(w_{i,j}) = 2$. For each of the $k' + 1$ copies of z agents, we set its approval set such that $l_z(w_{i,j}) = 4$ for all (i, j) where $(v_i, v_j) \in E$. We claim that a clique of size k exists in G if and only if a stable assignment of size k' exists in the GASP instance we created.

Proof of equivalence between instances. First, suppose that $C = \{v_1, v_2, \ldots, v_k\}$ is a clique of size k in G. We can construct a stable assignment of size k' as follows: (a) For k^2 copies of v_i, we assign them to a_i if $v_i \in C$ and to $a_\emptyset$ otherwise, (b) for two copies of $e_{i,j}$, we assign them to $w_{i,j}$ if $v_i \in C$ and $v_j \in C$ and to $a_\emptyset$ otherwise, and (c) copies of z are assigned to $a_\emptyset$. Note that this assignment assigns exactly $k^3 + 2\binom{k}{2} = k^3 + k(k-1) = k'$ agents to non-void activities. It is easy to verify that π is IR and stable, proof of which is created to due space.

Conversely, now suppose that π is a stable assignment of size k', and we show that there exists a clique of size k in G. If π assigns three or more agents to any $w_{i,j}$, then π must assign all copies of z to some activity (possibly $w_{i,j}$) or π would not be stable; yet we know that π is of size k' and there are $k' + 1$ copies of z, and therefore π can only assign two or fewer agents to each $w_{i,j}$. If π assigns two agents to some $w_{i,j}$, then those two agents must be the two copies of $e_{i,j}$ because no other agent approves the outcome $(w_{i,j}, 2)$. Furthermore, if π assigns the two copies of $e_{i,j}$ to $w_{i,j}$, then π must assign all k^2 copies of v_i to a_i and all k^2 copies of v_j to a_j – otherwise, π would not be stable. Let W be the set of activities of the form $w_{i,j}$ such that π assigns exactly two agents to $w_{i,j}$; if $|W| > \binom{k}{2}$, then there must be at least $k + 1$ indices that appear in elements of W, which implies that π

must assign agents to at least $k+1$ activities of the form a_i. This is a contradiction because π is of size k' but $(k+1)k^2 > k'$. Therefore, $|W| \leq \binom{k}{2}$. Now suppose $|W| < \binom{k}{2}$ instead. As argued earlier, π can assign to at most k activities of the form a_i, but $k^3 + 2|W| < k'$, which implies that π cannot be of size k' if $|W| < \binom{k}{2}$. Lastly, suppose $|W| = \binom{k}{2}$ (and from previous arguments, it is clear that the number of the indices that appear in the elements of W must be exactly k); without loss of generality, assume $W = \{w_{i,j} : 1 \leq i < j \leq k\}$ (by re-labeling) – this implies that π assigns k^2 copies of v_l to a_l if $1 \leq l \leq k$, but more importantly, it implies that $(v_i, v_j) \in E$ because we create $w_{i,j}$ if and only if there is an edge between v_i and v_j. That is, $C = \{v_1, v_2, \ldots, v_k\}$ is a clique in the original instance. This completes the proof of W[1]-hardness of the problem. □

Unlike the case of increasing preferences, if all agents have decreasing preferences the problem admits an FPT algorithm.

Theorem 10. *k-Stable-GASP is in FPT when all agents have decreasing preferences for all activities.*

Proof. We use Color Coding to reduce the k-Stable-GASP with decreasing preferences to a variant of the Vertex Cover problem. With probability which is exponentially small only in k, we color agents and activities "properly", and given a proper coloring we can find a stable assignment of size k in polynomial time in n, p yet exponential only in k.

Preliminaries. Suppose that a stable assignment of size k exists, and without loss of generality we know that it assigns k agents to l distinct activities (where $l \in [1,k]$), which can be done by checking every value in $[1,k]$. We first color agents and activities using l colors 1 through l, uniformly and independently at random (let $c(i)$ denote the color of agent i and $c(a_j)$ the color of activity a_j), and then fix the value of k_d for each $d \in [1,l]$ such that $\sum_{d \in [1,l]} k_d = k$. We say that the coloring c (together with l and k_d's) is compatible with a stable assignment π of size k (using l activities) if π assigns exactly k_d agents to an activity of color d for every $d \in [1,l]$. Given some coloring c, our algorithm will find a stable assignment compatible with c or determine that no such stable assignment exists. It is clear that any stable assignment (of size k) has at least one compatible coloring. With probability at least $(1/l)^{l+k}$ (which is exponentially small only in k), our randomized

coloring is a compatible coloring of a stable assignment of size k (provided that it exists); the algorithm can be easily be de-randomized using a family of k-perfect hash functions as shown in the work [2].

FPT Algorithm. We now proceed with fixed values of l and k_d's as well as some coloring c as described earlier. We will use the special color $l+1$ to mark the agents and activities that cannot be assigned/used in any stable assignment that is compatible with the given coloring c. Define $N_d = \{i \in N : c(i) = d\}$ and $A^*_d = \{a_j \in A^* : c(a_j) = d\}$ where $d \in [1, l+1]$; these subsets naturally partition N and A^* into $l+1$ subsets by their colors (at first N_{l+1} and A^*_{l+1} are empty, but we may re-color some agents and activities during the course of the algorithm). Let $N(a_j) = \{i \in N_{c(a_j)} : u_i(a_j) \geq k_{c(a_j)}\}$, which is the set of agents who have the same color as a_j and approve the size $k_{c(a_j)}$ for activity a_j (recall that agents have decreasing preferences, so we only need to check their upper-bound $u_i(a_j)$ for a given activity a_j). If $|N(a_j)| > k_{c(a_j)}$, then we label the activity a_j as "popular" because any compatible assignment must assign $k_{c(a_j)}$ agents of the same color to a_j, but more than $k_{c(a_j)}$ agents approve a_j for size $k_{c(a_j)}$. If any color $d \in [1, l]$ contains two or more popular activities, we reject the coloring because there is no stable assignment compatible with this coloring. To see why, if no agents are assigned to a popular activity of some color d, then due to compatibility there must exist at least one agent of the same color who is assigned to the void activity but approves the popular activity for size 1. Therefore, any stable, compatible assignment must assign k_d agents to a popular activity for color d (if any), but if there exist multiple popular activities of the same color, then no compatible assignment is stable. Without loss of generality (by re-coloring) let us assume that colors $[1, q]$ contain exactly one popular activity and colors $[q+1, l]$ contain non-popular activities (it is possible that $q = 0$ or $q = l$). To emphasize, we shall refer to colors in $[1, q]$ as "popular" colors and in $[q+1, l]$ as "unpopular" colors.

Let us now examine each color to decide whether we should reject the coloring or whether we can exclude some agents and/or activities from consideration (by re-coloring them as the special color, $l+1$). First, for each popular color $d \in [1, q]$ with a popular activity a_{j_d} (recall that there is exactly one popular activity for each popular color), we re-color all agents in $(N_d \setminus N(a_{j_d}))$ and all activities in $(A^*_d \setminus \{a_{j_d}\})$ as the special color

$(l+1)$ because they cannot be assigned/used in any stable assignment compatible with c as we argued earlier. Next, for each unpopular color $d \in [q+1,l]$, if there exist two distinct activities $a_j, a_{j'} \in A^*(d)$ such that $N(a_j) \neq N(a_{j'})$, then we reject the coloring; any compatible assignment must assign no agents to at least one of these two activities (assume that a_j is such activity), but at least one agent in $N(a_j)$ approves $(a_j, 1)$ (due to decreasing preferences) while she must be assigned to the void activity, which implies instability of the assignment. If the coloring is not rejected after these conditions are checked, then we have $N(a_j) = N(a_{j'})$ for all $a_j, a_{j'} \in A^*(d)$ where $d \in [q+1,l]$. Let us re-color all agents in $N(d) \setminus N(a_j)$ as $l+1$ where a_j is any activity in $A^*(d)$ for all $d \in [q+1,l]$. We then check for another condition for each unpopular color $d \in [q+1,l]$. Let $A'(d) = \{a_j \in A^*(d) : \exists i \in N_{l+1}, u_i(a_j) \geq 1\}$. If $A'(d)$ contains two or more activities, it is clear that the coloring must be rejected because agent i (who cannot be assigned to any activity under the given coloring) approves size 1 for the activities in $A'(d)$ but the assignment can only choose one activity from $A^*(d)$. Therefore, if $|A'(d)| \geq 2$ then we reject the coloring; otherwise, if $|A'(d)| = 1$, then we re-color all activities in $A^*(d) \setminus A'(d)$ as $l+1$ (as the only one in $A'(d)$ must be used for color d). If $|A'(d)| = 0$, this step has no effect for this color. Lastly, we now consider the agents of color $l+1$ (who must be assigned to the void activity by any stable assignment compatible with the given coloring). Let us define $k_{l+1} = 0$ for convenience (i.e., we do not assign any agents of color $l+1$ to any activities). For each color $d \in [1,l+1]$, if there exists some activity $a_j \in A^*(d)$ and some agent $i \in N(l+1)$ with $u_i(a_j) \geq k_d + 1$, then we reject the coloring because agent i is to be assigned to the void-activity, but she approves the outcome (a_j, k_d+1) as well as $(a_j, 1)$ (due to decreasing preferences), which means that regardless of whether a_j is used or not, no assignment would not be stable and compatible at the same time due to agent i. If the coloring has not been rejected, then we can now safely ignore all agents in $N(l+1)$ (as if they are non-existent) because stability constraint would not be violated by those agents.

We now proceed with the assumption that the coloring has not been rejected by our algorithm. Recall that we need to choose k_d agents among N_d where $d \in [1,q]$ to be assigned to the popular activity a_{j_d} while we know exactly which $k_{d'}$ agents must be assigned to one of the activities in $A^*_{d'}$ where $d' \in [q+1,l]$. For each popular color $d \in [1,q]$ define $N'_d = \{i \in N_d : \exists d' \in [1,l+1], u_i(a_j) \geq k_{d'} + 1 \text{ where } a_j \in A_{d'}\} \cup \{i \in N_d : \exists d' \in [q+1,l], |\{a_j \in$

$A_{d'} : u_i(a_j) \geq 1\}| \geq 2\}$. Any stable assignment compatible with c must assign all agents in N'_d to the popular activity a_{j_d}. Otherwise, if some agent i in N'_d is assigned to the void-activity instead, then the resulting assignment cannot be stable; if i is contained in the first set (on the right-hand-side of definition of N'_d) above, then i approves sizes of both $k_{d'}+1$ and 1 for some activity a_j, which implies that regardless of whether a_j is used or not, i would wish to join a_j instead of $a_\emptyset$, while if i is contained in the second set (on the right-hand-side of definition of N'_d), then i approves size 1 for at least two non-popular activities of the same color which implies that i would wish to join one of them that is not used. Therefore, if $|N'_d| > k_d$ for some $d \in [1,q]$ we must reject the coloring, and otherwise we must assign all agents in N'_d to a_{j_d}. Without loss of generality we can assume that $N'_d = \emptyset$ for all $d \in [1,q]$ (provided that the coloring is not rejected by this point) by assigning all such agents to the appropriate popular activity and then decreasing k_d by $|N'_d|$ before we proceed to the next step.

Now suppose that for some color $d \in [1,q]$ and agent $i \in N_d$ and some color $d' \in [q+1,l]$ and some activity $a_j \in A^*_{d'}$, we have $u_i(a_j) \geq 1$. If i is assigned to $a_\emptyset$ (instead of a_{d_j}) and a_j is not used (no agents is assigned to it), then the assignment cannot be stable as i approves $(a_j, 1)$. That is, any compatible, stable assignment must assign i to a_{d_j} and/or use activity a_j. If we consider agents in $X = \cup_{d\in[1,q]} N_d$ and activities in $Y = \cup_{d'\in[q+1,l]} A^*_{d'}$ as vertices and there is an edge between (i, a_j) if and only if $u_i(a_j) \geq 1$ where $i \in X$ and $a_j \in Y$ (as a bipartite graph), finding a compatible assignment is equivalent to finding a vertex cover such that it chooses exactly k_d vertices from each N_d with $d \in [1,q]$ and exactly 1 vertex from each $A^*_{d'}$ with $d' \in [q+1,l]$. Because the total number of vertices to be selected is bounded above by $k+l$, one can use a bounded search tree to determine whether a vertex cover of a small size exists or not in FPT time (i.e., exponential only in k but polynomial in n, p). If vertex i from X is chosen then we assign i to the popular activity of the same color and if vertex a_j from Y is chosen then we assign the agents of the same color to it. It is easy to verify that a compatible, stable assignment exists if and only if a vertex cover (with the aforementioned constraints) exists in this bipartite graph.

While we omit the details, it is straightforward to verify that our algorithm would not reject any coloring c which is compatible with at least one stable assignment. □

Lastly, we consider another special case of GASP when all activities are (pairwise) equivalent. In this case, the problem of finding a stable assignment becomes the problem of partitioning agents into groups where only group sizes matter (as all of the activities are identical to every agent). This restriction allows the problem for an FPT algorithm

Theorem 11. *k-Stable-GASP is in FPT if all (non-void) activities are equivalent.*

Proof. Let $p = |A^*|$ be the number of non-void activities which we assume are all equivalent. We use Color Coding to design a randomized FPT algorithm, which can easily be de-randomized using a family of k-perfect hash functions as shown in the work [2].

We can assume that $p \leq k+1$ because k agents can be assigned to at most k copies and having more than one extra copy to which no agents is assigned does not change the problem (this is because we are seeking a solution of size exactly k). For brevity, we only prove the claim when $p = k+1$ in this work, but it can be easily extended to the cases when $p < k+1$.

Preliminaries. Let $N = \{1,2,\ldots,n\}$ be the set of n agents and $A^* = \{a_1, a_2, \ldots, a_p\}$ be the set of p copies of the only activity when $p = k+1$. Recall that by definition of equivalent activities, every agent i has $S_i(a_j) = S_i(a_{j'})$ for all j, j'. We first fix l (the number of copies of the activity to be used by a stable assignment) which must be between 1 and k, and $k_1, k_2, \ldots, k_l$ which is the number of agents assigned to each of the l copies; for convenience we define $k_{l+1} = 0$ as there is at least one extra copy that would not be used by the assignment. The total number of possible values for l and k_d's are bounded above by $O(k^k)$, which is exponential only in k. After we fix l, we color all agents uniformly and independently at random using colors 1 through l; let $c(\cdot)$ be this coloring scheme and $c(i)$ denote the color of agent i. We say that coloring c (together with l and k_d's) and a stable assignment π of size k are *compatible* if π assigns exactly k_d agents of color d to activity a_d for $d \in [1,l]$.

FPT Algorithm. Our algorithm will find a stable assignment compatible with c or determine that no such stable assignment exists. Note that any stable assignment of size k has at least one compatible coloring, and the probability that a random coloring we choose is

compatible with some fixed stable assignment of size k is at least $(1/l)^k$ (as we must color those k agents correctly) which is exponentially small only in k.

Our algorithm first partitions agents of each color into several subsets, and check several necessary conditions for the coloring to be compatible with at least one stable assignment; if any of the conditions is not met, the coloring will be rejected by the algorithm (as there is no stable assignment compatible with the given coloring). For each color $d \in [1,l]$, the algorithm computes three subsets: $N_d = \{i \in N : c(i) = d\}$, $N_d^{\text{IR}} = \{i \in N_d : (a_d, k_d) \in S_i\}$, and $N_d^{\text{IN}} = \{i \in N_d : \exists d' \in [1,l+1] \text{ s.t. } (a_{d'}, k_{d'}+1) \in S_i\}$ (recall $k_{l+1} = 0$). If $|N_d^{\text{IR}}| < k_d$ for any $d \in [1,l]$, no stable assignment is compatible with c because assigning k_d agents to a_d would not be individually rational (i.e., not enough agents approve the outcome), so the coloring should be rejected in this case. If for some $d \in [1,l]$ the set $N_d^{\text{IN}} - N_d^{\text{IR}}$ is not empty but contains some agent i, then no stable assignment is compatible with c because a stable assignment cannot assign i to a_d (because $i \notin N_d^{\text{IR}}$) but i would wish to join $a_{d'}$ for some $d' \in [1,l+1]$ which would make the assignment not stable; hence the coloring should be rejected in this case. If $|N_d^{\text{IN}}| > k_d$, then at least one agent i in N_d^{IN} should be assigned to the void activity, but i would wish to join $a_{d'}$ for some $d' \in [1,l+1]$ which would make the assignment not stable; hence the coloring should be rejected. If the coloring is not rejected by any of the cases mentioned earlier, then we have the following three conditions for every color $d \in [1,l]$: (a) $|N_d^{\text{IR}}| \geq k_d$, (b) $|N_d^{\text{IN}}| \leq k_d$, and (c) $N_d^{\text{IR}} \subseteq N_d^{\text{IN}}$. Let us define X_d for each $d \in [1,l]$ as follows: X_d contains an arbitrary set of k_d agents from N_d^{IR} such that every agent in N_d^{IN} is contained in X_d. Note that this is always possible due to the three conditions mentioned above. We claim that an assignment π which assigns agents in X_d to a_d and all other agents to $a_\emptyset$ is a stable assignment compatible with c. To prove compatibility, all agents in X_d are by definition of color d and $|X_d| = k_d$ for all $d \in [1,l]$. To prove stability, first consider any agent i who is assigned to the void activity and suppose $d = c(i)$. Since $i \notin X_d$, we know that $i \notin N_d^{\text{IN}}$ by definition, and therefore there is no $d' \in [1,l+1]$ such that $(a_{d'}, k_{d'}+1) \in S_i$. Now consider any agent i who is assigned to a_d by π (thus $c(i) = d$). By definition $i \in X_d$ and thus $i \in N_d^{\text{IR}}$, which implies that $(a_d, k_d) \in S_i$. Therefore π is a stable assignment of size k, compatible with c.

Let us now prove that if there is at least one stable assignment that is compatible with c, then the algorithm does not reject the coloring. Let π be one such assignment and let

X_d be the set of agents assigned to a_d by π. Due to compatibility we have $|X_d| = k_d$ and $c(i) = d$ for all $i \in X_d$ for all $d \in [1,l]$; in particular, by definition $X_d \subseteq N^{\text{IR}_d}$ and thus the first condition (a) above holds for all $d \in [1,l]$. Due to stability of π, every agent i with $\pi(i) = a_\emptyset$ satisfies that $\nexists d' \in [1,l+1]$ such that $(a_{d'}, k_{d'}+1) \in S_i$. Therefore the conditions (b) and (c) above hold for all $d \in [1,l]$, which proves that the coloring c would not be rejected by the algorithm.

We have shown that if we begin with a coloring c (together with l and k_d's) that is compatible with at least one stable assignment, then our algorithm would find a stable assignment compatible with the coloring and that if no such assignment exists the coloring would be rejected. This is a Monte Carlo algorithm with probability of success at least $(1/k)^k$ and runtime bounded by $O((k^k)nk)$ (as our algorithm must enumerate all possible values of l and k_d's), which is polynomial in n but exponential only in k. □

4.3.3 k-EF-GASP: Finding Envy-free Assignments

Recall that k-EF-GASP is the problem of finding an envy-free assignment of size k. Similarly to how we showed that finding a stable assignment is computationally harder than finding an IR assignment under parameterization, we can show that finding an envy-free (EF) assignment is also computationally harder than finding an IR assignment under parameterization; again, this relationship is not apparent under the classic complexity classes (i.e., NP-hardness).

Theorem 12. *k-EF-GASP is $W[1]$-complete. The problem remains to be $W[1]$-complete even if each agent approves at most one size per activity.*

Proof. We reduce from the k-Clique problem. Let $G = (V,E)$ be an undirected graph and k be a parameter of the k-Clique instance.

Construction of `GASP` instance. Let $V = \{v_1, v_2, \ldots, v_n\}$, and for each vertex $v_i \in V$, we create activity a_i and k^2 copies of v_i as agents. For each edge $(v_i, v_j) \in E$, we create an activity $e_{i,j}$ and an agent $w_{i,j}$. This creates $|V|+|E|$ activities and $k^2|V|+|E|$ agents overall. For each copy of v_i, we set its approval set as $S_{v_i} = \{(a_i, k^2)\} \cup \{(e_{i,j}, 1) : (v_i, v_j) \in E\}$ and for each agent $w_{i,j}$ we set its approval set as $S_{w_{i,j}} = (e_{i,j}, 1)$. Let $k' = k^3 + \binom{k}{2} =$

$k^3 + k(k-1)/2$. We claim that a clique of size k exists in G if and only if an envy-free assignment of size k' exists in the GASP instance we created.

Proof of equivalence between instances. Suppose that π is an envy-free assignment of size k'. If π assigns any copy of v_i to some activity $e_{i,j}$, then π cannot be envy-free because $w_{i,j}$ wishes to be assigned to $e_{i,j}$ in place of the copy of v_i; furthermore, π cannot assign more than one agent to any $e_{i,j}$ as no other agent approves the activity with any other size than 1. If π assigns any copy of v_i to a_i, then it must assign all k^2 copies of v_i to a_i as those agents only approve a_i with size k^2. Because π is of size k', it is clear that π can only assign agents to at most k different activities of the form a_i. Now suppose π assigns some $w_{i,j}$ to $e_{i,j}$; due to envy-freeness, all k^2 copies of v_i must be assigned to a_i and all k^2 copies of v_j must be assigned to a_j; this implies that π can assign at most $\binom{k}{2}$ agents of the form $w_{i,j}$ to activities of the form $e_{i,j}$ (otherwise, π cannot be of size k' because $k^2(k+1) > k'$). Therefore, we conclude that π assigns k^3 agents of the form v_i to k activities of the form a_i (without loss of generality, assume those activities are $\{a_1, a_2, \ldots, a_k\}$) and that π assigns $w_{i,j}$ to $e_{i,j}$ if and only if $1 \leq i, j \leq k$ (all other agents are assigned to the void activity). This implies that the original instance contains a clique $C = \{v_1, v_2, \ldots, v_k\}$ as there is an edge (v_i, v_j) if $1 \leq i, j \leq k$.

To prove the converse, suppose that $C = \{v_1, v_2, \ldots, v_k\}$ is a clique in the original instance. Let π be an assignment such that π assigns k^2 copies of v_i to a_i if $i \leq k$ and $w_{i,j}$ to $e_{i,j}$ if $1 \leq i, j \leq k$ and assigns all other agents to the void activity. Clearly π is individually rational by the construction of approval sets; π is also envy-free because no copy of v_i with $i > k$ or $w_{i,j}$ with $i > k$ or $j > k$ wishes to replace any other agent who is assigned to a non-void activity. This completes the proof of $W[1]$-hardness.

Note that in our reduction each agent approves at most one size per activity, proving the second statement in the theorem.

To show completeness, one can reduce k-EF-GASP to the colored k-clique problem which is known to be W[1]-complete (we omit the details). □

Although k-EF-GASP is $W[1]$-complete, the problem may admit FPT algorithms if we assume that preferences of agents are restricted to increasing preferences or decreasing preferences. Unlike the case of stable assignments, we show that k-EF-GASP remains to be

$W[1]$-complete even if all agents have increasing preferences or all agents have decreasing preferences.

Theorem 13. *k-EF-GASP is $W[1]$-complete even if all agents have increasing preferences.*

sketch. A slight modification to the reduction we used in proof of Theorem 12 shows that k-EF-GASP is $W[1]$-hard even if all agents have increasing preferences. We construct the same instance, but we change the approval set of each agent such that if agent i approves an outcome (a,x) for some activity a and size x, then we let the agent approve all outcomes (a,x') with $x < x' \leq |N|$, which ensures that all agents have increasing preferences. In addition we create $k'+1$ copies of a dummy agent z such that z approves all outcomes $(e_{i,j},x)$ with $2 \leq x \leq |N|$ for all activities $e_{i,j}$ we create.

It is easy to see that if π is an envy-free assignment of size k', then π cannot assign any copy of z to any activity because if π assigns any copy of z to some $e_{i,j}$, then π must assign all copies of z to some activity (to avoid envy-freeness with respect to those copies) which results in the size of π being at least $k'+1$. This in turn implies that π cannot assign more than one agent to any $e_{i,j}$; if two or more agents are assigned to $e_{i,j}$, then all copies of z must be assigned to some activity (or they would be envious) due to increasing preferences. Lastly, this implies that π cannot assign any v_i to any $e_{i,j}$ because such assignment implies $w_{i,j}$ must also be assigned to $e_{i,j}$ (due to envy-freeness and increasing preferences), resulting in more than one agent being assigned to $e_{i,j}$. With this observation, the rest of the proof follows in a straightforward manner, and we omit details. Note that completeness follows from the fact that k-EF-GASP is in $W[1]$. □

Theorem 14. *k-EF-GASP is $W[1]$-complete even if all agents have decreasing preferences.*

sketch. A slight modification to the reduction we used in proof of Theorem 12 shows that k-EF-GASP is $W[1]$-hard even if all agents have decreasing preferences. We construct the same instance, but we change the approval set of each agent such that if agent i approves an outcome (a,x) for some activity a and size x, then we let the agent approve all outcomes (a,x') with $1 \leq x' < x$, which ensures that all agents have decreasing preferences. The rest of the proof follows in a straightforward manner, and we omit details. Note that completeness follows from the fact that k-EF-GASP is in $W[1]$. □

Theorem 15. *k-Stable-GASP is in FPT if all (non-void) activities are equivalent.*

Proof. Let $p = |A^*|$ and let all non-void activities be equivalent. We use Color Coding to design a randomized FPT algorithm, which can easily be de-randomized using a family of k-perfect hash functions as shown in the work [2]. Because there are only n agents we can assume that $p \leq k$ because k agents can be assigned to at most k copies (unlike the case of stability, extra copies have no effect in envy-freeness as agents are only envious of others who participate in some activity). For simplicity we only prove the claim when $p = k$ but it can be easily extended to the cases when $p < k$.

Preliminaries. Let $N = \{1, 2, \ldots, n\}$ be the set of n agents and $A^* = \{a_1, a_2, \ldots, a_p\}$ be the set of p copies of the only activity when $p = k$. Recall that by definition of equivalent activities, every agent i has $S_i(a_j) = S_i(a_{j'})$ for all j, j'. We first fix l (the number of copies of the activity to be used by an EF assignment) which must be between 1 and k, and $k_1, k_2, \ldots, k_l$ which is the number of agents assigned to each of the l copies. The total number of possible values for l and k_d's are bounded above by $O(k^k)$, which is exponential only in k. After we fix l, we color all agents uniformly and independently at random using colors 1 through l; let $c()$ be this coloring scheme and $c(i)$ denote the color of agent i. We say that coloring c (together with l and k_d's) and an EF assignment π of size k are *compatible* if π assigns exactly k_d agents of color d to activity a_d for $d \in [1, l]$.

Our algorithm will find an EF assignment compatible with c or determine that no such EF assignment exists. Note that any EF assignment of size k has at least one compatible coloring, and the probability that a random coloring we choose is compatible with some fixed EF assignment of size k is at least $(1/l)^k$ (as we must color those k agents correctly) which is exponentially small only in k.

FPT algorithm. Our algorithm first partitions agents of each color into several subsets, and check several necessary conditions for the coloring to be compatible with at least one EF assignment; if any of the conditions is not met, the coloring will be rejected by the algorithm (as there is no EF assignment compatible with the given coloring). For each color $d \in [1, l]$, the algorithm computes three subsets: $N_d = \{i \in N : c(i) = d\}$, $N_d^{\text{IR}} = \{i \in N_d : (a_d, k_d) \in S_i\}$, and $N_d^{\text{IN}} = \{i \in N_d : \exists d' \in [1, l] \text{ s.t. } (a_{d'}, k_{d'}) \in S_i\}$. If $|N_d^{\text{IR}}| < k_d$ for

any $d \in [1,l]$, no EF assignment is compatible with c because assigning k_d agents to a_d would not be individually rational (i.e., not enough agents approve the outcome), so the coloring should be rejected in this case. If for some $d \in [1,l]$ the set $N_d^{\text{IN}} - N_d^{\text{IR}}$ is not empty but contains some agent i, then no EF assignment is compatible with c because an EF assignment cannot assign i to a_d (because $i \notin N_d^{\text{IR}}$) but i would wish to join $a_{d'}$ for some $d' \in [1,l]$ which would make the assignment not envy-free; hence the coloring should be rejected in this case. If $|N_d^{\text{IN}}| > k_d$, then at least one agent i in N_d^{IN} should be assigned to the void activity, but i would wish to join $a_{d'}$ for some $d' \in [1,l]$ which would make the assignment not envy-free; hence the coloring should be rejected. If the coloring is not rejected by any of the cases mentioned earlier, then we have the following three conditions for every color $d \in [1,l]$: (a) $|N_d^{\text{IR}}| \geq k_d$, (b) $|N_d^{\text{IN}}| \leq k_d$, and (c) $N_d^{\text{IR}} \subseteq N_d^{\text{IN}}$. Let us define X_d for each $d \in [1,l]$ as follows: X_d contains an arbitrary set of k_d agents from N_d^{IR} such that every agent in N_d^{IN} is contained in X_d. Note that this is always possible due to the three conditions mentioned above. We claim that an assignment π which assigns agents in X_d to a_d and all other agents to $a_\emptyset$ is an EF assignment compatible with c. To prove compatibility, all agents in X_d are by definition of color d and $|X_d| = k_d$ for all $d \in [1,l]$. To prove stability, first consider any agent i who is assigned to the void activity and suppose $d = c(i)$. Since $i \notin X_d$, we know that $i \notin N_d^{\text{IN}}$ by definition, and therefore there is no $d' \in [1,l]$ such that $(a_{d'}, k_{d'}) \in S_i$. Now consider any agent i who is assigned to a_d by π (thus $c(i) = d$). By definition $i \in X_d$ and thus $i \in N_d^{\text{IR}}$, which implies that $(a_d, k_d) \in S_i$. Therefore π is an EF assignment of size k, compatible with c.

Let us now prove that if there is at least one EF assignment that is compatible with c, then the algorithm does not reject the coloring. Let π be one such assignment and let X_d be the set of agents assigned to a_d by π. Due to compatibility we have $|X_d| = k_d$ and $c(i) = d$ for all $i \in X_d$ for all $d \in [1,l]$; in particular, by definition $X_d \subseteq N^{\text{IR}_d}$ and thus the first condition (a) above holds for all $d \in [1,l]$. Due to stability of π, every agent i with $\pi(i) = a_\emptyset$ satisfies that $\nexists d' \in [1,l]$ such that $(a_{d'}, k_{d'}) \in S_i$. Therefore the conditions (b) and (c) above hold for all $d \in [1,l]$, which proves that the coloring c would not be rejected by the algorithm.

We have shown that if we begin with a coloring c (together with l and k_d's) that is compatible with at least one EF assignment, then our algorithm would find an EF assignment

compatible with the coloring and that if no such assignment exists the coloring would be rejected. This is a Monte Carlo algorithm with probability of success at least $(1/k)^k$ and runtime bounded by $O((k^k)nk)$ (as our algorithm must enumerate all possible values of l and k_d's), which is polynomial in n but exponential only in k. □

4.3.4 k-Perfect-GASP: Finding Perfect Assignments

Recall that k-Perfect-GASP is the problem of finding a perfect assignment that uses k activities out of p activities. Under this parameterization, GASP is W[2]-hard as the following theorem shows.

Theorem 16. *k-Perfect-GASP is $W[2]$-hard. The problem remains to be $W[2]$-hard even if all agents have increasing preferences or all agents have decreasing preferences.*

Proof. Consider an instance of the parameterized Dominating Set problem which consists of a graph $G = (V,E)$ and a parameter k, and asks whether there exists a dominating set D of size k ($D \subseteq V$ is a dominating set if for every node $v \in V$ either $v \in D$ or v has a neighbor in D). This problem is known to be $W[2]$-complete.

Let us create an instance of GASP as follows: Let $N = \{1,2,\ldots,n\}$ and $A^* = \{a_1,a_2,\ldots,a_n\}$ where $n = |V|$. For each agent i, define $S_i = \{(a_j,x) : ((v_i,v_j) \in E) \wedge (1 \leq x \leq n)\} \cup \{(a_i,x) : 1 \leq x \leq n\}$. Note that in this instance agents do not care about the number of participants, but the activities only. Finally we set the parameter of GASP to be equal to the parameter of the Dominating Set instance. Note that the size of the GASP instance we create is polynomial in n and k.

Let D be a dominating set of size k in the original instance. Let us construct a perfect assignment π as follows: For each agent i, if $v_i \in D$, then let $\pi(i) = a_i$; if $v_i \notin D$, then there exists some $v_j \in D$ such that $(v_i,v_j) \in E$ because D is a dominating set, and let $\pi(i) = a_j$. Clearly π is a perfect assignment that uses only k activities. Conversely, suppose that a perfect assignment π exists which uses exactly k activities. Let A' be the set of k activities to which at least one agent is assigned under π (note that $|A'| = k$). Let $D = \{v_i : a_i \in A'\}$, and we claim that D is a dominating set in G. For any node $v_i \notin D$, we know that π assigns agent i to some activity $a_j \in A'$ where $a_j \neq a_i$, and thus $v_j \in D$. Since π is individually rational, it implies that $(v_i,v_j) \in E$, and therefore D is a dominating set.

Note that in our reduction all agents have increasing preferences as well as decreasing preferences, proving the second statement of the theorem. □

It is not surprising that k-Perfect-GASP is the most complex problem being considered, as it pertains to the strongest solution concept. We do not know the exact complexity of k-Perfect-GASP when all activities are equivalent, and it remains to be an open problem (yet the problem is known to be NP-complete by Darmann et al. [14]).

4.4 Discussion and Future Work

In this work we investigated the parameterized complexity of the Group Activity Selection Problem (GASP) when parameterized by the size of the solution, for four different solution concepts and under various restrictions on inputs. Despite the fact that all problems being considered are NP-hard, we showed that some special cases of the problem admit efficient FPT algorithms. Our results indicate that the computational complexity of GASP varies when its input is restricted (imposed by special preferences of agents or uniformity of activities) or the solution concept changes, which is not distinguishable under the classic complexity. Our work leaves a few interesting open problems for future work. First, we do not know the exact complexity of k-Perfect-GASP with equivalent activities besides its NP-completeness. It would be intriguing if the problem is FPT. Second, the focus in this chapter is to exhibit any FPT algorithm; we have not tried hard to optimize the dependence on k. It would be interesting to do this and to show conditional lower bounds on how the runtime should depend on k, especially in the case of k-IR-GASP. Lastly, one can consider a different setting where agents have a strict ordering over the set of outcomes, instead of having approved outcomes that are equally preferred. Darmann recently proved several easiness and hardness results under this setting [13], but no parameterized complexity results are known yet.

Chapter 5

Complexity of Stable Invitations Problem

Chapter 6

Incentive Compatibility in Group Assignment Problems

Chapter 7

Open Problems

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