The Finite Element Method for Structural Mechanics*

Carlos X. Azua Gonzalez[1]

Prepared for guest lectures in the following institution

Host institution: Universidad Estatal del Sur de Manabi (UNESUM)[a]

[a]*Km $1\frac{1}{2}$ Vía Jipijapa-Noboa - Campus Los Ángeles, Ecuador*

Abstract

This document represents practical guidance of the material to be covered in the intended-to-be crash course, on the use of the computing software Mathcad, for structural mechanics applications. This document complements any other relevant guidance provided in the syllabus of the course.

1. An introduction to Mathcad

Installation. A free trial version of PTC Mathcad can be found in link below: https://alfasoft.science/en/shop/mathematics/mathcad-prime-7-trial/.

Please follow the distributor's instructions for installation before the course

- Download the install file (zip) to your computer.
- Unpack the installation file (zip).
- Double-click the file "setup.exe" to start the installation of Mathcad.
- Choose Trial/New License when asked.

URL: `http://unesum.edu.ec/` (Host institution: Universidad Estatal del Sur de Manabi (UNESUM))

[1]This is material prepared by the author, and the manuscript represents solely the author's views, i.e. this document does not represent any views of any affiliations where the author may be linked, directly or indirectly, at the moment the guest lectures may have taken place. Funding disclosure: No financial aid of any kind has been provided by the host institution for the preparation of this study material.

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- During the installation you will be asked to enter a license code. Leave this field blank and click "Next". You will get a fully functional 30-days version of Mathcad Prime.

Please visit the corresponding helpdesk for support support.alfasoft.com if you have any questions.

Usage. You will be shown, at the beginning of the course, a general overview of toolbars and some key features that may be useful for the course.

Dynamics of the learning process. The guest lecturer remains an instrument for your learning, i.e. you are responsible for your own learning!. Therefore, if you have a question, just interrupt by hanging your hand (even digitally is possible through most platforms for videoconferencing these days).

Content of the course.

1 **Introduction:** A brief description of the historical advancements of matrix analysis of solids and structures. An overview of mathcad environment for simple algebraic, matrix and symbolic operations. (3 hours of contact)

2 **Matrix form of equilibrium equations I:** Derivation of global equilibrium equations for Continua and its generalised weak form. Understanding finite element interpolations. (3 hours of contact)

3 **Matrix form of equilibrium equations II:** Derivation of the stiffness matrix for a 2D truss finite element. Understanding the Jacobian and Gaussian integration. Numerical evaluation of stiffness matrices. (3 hours of contact)

4 **Global assemblage of finite elements:** Understanding element topology, element connectivity and global matrix sparsity. Application to assemblage of truss elements. (3 hours of contact)

5 **Boundary conditions part I:** Understanding mixed boundary conditions and (force) reactions in structural mechanics. Hands on modifying stiffness matrices and load vectors for finite element pre-solution. (3 hours of contact)

6 **Boundary conditions part II:** Computation of (force) reactions in structures under mixed boundary conditions. Basic element post-solution. (3 hours of contact)

7 **Introductory dynamics:** Understanding mass matrix and stiffness matrix relations in free vibration problems. Introduction to the computation of eigenfrequencies using matrix analysis. (2 hours of contact)

2. Matrix operations

2.1. Defining matrices

Exercise 1:

Define the following matrices:

$$\mathbf{A} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \mathbf{B} := \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad ; \quad \mathbf{C} := \begin{bmatrix} 1 & \text{-3} \\ 3 & 4 \end{bmatrix} \tag{1}$$

For your own analysis:

- Which of these matrices are symmetric?
- Mention one example where dealing with symmetric matrices is beneficial in accelerating structural mechanics computer codes?
- Which of these matrices are singular? How can you prove formally that any of these matrices is singular?
- Can you mention one particular example where we use computations with singular matrices in structural analysis?

2.2. Matrix addition and subtraction

Exercise 2:

Compute the following addition and/or subtractions:
Use **A** and **B** from equation 1.

$$\begin{gathered} \mathbf{D} = \mathbf{A} + \mathbf{A} \;\text{ and verify }\; \mathbf{E} = 2 \cdot \mathbf{A} = \mathbf{D} \\ \mathbf{F} = \mathbf{A} + \mathbf{B} \\ \mathbf{G} = \mathbf{A} - \mathbf{B} \\ \mathbf{H} = \mathbf{F} + \mathbf{G} \\ \mathbf{J} = \tfrac{1}{2}\mathbf{H} \text{ , can you prove } \mathbf{J} \text{ is a multiple of } \mathbf{A} \text{ ?} \end{gathered} \tag{2}$$

For your own analysis:

- Is $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ admissible as an identity ?

2.3. Matrix multiplication

Exercise 3:

Compute the following matrix multiplications:

Use $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ from equation 1.

$$\begin{gathered}
\mathbf{D}_m = \mathbf{A}\cdot\mathbf{A} \\
\mathbf{E}_m = 2\cdot\mathbf{D}_m\cdot\mathbf{A} \text{ , and verify } \tfrac{1}{2}\mathbf{E}_m = \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A} \\
\mathbf{F}_m = \mathbf{A}\cdot\mathbf{B} \\
\mathbf{G}_m = \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A} \\
\mathbf{H}_m = \mathbf{B}\cdot\mathbf{C} \\
\mathbf{J}_m = \mathbf{C}\cdot\mathbf{B} \text{ , can you prove } \mathbf{J}_m \text{ is a not multiple of } \mathbf{H}_m \text{ ?}
\end{gathered} \tag{3}$$

For your own analysis:

- Can you mention one special characteristic of $\mathbf{A}$ aside from being a diagonal matrix?
- How is this type of matrix labeled (referring to matrix $\mathbf{A}$)?

2.4. Transpose of a matrix

Exercise 4:

Compute the following operations involving the transpose of a matrix:

Use $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ from equation 1.

$$\begin{gathered}
\mathbf{D}_t = (\mathbf{A}\cdot\mathbf{A})^T \\
\mathbf{E}_t = (2\cdot\mathbf{D}_t)^T \text{ , and verify } \mathbf{E}_t = 2\cdot\mathbf{A}\cdot\mathbf{A} \\
\mathbf{F}_t = \mathbf{A}^T\cdot\mathbf{B} \\
\mathbf{G}_t = \mathbf{A}\cdot\mathbf{B}^T\cdot\mathbf{A} \\
\mathbf{H}_t = \mathbf{B}\cdot\mathbf{C}^T \\
\mathbf{J}_t = \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A} \text{ , can you prove } \mathbf{J}_t \text{ is equal to } (\mathbf{G}_t)^{\mathrm{T}} \text{ ?}
\end{gathered} \tag{4}$$

For your own analysis:

- Can you prove $\mathbf{A}^T = \mathbf{A}^{-1}$?
- Is the previous statement ($\mathbf{A}^T = \mathbf{A}^{-1}$) an identity or a particular case?

2.5. Diagonal matrix and identity matrix

Exercise 5:

Compute the following operations involving diagonal and identity matrices: Use $\mathbf{A}$ from equation 1.

$$\begin{gathered}
\mathbf{D}_d := \text{diag}\{1, 0, 1\} \\
\mathbf{E}_d = (5 \cdot \mathbf{D}_d)^T \text{ , and verify } (\mathbf{E}_d)^T = 5 \cdot \mathbf{D}_d \\
\mathbf{F}_d = \mathbf{A}^T \cdot \mathbf{A} \text{ , can you prove } \mathbf{F}_d \text{ is equal to the identity matrix ?} \\
\mathbf{I}_{3\times3} := \text{diag}\{1, 1, 1\} \\
\mathbf{G}_g = 3 \cdot \mathbf{I}_{3\times3}
\end{gathered} \tag{5}$$

For your own analysis:

- Is the matrix $\mathbf{A}$ an identity matrix? can you provide a 'practical' proof using an example in Mathcad (no formal proof required)?

2.6. Practical calculation of determinants

Exercise 6:

Compute the following determinants by hand and using a computing tool: Notice the convention states $\mathbf{I}_{3\times3}$ is the identity matrix (dimensions 3 by 3). Use $\mathbf{A}$ from equation 1.

$$\begin{gathered}
a_d = \det(\mathbf{A}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \\
d_d = \det(\text{diag}\{1, 0, 1\}) \\
e_d = \det(\mathbf{I}_{3\times3}) \\
f_d = \det(3 \cdot \mathbf{I}_{3\times3}) \\
g_d = \det(\mathbf{I}_{3\times3}^T) \text{ , can you verify } g_d = e_d = \det(\mathbf{I}_{3\times3})?
\end{gathered} \tag{6}$$

For your own analysis:

- Can you explain the importance of the determinant in evaluating the existence of an inverse matrix?

- Can you explain the importance of the determinant in computing eigenvalues (e.g. to compute principal directions in structural engineering)?

2.7. Inverse of a matrix

Exercise 7:

Compute the inverse of the following matrices by hand or use a computer: Notice the convention states $\mathbf{I}_{3\times 3}$ is the identity matrix (dimensions 3 by 3). Use $\mathbf{A}$ from equation 1.

$$\begin{aligned}
\mathbf{D}_i &= (\mathbf{A})^{-1} \\
\mathbf{E}_i &= (\text{diag}\{1,0,1\})^{-1} \\
\mathbf{F}_i &= (\mathbf{I}_{3\times 3})^{-1} \\
\mathbf{G}_i &= (3\cdot\mathbf{I}_{3\times 3})^{-1} \\
\mathbf{H}_i &= (\mathbf{I}^T_{3\times 3})^{-1} \text{ , can you verify } \mathbf{H}_i = 3\cdot\mathbf{G}_i?
\end{aligned} \tag{7}$$

For your own analysis:

- Can you explain the use of inverse matrix operations, e.g. to solve a linear algebraic system of equations?

- Can you relate the use of inverse matrix operations in solving assembled equations of equilibrium in structural engineering (hint: remember the generalised principle of virtual work $\mathbf{K}_g\cdot\mathbf{u}_g = \mathbf{F}^{ext}_g$)?

2.8. Vector inner and cross product

Exercise 8:

Compute the following vector operations by hand or use the computer:

Notice the convention in use treats vectors in a matrix format.

Therefore, inner product of $\mathbf{v}$ between the same vector, follows from $\mathbf{v}^T \cdot \mathbf{v}$.

$$\begin{gathered}
\text{define the following vector, } \mathbf{u} := [1, 0, 0]^T \\
\text{define the following vector, } \mathbf{v:} = [0, 1, 0]^T \\
\text{compute inner product, } d_v = \mathbf{u}^T \cdot \mathbf{u} \\
\text{compute inner product, } e_v = \mathbf{u}^T \cdot \mathbf{v} \\
\text{compute the euclidean norm } ||\mathbf{u}|| = \sqrt{(}\mathbf{u}^T \cdot \mathbf{u}) \\
\text{compute the cross product } \mathbf{f}_v = \mathbf{u} \times \mathbf{v}
\end{gathered} \tag{8}$$

For your own analysis:

- What is the angle between $\mathbf{f}_v$ and the plane formed by its 'parent' vectors $\mathbf{u}$ and $\mathbf{v}$?
- Can you compute the angle between $\mathbf{u}$ and $\mathbf{v}$?
- Can you mention one application of the dot product in computing mechanics related terms aside from euclidean norm of physically-meaningful vectors?

2.9. Eigenvalue problems

Notice the eigenvalues λ of $\mathbf{M}_e$ comply with the equation:

$$\mathbf{M}_e \cdot \mathbf{x}_e = \lambda \cdot \mathbf{x}_e \tag{9}$$

The eigenvalues are found by solving for the characteristic equation of $\mathbf{M}_e$:

$$det(\mathbf{M}_e - \lambda \cdot \mathbf{I}) = 0 \tag{10}$$

Exercise 9:

Compute the eigenvalues for the following matrices using the computer:

$$
\begin{aligned}
\boldsymbol{\sigma}_e &:= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\
\boldsymbol{\sigma}_f &:= \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix} \\
\boldsymbol{\sigma}_g &:= \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \\
\boldsymbol{\sigma}_h &:= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
\tag{11}
$$

For your own analysis:

- Can you explain what is the physical meaning of eigenvectors of the stress matrix $\boldsymbol{\sigma}$ (commonly referred as Cauchy stress tensor in solid mechanics)?
- Can you explain what is the physical meaning of the corresponding eigenvalues of the same stress matrix $\boldsymbol{\sigma}$?
- Are the eigenvectors of the stress matrix $\boldsymbol{\sigma}$ perpendicular to each other?

3. Structural analysis for mdof systems composed of 1-D bar elements

3.1. Elastic stiffness matrix of 1-D bar element

Exercise 10:

For the 1-D bar element shown, carry out the following tasks:

For simplicity, consider the axis of reference is aligned to the axis of the bar.

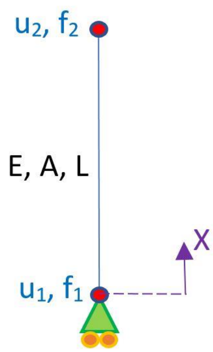

Figure 1: Single 1-D bar element under constraint displacement at node 1 and known displacement at node 2.

- Compute the element local stiffness matrix $\mathbf{K}^{local}$
  hint: $\mathbf{F}^{ext,local}=\mathbf{K}^{local}\cdot\mathbf{u}^{local} \rightarrow \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{E\cdot A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\cdot\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$
- Restrain the displacement $u_1 = 0$ $[mm]$ and apply a (compressive) displacement $u_2 = -1$ $[mm]$, then compute the reaction at the bottom of the element.
- Calculate by hand the elastic uniaxial strain of the bar and the corresponding uniaxial stress and axial force, and check if the reaction computed by matrix analysis coincides with the hand calculations.
- Can you verify that the element is in equilibrium?
- If the stiffness matrix is singular, what do we need to do to solve the system for mixed boundary conditions (e.g. restraint displacement at support and known force at the other extreme)?

where $A = 400$ $[mm]$ $\times$ 400 $[mm]$, $L = 2500$ $[mm]$, $E = 30000$ N/mm^2.

Exercise 11:

For the mdof system given, carry out the following tasks:

For simplicity, consider the axis of reference is aligned to the axis of the bar.

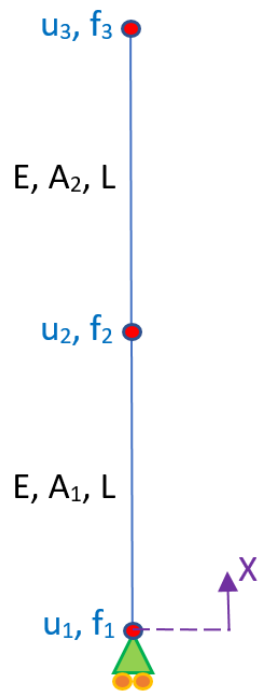

Figure 2: Two 1-D bar elements under constraint displacement at node 1 and known force at nodes 2 & 3.

- Discretise and number nodes & elements 'locally' with corresponding mapping to global numbering, using two elements as previously described.
- Compute the elastic stiffness matrix of each element.
- Assemble local matrix components in corresponding global positions
- Carry static hand calculations to determine the displacement at the top.

- Carry static calculation using mixed boundary conditions (use any method of your preference), and check the results against hand calculations.

where:

$A_1 = 400\ [mm] \times 400\ [mm]$, $A_2 = 250\ [mm] \times 250\ [mm]$, $L = 2500\ [mm]$, $P = 30000\ [N]$, $E = 30000\ N/mm^2$. The nodal forces are defined as $f_3 = -\frac{P}{2}$ and $f_2 = -P$.

3.3. Limitations of the current introductory material

So far, the participants should have obtained a good grip on matrix operations in the computing tool (i.e. Mathcad software), and the participants will have gained some exposure to using these tools for small structural systems. The students would have enough experience, at this stage, to implement other element types (e.g. beams). Nonetheless, the students should be aware that the automation of processes in structural mechanics requires more advanced algorithmic development (e.g. using the computing tool or programming languages), however, this is beyond the scope of this course.

4. Further recommended reading

There are various sources available to keep learning computer programming in structural mechanics, and in general for programming the Finite Element Method (with applications to structural, solid mechanics and geomechanics).

For further reference on non-linear analysis, the following reference is a good start: De Borst et al. (2012).

5. Governing equation: Momentum balance in Continua

Let a solid body of differential volume $\delta V = \delta x \delta y \delta z$ be subject of an acceleration $\ddot{\mathbf{u}}$ due to the action of a stress field $\boldsymbol{\sigma}$ and body forces $\mathbf{b}$. The governing equation for momentum balance of the solid reads as follows:

$$\mathrm{Div}[\boldsymbol{\sigma}] + \mathbf{b} = \rho \ddot{\mathbf{u}} \tag{12}$$

where $\mathrm{Div}(\cdot)$ denotes the divergence operator. The divergence over the stress tensor simplifies to $\mathrm{Div}[\boldsymbol{\sigma}] = \left(\nabla^{\mathrm{T}} \cdot \boldsymbol{\sigma}\right)^{\mathrm{T}}$ where $\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}]^{\mathrm{T}}$. N.B.: The reader is left to prove equation 12, e.g. using Newton's second law over the differential volume δV.

The weak form of the governing equation is obtained when the equation 12 is multiplied by a perturbation of the deformation field $\delta \mathbf{u}$ using the dot product and by integrating over an elementary volume Ω:

$$\int_{\Omega} \delta \mathbf{u}^{\mathrm{T}} \cdot \big(\mathrm{Div}[\boldsymbol{\sigma}] + \mathbf{b}\big) d\Omega = \int_{\Omega} \delta \mathbf{u}^{\mathrm{T}} \cdot (\rho \ddot{\mathbf{u}}) d\Omega \tag{13}$$

Using integration by parts and doing some rearrangement, the first integral term reads $\int_{\Omega} (\delta \mathbf{u}^{\mathrm{T}} \cdot \mathrm{Div}[\boldsymbol{\sigma}]) d\Omega = \int_{\Omega} \mathrm{Div}[\boldsymbol{\sigma} \cdot \delta \mathbf{u}] d\Omega - \int_{\Omega} \delta \boldsymbol{\varepsilon} : \boldsymbol{\sigma} d\Omega$, with $\varepsilon = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}})$. Replacing this first term in equation 13 reads:

$$\int_{\Omega} \mathrm{Div}[\boldsymbol{\sigma} \cdot \delta \mathbf{u}] d\Omega + \int_{\Omega} \delta \mathbf{u}^{\mathrm{T}} \cdot \mathbf{b} d\Omega = \int_{\Omega} \delta \boldsymbol{\varepsilon} : \boldsymbol{\sigma} d\Omega + \int_{\Omega} \delta \mathbf{u}^{\mathrm{T}} \cdot (\rho \ddot{\mathbf{u}}) d\Omega \tag{14}$$

Further, applying the divergence theorem to the first term on the left hand side $\int_{\Omega} \mathrm{Div}[\boldsymbol{\sigma} \cdot \delta \mathbf{u}] d\Omega = \int_{\Gamma} \left(\delta \mathbf{u}^{\mathrm{T}} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}\right) d\Gamma = \int_{\Gamma_\sigma} \left(\delta \mathbf{u}^{\mathrm{T}} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}\right) d\Gamma$, where this has been simplified to an integral over the surface without deformation constraints Γ_σ, noting that $\Gamma \equiv \Gamma_\sigma \cup \Gamma_u$. Since the external tractions are defined as $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ with $\mathbf{n}$ being the normal to Γ, the generalised weak form reads as follows:

$$\underbrace{\int_{\Omega} \delta \mathbf{u}^{\mathrm{T}} \cdot (\rho \ddot{\mathbf{u}}) d\Omega}_{\text{inertial energy}} + \underbrace{\int_{\Omega} \delta \boldsymbol{\varepsilon} : \boldsymbol{\sigma} d\Omega}_{\text{strain energy}} = \underbrace{\int_{\Omega} \delta \mathbf{u}^{\mathrm{T}} \cdot \mathbf{b} d\Omega}_{\text{work of body forces}} + \underbrace{\int_{\Gamma_\sigma} \delta \mathbf{u}^{\mathrm{T}} \cdot \mathbf{t} d\Gamma}_{\text{work due to tractions}} \tag{15}$$

6. Finite Element discretisation

The essence of Finite Element Methods is the sub-division of problem domains into smaller element domains, where the solution field is known in nodal positions and approximated elsewhere within the elements by using interpolation functions, so called shape functions.

The following sections make a particular derivation of a truss element in 2D.

6.1. Interpolation in a 2D truss element

Interpolation is done in standard form assuming iso-parametric shape functions, i.e. the solution field and the geometric space are interpolated with the same functions. If interpolation is done conveniently in local coordinates the following expressions apply:

$$\mathbf{u}' = \sum_{inode=1}^{nnode} N_{inode} \mathbf{u}'_{inode} \tag{16}$$

$$x' = \sum_{inode=1}^{nnode} N_{inode} x'_{inode} \tag{17}$$

where $\mathbf{u}'$ is the local displacement field within an element, and x' is the position along the local axis that, for mathematical convenience, is set to vary in the range $x' \in [-l/2, l/2]$ with l being the length of the element.

Projecting the nodal Cartesian displacements into the local axes using dot product leads to the following transformation operation:

$$\mathbf{u}'_{el} = \mathbf{T}_u \mathbf{u}_{el} \tag{18}$$

Replacing equation 18 into equation 16 leads to the following expression for the local displacement vector within an element:

$$\mathbf{u}' = \mathbf{N}\mathbf{T}_u \mathbf{u}_{el} \tag{19}$$

6.2. Deformation gradients in a 2D truss element

The strain along the element axis $\varepsilon_{x'x'}$, assuming small deformations, is defined by taking the spatial derivative of the axial deformation $u'_{x'}$ from equation 19 as follows:

$$\varepsilon_{x'x'} = \mathbf{B}\mathbf{T}_u\mathbf{u}_{el} \tag{20}$$

where $\mathbf{B}$ is the strain-displacement operator that contains spatial gradient of the shape functions considering the local axis.

6.3. Constitutive equation

The constitutive equation in its simplest form follows from a linear elastic relationship between the axial stress $\sigma_{x'x'}$ and the strain $\varepsilon_{x'x'}$ using equation 20 as follows:

$$\sigma_{x'x'} = E\mathbf{B}\mathbf{T}_u\mathbf{u}_{el} \tag{21}$$

6.4. Finite Element form of equilibrium

The equilibrium expression as in equation 15 is re-written to compute work-conjugated multiplications with terms being transformed consistently into local coordinates:

$$\int_\Omega (\delta\mathbf{u}')^{\mathrm{T}} \cdot (\rho\ddot{\mathbf{u}}') d\Omega + \int_\Omega \delta\varepsilon_{x'x'}\sigma_{x'x'} d\Omega = \int_\Omega (\delta\mathbf{u}')^{\mathrm{T}} \cdot \mathbf{b}' d\Omega + \int_{\Gamma_\sigma} (\delta\mathbf{u}')^{\mathrm{T}} \cdot \mathbf{t}' d\Gamma \tag{22}$$

where $\mathbf{b}' = \mathbf{T}_b\mathbf{b}$ stands for the transformed body forces, $\mathbf{t}'$ stands for the local tractions.

Using equations 19, 20 and 21 in equation 22 results in the next expression:

$$\delta\mathbf{u}_{el}^{\mathrm{T}} \underbrace{\left(\mathbf{T}_u^{\mathrm{T}} \int_\Omega \rho\mathbf{N}^{\mathrm{T}}\mathbf{N} d\Omega \mathbf{T}_u\right)}_{\mathbf{M}_{el}} \ddot{\mathbf{u}}_{el} + \delta\mathbf{u}_{el}^{\mathrm{T}} \underbrace{\left(\mathbf{T}_u^{\mathrm{T}} \int_\Omega \mathbf{B}^{\mathrm{T}} E\mathbf{B} d\Omega \mathbf{T}_u\right)}_{\mathbf{K}_{el}} \mathbf{u}_{el} = \delta\mathbf{u}_{el}^{\mathrm{T}}\mathbf{F}_{ext}^{el} \tag{23}$$

Further, the term $\delta\mathbf{u}_{el}^{\mathrm{T}}$ drops out of the equation since this is an arbitrary perturbation and leads to the following equation of motion:

$$\mathbf{M}_{el}\ddot{\mathbf{u}}_{el} + \mathbf{K}_{el}\mathbf{u}_{el} = \mathbf{F}_{ext}^{el} \tag{24}$$

where the mass matrix $\mathbf{M}_{el}$, the stiffness matrix $\mathbf{K}_{el}$ and the external load vector $\mathbf{F}_{ext}^{el}$ are defined as follows for the element:

$$\mathbf{M}_{el} = \mathbf{T}_u^{\mathrm{T}} \int_{\Omega} \rho \mathbf{N}^{\mathrm{T}} \mathbf{N} d\Omega \mathbf{T}_u \tag{25}$$

$$\mathbf{K}_{el} = \mathbf{T}_u^{\mathrm{T}} \int_{\Omega} \mathbf{B}^{\mathrm{T}} E \mathbf{B} d\Omega \mathbf{T}_u \tag{26}$$

$$\mathbf{F}_{ext}^{el} = \mathbf{T}_u^{\mathrm{T}} \int_{\Omega} \mathbf{N}^{\mathrm{T}} \mathbf{T}_b \mathbf{b} d\Omega + \mathbf{T}_u^{\mathrm{T}} \int_{\Gamma_\sigma} \mathbf{N}^{\mathrm{T}} \mathbf{t}' d\Gamma + \mathbf{T}_u^{\mathrm{T}} \mathbf{F}_p' \tag{27}$$

where the possibility of local point forces $\mathbf{F}_p'$ applied at the nodes is also considered.

Often a numerical damping term needs to be considered. If the material behaves nonlinearly or there if the boundaries are viscous, then physically based damping occurs. Following such reasoning for a damping term, the full equation of motion in matrix form is as follows:

$$\mathbf{M}_{el}\ddot{\mathbf{u}}_{el} + \mathbf{C}_{el}\dot{\mathbf{u}}_{el} + \mathbf{K}_{el}\mathbf{u}_{el} = \mathbf{F}_{ext}^{el} \tag{28}$$

where $\mathbf{C}$ is a damping matrix.

6.5. Finite Element assemblage: quasi-static analysis

For the purpose of this course, the structures will be analysed under a quasi-static condition, i.e. the structure is considered to be loaded with a very low frequency and the analysis provides the steady-state solution after a long period of time when the structure is essentially free of vibrations. In this regard, the analysis provides the final deformation of the structure due to a load that remains long enough, without considering time dependent effects.

Equation 28 takes the form of the principle of virtual work under quasi-static conditions as follows:

$$\mathbf{F}_{int}^{el} = \mathbf{F}_{ext}^{el} \tag{29}$$

$$\mathbf{F}_{int}^{el} = \underbrace{\mathbf{T}_u^{\mathrm{T}} \int_{\Omega} \mathbf{B}^{\mathrm{T}} \sigma_{x'x'} d\Omega}_{\mathbf{K}_{el}\mathbf{u}_{el}} \tag{30}$$

Accounting for the fact that the response of the structural system follows from the combined action of the finite elements being loaded, the assemblage of the matrix and load vector is as follows:

$$\underbrace{\left(\bigcup_{ielem=1}^{nelem} \mathbf{K}_{el} \right)}_{\mathbf{K}_g} \mathbf{u}_g = \underbrace{\bigcup_{ielem=1}^{nelem} \mathbf{F}_{ext}^{el}}_{\mathbf{F}_{ext}} \tag{31}$$

6.6. An introduction to eigenfrequency analysis

The equation of motion for the free vibration problem of the structural system, i.e. after global assemblage of equation 28 under no external loads, is simplified under the assumption that the nodal displacements follow the periodic form $\mathbf{u}_g = \tilde{\mathbf{u}}_g \cdot e^{i\omega t}$ as follows:

$$\left(-\omega^2 \mathbf{M}_g + i\omega \mathbf{C}_g + \mathbf{K}_g \right) \tilde{\mathbf{u}}_g \cdot e^{i\omega t} = \mathbf{0} \tag{32}$$

where further simplification is obtained when the structural system is assumed to be undamped as follows:

$$\left(-\omega^2 \mathbf{M}_g + \mathbf{K}_g \right) \tilde{\mathbf{u}}_g \cdot e^{i\omega t} = \mathbf{0} \tag{33}$$

where the non-trivial solution for the displacements $\mathbf{u}_g$ is obtained when the following generalised eigenvalue problem is solved:

$$\det\left(\mathbf{M}_g^{-1} \mathbf{K}_g - \omega^2 \mathbf{I} \right) = 0 \tag{34}$$

In general, there is as many (angular) eigenfrequencies $\omega_i = 2\pi f_i$ as the total number of degrees of freedom of the system, and each eigenfrequency is associated with an eigenmode $\mathbf{u}_g^{\omega_i}$ and a frequency f_i.

In practice, only a few of these eigenfrequencies might be of importance from an engineering perspective, e.g. the lowest frequencies which could be matched by the frequency content of an earthquake loading. Therefore, advanced methods exist to extract a particular range of eigenfrequencies with less computational cost.

The study of such eigenfrequencies and vibration modes is of general interest in structural engineering, since vibration comfort criteria must be complied according to codes of practice, and resonance during high-amplitude long-duration loading events must be avoided at all costs. In fact, when the input frequencies coincide with that of the eigenfrequencies of the structure, it is important that the dynamic features of the structure be changed during the design to avoid resonance.

7. Appendix A

7.1. Shape functions

The shape functions for a line element with two nodes are as follows:

$$N_1(\xi) = \frac{1}{2}(1 - \xi) \tag{35}$$

$$N_2(\xi) = \frac{1}{2}(1 + \xi) \tag{36}$$

The derivatives of the shape functions are as follows:

$$\frac{\partial N_1(\xi)}{\partial \xi} = -\frac{1}{2} \tag{37}$$

$$\frac{\partial N_2(\xi)}{\partial \xi} = \frac{1}{2} \tag{38}$$

The Cartesian derivatives of the shape functions are as follows:

$$\left\{ \begin{array}{cc} \frac{\partial N_1(\xi)}{\partial x'} & \frac{\partial N_2(\xi)}{\partial x'} \end{array} \right\} = \mathbf{J}^{-1} \cdot \left\{ \begin{array}{cc} \frac{\partial N_1(\xi)}{\partial \xi} & \frac{\partial N_2(\xi)}{\partial \xi} \end{array} \right\} \tag{39}$$

In addition, the matrix $\mathbf{N}$ that contains shape functions is defined as follows:

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \tag{40}$$

7.2. Jacobian

The Jacobian for transforming a line space is as follows:

$$\mathbf{J} = \frac{\partial x'}{\partial \xi} = \left\{ \begin{array}{cc} \frac{\partial N_1(\xi)}{\partial \xi} & \frac{\partial N_2(\xi)}{\partial \xi} \end{array} \right\} \cdot \left\{ \begin{array}{c} x'_1 \\ x'_2 \end{array} \right\} \tag{41}$$

where for convenience the nodal local positions are assigned as $x'_1 = -l/2$ and $x'_2 = l/2$.

7.3. Strain-displacement matrix

The strain-displacement matrix for a two-noded truss element is as follows:

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x'} & 0 & \frac{\partial N_2}{\partial x'} & 0 \end{bmatrix} \tag{42}$$

7.4. Transformation matrices

The transformation matrix for displacements in a two-noded truss element reads as follows:

$$\mathbf{T}_u = \begin{bmatrix} cos(\theta) & sin(\theta) & 0 & 0 \\ -sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & cos(\theta) & sin(\theta) \\ 0 & 0 & -sin(\theta) & cos(\theta) \end{bmatrix} \tag{43}$$

The transformation matrix for body forces in a two dimensional truss element reads as follows:

$$\mathbf{T}_b = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \tag{44}$$

7.5. Gaussian quadrature

The Gaussian integration of a function $f(x')$ within the domain Γ is computed as follows:

$$\int_\Gamma f(x')dx' = \int_{-1}^{1} f\big(x'(\xi)\big)\det(\mathbf{J})d\xi = \sum_{i_p=1}^{n_p} f\big(x'(\xi_{i_p})\big) \cdot \det(\mathbf{J}) \cdot w_{i_p} \tag{45}$$

where the integration coefficients w_{i_p} and locations ξ_{i_p} are as follows:

Point-rule	Positions	Weighting
1	0.0	2.0
2	$[-1/\sqrt{3}, 1/\sqrt{3}]$	$[1.0, 1.0]$
3	$[-\sqrt{3/5}, 0.0, \sqrt{3/5}]$	$[5/9, 8/9, 5/9]$

N.B.: n_p-point integration is exact over a function that can be fitted with a polynomial of order $(2 \cdot n_p - 1)$ or lower.

7.6. Modifying global stiffness matrix to impose boundary conditions

The stiffness matrix and the load vector get modified to ensure essential boundary conditions are prescribed as follows:

$$
\begin{array}{c}
\begin{matrix} \text{1-st} & \text{2-nd} & \cdots & r\text{-th} & \cdots & \cdots & n\text{-th} \end{matrix} \\
\underbrace{\begin{bmatrix}
K_{11} & K_{12} & \cdots & 0 & \cdots & \cdots & K_{1n} \\
K_{21} & K_{22} & \cdots & 0 & \cdots & \cdots & K_{2n} \\
\vdots & & \ddots & \vdots & & & \vdots \\
0 & 0 & \cdots & K_{rr} & \cdots & 0 & 0 \\
 & & & \vdots & \ddots & & \\
 & & & 0 & & \ddots & \\
\text{symm} & & & 0 & & & K_{nn}
\end{bmatrix}}_{\mathbf{K}_g^{mod}}
\end{array}
\underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \\ \vdots \\ \vdots \\ u_n \end{Bmatrix}}_{\mathbf{u}_g}
= \mathbf{F}_{ext}^{mod} \tag{46}
$$

where u_r is a restrained or prescribed displacement, $n = ntdof$ with $ntdof$ standing for the total number of degrees of freedom of the assembled structural system. The modification of the load vector is as follows:

$$
\mathbf{F}_{ext}^{mod} = \underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_r \\ \vdots \\ \vdots \\ F_n \end{Bmatrix}}_{\mathbf{F}_{ext}} + \underbrace{\begin{Bmatrix} -\sum^{n_{pu}} K_{1r} \cdot u_r \\ -\sum^{n_{pu}} K_{2r} \cdot u_r \\ \vdots \\ -F_r + K_{rr} \cdot u_r \\ \vdots \\ \vdots \\ -\sum^{n_{pu}} K_{nr} \cdot u_r \end{Bmatrix}}_{\mathbf{F}_{ext}^{add}} \tag{47}
$$

where n_{pu} stands for the number of prescribed displacements and $\mathbf{F}_{ext}^{add}$ is added to the original load vector to maintain balance in the original set of equations except for the ones associated with nodal equilibrium under prescribed displacements.

Note that the solution of the modified system of equations for the assembled system, i.e. $\mathbf{u}_g = (\mathbf{K}_g^{mod})^{-1} \cdot \mathbf{F}_{ext}^{mod}$, is used to compute reactions as follows:

$$\mathbf{F}_{reac} = \mathbf{K}_g \cdot \mathbf{u}_g - \mathbf{F}_{ext} \tag{48}$$

References

De Borst, R., Crisfield, M.A., Remmers, J.J.C., Verhoosel, C.V., 2012. Nonlinear Finite Element Analysis of Solids and Structures. Wiley.