INSTITUTE OF ECONOMIC STUDIES, FACULTY OF SOCIAL SCIENCES
CHARLES UNIVERSITY IN PRAGUE (*established 1348*)

ROBUST STATISTICS AND ECONOMETRICS

INSTITUTE OF ECONOMIC STUDIES
FACULTY OF SOCIAL SCIENCES
CHARLES UNIVERSITY IN PRAGUE

JAN ÁMOS VÍŠEK

Week 7

Content of lecture

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   - From basic econometrics
   - Repetition from the previous lecture

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   - Deleting some observations
   - Frustrations and rebirths
   - Depressing the influence of some observations

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One really devilish snag - ceteris paribus

Holding other factors fixed - Ceteris paribus:

Jeffrey Wooldridge discusses an employment of regression model as a tool for emulating the CETERIS PARIBUS, as follows:

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$$y_i = \beta_0^0 + \beta_1^0 \cdot x_{i1} + \beta_2^0 \cdot x_{i2} + ... + \beta_p^0 \cdot x_{ip} + u_i \quad i = 1, 2, ..., n$$

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usually $\Delta x_{i1} = 1$.

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Warning - THIS INFORMATION IS OF LIMITED RELEVANCE AND IT MAY BE EVEN TOTALLY MISLEADING!

Misunderstanding the basic ideas can yield catastrophic conclusions

A regression for a “club of good health”

$$\begin{aligned} \textit{Time Total} &= -3.62 + 1.27 \cdot \textit{Weight} + 0.53 \cdot \textit{Puls} \\ &\quad -0.51 \cdot \textit{Strength} + 3.90 \cdot \textit{Time per quarter of mile} + u_i. \end{aligned}$$

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Then we can (frequently?) meet with a conclusion of type:

As the estimate of regression coefficient for Strength is negative,
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or (even)

Although the coefficient of determination is small,
the polarities of the estimated coefficient corresponds to our ideas.

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This first assertion from the previous slide can be true,

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WHY ? ?

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It will be our second r. v. .

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Put $X_1 = -X_2^k$. This will be our first r. v..
Evidently - due to symmetry - $\mathbb{E}[X_1 \cdot X_2] = 0$. Then

$$cov(X1, X2) = \mathbb{E}(X_1 \dot{X}_2) - \mathbb{E}X_1 \cdot \mathbb{E}X_2 = 0$$

as $\mathbb{E}X_2 = 0$, i. e. X_1 and X_2 are not correlated.

Naive interpretation can be misleading

Consider regression model

$$Y = 1 + X_1 + X_2 + \varepsilon$$

for $k = 20$ (say), i. e. $X_1 = -X_2^{40}$. Then:

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for $k = 20$ (say), i. e. $X_1 = -X_2^{40}$. Then:

If $X_2 >> 1$, an increase of it X_2
yields a decrease of Y (despite of positive sign of X_2).

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The First Estimator with 50% Breakdown Point

Repeated medians

Siegel, A. F. (1982): Robust regression using repeated medians.
Biometrica, 69, 242 - 244.

$$\hat{\beta}^{(j)} = \underset{i_1=1,2,\ldots,n}{\text{med}} \left(\ldots \left(\underset{i_{p-1}=1,2,\ldots,n}{\text{med}} \left(\underset{i_p=1,2,\ldots,n}{\text{med}} \left(\hat{\beta}_j \left(i_1, i_2, \ldots, i_p \right) \right) \right) \right) \right)$$

(requiring approx. n^p evaluations of model and orderings of estimates of coefficients - nearly surely never implemented)

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Content

The first solution broke the mystery and implied a chain of others

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Journal of Amer. Statist. Association 79, pp. 871-880.

the Least Median of Squares

$$\hat{\beta}^{(LMS,n,h)} = \underset{\beta \in R^p}{\arg\min}\; r^2_{(h)}(\beta) \qquad \frac{n}{2} < h \leq n,$$

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2. *scale- and regression equivariant*

(without any studentization of residuals).

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(without any studentization of residuals).

Main disadvantage $\sqrt[3]{n}\left(\hat{\beta}^{(LMS,n,h)} - \beta^0\right) = \mathcal{O}_p(1)$ (other will be discussed later).

Let's remove the deficiency of LMS

Hampel, F. R., E. M. Ronchetti, P. J. Rousseeuw, W. A. Stahel (1986):
Robust Statistics – The Approach Based on Influence Functions.
New York: J.Wiley *&* Son.

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$$\hat{\beta}^{(LTS,n,h)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{h} r_{(i)}^2(\beta) \qquad \frac{n}{2} < h \leq n,$$

(Notice the order of words, remember there is also the Trimmed Least Squares.)

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(Please, remember the optimal value of h).

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3. $\sqrt{n}\left(\hat{\beta}^{(LTS,n,h)} - \beta^0\right) = \mathcal{O}_p(1)$

Let's increase the efficiency with simultaneously keeping high breakdown point

Rousseeuw, P. J., V. Yohai (1984):
Robust regressiom by means of S-estimators.
Lecture Notes in Statistics No. 26 Springer Verlag, New York, 256-272.

S-estimators

$$\hat{\beta}^{(S,n,\rho)} = \underset{\beta \in R^p}{\arg\min} \left\{ \sigma \in R^+ : \sum_{i=1}^{n} \rho\left(\frac{r_i(\beta)}{\sigma}\right) = b \right\}$$

where $b = \boldsymbol{E}\rho\left(\frac{e_i}{\sigma_0}\right)$ with $\sigma_0^2 = \boldsymbol{E}e_1^2$ (for ρ see next slide).

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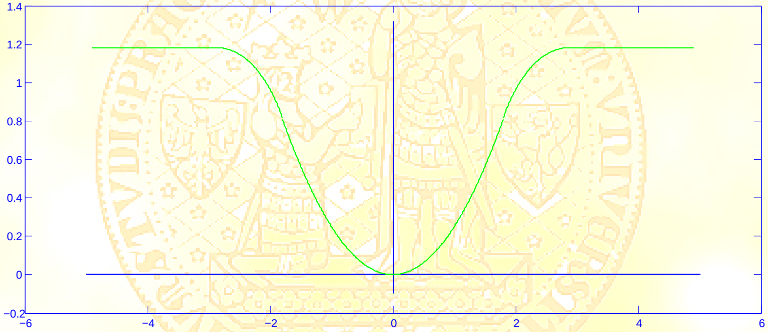
1. the breakdown point equal to 50%,
2. *scale- and regression equivariant,*
3. $\sqrt{n}\left(\hat{\beta}^{(S,n,\rho)} - \beta^0\right) = \mathcal{O}_p(1)$,
4. much better utilization of information from data,
   i. e. higher efficiency than LTS.

Peter Rousseeuw's objective function ρ

$$\rho : (-\infty, \infty) \to (0, \infty), \ \rho(x) = \rho(-x), \ \rho(0) = 0, \rho(x) = c \ \text{ for } \ x > d.$$



Finally - the victory

We have evidently reached something which is

“BOMB und IDIOTEN SICHER”.

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But maybe that it was only an illusion !!

Finally - the victory

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But maybe that it was only an illusion !!
It appeared that there is an “inborn” disadvantage
which is common to all robust estimator with high breakdown point.

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A shock and frustration - Engine Knock Data

Hettmansperger, T. P., S. J. Sheather (1992):
A Cautionary Note on the Method of Least Median Squares.
The American Statistician 46, 79–83.

Engine Knock Data ($n = 16, p = 4, h = 11$)

c	x_1	x_2	x_3	x_4	y
1	13.3	13.9	31	697	84.4
2	13.3	14.1	30	697	84.1
3	13.4	15.2	32	700	88.4
4	12.7	13.8	31	669	84.2
⋮	⋮	⋮	⋮	⋮	⋮
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x_1 is spark timing x_2 air/fuel ratio
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In fact they worked with two data sets.

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Let's call these data "Correct".

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Let's call these data "Damaged".

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15	12.9	15.1	36	721	93.3
16	12.7	15.9	37	696	93.1

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature
y engine knock number

A shock and frustration

Hettmansperger, T. P., S. J. Sheather (1992):
A Cautionary Note on the Method of Least Median Squares.
The American Statistician 46, 79–83.

Engine Knock Data $(n = 16, p = 4, h = 11)$

c	x_1	x_2	x_3	x_4	y
1	13.3	13.9	31	697	84.4
					84.1
					88.4
					84.2
⋮	⋮	⋮	⋮	⋮	⋮
14	12.7	16.1	35	649	93.0
15	12.9	15.1	36	721	93.3
16	12.7	15.9	37	696	93.1

Let's verify that
$h = [\frac{n}{2}] + [\frac{p+1}{2}] = 11.$

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature
y engine knock number

A shock and frustration

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The American Statistician 46, 79–83.

Engine Knock Data $(n = 16, p = 4, h = 11)$

c	x_1	x_2	x_3	x_4	y
1	13.3	13.9	31	697	84.4
⋮	⋮	⋮	⋮	⋮	⋮
14	12.7	16.1	35	649	93.0
15	12.9	15.1	36	721	93.3
16	12.7	15.9	37	696	93.1

An example of the real data, indicating a high sensitivity of the estimator with high breakdown point (LMS) to the shift of one observation.

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature
y engine knock number

A shock and frustration

Hettmansperger, T. P., S. J. Sheather (1992):
A Cautionary Note on the Method of Least Median Squares.
The American Statistician 46, 79–83.

Engine Knock Data $(n = 16, p = 4, h = 11)$

c	x_1	x_2	x_3	x_4	y

The values of $\hat{\beta}^{(LMS,n,h)}$ by "elemental" algorithm !
(still included in some packages - see the next slide)

Data	Interc.	SPARK	AIR	INTK	EXHS.
Correct data ($x_{22} = 14.1$)	30.08	0.21	2.90	0.56	-0.01
Damaged data($x_{22} = 15.1$)	-86.50	4.59	1.21	1.47	0.07

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature
y engine knock number

What was the algorithm of computing the estimate?

1. Select randomly an elemental set of p points and fit a regression plane to them.

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2. Compute all squared residuals and find the h-th smallest.

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1. Select randomly an elemental set of p points and fit a regression plane to them.
2. Compute all squared residuals and find the h-th smallest.
3. Repeat it "10 000" times and select that model (among these "10 000") with smallest h-th squared residual.

An improvement of the algorithm - a geometric characterization

Joss, J., A. Marazzi (1990):
Probabilistic algorithms for LMS regression.
Computational Statistics & Data Analysis 9, 123-134.

An improvement of the algorithm - a geometric characterization

Joss, J., A. Marazzi (1990):
Probabilistic algorithms for LMS regression.
Computational Statistics & Data Analysis 9, 123-134.

The geometric characterization
of exact solution of LMS extremal problem:

The exact solution has at least
$p + 1$ residuals of the same (absolute) value.

An improvement of the algorithm - a geometric characterization

1. Select randomly an elemental set of p points and fit a regression plane to them.

An improvement of the algorithm - a geometric characterization

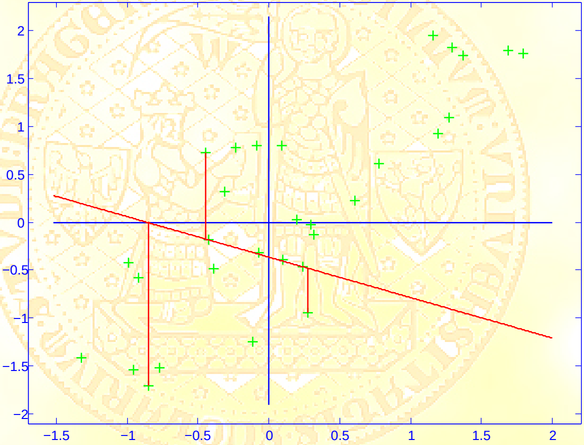
1. Select randomly an elemental set of p points and fit a regression plane to them.
2. Perform (repeatedly) its shift and rotation to decrease the value of the h-th squared residual and to reach the geometric representation.

An improvement of the algorithm - a geometric characterization

1. Select randomly an elemental set of p points
   and fit a regression plane to them.
2. Perform (repeatedly) its shift and rotation
   to decrease the value of the h-th squared residual
   and to reach the geometric representation.
3. Repeat it “10 000” times
   and select that model (among these “10 000”)
   with smallest h-th squared residual.

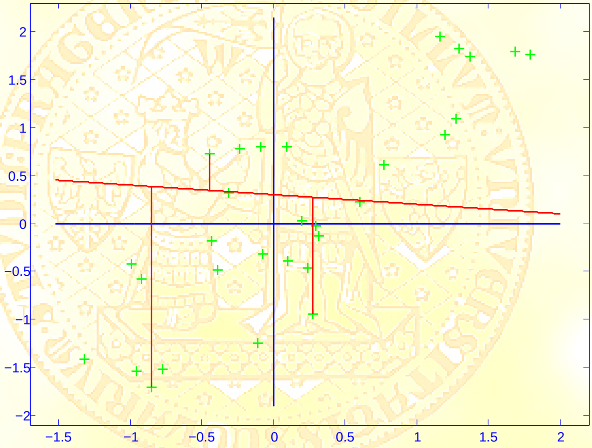
A geometric characterization

Unlucky selection of starting points



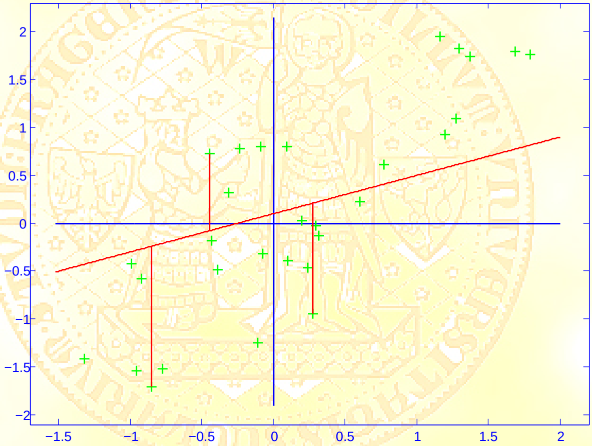
A geometric characterization

Starting shifting and spinning the line



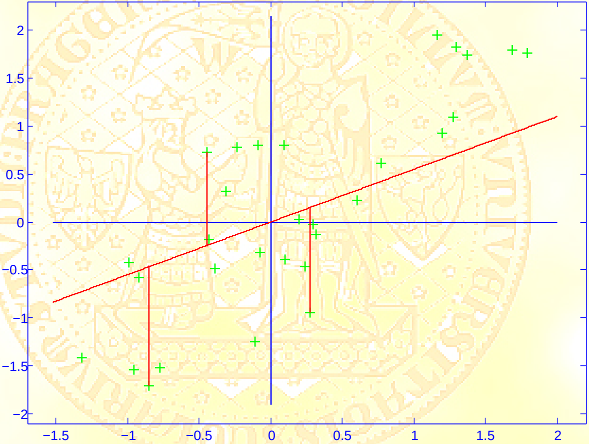
A geometric characterization

Continuing shifting and spinning the line



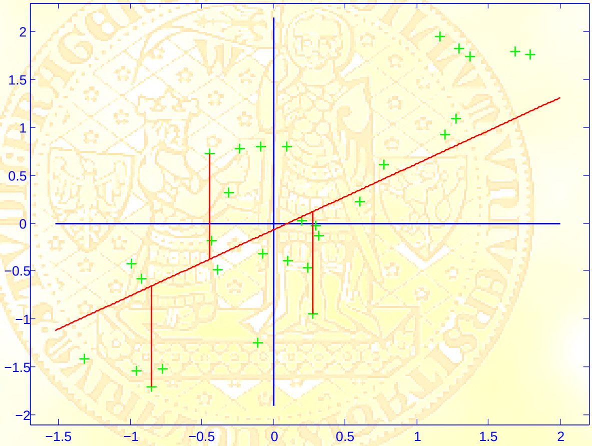
A geometric characterization

Nearly reaching the geometric characterization



A geometric characterization

Reaching the geometric characterization



A substantial improvement of the algorithm

- an employment of simplex method

Boček, P., P. Lachout (1993):
Linear programming approach to LMS-estimation.
Memorial volume of Comput. Statist. & Data Analysis 19(1995), 129 - 134.

A description is a bit complicated - it requires
to be familiar with a dual form of simplex method.

Hettmansperger, T. P., S. J. Sheather (1992):
A Cautionary Note on the Method of Least Median Squares.
The American Statistician 46, 79–83.

Engine Knock Data ($n = 16$, $p = 4$, $h = 11$)

c	x_1	x_2	x_3	x_4	y
15	12.9	15.1	36	721	93.3
16	12.7	15.9	37	696	93.1

The value of $\hat{\beta}^{(LMS,n,h)}$ by Boček-Lachout algorithm.

Data	Interc.	SPARK	AIR	INTK	EXHS.
Correct data ($x_{22} = 14.1$)	30.04	0.14	3.08	0.46	-0.01
Damaged data($x_{22} = 15.1$)	48.38	-0.73	3.36	0.23	-0.01

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature

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The difference between these two models is much lower.

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The difference between these two models is much lower.
So, the effect announced by H-S was a consequence of the bad algorithm.

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature
y engine knock number

Hettmansperger, T. P., S. J. Sheather (1992):
A Cautionary Note on the Method of Least Median Squares.
The American Statistician 46, 79–83.

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The value of $\hat{\beta}^{(LMS,n,h)}$ by Boček-Lachout algorithm.

Data	Interc.	SPARK	AIR	INTK	EXHS.
Correct data ($x_{22} = 14.1$)	30.04	0.14	3.08	0.46	-0.01

BUT THIS CONCLUSION - ALTHOUGH TRUE - WAS MISLEADING.

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature

Hettmansperger, T. P., S. J. Sheather (1992):
A Cautionary Note on the Method of Least Median Squares.
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WHY?

x_1 is spark timing $\quad x_2$ air/fuel ratio
x_3 intake temperature $\quad x_4$ exhaust temperature

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A Cautionary Note on the Method of Least Median Squares.
The American Statistician 46, 79–83.

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c	x_1	x_2	x_3	x_4	y

The value of $\hat{\beta}^{(LMS,n,h)}$ by Boček-Lachout algorithm.

Data	Interc.	SPARK	AIR	INTK	EXHS.
Correct data ($x_{22} = 14.1$)	30.04	0.14	3.08	0.46	-0.01

The correct conclusion is:

THE LARGE DIFFERENCE BETWEEN THE ESTIMATES WAS PARTIALLY DUE TO THE BAD ALGORITHM.

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature
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2	13.3	14.1	30	697	84.1
3	13.4	15.2	32	700	88.4
4	12.7	13.8	31	669	84.2
⋮	⋮	⋮	⋮	⋮	⋮
14	12.7	16.1	35	649	93.0
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Engine Knock Data $(n = 16, p = 4, h = 11)$

Realize that $\binom{16}{11} = 4368$, so that we can compute $\hat{\beta}^{(LTS,16,11)}$ exactly, just computing $\hat{\beta}^{(OLS,11)}$ for all subsamples of size 11 and select the "best" one.

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
⋮	⋮	⋮	⋮	⋮	⋮
14	12.7	16.1	35	649	93.0
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This is the exact value of $\hat{\beta}^{(LTS,n,h)}$!

Data	Interc.	SPARK	AIR	INTK	EXHS.
Correct data ($x_{22} = 14.1$)	35.11	-0.028	2.949	0.477	-0.009
Damaged data ($x_{22} = 15.1$)	-88.7	4.72	1.06	1.57	0.068

x_1 is spark timing $\quad x_2$ air/fuel ratio
x_3 intake temperature $\quad x_4$ exhaust temperature
y engine knock number

Hettmansperger, T. P., S. J. Sheather (1992):
A Cautionary Note on the Method of Least Median Squares.
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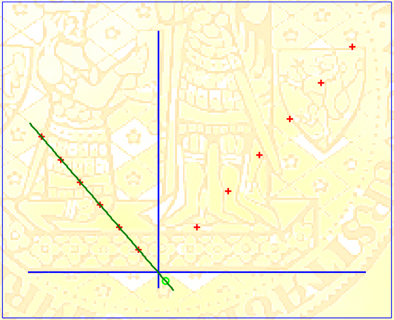
x_1 is spark timing x_2 air/fuel ratio

Víšek, J.Á (1994): A cautionary note on the method
of Least Median of Squares reconsidered.
Transactions of the Twelfth Prague Conference 1994, 254 - 259.

An (academic) explanation by a shift of "inlier"

SENSITIVITY OF ANY HIGH-BREAKDOWN-POINT ESTIMATOR
TO A SMALL CHANGE OF DATA

Model for the majority of data



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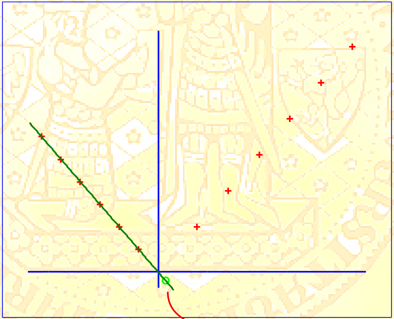
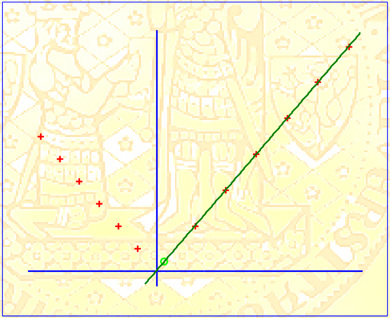


We are going to shift up this point “ o ”.

An (academic) explanation by a shift of “inlier”

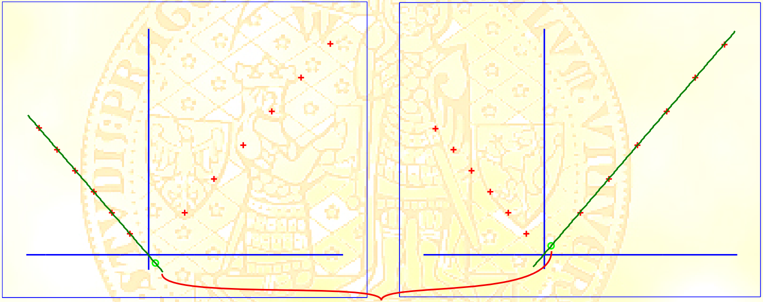
SENSITIVITY OF ANY HIGH-BREAKDOWN-POINT ESTIMATOR
TO A SMALL CHANGE OF DATA

Again model for the majority of data



An (academic) explanation by a shift of "inlier"

In both cases the model is for the majority of data



Notice: *The closer* the point (" ^{o} ") is to the *y*-axe,
the smaller shift causes the "switch" of the model.

Final conclusion - frustration

We have built up the theory on the sand not on a solid rock base.

Final conclusion - frustration

We have built up the theory on the sand not on a solid rock base.

Is it really so ?

ANALYSIS OF THE EXPORT FROM THE CZECH REPUBLIC TO EU IN 1994

Number of industries 91

X_ℓ	-	*export from i-th industry,*
US_ℓ	-	*number of university-passed employees in the i-th industry,*
HS_ℓ	-	*nuber of high school-passed employees in the i-th industry,*
VA_ℓ	-	*value added in the i-th industry,*
K_ℓ	-	*capital in the i-th industry,*
CR_ℓ	-	*percentage of market occupied by 3 largest producers,*
$TFPW_\ell$	-	*by wages normed productvity in the i-th industry,*
Bal_ℓ	-	*Balasa index in the i-th industry„*
DP_ℓ	-	*cost discontinuity in 1993 in the i-th industry*
	-	*etc., about 20 explanatory variables*

Analysis of the export from the Czech Republic to EU in 1994

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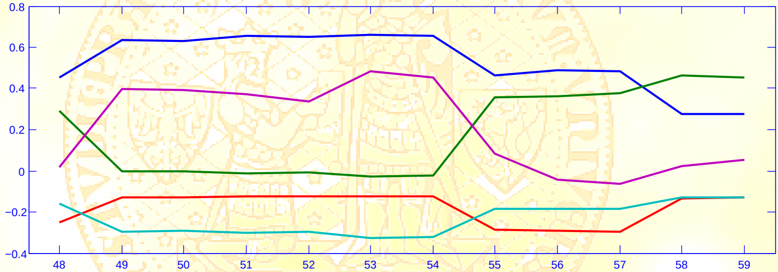
	-	*etc., about 20 explanatory variables*

No reasonable model by OLS - coefficient of determination 0.28

ANALYSIS OF THE EXPORT FROM THE CZECH REPUBLIC TO EU IN 1994 BY MEANS OF THE *least trimmed squares*.



The development of the estimates of regression coefficients. The blue line represents $\hat{\beta}_1^{(LTS,n,h)}$ (down-scaledby $\frac{1}{10}$), the purple is $\hat{\beta}_8^{(LTS,n,h)}$, the green is $\hat{\beta}_3^{(LTS,n,h)}$, the red is $\hat{\beta}_4^{(LTS,n,h)}$ and light blue (the lowest curve) is $\hat{\beta}_6^{(LTS,n,h)}$ (down-scaled again by $\frac{1}{10}$). There is an evidnet break at 54.

ANALYSIS OF THE EXPORT FROM THE CZECH REPUBLIC TO EU IN 1994 BY MEANS OF THE *least trimmed squares*

has found:

MAIN SUBGROUP

with number of industries 54 and model

$$\frac{X_\ell}{S_\ell} = 4.64 - 0.032 \cdot \frac{US_\ell}{VA_\ell} - 0.022 \cdot \frac{HS_\ell}{VA_\ell} - 0.124 \cdot \frac{K_\ell}{VA_\ell} + 1.035 \cdot CR_\ell$$
$$-3.199 \cdot TFPW_\ell + 1.048 \cdot BAL_\ell + 0.452 \cdot DP_\ell + \varepsilon_\ell$$

X_ℓ	-	*export from i-th industry,*
US_ℓ	-	*number of university-passed employees in the i-th industry,*
HS_ℓ	-	*nuber of high school-passed employees in the i-th industry,*
VA_ℓ	-	*value added in the i-th industry,*
K_ℓ	-	*capital in the i-th industry,*
CR_ℓ	-	*percentage of market occupied by 3 largest producers,*
$TFPW_\ell$	-	*by wages normed productvity in the i-th industry,*
Bal_ℓ	-	*Balasa index in the i-th industry„*
DP_ℓ	-	*cost discontinuity in 1993 in the i-th industry*

with coefficient of determination 0.97 and stable submodels

ANALYSIS OF THE EXPORT FROM THE CZECH REPUBLIC TO EU IN 1994 BY MEANS OF THE *least trimmed squares*

has found:

COMPLEMENTARY SUBGROUP

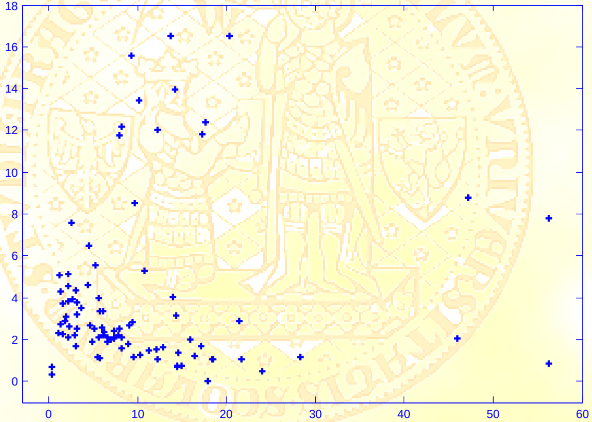
with number of industries 33 and model

$$\frac{X_\ell}{S_\ell} = -0.634 + 0.089 \cdot \frac{US_\ell}{VA_\ell} + 0.235 \cdot \frac{HS_\ell}{VA_\ell} + 0.249 \cdot \frac{K_\ell}{VA_\ell} + 1.174 \cdot CR_\ell$$
$$+0.690 \cdot TFPW_\ell + 2.691 \cdot BAL_\ell - 0.051 \cdot DP_\ell + \varepsilon_\ell$$

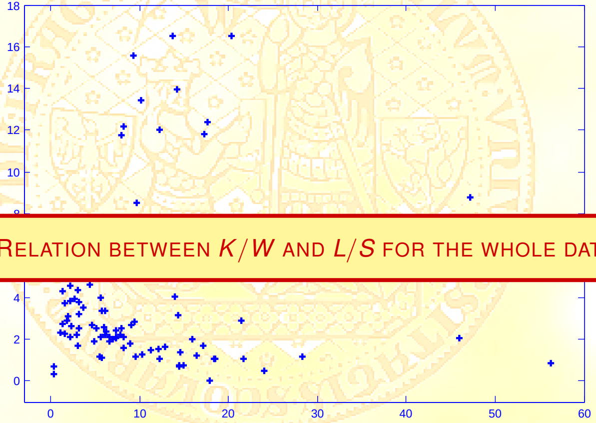
X_ℓ	-	*export from i-th industry,*
US_ℓ	-	*number of university-passed employees in the i-th industry,*
HS_ℓ	-	*nuber of high school-passed employees in the i-th industry,*
VA_ℓ	-	*value added in the i-th industry,*
K_ℓ	-	*capital in the i-th industry,*
CR_ℓ	-	*percentage of market occupied by 3 largest producers,*
$TFPW_\ell$	-	*by wages normed productvity in the i-th industry,*
Bal_ℓ	-	*Balasa index in the i-th industry„*
DP_ℓ	-	*cost discontinuity in 1993 in the i-th industry*

with coefficient of determination 0.93 and stable submodels

Analysis of the export from the Czech republic to EU in 1994 by means of the *least trimmed squares*.

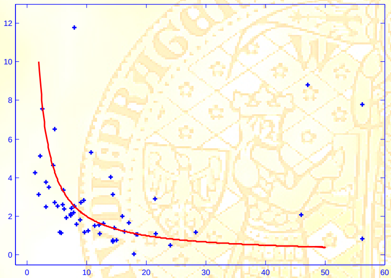


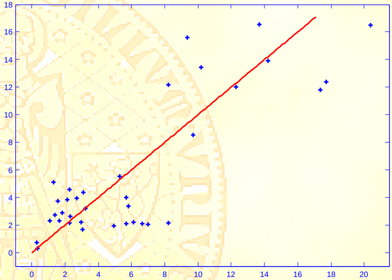
Analysis of the export from the Czech republic to EU in 1994 by means of the *least trimmed squares*.



Relation between K/W and L/S for the whole data.

ANALYSIS OF THE EXPORT FROM THE CZECH REPUBLIC TO EU IN 1994 BY MEANS OF THE *least trimmed squares*.

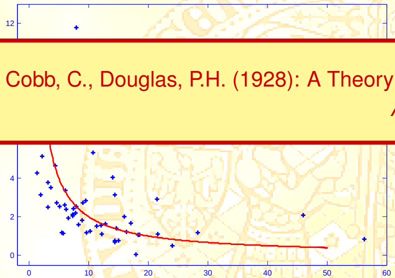


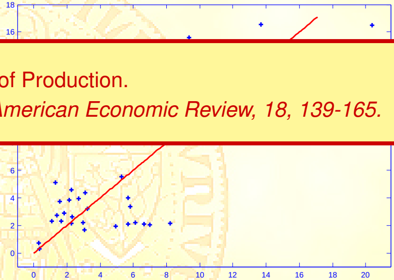


RELATION BETWEEN K/W AND L/S FOR THE *Main subpopulation* (LEFT PICTURE)

AND FOR THE *Complementary subpopulation* (RIGHT PICTURE).

ANALYSIS OF THE EXPORT FROM THE CZECH REPUBLIC TO EU IN 1994 BY MEANS OF THE *least trimmed squares*.





Cobb, C., Douglas, P.H. (1928): A Theory of Production. *American Economic Review, 18, 139-165.*

RELATION BETWEEN K/W AND L/S FOR THE *Main subpopulation* (LEFT PICTURE)

AND FOR THE *Complementary subpopulation* (RIGHT PICTURE).

Finally - all after - the victory

We haven't reached something which is

"BOMB und IDIOTEN SICHER"

but which is the powerful tool, if it is used with a care.

Content

The least weighted squares

Residuals $\quad \forall \beta \in R \quad \to \quad r_i(\beta) = Y_i - X_i'\beta$
Order statistics of squared residuals, i. e.

$$r^2_{(1)}(\beta) \leq r^2_{(2)}(\beta) \leq \ldots \leq r^2_{(n)}(\beta)$$

The least weighted squares

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Order statistics of squared residuals, i. e.

$$r^2_{(1)}(\beta) \le r^2_{(2)}(\beta) \le ... \le r^2_{(n)}(\beta)$$

Definition

Let $w(u): [0,1] \rightarrow [0,1], w(0) = 1$, (nonincreasing). Then

$$\hat{\beta}^{(LWS,n,w)} = \arg\min_{\beta \in R^p} \sum_{i=1}^n w\left(\frac{i-1}{n}\right) r^2_{(i)}(\beta)$$

will be called *the least weighted squares (LWS)*.

The least weighted squares

Residuals $\quad \forall \beta \in R \quad \rightarrow \quad r_i(\beta) = Y_i - X_i'\beta$
Order statistics of squared residuals, i. e.

$$r^2_{(1)}(\beta) \le r^2_{(2)}(\beta) \le \ldots \le r^2_{(n)}(\beta)$$

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Let $w(u): \; [0,1] \;\rightarrow\; [0,1], w(0) = 1$, (nonincreasing). Then

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- see one of the next slides.

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Víšek, J. Á. (2000): Regression with high breakdown point.
Robust 2000 (eds. Antoch, J. Dohnal, G.), 324 - 356.

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Notice:
The smallest residual obtains the largest weight

and vice versa

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To be able to answer it, let's make some preparatory steps.

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First of all, let's recall that there is the classical:

The weighted least squares

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Notice the order of words - it hints that

the weights are assigned to the observations by some external rule,

e. g. a geometric one, by which we select magnitudes of weights.

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Kmenta, J. (1986): *Elements of econometrics.* Macmillan Publishing Company, New York.

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Notice also that we have the formula for the estimator -
- hence it can be easy implemented and computed.

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How did we find the formula $\hat{\beta}^{(WLS,n,w)} = (X'WX)^{-1} X'WY$?

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We can rewrite the definition

$$\hat{\beta}^{(WLS,n,w)} = \underset{\beta \in R^p}{\arg\min} \sum_{i=1}^{n} w_i r_i^2(\beta) = \sum_{i=1}^{n} \left(\sqrt{w}_i r_i(\beta)\right)^2.$$

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Hence, considering transformed variables

$$\tilde{Y}_i = \sqrt{w_i} \cdot Y_i \qquad \text{and} \qquad \tilde{X}_i = \sqrt{w_i} \cdot X_i,$$

we have

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Finally, $\underline{\hat{\beta}^{(WLS,n,w)}(Y, X) = \hat{\beta}^{(OLS,n)}(\tilde{Y}, \tilde{X})}$.

Recalling the background of our model

Remenber that we consider the regression model

$$Y_i = X_i' \beta^0 + \varepsilon_i \quad \text{for} \quad i = 1, 2, \ldots$$

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We will need it a bit later.

Continuing preparatory considerations

1. Denote by Π the set of all permutations of numbers $\{1, 2, ..., n\}$ and its elements by $\boldsymbol{\pi}$,

   i. e. $\boldsymbol{\pi} \in \Pi \to \boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_n), \pi_i \in \{1, 2, ..., n\}$ and $\pi_i \neq \pi_j$ for $i \neq j$.

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6. Then (1) implies
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We have from previous slide

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Finally, keep in mind that

$$\boxed{S_n(\hat{\beta}^{(WLS,n,w,\hat{\boldsymbol{\pi}})}, \hat{\boldsymbol{\pi}}, \omega) = \min_{\beta \in R^p} \left\{ \min_{\boldsymbol{\pi} \in \Pi} \sum_{i=1}^{n} w_i r_{\pi_i}^2(\beta, \omega) \right\}}$$

(we will need it at the end of our considerations).

Continuing preparatory considerations

Now fix $\beta \in R^p$, put $w_i = w\left(\frac{i-1}{n}\right)$ and realize that

$$\begin{array}{ccccccccc} & w_1 & \geq & w_2 & \geq & \dots & \geq & w_n & \\ \text{and} & & & & & & & & \\ & r^2_{(1)}(\beta) & \leq & r^2_{(2)}(\beta) & \leq & \dots & \leq & r^2_{(n)}(\beta). & \end{array} \tag{2}$$

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It implies that for any $\beta \in R^p$

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Assume that not, i.e. the minimum

$$\min_{\pi \in \Pi} \sum_{i=1}^{n} w_i r^2_{\pi_i}(\beta)$$

is realized for another permutation of observations than given by (2)
and try to show a contradiction - which would prove (3)!!

Continuing preparatory considerations

An exhibition of contradiction: If the assertion doesn't hold, there is a pair $1 \leq i \leq j \leq n$ such that

$$w_i \cdot r^2_{(i)}(\beta) + w_j \cdot r^2_{(j)}(\beta) > w_i \cdot r^2_{(j)}(\beta) + w_j \cdot r^2_{(i)}(\beta) \tag{4}$$

and

$$w_j = w_i - a, \quad a \geq 0 \quad \text{and} \quad r^2_{(j)}(\beta) = r^2_{(i)}(\beta) + b, \quad b \geq 0. \tag{5}$$

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Plugging the values (5) into (4)

$$w_i \cdot r^2_{(i)}(\beta) + (w_i - a) \cdot (r2_{(i)}(\beta) + b) > w_i \cdot (r^2_{(i)}(\beta) + b) + (w_i - a) \cdot r^2_{(i)}(\beta),$$

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gives evidently

$$2 \cdot w_i \cdot r^2_{(i)}(\beta) + w_i \cdot b - a \cdot r^2_{(i)}(\beta) - a \cdot b > 2 \cdot w_i \cdot r^2_{(i)}(\beta) + w_i \cdot b - a \cdot r^2_{(i)}(\beta)$$

a contradiction (except of $a = 0$ and $b = 0$).

Final conclusion

We had (we have proved it several slides backwards)

$$S_n(\hat{\beta}^{(WLS,n,w,\hat{\pi})},\hat{\pi}) = \min_{\beta \in R^p} \left\{ \min_{\pi \in \Pi} \sum_{i=1}^{n} w_i r^2_{\pi_i}(\beta) \right\}$$

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and we have just proved that for any $\beta \in R^p$

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It implies

$$\hat{\beta}^{(WLS,n,w,\hat{\pi})} = \hat{\beta}^{(LWS,n,w)}.$$

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$$\min_{\pi \in \Pi} \sum_{i=1}^{n} w_i r_{\pi_i}^2(\beta) = \sum_{i=1}^{n} w_i r_{(i)}^2(\beta).$$

Plugging the second row to the first one

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It implies

$$\hat{\beta}^{(WLS,n,w,\hat{\pi})} = \hat{\beta}^{(LWS,n,w)}.$$

Q.E.D.

Final conclusion

We had (we have proved it several slides backwards)

$$S_n(\hat{\beta}^{(WLS,n,w,\hat{\boldsymbol{\pi}})}, \hat{\boldsymbol{\pi}}) = \min_{\beta \in R^p} \left\{ \min_{\boldsymbol{\pi} \in \Pi} \sum_{i=1}^{n} w_i r^2_{\pi_i}(\beta) \right\}$$

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n n

Realize that $\hat{\pi} = \hat{\pi}(\omega)$.

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Q.E.D.

A technical trick - Jaroslav Hájek, rank tests

Ranks $\rho(\beta, j)$ of the squared residuals: We put

$$\boxed{\rho(\beta, j) = i \in \{1, 2, ..., n\} \quad \text{if} \quad r_j^2(\beta) = r_{(i)}^2(\beta)}$$

Firstly - read what it means, secondly - explain what it allows.

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The last form of definition says:
The method decides itself which weight is assigned to which residual -
- we can speak about an implicit weighting.

Let's return once again to the question: Does LWS exist at all ?

There is always (for fixed $n \in N$) a solution of the extremal problem

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Notice that when we want to find LWS, we are looking for

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We will need it a bit later.

An excursion to the history - once again.

Definition of $\hat{\beta}^{(}WLS, n, w)$ in the matrix form -
- it'll give an answer how we have found a formula for it.

Let $w_i \in [0, 1], i = 1, 2, ..., n$ be weights
and $W = \mathrm{diag}(w_1, w_2, ..., w_n)$ a diagonal matrix. Then

$$\hat{\beta}^{(WLS,n,w)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} w_i r_i^2(\beta) = (Y - X\beta)' \, W \, (Y - X\beta) \tag{6}$$

will be called *the Weighted Least Squares (WLS)*.

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Remember the third row from bottom.

An excursion to the history - continued

So, we have the normal equations for WLS:

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An excursion to the history - continued

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Let us take it into account

together with the last conclusion on one from the previous slides -

- see the next slide.

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1. On some of the previous slides we had:

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3. Hence:

   The estimator $\hat{\beta}^{(LWS,n,w)}$ is one of
   the solutions of the normal equations

$$\sum_{i=1}^{n} w\left(\frac{\rho(\beta,i)-1}{n}\right) X_i (Y_i - X_i'\beta) = 0.$$

Have we helped to ourselves?

Have we obtained something which can be useful for establishing the properties of $\hat{\beta}^{(LWS,n,w)}$?

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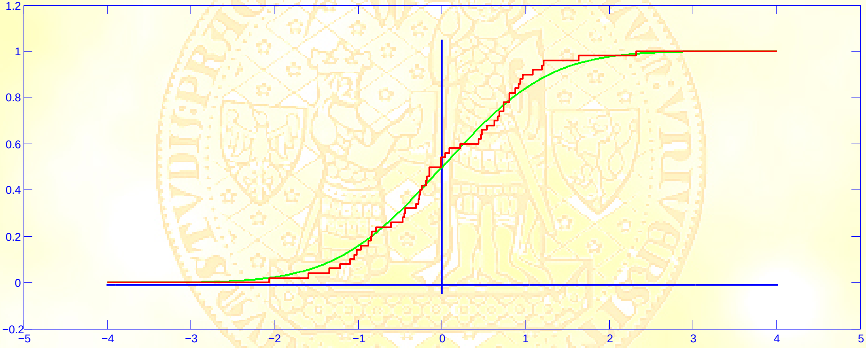
It seems that no but the correct answer is: YES !!

Let's realize what is the rank of the squared residual $\rho(\beta, i)$.

For a hint see the next slide !

On the second lecture we saw the graph of empirical d.f.:



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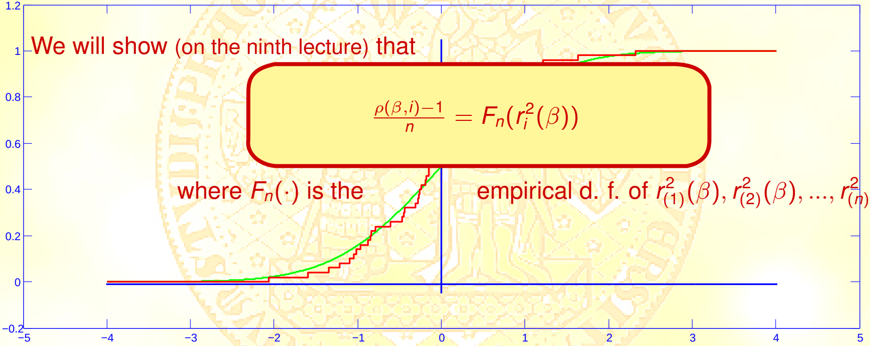

We will show (on the ninth lecture) that

$$\frac{\rho(\beta,i)-1}{n} = F_n(r_i^2(\beta))$$

where $F_n(\cdot)$ is the empirical d. f. of $r_{(1)}^2(\beta), r_{(2)}^2(\beta), \ldots, r_{(n)}^2(\beta)$.

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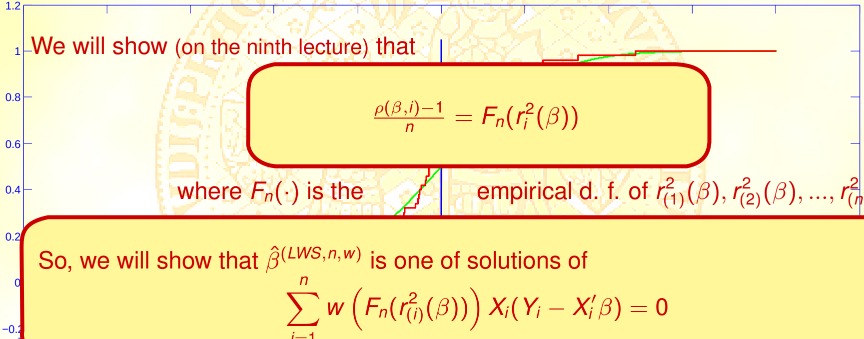

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So, we will show that $\hat{\beta}^{(LWS,n,w)}$ is one of solutions of

$$\sum_{i=1}^{n} w\left(F_n(r_{(i)}^2(\beta))\right) X_i(Y_i - X_i'\beta) = 0$$

and employing the fact that e. d. f. converges to the underlying d. f., we can ...

THANKS FOR ATTENTION