INSTITUTE OF ECONOMIC STUDIES, FACULTY OF SOCIAL SCIENCES
CHARLES UNIVERSITY IN PRAGUE (*established 1348*)

ROBUST STATISTICS AND ECONOMETRICS

INSTITUTE OF ECONOMIC STUDIES
FACULTY OF SOCIAL SCIENCES
CHARLES UNIVERSITY IN PRAGUE

JAN ÁMOS VÍŠEK

Week 3

Content of lecture

1. Repetiton is mother of wisdom(Jan Amos Komensky)

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2. Examples of influence function

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2. Examples of influence function
3. Classical and modern requirements on the point estimator

Content

A brief repetition of some points from previous lectures

A problem of the classical statistics and econometrics

A tacit hope in ingnoring deviations from ideal models was that they would not matter; that statistical procedures which were optimal under strict model would still be approximately optimal under the approximate model. Unfortunately, it turned out that this hope was often drastically wrong; even mild deviations often have much larger effects than were anticipated by most statisticians.

John W. Tukey (1960)

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(And we gave two examples - Ronald Aylmer Fisher and Peter Huber.)

What we want to achieve by robust methods?

The main goals of robust statistics

1. To describe the structure best fitting the bulk of data.
2. To identify deviating data points (outliers) or deviating substructures for further treatment, if desired.
3. To identify and give a warning about highly influential data points (leverage points).
4. To deal with unsuspected serial correlation, or more generally, with deviations from the assumed correlation structures.

Which types of problems we can meet with?

The four main types of deviations from the strict parametric model

1. The occurence of gross errors.
2. Rounding and grouping.
3. The model may have been conceived as an approximation anyway, e.g., by virtue of CLT.
4. Apart of distributional assumptions, the assumption of independence (or of some specific correlation structure) may only be approximately fulfilled.

How have we attempted to cope with these tasks ?

Three approaches:

1. Huber's alternative to classical point estimation via neighbourhoods.
2. Huber's alternative to classical testing hypotheses via capacities.
3. Hampel's infinitesimal approach via Prokhorov metric and influence function.

Hampel's approach - a bit more mathematics

The Hampel's approach is based on two basic ideas and a nice fact:

1. The first idea - any estimator can be interpreted as a function T (say) from the space of all distribution functions $\mathcal{H}$ to the parameter space Θ (say).
2. The second idea - the function T can be studied by an infinitesimal calculus of limits, derivaties, integrals, etc.
3. A nice fact - the Kolmogorov-Smirnov result - the empirical d.f. converge uniformly to the "true" underlying one.

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Making preparation steps for explanation of Hampel's approach

Recalling Taylor's expansion for a real function of real variable

1. The real function of one real variable $f(x)$

   $\rightarrow$ Taylor's expansion of $f(x) = f(x^0) + f'(x^0) \cdot (x - x^0) + \ldots$.

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2. Let's recall the derivative of
   the function $f(x)$ at a given point x_0,

$$f'(x^0) = \lim_{\delta \to 0} \frac{f(x^0 + \delta) - f(x^0)}{\delta}$$

$\rightarrow$ the derivative offers an information about
the behaviour of the function in a neighbourhood of x_0.

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Recalling Taylor's expansion for a real function of finitely-dimensional variable

1. The real function of several real variables $f(x_1, x_2, ..., x_p)$

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2. Let's recall again the partial derivative of the function $f(x)$ at the point x^0 along the j-th coordinate, i e.

$$\frac{\partial f(x^0)}{\partial x_j} = \lim_{\delta_j \to 0} \frac{f(x^0 + \Delta_j) - f(x^0)}{\delta_j}$$

where $\Delta_j = (0, 0, ..., \delta_j, ..., 0)'$.

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3. Realize that $\max_{j=1,2,...p} \left| \frac{\partial f(x^0)}{\partial x_j} \right|$ is a hint about the behaviour of the function in a neigbourhood of x^0.

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   the function $f(x)$ at the point x^0 along the j-th coordinate, i e.

   Realize that when computing $\frac{\partial f(x^0)}{\partial x_j}$, we change only one coordinate,
   i. e. we compute the derivative in one direction.

3. Realize that $\max_{j=1,2,...p} \left| \frac{\partial f(x^0)}{\partial x_j} \right|$ is a hint about
   the behaviour of the function in a neigbourhood of x^0.

Making preparation steps for explanation of Hampel's approach

Let's think about the partial derivative once again - a bit alternative approach.

Consider the partial derivative of
the function $f(x)$ at the point x^0 along the j-th coordinate, i e.

$$\frac{\partial f(x^0)}{\partial x_j} = \lim_{\delta \to 0} \frac{f(x^0 + \delta \cdot \Delta_j) - f(x^0)}{\delta}$$

where $\Delta_j = (0, 0, ..., 1, ..., 0)'$ - the unit is on the j-th position.

Defining the influence function

Generalizing Taylor's expansion for a real function of uncountably-dimensional variable

1. Denote a degenerated (at the point x) d. f. Δ_x
$\rightarrow \quad \Delta_x(v) = 0$ for $v \leq x$, $\Delta_x(v) = 1$ otherwise.

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Notice that the influence function $IF(x, T, F)$ has three arguments:

1. the point x at which the contamination is assumed,
2. the functional T in question

and finally

3. the d. f. F, as the point of space $\mathcal{H}$.

of the functional T at the d. f. F.

Explaining the role of influence function - the most important thing

What is the influence function good for?

1. Under some technical conditions

$$T(F_n) \cong T(F) + \int IF(x, F, T)\mathrm{d}F_n(x) + remainder_1,$$

i. e

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2. It means that if we add new observation, say x_{n+1},
the value of estimator changes approximately from

$$T(F) + \frac{1}{n}\sum_{i=1}^{n} IF(x_i, F, T) \quad \text{to} \quad T(F) + \frac{1}{n+1}\sum_{i=1}^{n+1} IF(x_i, F, T).$$

Explaining the role of influence function - an extra bonus

Asymptotic normality of estimator follows from ... .

1. We had, under some technical conditions

$$T(F_n) \cong T(F) + \frac{1}{n}\sum_{i=1}^{n} IF(x_i, F, T) + remainder_1$$

or equivalently

$$\sqrt{n}\,(T(F_n) - T(F)) \cong \frac{1}{\sqrt{n}}\sum_{i=1}^{n} IF(x_i, F, T) + remainder_2. \qquad (1)$$

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Please, keep the last two slides in mind for a moment.

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And that is what we'll discuss today:

Plan for the rest of today talk

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Prior to it, let's recall the idea of interpreting the point estimator
as a function (functional) of empirical distribution function.

We had at the second lecture:

The Hampel approach

Estimator as a function of distribution function

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2. Let $F_n(.) \in \mathcal{H}$ be an empirical d. f. corresponding to the observations $x_1, x_2, ..., x_n$, then $T(F_n) = \int x \mathrm{d}F_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$
(because $F_n(x)$has positive $\mathrm{d}F_n(x)$ of size $\frac{1}{n}$ just at the points $x_1, x_2, ..., x_n$).

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3. If we plug-in instead of empirical d. f. the underlying d. f. F,
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which is a theoretical counterpart to the estimator.

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4. Typically, for any estimator we have a theoretical conterpart so that we can write $\hat{\theta}^{(n)} = T_n(F_n)$ and $\theta = T(F)$, where F_n is the empirical d. f. corresponding to the underlying d. f..

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   Do you remember?

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The Hampel approach

Estimator as a function of distribution function - another example

1. We could consider also $s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$
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3. Let $F_n(.) \in \mathcal{H}$ be an empirical d. f. corresponding to the observations $x_1, x_2, ..., x_n$ and put $T^{(2)}(F_n) = \frac{n}{n-1}\int x^2 \mathrm{d}F_n(x) = \frac{1}{n-1}\sum_{i=1}^{n} x_i^2$.

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4. Then we have $s_n^2 = T^{(2)}(F_n) - \frac{n}{n-1}T^{(1)}(F_n)$.
5. If we plug-in instead of empirical d. f. the underlying d. f. F, we obtain a theoretical counterpart to the estimator.

The Hampel approach - generalization of previous examples

Usually we can find a function $h(x)$ (say) so that we have

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1. for $T^{(1)}(F_n)$ the function $h^{(1)}(x) = x$

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2. for $T^{(2)}(F_n)$ the function $h^{(2)}(x) = \frac{n}{n-1}x^2$.

Content

Recalling notation

Let's recall that for the standard normal distribution we use usually Φ and for its density ϕ.

Returning to the definition of influence function

Fix a functional $T : \mathcal{H} \rightarrow R$ (now T is given by $h(x) = x$) and consider the partial derivative
of the functional T at the point F along the x-th coordinate, i. e.

$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

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4. Finally, $IF(x, T, \Phi) = \lim_{\delta \to 0} \frac{\delta \cdot x}{\delta} = x$.

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And compute the $IF(x, T, \Phi_{\mu,\sigma^2})$.

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Content

Preparing to enlarge the set of requirements on (robust) estimators

Recalling the classical requirements on estimators

1. Unbiasedness

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Let's discuss them from the point of view of robust procedures -
- we know already enough about it to be able to do it.

6. Scale- and regression-equivariance
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Let's start with admissibility, recalling its definition.

5. Efficiency
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Definition of Mean Square Error (MSE):

Let $\hat{\theta}$ be an estimator, then

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Definition of admissible estimator:

Let $\hat{\theta}$ be an estimator, then we say that $\hat{\theta}$ is admissible if there is not an estimator better than $\hat{\theta}$.

(And we assume that it holds independently on number of observations.)

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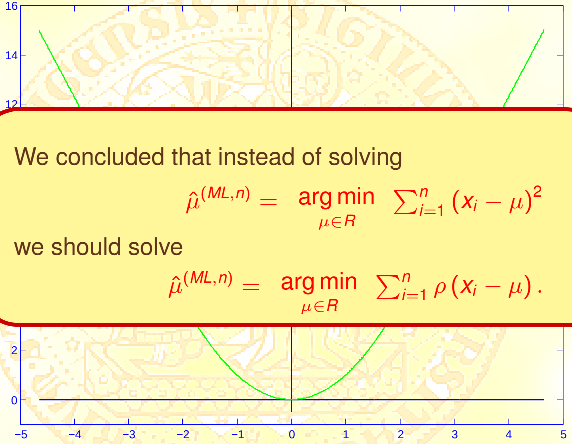
Now, let's return to the first lecture and discuss the unbiasedness.

Recalling that we found on the first lecture



We concluded that instead of solving

$$\hat{\mu}^{(ML,n)} = \arg\min_{\mu \in R} \sum_{i=1}^{n} (x_i - \mu)^2$$

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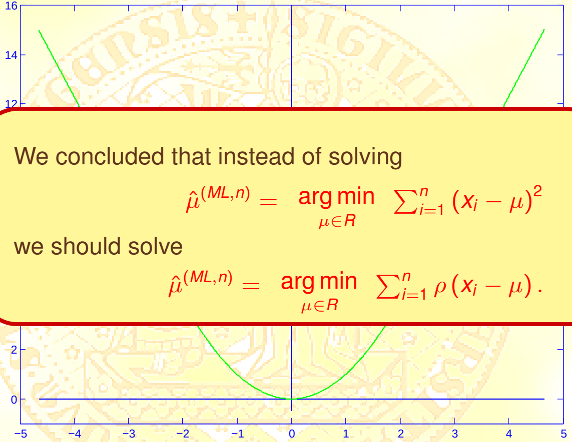

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Notice the bottom line in the frame !!

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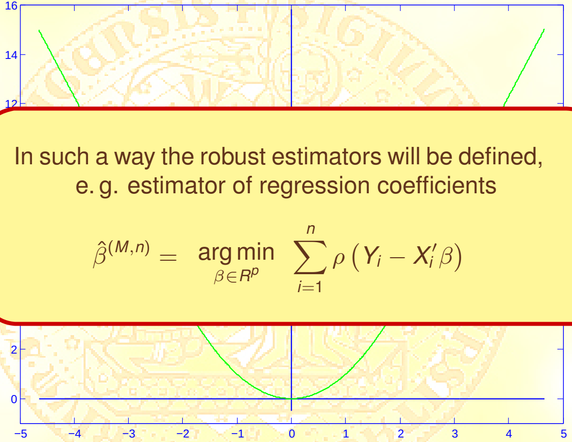

In such a way the robust estimators will be defined, e. g. estimator of regression coefficients

$$\hat{\beta}^{(M,n)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} \rho\left(Y_i - X_i'\beta\right)$$

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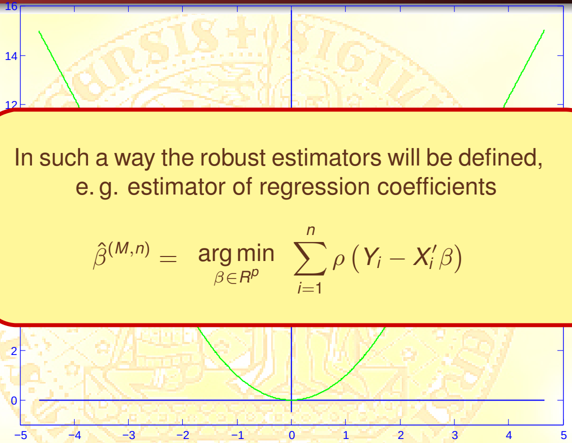

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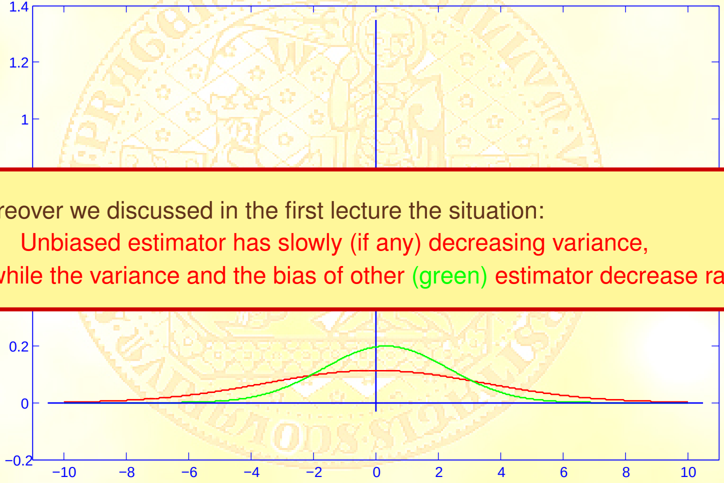
But then we cannot (typically) find a formula for (robust) estimators
and hence we cannot prove (compute ?!?) unbiasedness.

Possible density of unbiased and biased estimator

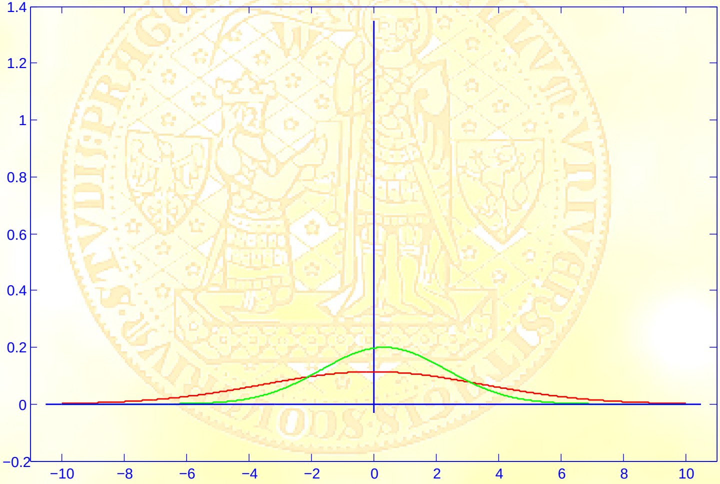


Moreover we discussed in the first lecture the situation:

Unbiased estimator has slowly (if any) decreasing variance,

while the variance and the bias of other (green) estimator decrease rapidly.

Notice decreasing variance and bias



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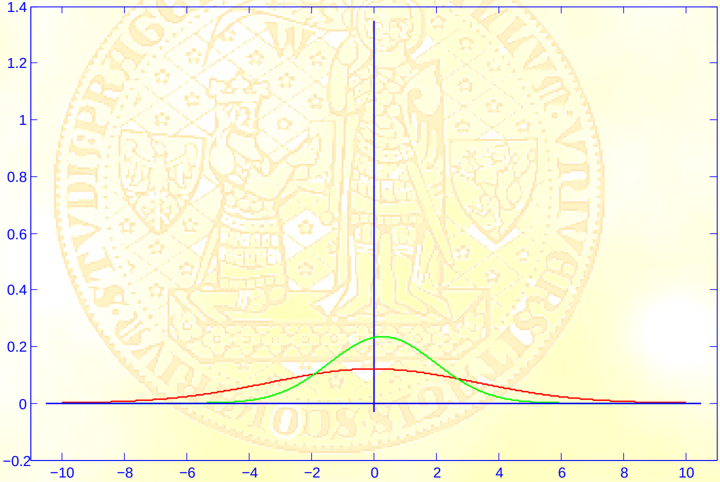

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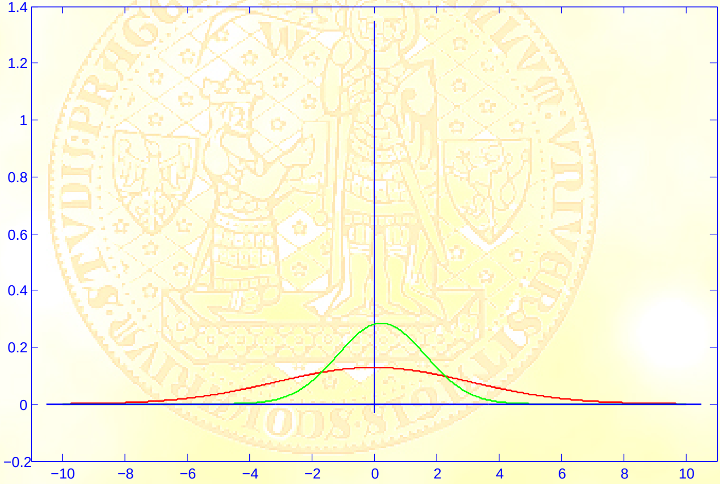

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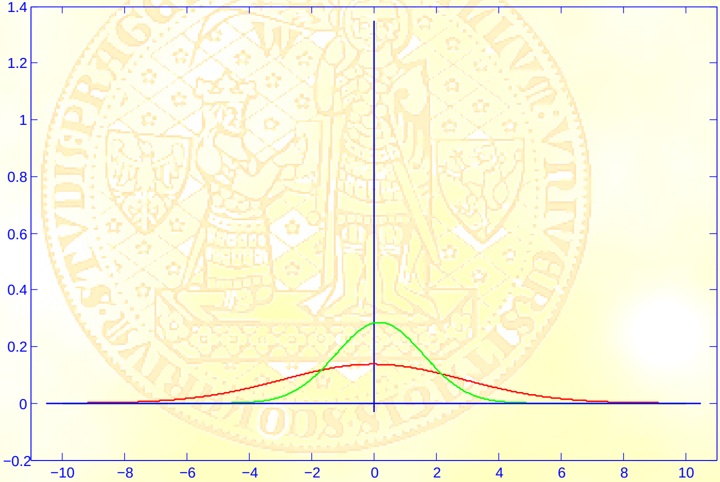

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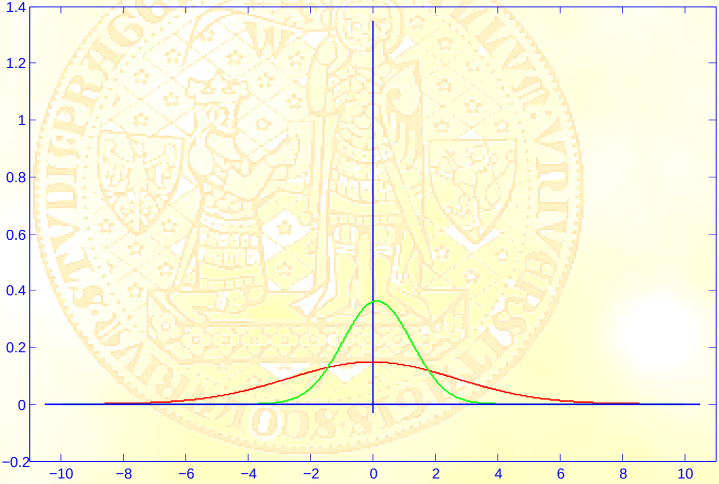

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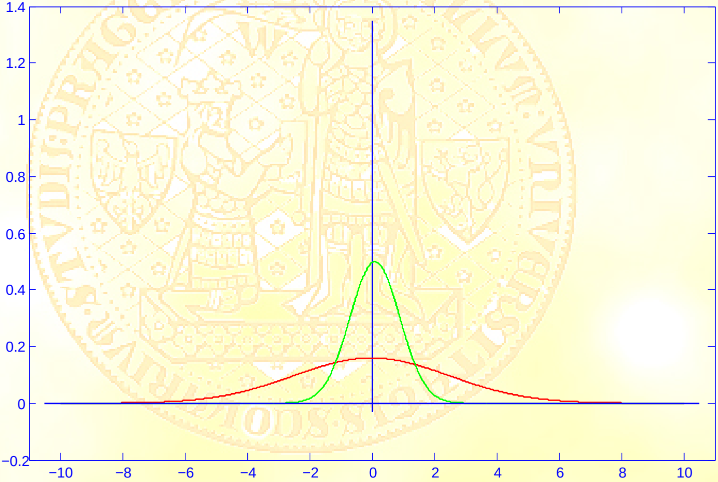

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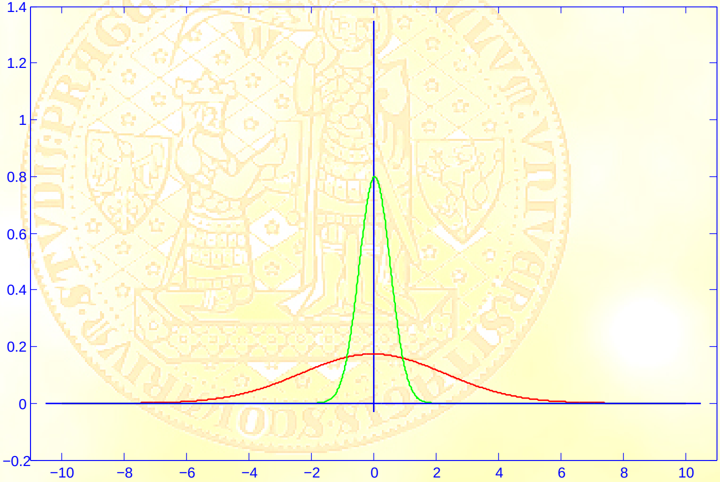

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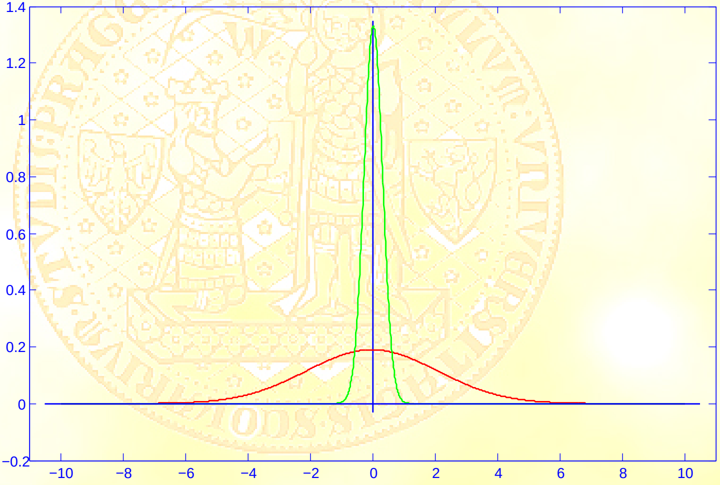

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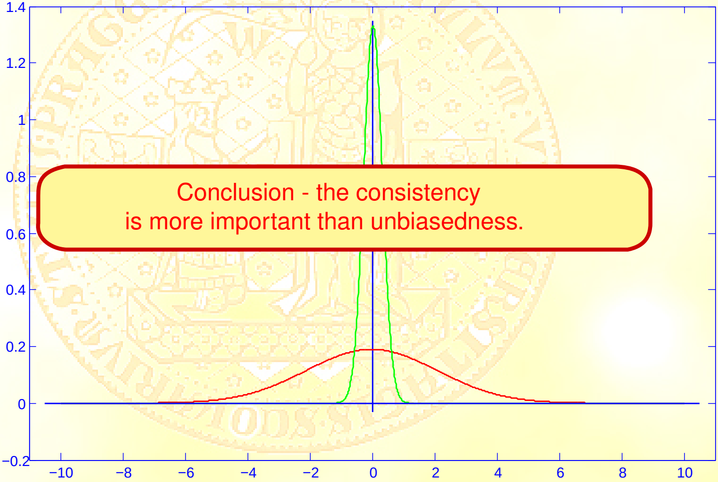
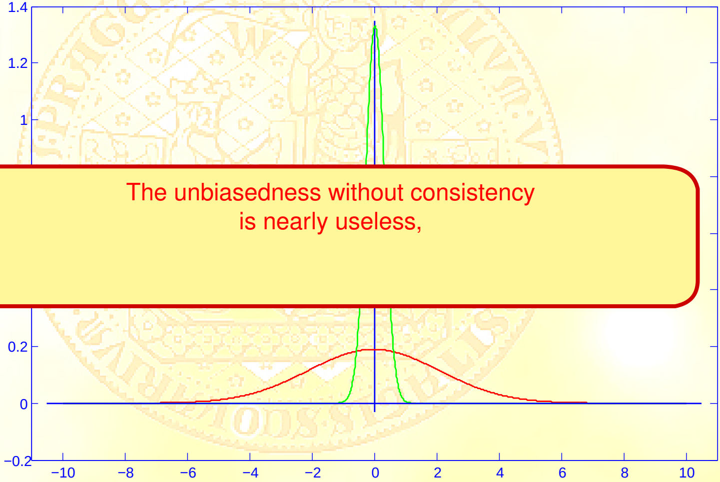


Conclusion - the consistency
is more important than unbiasedness.

Notice decreasing variance and bias



The unbiasedness without consistency
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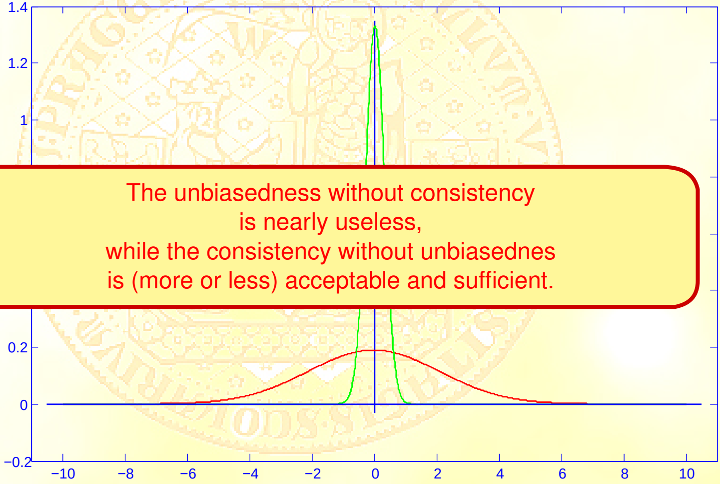

The unbiasedness without consistency
is nearly useless,
while the consistency without unbiasednes
is (more or less) acceptable and sufficient.

Preparing to enlarge the set of requirements on (robust) estimators

Nearly concluding:

The requirements overtaken from the classical statistics

1. Consistency (typically weak, i. e. in probability)

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We still didn't discuss scale- and regression-equivariance

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We still didn't discuss scale- and regression-equivariance
- so let's do it.

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5. Scale- and regression-equivariance

Framework: $Y_i = X_i' \beta^0 + e_i$

$i = 1, 2, \ldots, n$

Equivariance of $\hat{\beta}^{(n)}$

$$\hat{\beta}(Y, X) : M(n, p+1) \to R^p$$

scale-equivariant : $\forall c \in R^+ \quad \hat{\beta}(cY, X) = c\hat{\beta}(Y, X)$

regression-equivariant : $\forall b \in R^p \quad \hat{\beta}(Y + Xb, X) = \hat{\beta}(Y, X) + b$

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$$\hat{\beta}^{(L_1,n)} = \arg\min_{\beta \in R^p} \sum_{i=1}^n |Y_i - X_i'\beta|$$

Framework: $Y_i = X_i'\beta^0 + e_i$

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Equivariance - invariance of $\hat{\sigma}^2$

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Examples : $s_n^2 = \frac{1}{n-p} \sum_{i=1}^n r_i^2(\hat{\beta}^{(OLS,n)})$

What is the equivariance of $\hat{\beta}^{(n)}$ good for ?

1. When the units of measurement have been changed,
we don't need to recalculate the estimator
- we just shift the decimal point
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1. When the units of measurement have been changed,
   we don't need to recalculate the estimator
   - we just shift the decimal point
   (we are used to it from classical statistics).
2. The requirement of invariance and equivariance
   removed superefficiency.

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Finally, concluding:

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And now we add some others which correspond to the spirit of the discussion we have passed up to this moment.

4. Loss of efficiency as small as possible
5. Scale- and regression-equivariance

Enlarging the set of requirements on the point estimator

Returning to IF once again

Let's recall that if we add new observation, say x_{n+1},
the value of estimator changes from

$$T(F) + \frac{1}{n}\sum_{i=1}^{n} IF(x_i, F, T) \quad \text{to} \quad T(F) + \frac{1}{n+1}\sum_{i=1}^{n+1} IF(x_i, F, T).$$

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Influence function $IF(x, F, T)$ (IF) predetermines or predestinates (many) properties of estimator.

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Influence function $IF(x, F, T)$ (IF) predetermines or predestinates (many) properties of estimator.

So, let's define a couple of new requirements by it.

We should depress the influence of uotlying observations

Hampel's approach - characteristics of the functional T at the d. f. F

- Clearly,

$$\gamma^* = \sup_{x \in R} |IF(x, T, F)|$$

represents a maximal possible contribution of observation x to the value of the functional T provided the d. f. which generated data was F.

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- γ^* is called *gross-error sensitivity*.

We should depress the influence of large number of small shifts

Hampel's approach - characteristics of the functional T at the d. f. F

- Similarly, the maximal Lipschitz ratio

$$\lambda^* = \sup_{x,y \in R} \left| \frac{IF(x,T,F) - IF(y,T,F)}{x-y} \right|$$

represents a maximal possible contribution to the value of the functional T provided the d. f. which generated data was F by a rounding observation x.

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- λ^* is called *local-shift sensitivity*.

Some extremely remote observations shouldn't be taken into account at all

Hampel's approach - characteristics of the functional T at the d. f. F

- Finally,

$$\rho^* = \inf\left\{r \in R^+ \ :\ IF(x, T, F) = 0, \ |x| > r\right\}$$

represents a value such that any observation which is in absolute value larger then ρ^* brings no contribution to the value of the functional T provided the d. f. which generated data was F.

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- ρ^* is called *rejection point*.

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Finally, there are also a couple of practical requirements. All these topics will be discussed in the next lectures.

THANKS FOR ATTENTION