Topic 6: Differentiation

Jacques Text Book (edition 4):

Chapter 4

1. Rules of Differentiation
2. Applications

Differentiation is all about measuring change!

Measuring change in a linear function:

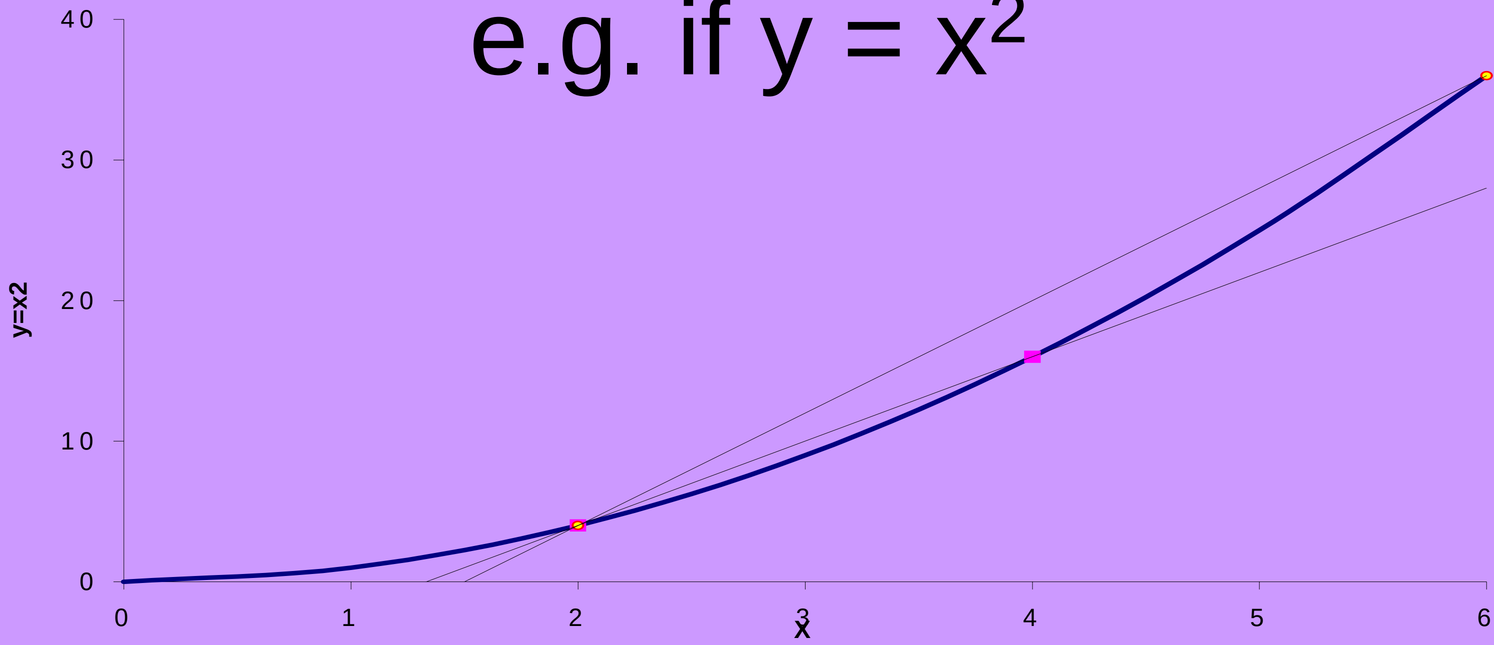
$$y = a + bx$$

a = intercept

b = constant slope i.e. the impact of a unit change in x on the level of y

$$\mathbf{b} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

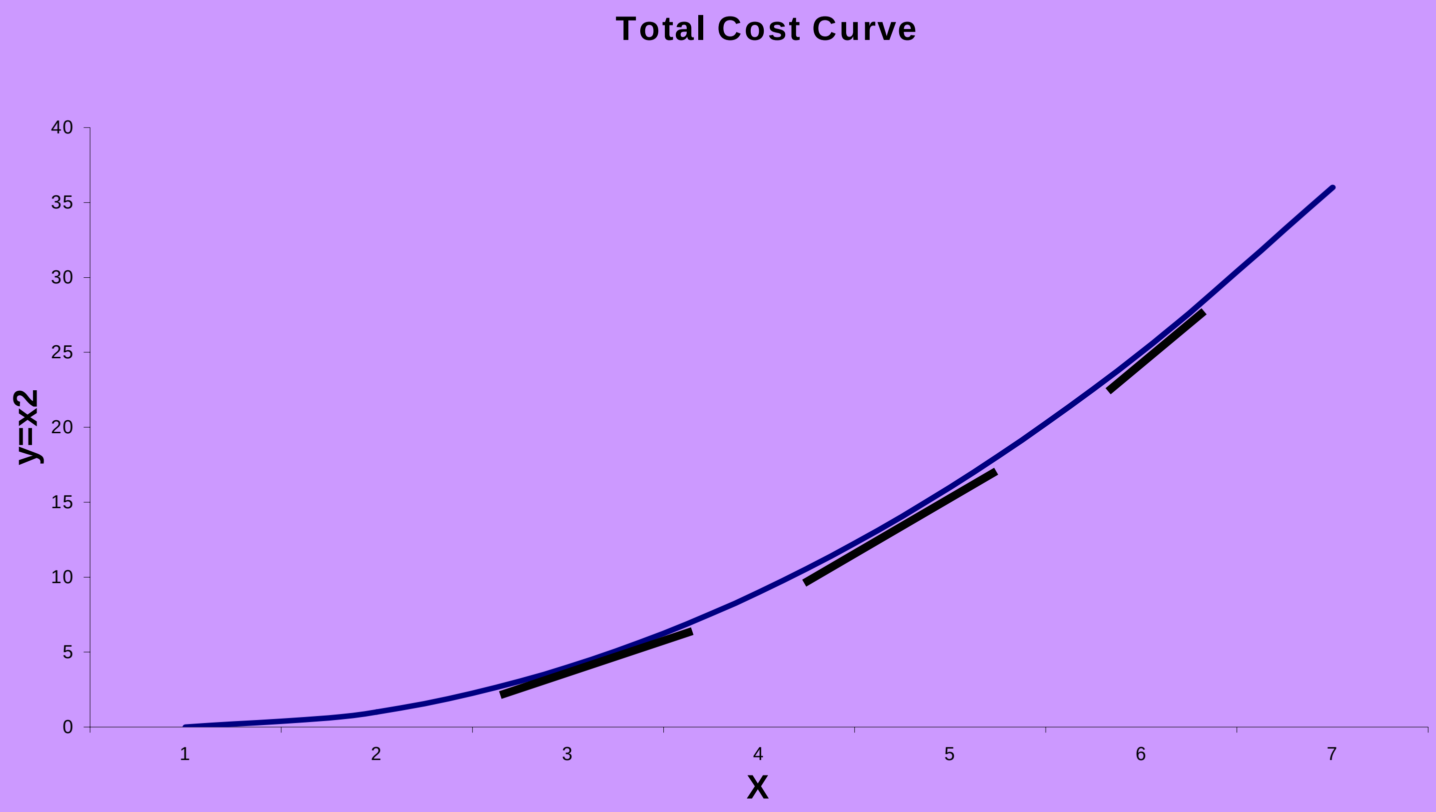
If the function is non-linear:
e.g. if y = x^2



$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$ gives slope of the *line* connecting 2 points (x_1, y_1) and (x_2,y_2) on a curve

- (2,4) to (4,16): slope = $^{(16-4)}/_{(4-2)}$ = 6
- (2,4) to (6,36): slope = $^{(36-4)}/_{(6-2)}$ = 8

The slope of a curve is equal to the slope of the line (or tangent) that touches the curve at that point



which is different for different values of x

Example:A firms cost function is $Y = X^2$

X	ΔX	Y	ΔY
0		0	
1	+1	1	+1
2	+1	4	+3
3	+1	9	+5
4	+1	16	+7

$Y = X^2$

$Y+\Delta Y = (X+\Delta X)^2$

$Y+\Delta Y = X^2+2X.\Delta X+\Delta X^2$

$\Delta Y = X^2+2X.\Delta X+\Delta X^2 - Y$

since $Y = X^2 \quad \Rightarrow \quad \Delta Y = 2X.\Delta X+\Delta X^2$

$$\frac{\Delta Y}{\Delta X} = 2X+\Delta X$$

The slope depends on X and ΔX

The slope of the graph of a function is called the derivative of the function

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- The process of differentiation involves letting the change in x become arbitrarily small, i.e. letting Δ x $\to$ 0
- e.g if = 2X+ΔX and ΔX $\to$0
- $\Rightarrow$ = 2X in the limit as ΔX $\to$0

the slope of the non-linear function $Y = X^2$ is 2X

- the slope tells us the change in y that results from a very small change in X
- We see the slope varies with X

e.g. the curve at X = 2 has a slope = 4

and the curve at X = 4 has a slope = 8

- In this example, the slope is steeper at higher values of X

Rules for Differentiation (section 4.3)

1. The Constant Rule

If y = c where c is a constant,

$$\frac{dy}{dx} = 0$$

e.g. y = 10 then $\frac{dy}{dx} = 0$

2. The Linear Function Rule

If $y = a + bx$

$$\frac{dy}{dx} = b$$

e.g. $y = 10 + 6x$ then $\frac{dy}{dx} = 6$

3. The Power Function Rule

If $y = ax^n$, where a and n are constants

$$\frac{dy}{dx} = n.a.x^{n-1}$$

i) y = 4x => $\frac{dy}{dx} = 4x^0 = 4$

ii) y = $4x^2$ => $\frac{dy}{dx} = 8x$

iii) y = $4x^{-2}$ => $\frac{dy}{dx} = -8x^{-3}$

4. The Sum-Difference Rule

If $y = f(x) \pm g(x)$

$$\frac{dy}{dx} = \frac{d[\,f(x)]}{dx} \pm \frac{d[\,g(x)]}{dx}$$

If y is the sum/difference of two or more functions of x:

differentiate the 2 (or more) terms separately, then add/subtract

(i) $y = 2x^2 + 3x$ then $\frac{dy}{dx} = 4x + 3$

(ii) $y = 5x + 4$ then $\frac{dy}{dx} = 5$

5. The Product Rule

If y = $u.v$ where u and v are functions of x,
($u = f(x)$ *and* $v = g(x)$) Then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Examples

If $y = u.v$ $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

i) $y = (x+2)(ax^2+bx)$

$$\frac{dy}{dx} = (x+2)(2ax+b) + (ax^2+bx)$$

ii) $y = (4x^3-3x+2)(2x^2+4x)$

$$\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \left(4\boldsymbol{x}^3 - 3\boldsymbol{x} + 2\right)\left(4\boldsymbol{x} + 4\right) + \left(2\boldsymbol{x}^2 + 4\boldsymbol{x}\right)\left(12\boldsymbol{x}^2 - 3\right)$$

6. The Quotient Rule

- If y = *u/v*   where *u* and *v* are functions of x (*u = f(x) and v = g(x)*   )   Then

$$\frac{dy}{dx} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$$

$$If \quad y = \frac{u}{v} \qquad then \qquad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1

$$y = \frac{(x+2)}{(x+4)}$$

$$\frac{dy}{dx} = \frac{(x+4)(1) - (x+2)(1)}{(x+4)^2} = \frac{-2}{(x+4)^2}$$

7. The Chain Rule (Implicit Function Rule)

- If y is a function of v, and v is a function of x, then y is a function of x and

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

Examples $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$

i) $y = (ax^2 + bx)^{½}$

let $v = (ax^2 + bx)$, so $y = v^{½}$

$$\frac{dy}{dx} = \frac{1}{2}\left(ax^2 + bx\right)^{-\frac{1}{2}}.\left(2ax + b\right)$$

ii) $y = (4x^3 + 3x – 7)^4$

let $v = (4x^3 + 3x – 7)$, so $y = v^4$

$$\frac{dy}{dx} = 4\left(4x^3 + 3x - 7\right)^3.\left(12x^2 + 3\right)$$

8. The Inverse Function Rule

$$\text{If } x = f(y) \text{ then } \frac{dy}{dx} = \frac{1}{dx/dy}$$

- Examples

i) $x = 3y^2$ then

$$\frac{dx}{dy} = 6y \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{6y}$$

ii) $y = 4x^3$ then

$$\frac{dy}{dx} = 12x^2 \quad \text{so} \quad \frac{dx}{dy} = \frac{1}{12x^2}$$

Differentiation in Economics
Application I

- Total Costs = TC = FC + VC
- Total Revenue = TR = P * Q
- π = Profit = TR – TC
- Break even: $\pi = 0$, or TR = TC
- Profit Maximisation: MR = MC

Application I: Marginal Functions (Revenue, Costs and Profit)

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  Calculating Marginal Functions

$$MR = \frac{d(TR)}{dQ}$$

$$MC = \frac{d(TC)}{dQ}$$

Example 1

- A firm faces the demand curve P=17-3Q
- (i) Find an expression for TR in terms of Q
- (ii) Find an expression for MR in terms of Q

Solution:

TR = P.Q = 17Q – 3Q^2

$$MR = \frac{d(TR)}{dQ} = 17 - 6Q$$

Example 2

A firms total cost curve is given by

$$TC=Q3- 4Q^2+12Q$$

(i) Find an expression for AC in terms of Q

(ii) Find an expression for MC in terms of Q

(iii) When does AC=MC?

(iv) When does the slope of AC=0?

(v) Plot MC and AC curves and comment on the economic significance of their relationship

Solution

(i) TC $= Q^3 - 4Q^2 + 12Q$

Then, **AC** $= {}^{TC}/_{Q} = Q^2 - 4Q + 12$

(ii) MC = $\dfrac{d(TC)}{dQ} = 3Q^2 - 8Q + 12$

(iii) When does AC = MC?

$Q^2 - 4Q + 12 \ \mathbf{=} \ 3Q^2 - 8Q + 12$

$\Rightarrow Q = 2$

Thus, AC = MC when Q = 2

Solution continued….

(iv) When does the slope of AC = 0?

$$\frac{d\left(AC\right)}{dQ} = 2Q - 4 = 0$$

$\Rightarrow$ Q = 2 when slope AC = 0

(v) Economic Significance?

MC cuts AC curve at minimum point…

9. Differentiating Exponential Functions

If $y = \exp(x) = e^{x}$ where $e = 2.71828\ldots.$

$$\text{then } \frac{dy}{dx} = e^{x}$$

More generally,

If $y = Ae^{rx}$

$$\text{then } \frac{dy}{dx} = rAe^{rx} = ry$$

Examples

1) $y = e^{2x}$ then $\dfrac{dy}{dx} = 2e^{2x}$

2) $y = e^{-7x}$ then $\dfrac{dy}{dx} = -7e^{-7x}$

10. Differentiating Natural Logs

Recall if $\mathbf{y = e^x}$ then $\mathbf{x = \log_e y = \ln y}$

- If $y = e^x$ then $\frac{dy}{dx} = e^x = y$

- **From The Inverse Function Rule**

$$y = e^x \Rightarrow \frac{dx}{dy} = \frac{1}{y}$$

- Now, if $\mathbf{y = e^x}$ this is equivalent to writing
  $\mathbf{x = \ln y}$

- Thus, $\mathbf{x = \ln y} \Rightarrow \frac{dx}{dy} = \frac{1}{y}$

More generally,

if $\mathbf{y = \ln x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

NOTE: the derivative of a natural log function does not depend on the co-efficient of x

Thus, if $\mathbf{y = \ln mx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

Proof

- if $y = \ln mx \qquad m>0$
- **Rules of Logs** $\Rightarrow y = \ln m + \ln x$
- **Differentiating** (Sum-Difference rule)

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

Examples

1) $y = \ln 5x \quad (x>0) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

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2) $y = \ln(x^2+2x+1)$

let $v = (x^2+2x+1)$ so $y = \ln v$

Chain Rule: $\Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{dv} \cdot \dfrac{dv}{dx}$

$$\frac{dy}{dx} = \frac{1}{x^2 + 2x + 1} . \left(2x + 2\right)$$

$$\frac{dy}{dx} = \frac{\left(2x + 2\right)}{\left(x^2 + 2x + 1\right)}$$

3) $y = x^4 \ln x$

Product Rule: $\Rightarrow$

$$\frac{dy}{dx} = x^4 \frac{1}{x} + \ln x . 4x^3$$

$$= x^3 + 4x^3 \ln x \quad = x^3 \left(1 + 4 \ln x\right)$$

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4) $y = \ln(x^3(x+2)^4)$

Simplify first using rules of logs

$\Rightarrow y = \ln x^3 + \ln(x+2)^4$

$\Rightarrow y = 3\ln x + 4\ln(x+2)$

$$\frac{dy}{dx} = \frac{3}{x} + \frac{4}{x+2}$$

Applications II

- how does demand change with a change in price……
- e_d= $$\frac{proportional\ change\ in\ demand}{proportional\ change\ in\ price}$$

$$= \frac{\Delta Q}{Q} \Big/ \frac{\Delta P}{P} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}$$

Point elasticity of demand

$$e_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

e_d is negative for a downward sloping demand curve

–Inelastic demand if $| e_d |<1$

–Unit elastic demand if $| e_d |=1$

–Elastic demand if $| e_d |>1$

Example 1

Find e_d of the function $Q = aP^{-b}$

$$\mathbf{e_d} = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

$$e_d = -baP^{-b-1} \cdot \frac{P}{aP^{-b}}$$

$$= \frac{-baP^{-b}}{P} \cdot \frac{P}{aP^{-b}} = -b$$

e_d at all price levels is –b

Example 2

If the (inverse) Demand equation is

$P = 200 - 40\ln(Q+1)$

Calculate the price elasticity of demand when $Q = 20$

- Price elasticity of demand: $\mathbf{e_d} = \frac{dQ}{dP} \cdot \frac{P}{Q} \quad < 0$
- P is expressed in terms of Q,

$$\frac{dP}{dQ} = -\frac{40}{Q+1}$$

- Inverse rule $\Rightarrow \frac{dQ}{dP} = -\frac{Q+1}{40}$
- Hence, $\mathbf{e_d} = -\frac{Q+1}{40} \cdot \frac{P}{Q} \quad < 0$
- Q is 20 $\Rightarrow \mathbf{e_d} = -\frac{21}{40} \cdot \frac{78.22}{20} = -2.05$

(where $P = 200 - 40\ln(20+1) = 78.22$)

Application III: Differentiation of Natural Logs to find *Proportional* Changes

The derivative of log(f(x)) ≡ $^{f'(x)}/_{f(x)}$, or the proportional change in the variable x

i.e. y = f(x), then the proportional Δ x

$$= \frac{dy}{dx} \cdot \frac{1}{y} = \frac{d(\ln y)}{dx}$$

Take logs and differentiate to find proportional changes in variables

1) Show that if $y = x^{\alpha}$, then $\frac{dy}{dx} \cdot \frac{1}{y} = \frac{\alpha}{x}$

and this $\equiv$ derivative of ln(y) with respect to x.

Solution:

$$\frac{dy}{dx} \cdot \frac{1}{y} = \frac{1}{y} . \alpha\, x^{\alpha - 1}$$

$$= \frac{1}{y} . \alpha \, \frac{x^{\alpha}}{x}$$

$$= \frac{1}{y} . \alpha \, . \frac{y}{x}$$

$$= \frac{\alpha}{x}$$

Solution Continued…

Now $\ln y = \ln x^{\alpha}$

Re-writing $\Rightarrow \ln y = \alpha \ln x$

$$\Rightarrow \frac{d(\ln y)}{dx} = \alpha . \frac{1}{x} = \frac{\alpha}{x}$$

Differentiating the ln y with respect to x gives the proportional change in x.

Example 2: If Price level at time t is $P(t) = a+bt+ct^2$ Calculate the rate of inflation.

Solution:

The inflation rate at t is the proportional change in p

$$\frac{1}{P(t)}.\frac{dP(t)}{dt}=\frac{b+2ct}{a+bt+ct^2}$$

Alternatively,

differentiating the log of P(t) wrt t directly

$\ln P(t) = \ln(a+bt+ct^2)$

where $v = (a+bt+ct^2)$ so $\ln P = \ln v$

Using chain rule,

$$\frac{d(\ln P(t))}{dt}=\frac{b+2ct}{a+bt+ct^2}$$