INSTITUTE OF ECONOMIC STUDIES, FACULTY OF SOCIAL SCIENCES
CHARLES UNIVERSITY IN PRAGUE (*established 1348*)

ROBUST STATISTICS AND ECONOMETRICS

INSTITUTE OF ECONOMIC STUDIES
FACULTY OF SOCIAL SCIENCES
CHARLES UNIVERSITY IN PRAGUE

JAN ÁMOS VÍŠEK

Week 4

Content of lecture

1. Repetiton is mother of wisdom(Jan mos Komensk)

Content of lecture

1. Repetiton is mother of wisdom(Jan mos Komensk)
2. The breakdown point

Content of lecture

1. Repetiton is mother of wisdom(Jan mos Komensk)
2. The breakdown point
3. Specification of robustness characteristics for classical estimators

Content

A brief repetition of some points from previous lectures

We have concluded:

The requirements overtaken from the classical statistics

1. Consistency (typically weak, i. e. in probability)
2. $\sqrt{n}$-consistency (root-n-consistency)
3. Asymptotic normality
4. Loss of efficiency as small as possible
5. Scale- and regression-equivariance

A brief repetition of some points from previous lectures

Then we added:

The requirements enlarging the classical paradigma

1. Gross-error sensitivity

$$\gamma^* = \sup_{x \in R} |IF(x, T, F)|$$

A brief repetition of some points from previous lectures

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$$\lambda^* = \sup_{x,y \in R} \left| \frac{IF(x, T, F) - IF(y, T, F)}{x - y} \right|$$

A brief repetition of some points from previous lectures

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3. Rejection point
$$\rho^* = \inf \left\{ r \in R^+ \; : \; IF(x, T, F) = 0, \;\; |x| > r \right\}$$

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$$\rho^* = \inf \left\{ r \in R^+ \ : \ IF(x, T, F) = 0, \ \ |x| > r \right\}$$
4. Breakdown point - will be discused as the first topic today

A brief repetition of some points from previous lectures

Then we added$_{(continued)}$:

The requirements enlarging the classical paradigma

1. Tight algorithm and reliable implementation
   - invention and verification

A brief repetition of some points from previous lectures

Then we added$_{(\textit{continued})}$:

The requirements enlarging the classical paradigma

1. Tight algorithm and reliable implementation
   - invention and verification

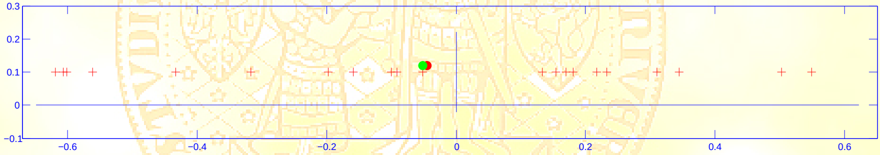
2. Good heuristic - to convince people to employ it

Content

First of all some inspiration

Observe the mean • and the median •

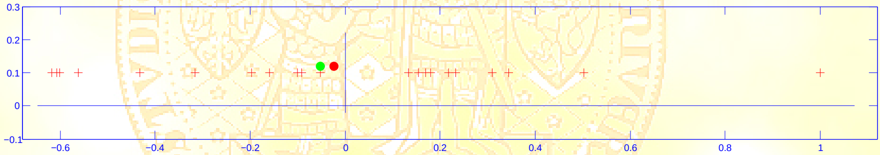
Non-contaminated data



First of all some inspiration

Observe the mean ● and the median ●

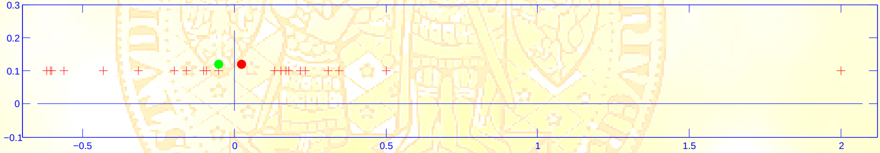
Contamination at point 1



First of all some inspiration

Observe the mean • and the median •

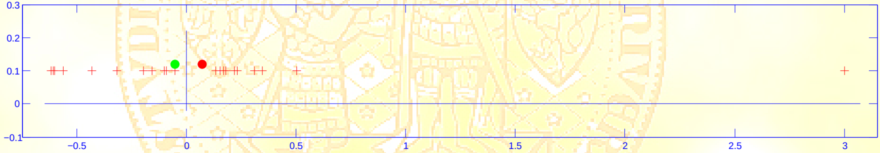
Contamination at point 2



First of all some inspiration

Observe the mean ● and the median ●

Contamination at point 3



First of all some inspiration

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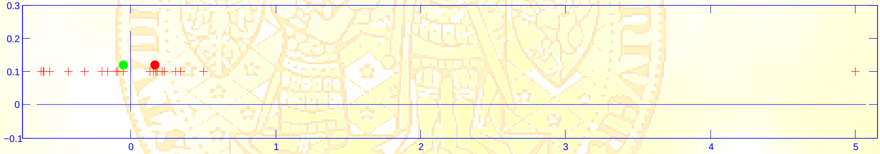
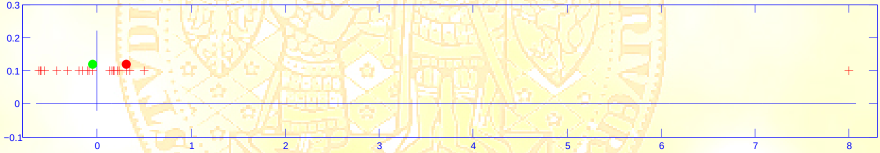
Contamination at point 5



First of all some inspiration

Observe the mean • and the median •

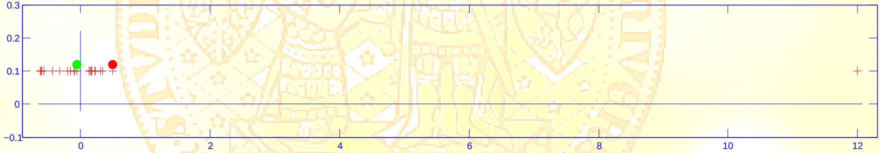
Contamination at point 8



First of all some inspiration

Observe the mean • and the median •

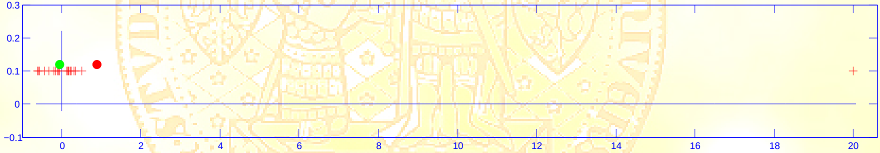
Contamination at point 12



First of all some inspiration

Observe the mean • and the median •

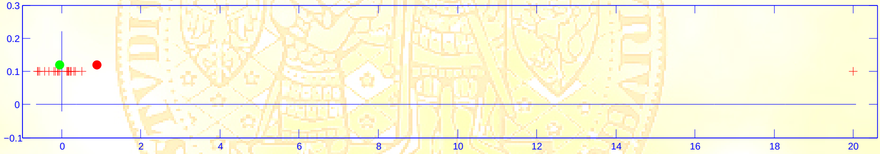
Contamination at point 20



First of all some inspiration

Observe the mean • and the median •

Contamination at point 20

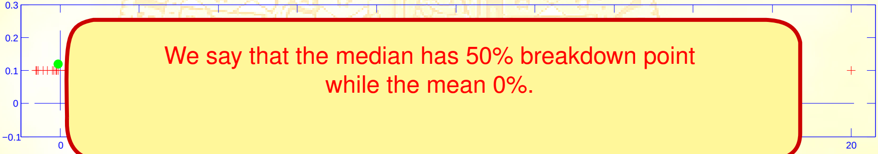


We would have to move 50% of observations
to see any impact on the median.

First of all some inspiration

Observe the mean • and the median •

Contamination at point 20



We say that the median has 50% breakdown point
while the mean 0%.

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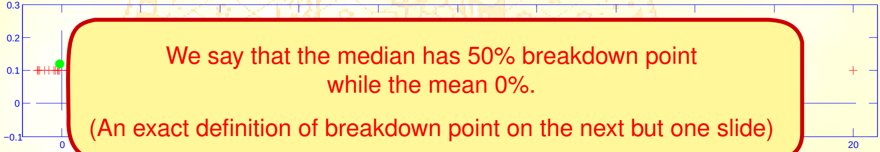
First of all some inspiration

Observe the mean • and the median •

Contamination at point 20



We say that the median has 50% breakdown point
while the mean 0%.

(An exact definition of breakdown point on the next but one slide)

We would have to move 50% of observations
to see any impact on the median.

Then there is of course a question:

Why we use more frequently mean than median?

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$$\lim_{n\to\infty} \frac{\mathrm{var}_\Phi\left(\bar{x}^{(n)}\right)}{\mathrm{var}_\Phi\left(median^{(n)}\right)} = 0.6....$$

Hampel's approach - characteristics of the functional T at the d. f. F

- An overall characteristic of the functional (the estimator) is

$$\varepsilon^* = \sup \Big\{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, \quad K_\varepsilon \ \textit{compact} \\ \pi(F,G) < \varepsilon \ \Rightarrow \ G(\{T_n \in K_\varepsilon\}) \underset{n\to\infty}{\longrightarrow} 1 \Big\}$$

where $\pi(F,G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$ and T_n is an empirical counterpart to the functional T.

Hampel's approach - characteristics of the functional T at the d. f. F

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- ε^* is called *breakdown point*

(explanation of Prokhorov metric is on one of the next slides, then finite sample breakdown point).

Hampel's approach - characteristics of the functional T at the d. f. F

- An overall characteristic of the functional (the estimator) is

ε^*

It is not trivial to understand this definition -
- we shall try after some preparation.

where $\pi(F, G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$
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- ε^* is called *breakdown point*.

Hampel's approach - characteristics of the functional T at the d. f. F

An overall characteristic of the functional (the estimator) is

The definition contains new notions:
compact set and Prokhorov metric.

ε^* is called *breakdown point*.

Hampel's approach - characteristics of the functional T at the d. f. F

An overall characteristic of the functional (the estimator) is

The definition contains new notions:
compact set and Prokhorov metric.

1. Firstly, what is a compact set? And what is good for ?

ε^* is called *breakdown point*.

Hampel's approach - characteristics of the functional T at the d. f. F

An overall characteristic of the functional (the estimator) is

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compact set and Prokhorov metric.

1. Firstly, what is a compact set? And what is good for ?
2. Secondly, an inspiration why we need Prokhorov metric.

ε^* is called *breakdown point*.

Hampel's approach - characteristics of the functional T at the d. f. F

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The definition contains new notions:
compact set and Prokhorov metric.

1. Firstly, what is a compact set? And what is good for ?
2. Secondly, an inspiration why we need Prokhorov metric.
3. Thirdly, an explanation what Prokhorov metric is.

ε^* is called *breakdown point*.

Hampel's approach - characteristics of the functional T at the d. f. F

- An overall characteristic of the functional (the estimator) is

Then we return to the definiton of breakdown point.

where $\pi(F, G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$
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Hampel's approach - characteristics of the functional T at the d. f. F

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1. Firstly, we explain the sense of it.

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Hampel's approach - characteristics of the functional T at the d. f. F

- An overall characteristic of the functional (the estimator) is

Then we return to the definiton of breakdown point.

1. Firstly, we explain the sense of it.
2. Then we try to read the mathematics.

where $\pi(F, G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$
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- ε^* is called *breakdown point*.

So, let us start with compact set

- Open and closed sets
  C is closed set if for any sequence $\{x_n\}_{n=1}^{\infty} \subset C$ such that

$$\exists \left(\lim_{n\to\infty} x_n = x_0 \right) \quad \Rightarrow \quad x_0 \in C.$$

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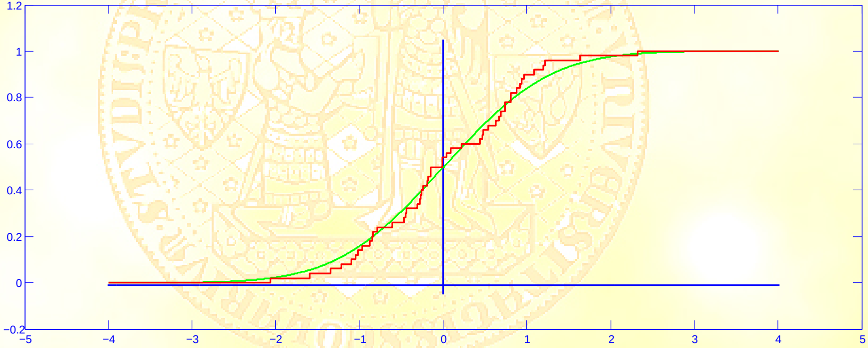
- C is compact if it is closed and $\forall \ (x \in C) \quad \|x\| < K < \infty$.
- The sense of compact sets and the use of compactness:
  Any open cover contains a finite subcover.

Recalling Kolmogorov-Smirnov metric

$$D(F,G) = \sup_{-\infty<x<\infty} |F(x) - G(x)|$$



Recalling Kolmogorov-Smirnov metric

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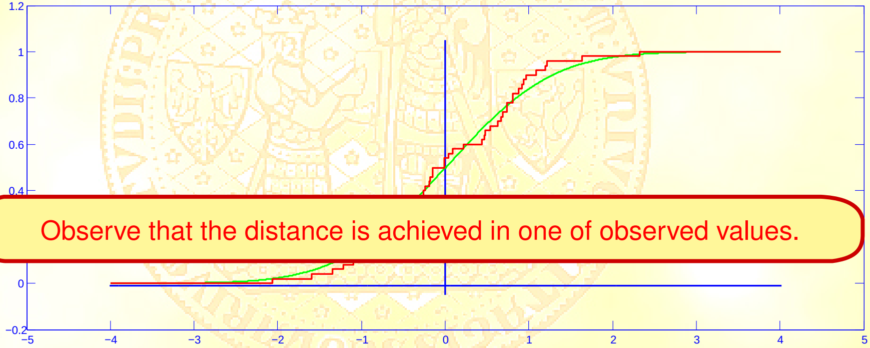
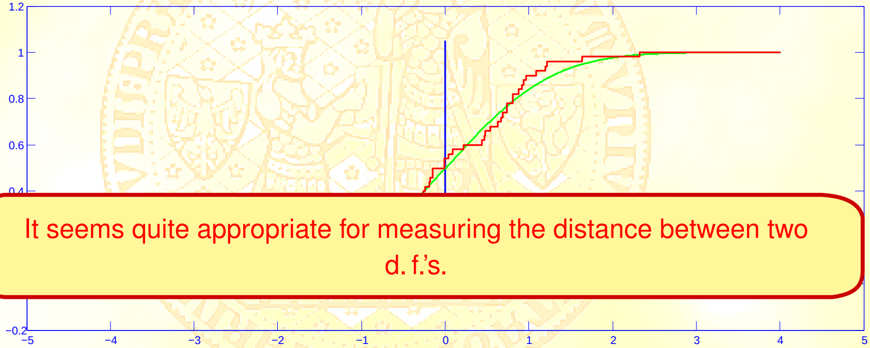


Observe that the distance is achieved in one of observed values.

Recalling Kolmogorov-Smirnov metric

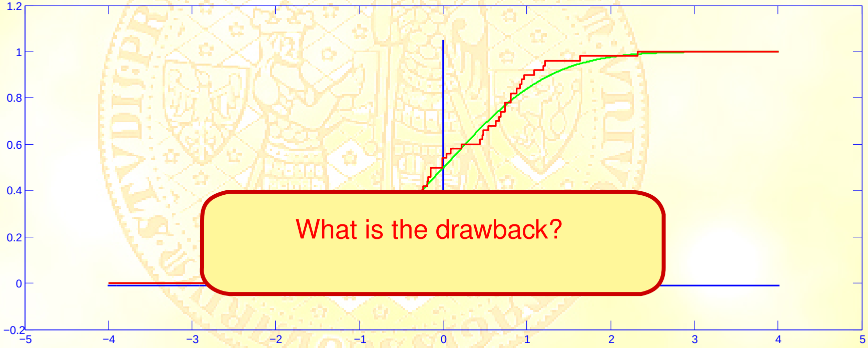
$$D(F,G) = \sup_{-\infty<x<\infty} |F(x) - G(x)|$$



It seems quite appropriate for measuring the distance between two d. f.'s.

Recalling Kolmogorov-Smirnov metric

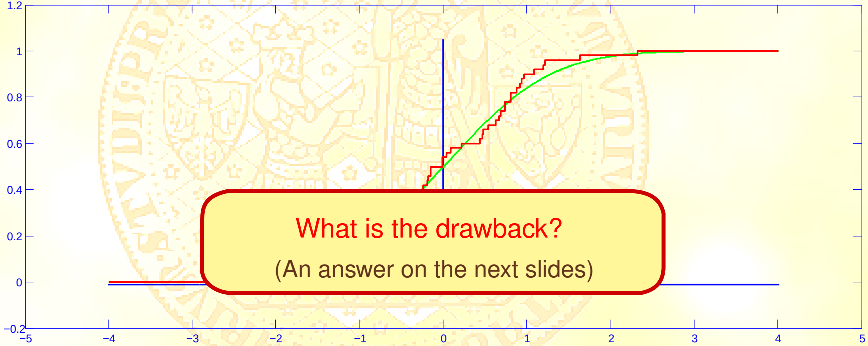
$$D(F, G) = \sup_{-\infty<x<\infty} |F(x) - G(x)|$$



What is the drawback?

Recalling Kolmogorov-Smirnov metric

$$D(F,G) = \sup_{-\infty<x<\infty} |F(x) - G(x)|$$

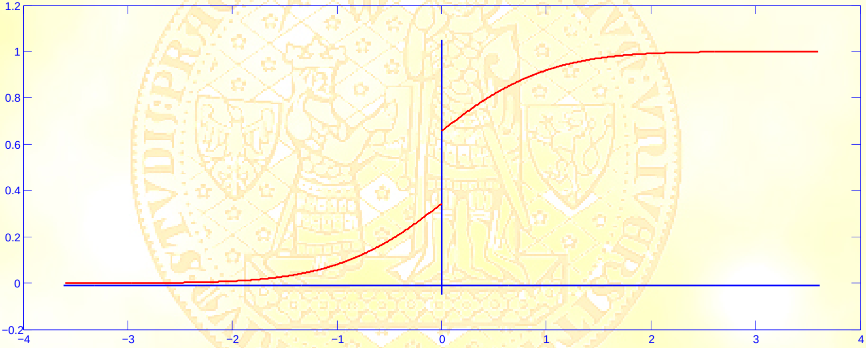


What is the drawback?

(An answer on the next slides)

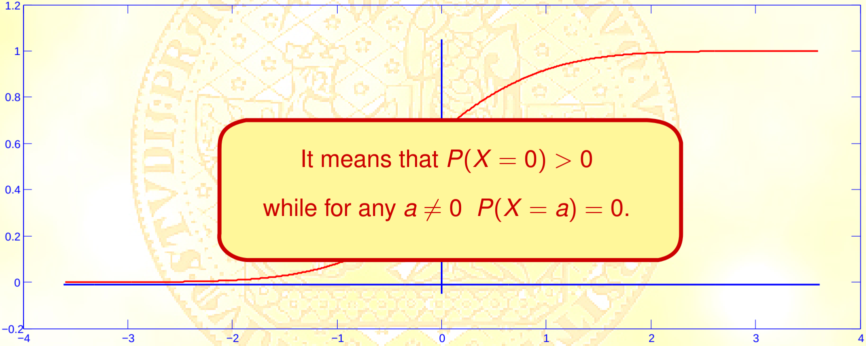
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Distribution function with a jump in the origine



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Distribution function with a jump in the origine

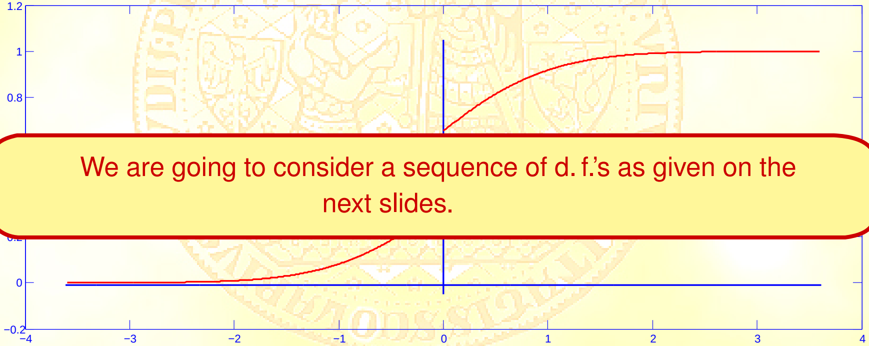


It means that $P(X = 0) > 0$

while for any $a \neq 0$ $P(X = a) = 0$.

Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

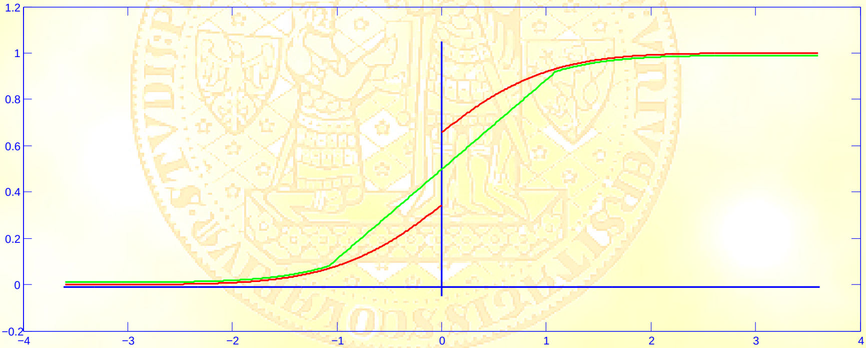
D. f. with a jump in the origine .



We are going to consider a sequence of d. f.'s as given on the next slides.

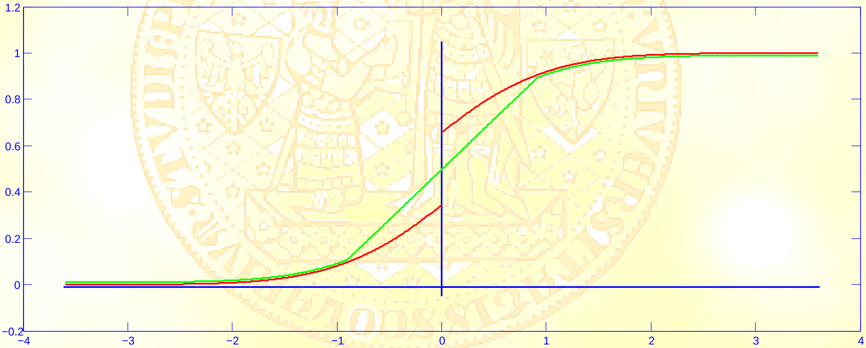
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 2.2 .



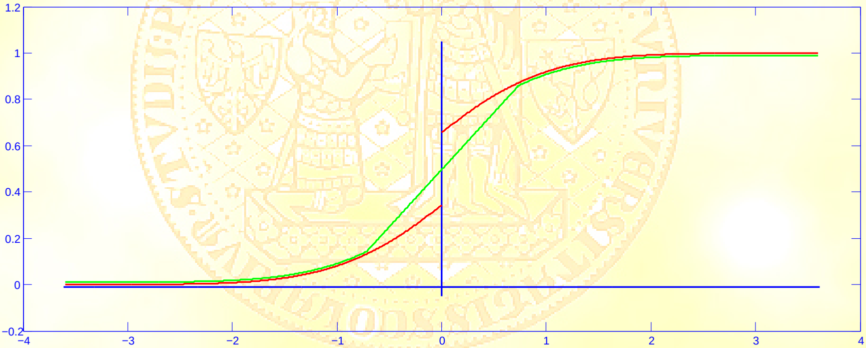
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 1.8 .



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

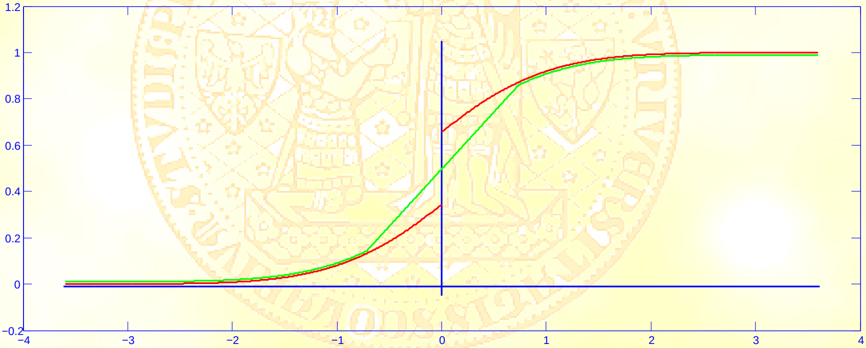
Width of linear part 1.5.



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

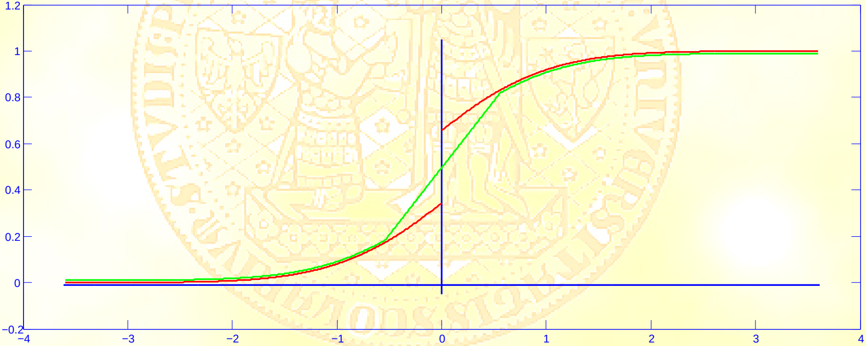
Width of linear part 1.5.

Could You say what is the K-S distance
between green and red d. f.?



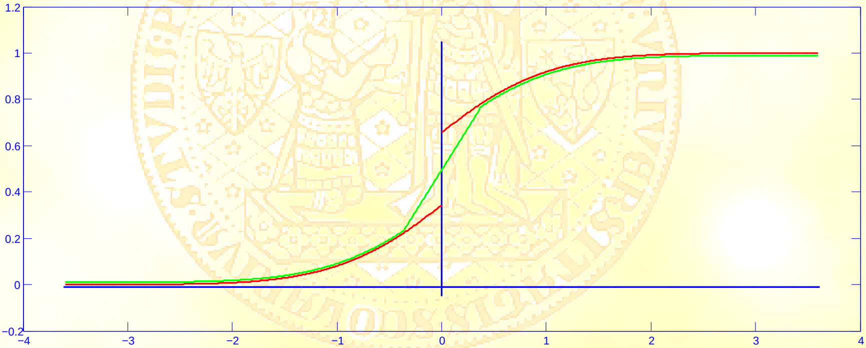
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 1.1.



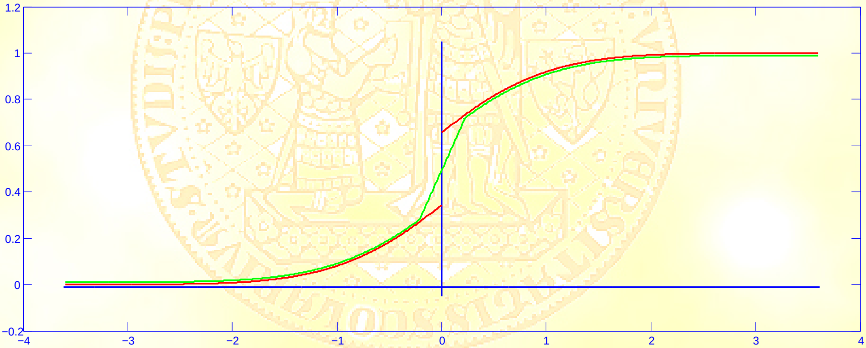
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.8.



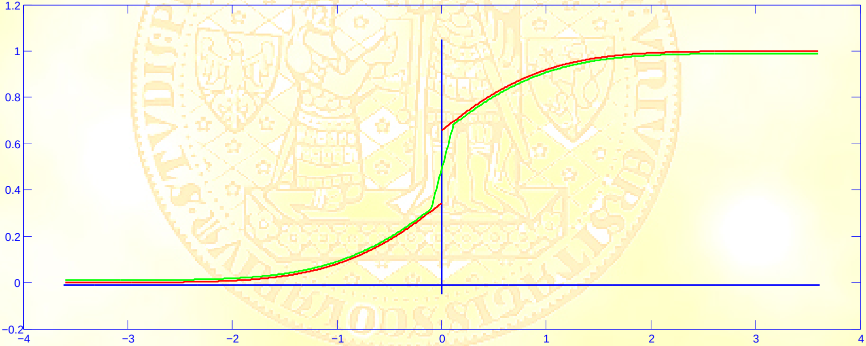
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.4.



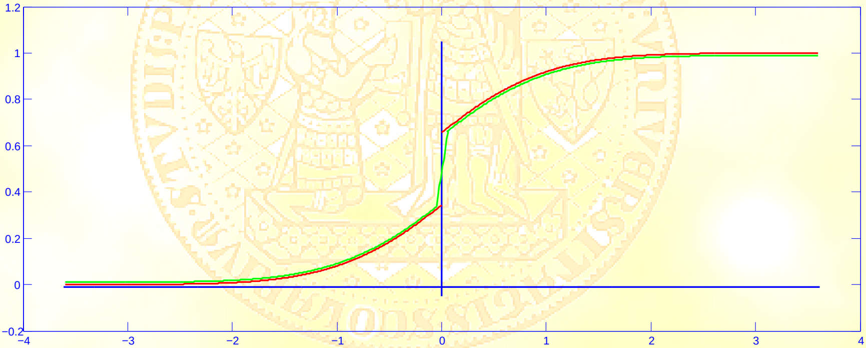
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.3.



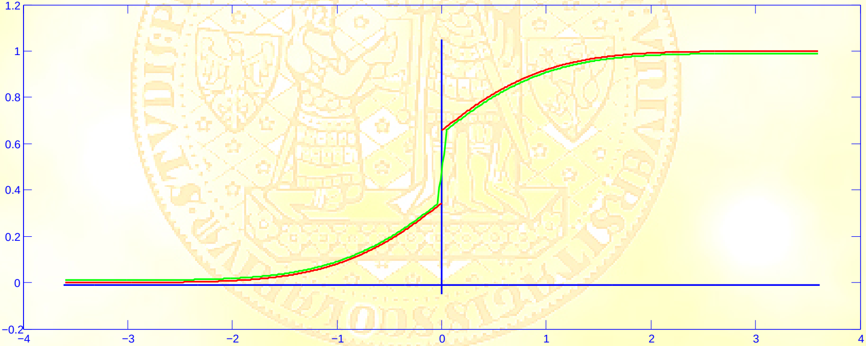
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.2.



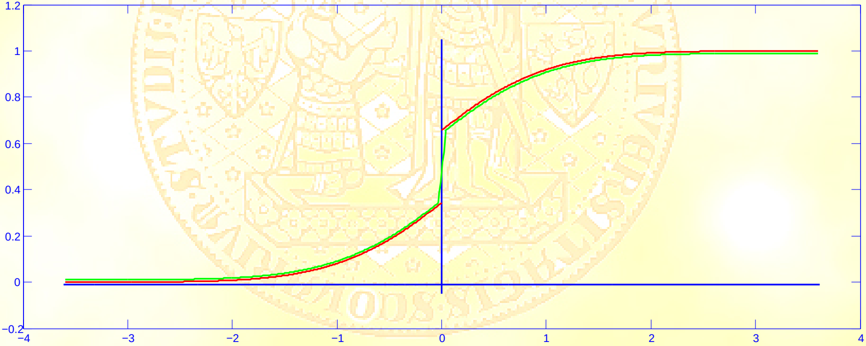
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.1.



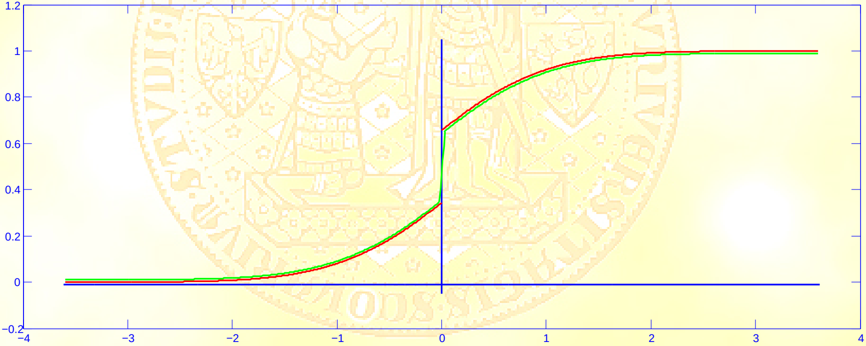
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.08.



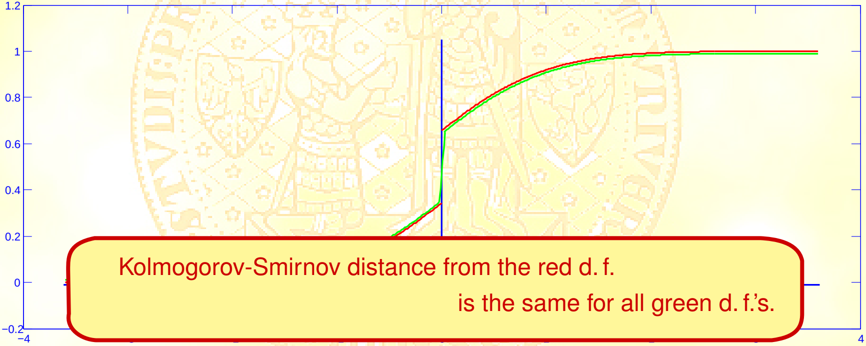
Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.06.



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

Width of linear part 0.06.



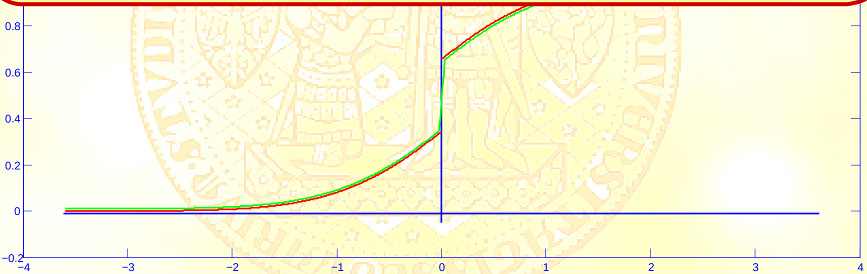
Kolmogorov-Smirnov distance from the red d. f.

is the same for all green d. f.'s.

Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

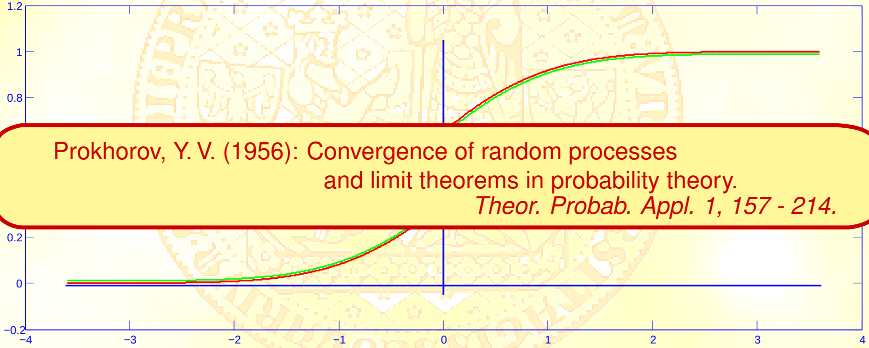
Width of linear part 0.06.

Kolmogorov-Smirnov distance need not be appropriate
for measuring distance between d.f.'s in all situations.



Intuitive convergence of d. f.'s - but the Kolmogorov-Smirnov distance

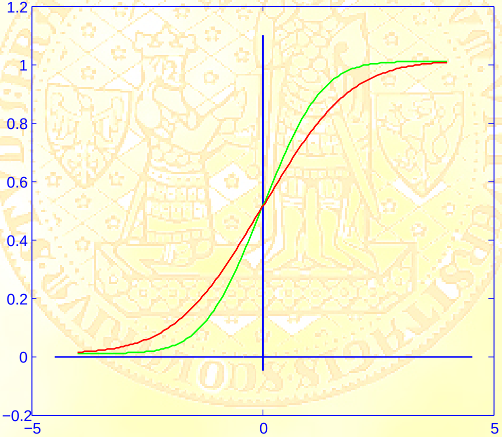
Width of linear part 0.06.



Prokhorov, Y. V. (1956): Convergence of random processes and limit theorems in probability theory. *Theor. Probab. Appl. 1, 157 - 214.*

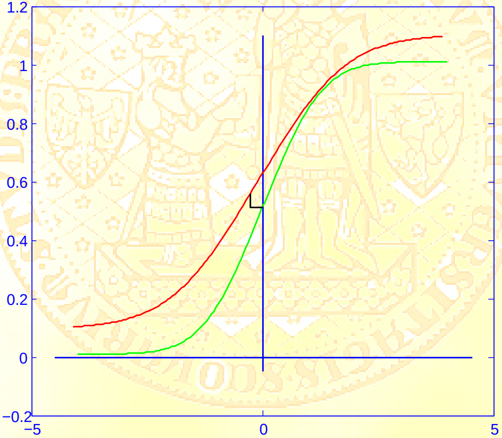
Explaining Prohorov distance (notice different transcription)

Coinsider two d. f.'s



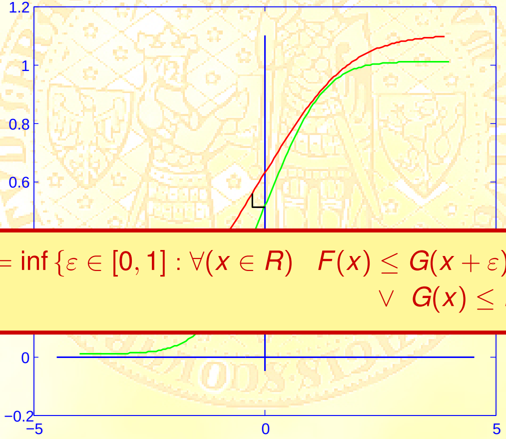
Explaining Prokhorov distance

Prokhorov metric



Explaining Prokhorov distance

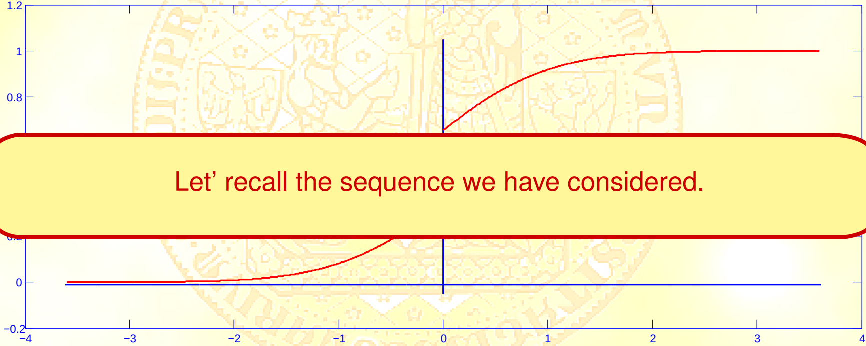
Prokhorov metric



$$\pi(F,G) = \inf\{\varepsilon \in [0,1] : \forall (x \in R) \quad F(x) \le G(x+\varepsilon) + \varepsilon \ \vee \\ \vee \ G(x) \le F(x+\varepsilon) + \varepsilon\}$$

Comparing Kolmogorov-Smirnov and Prokhorov metrics

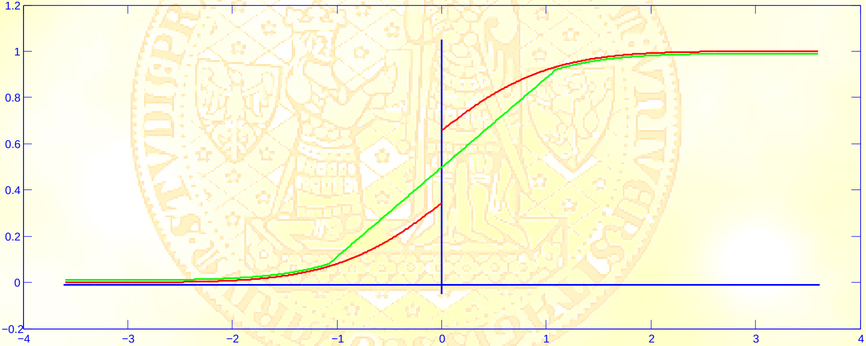
D. f. with a jump in the origine .



Let' recall the sequence we have considered.

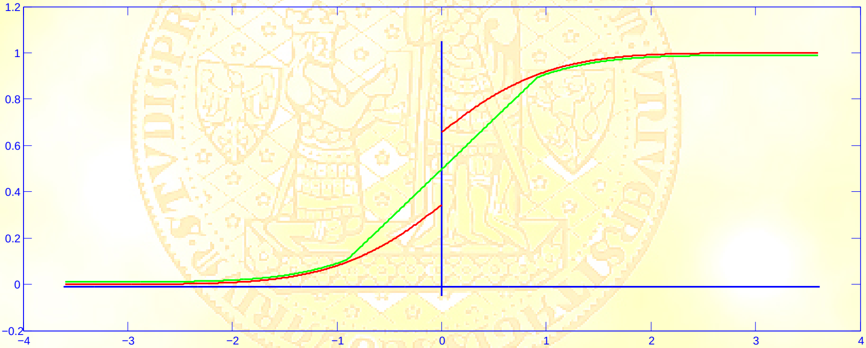
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 2.2 .



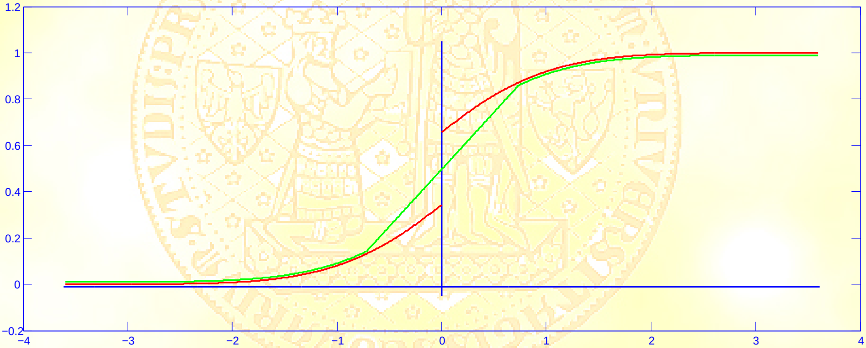
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 1.8 .



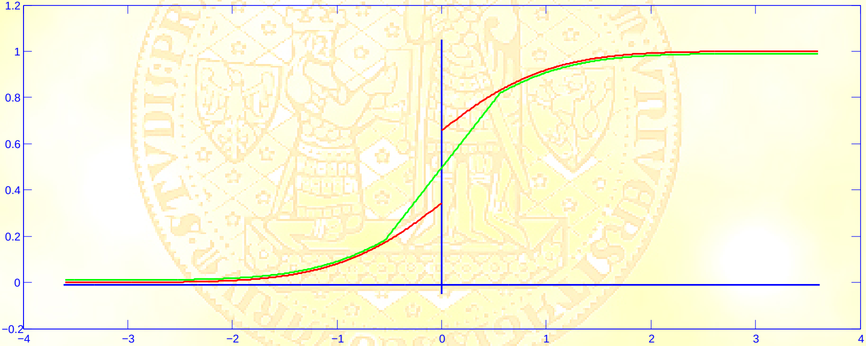
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 1.5.



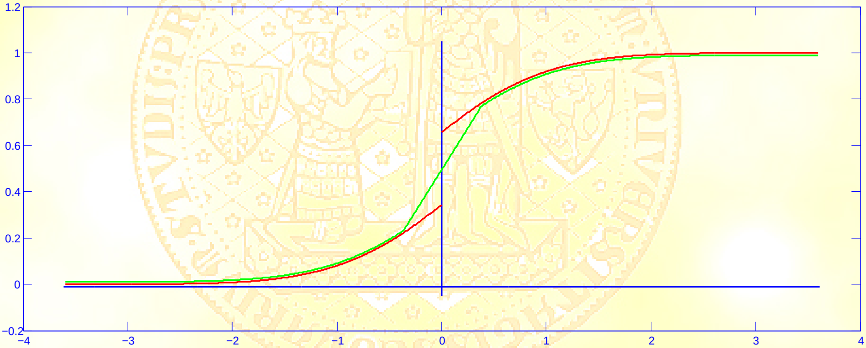
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 1.1.



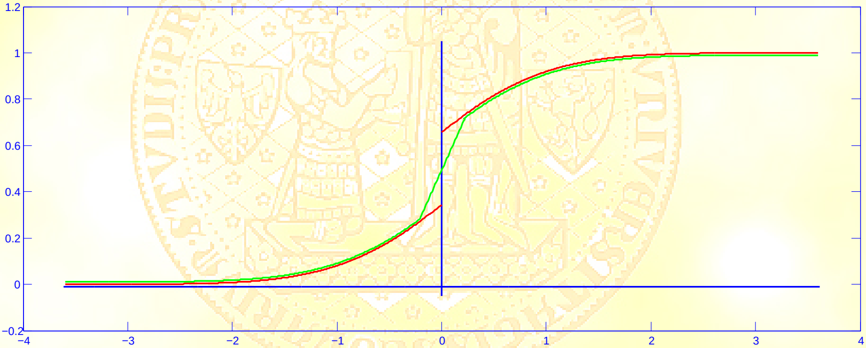
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.8.



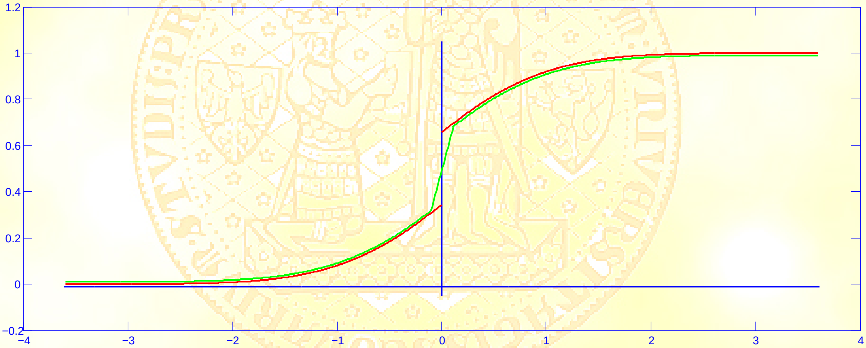
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.4.



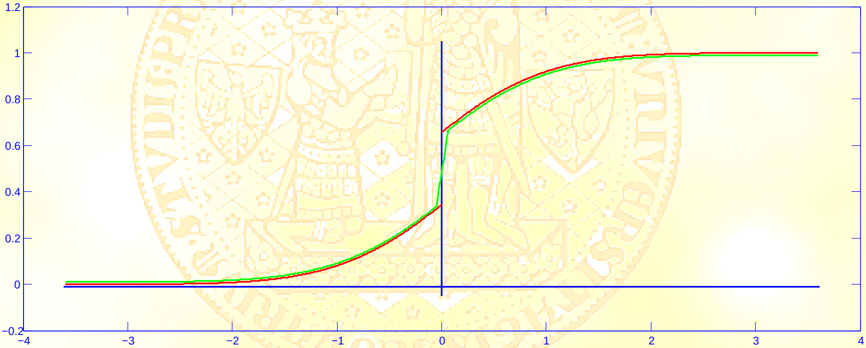
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.3.



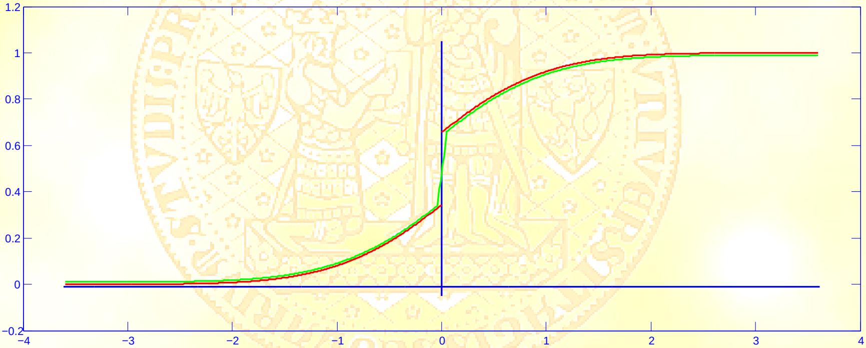
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.2.



Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.1.

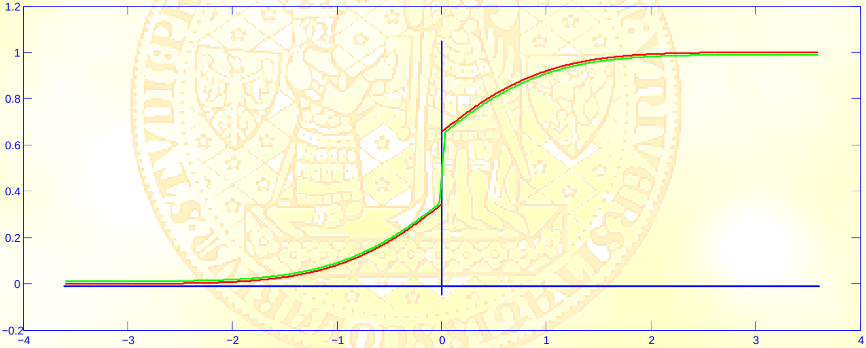


Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.08.

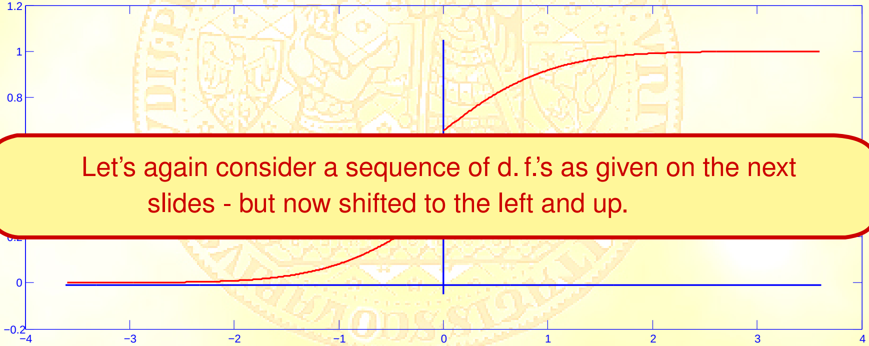
Comparing Kolmogorov-Smirnov and Prokhorov metrics

Width of linear part 0.06.



The convergence in Prokhorov metrics corresponds with intuitive idea of it.

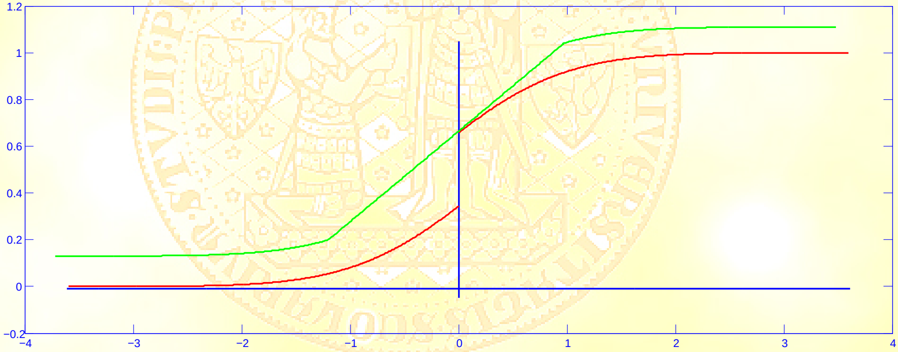
D. f. with a jump in the origine .



Let's again consider a sequence of d. f.'s as given on the next slides - but now shifted to the left and up.

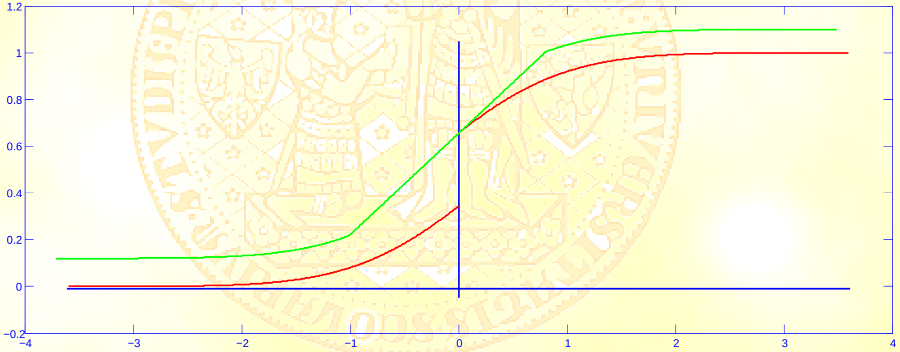
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 2.2 .



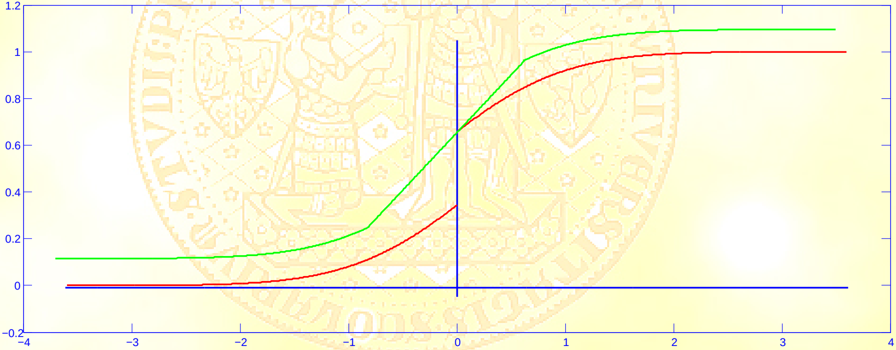
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 1.8 .



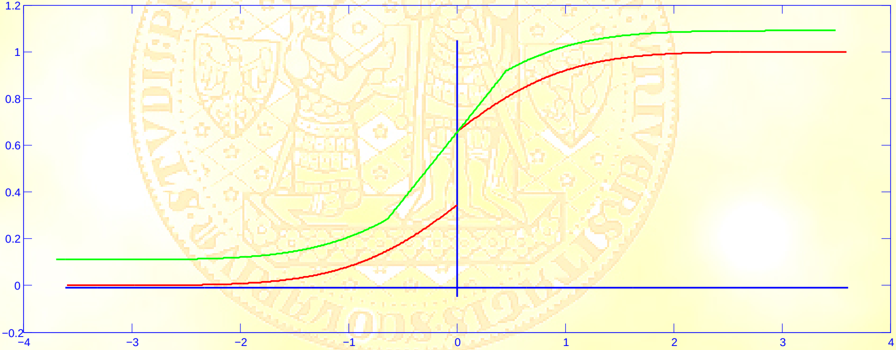
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 1.5.



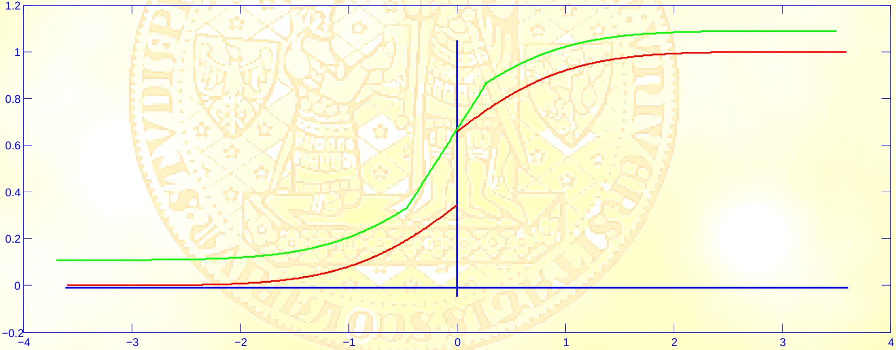
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 1.1.



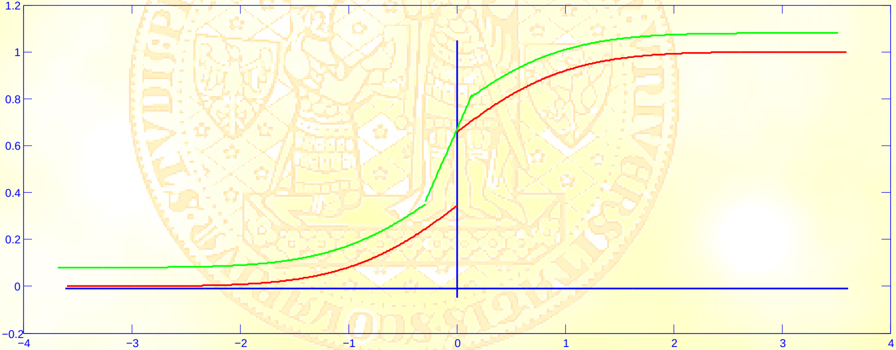
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.8.



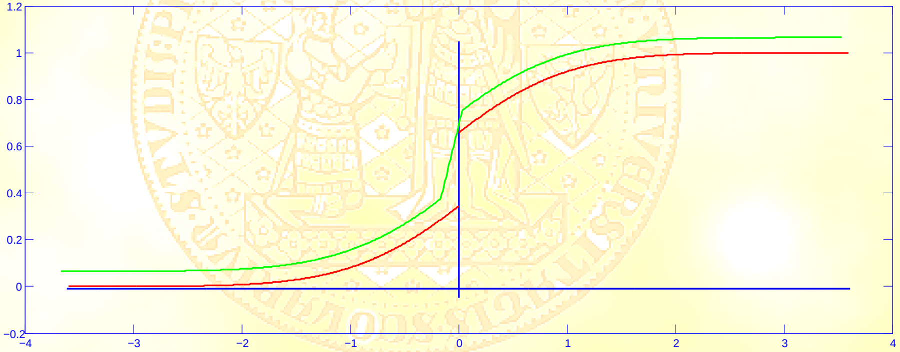
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.4.



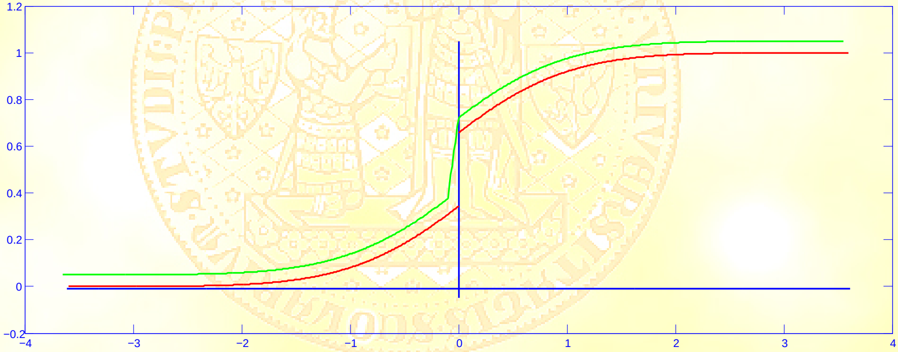
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.3.



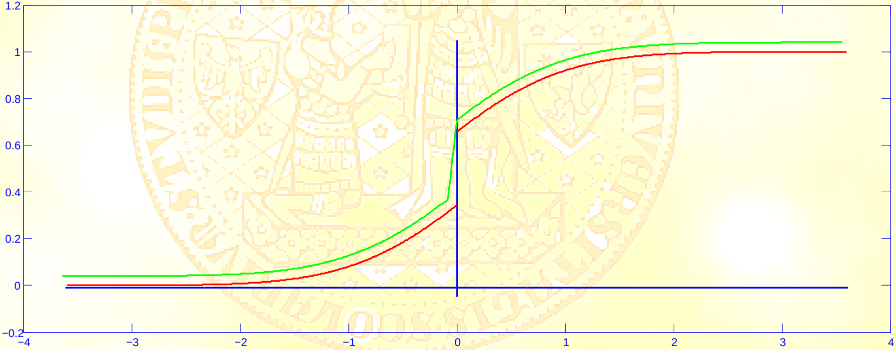
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.2.



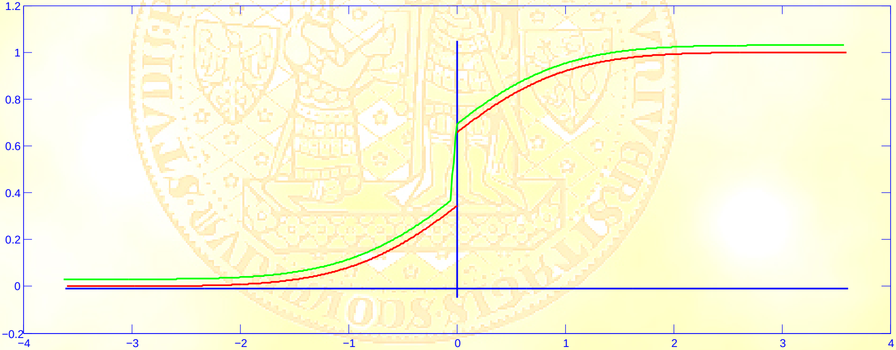
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.1.



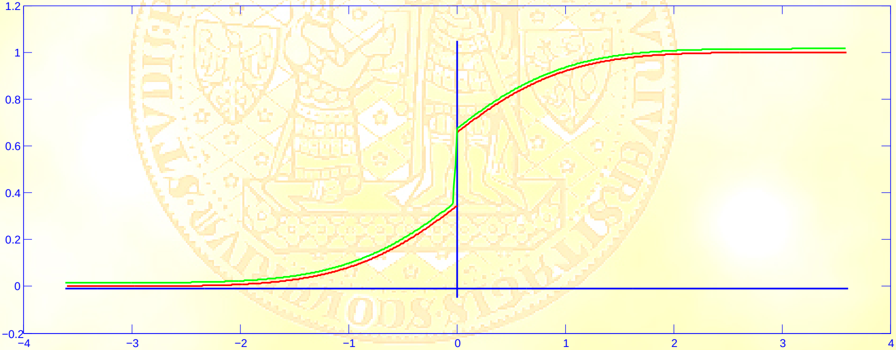
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.08.



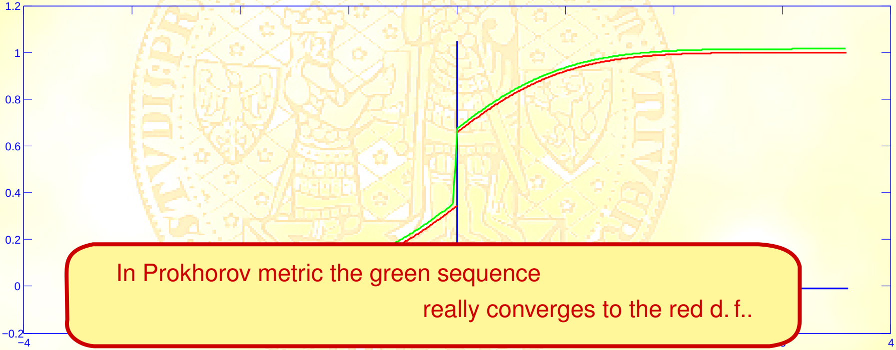
The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.06.



The convergence in Prokhorov metrics corresponds with intuitive idea of it.

Width of linear part 0.06.



In Prokhorov metric the green sequence really converges to the red d. f..

The global empirical characteristc of estimator.

Hampel's approach - characteristics of the functional T at the d. f. F

- *Breakdown point* - "finite" sample version

$$x_1, x_2, ..., x_n \quad \Rightarrow \quad T_n(x_1, x_2, ..., x_n)$$

The global empirical characteristc of estimator.

Hampel's approach - characteristics of the functional T at the d. f. F

- *Breakdown point* - "finite" sample version

$$x_1, x_2, ..., x_n \quad \Rightarrow \quad T_n(x_1, x_2, ..., x_n)$$

- Find maximal m_n such that for any

$$y_1, y_2, ..., y_{m_n} \quad \Rightarrow \quad |T_n(x_1, x_2, ..., x_{n-m_n}, y_1, y_2, ..., y_{m_n})| < \infty$$

$$\left(0 < T_n(x_1, x_2, ..., x_{n-m_n}, y_1, y_2, ..., y_{m_n}) < \infty \text{ - for scale } \right).$$

The global empirical characteristc of estimator.

Hampel's approach - characteristics of the functional T at the d. f. F

- *Breakdown point* - "finite" sample version

$$x_1, x_2, ..., x_n \quad \Rightarrow \quad T_n(x_1, x_2, ..., x_n)$$

- Find maximal m_n such that for any

$$y_1, y_2, ..., y_{m_n} \quad \Rightarrow \quad |T_n(x_1, x_2, ..., x_{n-m_n}, y_1, y_2, ..., y_{m_n})| < \infty$$

($0 < T_n(x_1, x_2, ..., x_{n-m_n}, y_1, y_2, ..., y_{m_n}) < \infty$ - for scale).

- Put

$$\varepsilon^* = \lim_{n\to\infty} \frac{m_n}{n}$$

(we'll return to it later on,

now let's return to the exact definition of the breakdown point).

Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

- An overall characteristic of the functional (the estimator) is

$$\varepsilon^* = \sup \Big\{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, \quad K_\varepsilon \quad \textit{compact} \\ \pi(F,G) < \varepsilon \;\Rightarrow\; G(\{T_n \in K_\varepsilon\}) \underset{n\to\infty}{\longrightarrow} 1 \Big\}$$

where $\pi(F,G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$ and T_n is an empirical counterpart to the functional T.

Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

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where $\pi(F,G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$
and T_n is an empirical counterpart to the functional T.

- ε^* is called *breakdown point*.

Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

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$$\varepsilon^* = \sup\left\{\varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, \quad K_\varepsilon \quad \textit{compact} \right.$$

$$\left. \pi(F, G) < \varepsilon \;\Rightarrow\; G(\{T_n \in K_\varepsilon\}) \underset{n\to\infty}{\longrightarrow} 1 \right\}$$

where $\pi(F, G)$ is the *Prokhorov metric* of $F(x)$ and $G(x)$
and T_n is an empirical counterpart to the functional T.

- ε^* is called *breakdown point*.

Let's rewrite the mathematical part of definition on the next slide.

Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup\Big\{\varepsilon \le 1 : \exists K_\varepsilon \subset \Theta,\ \ K_\varepsilon \ \ \textit{compact}$$
$$\pi(F,G) < \varepsilon \ \Rightarrow \ G\left(\{T_n \in K_\varepsilon\}\right) \ \underset{n\to\infty}{\longrightarrow} \ 1\Big\}$$

Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup \Big\{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, \quad K_\varepsilon \quad \textit{compact} \\ \pi(F,G) < \varepsilon \;\Rightarrow\; G(\{T_n \in K_\varepsilon\}) \underset{n\to\infty}{\longrightarrow} 1 \Big\}$$

1. Let's assume that $T \in R$, so it is connected with a parameter of F.

Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup \Big\{ \varepsilon \leq 1 : \exists K_\varepsilon \subset \Theta, \quad K_\varepsilon \ \ \textit{compact}$$
$$\pi(F,G) < \varepsilon \ \Rightarrow \ G(\{T_n \in K_\varepsilon\}) \underset{n\to\infty}{\longrightarrow} 1 \Big\}$$

1. Let's assume that $T \in R$, so it is connected with a parameter of F.
2. K_ε could be sufficiently wide interval - for this moment.

Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup\left\{\varepsilon \le 1 : \exists K_\varepsilon \subset \Theta,\ \ K_\varepsilon \ \ \textit{compact} \right.$$
$$\left. \pi(F,G) < \varepsilon \ \Rightarrow \ G(\{T_n \in K_\varepsilon\}) \ \underset{n\to\infty}{\longrightarrow} \ 1\right\}$$

1. Let's assume that $T \in R$, so it is connected with a parameter of F.
2. K_ε could be sufficiently wide interval - for this moment.
3. The definition says that T_n is so "good" for estimating "F" that whenever G is sufficiently close to F, T_n converges in probability with respect to G to something finite.

Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup\Big\{\varepsilon \le 1 : \exists K_\varepsilon \subset \Theta,\ \ K_\varepsilon \ \ \textit{compact}$$
$$\pi(F,G) < \varepsilon \ \Rightarrow\ G(\{T_n \in K_\varepsilon\}) \underset{n\to\infty}{\longrightarrow} 1\Big\}$$

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Let's read the definiton of breakdown point

Hampel's approach - characteristics of the functional T at the d. f. F

$$\varepsilon^* = \sup\Big\{\varepsilon \le 1 : \exists K_\varepsilon \subset \Theta,\ \ K_\varepsilon \ \ \textit{compact} \qquad \pi(F,G) < \varepsilon \ \Rightarrow \ G(\{T_n \in K_\varepsilon\}) \ \underset{n\to\infty}{\longrightarrow} \ 1\Big\}$$

1. Let's assume that $T \in R$, so it is connected with a parameter of F.
2. K_ε could be sufficiently wide interval - for this moment.
3. The definition says that T_n is so "good" for estimating "F" that whenever G is sufficiently close to F, T_n converges in probability with respect to G to something finite.

That's all.

Content

Specifying the “robustness” characteristics for the location parameter

At the beginning of the third lecture
we have computed influence function for $T(\Phi) = \boldsymbol{E}_\Phi Z$:

Recalling definition of influence function is :

Fix a functional $T : \mathcal{H} \rightarrow R$ .....

$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\Big((1-\delta)F(.) + \delta \cdot \Delta_x\Big) - T\Big(F(.)\Big)}{\delta}.$$

1. Fix $T(\Phi) = \boldsymbol{E}_\Phi(Z) = \int Z d\Phi = \int z \cdot \phi(z) dz$.
2. $T\Big(\Phi(.)\Big) = \frac{1}{\sqrt{2\pi}} \int z \cdot \exp\left\{-\frac{z^2}{2}\right\} dz = 0$
3. $T\Big((1-\delta)\Phi(.) + \delta \cdot \Delta_x\Big)$
   $= \int z \left\{(1-\delta)\frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{z^2}{2}\right\} + \delta \cdot \Delta_x\right\} dz = (1-\delta) \cdot 0 + \delta \cdot x.$
4. Finally, $IF(x, T, \Phi) = \lim_{\delta \to 0} \frac{\delta \cdot x}{\delta} = x$.

Specifying the “robustness” characteristics for the location parameter

We easy verify that the same computation can be done whenever r. v. Z has finite mean value $T(F) = \boldsymbol{E}_F Z = \mu \in R$:

Fix a functional $T : \mathcal{H} \to R$ ...

$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

Specifying the “robustness” characteristics for the location parameter

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$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

1. Fix $T(F) = \boldsymbol{E}_F(Z) = \int Z \mathrm{d}\Phi = \int z \cdot f(z) \mathrm{d}z$.

Specifying the “robustness” characteristics for the location parameter

We easy verify that the same computation can be done
whenever r. v. Z has finite mean value $T(F) = \boldsymbol{E}_F Z = \mu \in R$:

Fix a functional $T : \mathcal{H} \rightarrow R$ ...

$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

1. Fix $T(F) = \boldsymbol{E}_F(Z) = \int Z \mathrm{d}\Phi = \int z \cdot f(z) \mathrm{d}z$.
2. $T\left(F(.)\right) = \int z \cdot f(z) \mathrm{d}z = \mu$

Specifying the “robustness” characteristics for the location parameter

We easy verify that the same computation can be done
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$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

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2. $T\left(F(.)\right) = \int z \cdot f(z) \mathrm{d}z = \mu$
3. $T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right)$
$= \int z \left\{(1-\delta) f(x) + \delta \cdot \Delta_x\right\} \mathrm{d}z = (1-\delta) \cdot \mu + \delta \cdot x.$

Specifying the “robustness” characteristics for the location parameter

We easy verify that the same computation can be done whenever r. v. Z has finite mean value $T(F) = \boldsymbol{E}_F Z = \mu \in \boldsymbol{R}$:

Fix a functional $T : \mathcal{H} \rightarrow R$ ...

$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

1. Fix $T(F) = \boldsymbol{E}_F(Z) = \int Z \mathrm{d}\Phi = \int z \cdot f(z) \mathrm{d}z$.
2. $T\left(F(.)\right) = \int z \cdot f(z) \mathrm{d}z = \mu$
3. $T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right)$
$= \int z \left\{(1-\delta)f(x) + \delta \cdot \Delta_x\right\} \mathrm{d}z = (1-\delta) \cdot \mu + \delta \cdot x.$
4. Finally, $IF(x, T, F) = \lim_{\delta \to 0} \frac{\delta \cdot (-\mu + x)}{\delta} = -\mu + x.$

Specifying the “robustness” characteristics for the location parameter

Hence the “robustness” characteristics of $T(F) = \boldsymbol{E}_F(X)$ are:

1. The gross error sensitivity $\gamma^* = \sup_{x \in R} |IF(x, T, F)| = \infty$.

Specifying the “robustness” characteristics for the location parameter

Hence the “robustness” characteristics of $T(F) = \boldsymbol{E}_F(X)$ are:

1. The gross error sensitivity $\gamma^* = \sup_{x \in R} |IF(x, T, F)| = \infty$.
2. The local-shift sensitivity $\lambda^* = \sup_{x,y \in R} \left| \frac{IF(x,T,F) - IF(y,T,F)}{x - y} \right| = 1$.

Specifying the “robustness” characteristics for the location parameter

Hence the “robustness” characteristics of $T(F) = \boldsymbol{E}_F(X)$ are:

1. The gross error sensitivity $\gamma^* = \sup_{x \in R} |IF(x, T, F)| = \infty$.
2. The local-shift sensitivity $\lambda^* = \sup_{x,y \in R} \left| \frac{IF(x,T,F) - IF(y,T,F)}{x-y} \right| = 1$.
3. The rejection point $\rho^* = \inf \left\{ r \in R^+ \ : \ IF(x, T, F) = 0, \ \ |x| > r \right\} = \infty$.

Specifying the “robustness” characteristics for the location parameter

Hence the “robustness” characteristics of $T(F) = \boldsymbol{E}_F(X)$ are:

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3. The rejection point $\rho^* = \inf \left\{ r \in R^+ \ : \ IF(x, T, F) = 0, \ \ |x| > r \right\} = \infty$.
4. The breakdown point $\varepsilon^* = 0$

(the last characteristic is “derived heuristically”
from the finite version of breakdown point).

Specifying the “robustness” characteristics for the scale parameter

At the beginning of the third lecture
we have also computed influence function for $T(\Phi) = \boldsymbol{E}_\Phi X^2$:

The Recalling again definition of influence function is :
Fix a functional $T : \mathcal{H} \rightarrow R$ .....

$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\Big((1-\delta)F(.) + \delta \cdot \Delta_x\Big) - T\Big(F(.)\Big)}{\delta}.$$

1. Fix $T(\Phi) = \boldsymbol{E}_\Phi(X^2) = \int X^2 \mathrm{d}\Phi = \int z^2 \cdot \phi(z)\mathrm{d}z$.
2. $T\Big(\Phi(.)\Big) = \frac{1}{\sqrt{2\pi}} \int z^2 \cdot \exp\left\{-\frac{z^2}{2}\right\} \mathrm{d}z = 1$
3. $T\Big((1-\delta)\Phi(.) + \delta \cdot \Delta_x\Big)$
$= \int z^2 \left\{(1-\delta)\frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{z^2}{2}\right\} + \delta \cdot \Delta_x\right\} \mathrm{d}z = (1-\delta) \cdot 1 + \delta \cdot x^2.$
4. Finally, $IF(x, T, \Phi) = \lim_{\delta \to 0} \frac{(1-\delta) \cdot 1 + \delta \cdot x^2 - 1}{\delta} = -1 + x^2.$

Specifying the “robustness” characteristics for the scale parameter

We easy verify that the same computation can be done whenever r. v. Z has finite variance $T(F) = \boldsymbol{E}_F\,(Z - \boldsymbol{E}Z)^2 = \sigma^2 \in R^+$:

Fix a functional $T\ :\ \mathcal{H}\ \ \to\ \ R$ ...

$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

Specifying the “robustness” characteristics for the scale parameter

We easy verify that the same computation can be done whenever r. v. Z has finite variance $T(F) = \boldsymbol{E}_F\left(Z - \boldsymbol{E}Z\right)^2 = \sigma^2 \in R^+$:

Fix a functional $T : \mathcal{H} \rightarrow R$ ...

$$IF(x, T, F) = \lim_{\delta \to 0} \frac{T\left((1-\delta)F(.) + \delta \cdot \Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

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1. Fix $T(F) = \boldsymbol{E}_F(Z - \boldsymbol{E}Z)^2 = \int (Z - \boldsymbol{E}Z)^2 \, \mathrm{d}F = \int (z - \boldsymbol{E}Z)^2 \cdot f(z)\mathrm{d}z.$

Specifying the “robustness” characteristics for the scale parameter

We easy verify that the same computation can be done whenever r. v. Z has finite variance $T(F) = \boldsymbol{E}_F\,(Z - \boldsymbol{E}Z)^2 = \sigma^2 \in R^+$:

Fix a functional $T : \mathcal{H} \rightarrow R$ ...

$$IF(x,T,F) = \lim_{\delta\to 0} \frac{T\left((1-\delta)F(.) + \delta\cdot\Delta_x\right) - T\left(F(.)\right)}{\delta}.$$

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$= \int (z - \boldsymbol{E}Z)^2 \left\{(1-\delta)f(z) + \delta \cdot \Delta_x\right\} \mathrm{d}z = (1-\delta) \cdot \sigma^2 + \delta \cdot (x - \boldsymbol{E}Z)^2.$

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4. Finally, $IF(x, T, F) = \lim_{\delta \to 0} \frac{\delta \cdot \left(-\sigma^2 + (x - \boldsymbol{E}Z)^2\right)}{\delta} = -\sigma^2 + (x - \boldsymbol{E}Z)^2$.

Specifying the “robustness” characteristics for the scale parameter

Hence the “robustness” characteristics of $T(F) = \boldsymbol{E}_F(Z - \boldsymbol{E}Z)^2$ are:

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4. The breakdown point $\varepsilon^* = 0$

(the last characteristic is again "derived heuristically"
from the finite version of breakdown point).

THANKS FOR ATTENTION