ECS 132 Spring 2024: Assignment 2
Solution

1 Problem

Consider a universe where it is equally likely that a child is born with brown or green eyes. A couple has two children.

1. What is the probability that both have green eyes?
2. What is the probability that both have green eyes if the older of the two has green eyes?

Solution

1. The sample space $S = \{(B,G),(B,B),(G,B),(G,G)\}$ where the first element of the tuple is the elder child. There is only one state with both G. So $P(\text{both green}) = 1/4$.
2. We have the same initial sample space, but given the information that the older of the two child is a girl, the constrained sample space becomes $S_c = \{(G,B),(G,G)\}$. One of out the two states has both G, so the probability $P(\text{both green}) = 1/2$.

2 Problem

A person has n keys, of which only one will open her door.

1. If she tries the keys at random, discarding those that don't work, what is the probability that she will open the door on her kth try?
2. If she does not discard previously tried keys what is the probability that she will open the door on her kth try?

Solution

1. On the first try, the probability of getting the right key is $\frac{1}{n}$ and of not getting the right key is $\frac{n-1}{n}$. On the second try, there are $n-1$ remaining keys so the probability of getting the right key is $\frac{1}{n-1}$ and of not getting the right key is $\frac{n-2}{n-1}$. On the kth try there are $n-k+1$ remaining keys. Let X denote the random variable which is the number of attempts to get the right key. Hence,

$$P\{X=k\} = \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\cdots\left(\frac{n-k+1}{n-k+2}\right)\left(\frac{1}{n-k+1}\right)$$

2. What if she does not discard previously tried keys?

   On each try the probability that she will get the right key is $1/n$. Let X be the random variable which denotes the number of attempts needed to get the right key. The probability that she will get the right key on the kth try implies that she did not get the right key on the first $k-1$ attempts. Hence,

$$P\{X = k\} = \frac{1}{n}\left(\frac{n-1}{n}\right)^{k-1} \quad \text{where } k = 1, 2, \ldots$$

3 Problem

Consider that three types of requests arrive at an IT help desk: urgent (colored red), important (colored green), and routine (colored black). What is the probability that the IT specialist receives 10 red requests among the first 20 requests received on a particular day?

Solution

Treat this as a binomial process. Filling each slot is a Bernoulli process where the probability of "success" (getting red) is 1/3. There are 20 independent slots to fill, and we want to know what is the probability that 10 of them are red. Considering how many independent Bernoulli slots are successful is a Binomial process with $p = 1/3$ and $n = 20$ evaluated at $k = 10$.

$$P(10 \text{ reds}) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{20}{10}\left(\frac{1}{3}\right)^{10}\left(\frac{2}{3}\right)^{10} = 0.054259$$

4 Problem

Bridge is a card game in which the deck of 52 cards is randomly and equally divided among the 4 players. So each player gets 13 cards. What is the probability that one player receives all the 13 hearts?

As an aside, did you know that one of the greatest bridge players in the world is a (retired) faculty of the Computer Science Department here at UCDavis? Yes! Prof. Charles U. (Chip) Martel who retired a few years ago but still does research (`https://web.cs.ucdavis.edu/~martel/main/` has been the world champion many times [1].

Solution

Let H_i denote the event that hand i has all 13 hearts, $i = 1, 2, 3, 4$. There is only one way for that hand to arise. The number of possible distinct 5 card hands is $\binom{52}{13}$. So $P(H_i)$ is given by

$$P(H_i) = \frac{1}{\binom{52}{13}} \quad i = 1, 2, 3, 4$$

Note there are four players and any one of them could get the hand. But if player 1 gets the had with all hearts, then no other play can get such a hand, so the events $H_i, i = 1, 2, 3, 4$ are mutually exclusive. We

[1] `https://alum.mit.edu/slice/meet-chip-martel-75-one-worlds-greatest-bridge-players`

need to calculate $P(\cup_{i=1}^4 H_i)$, the probability that any of the four players received the hand.

$$\begin{aligned} P(\cup_{i=1}^4 H_i) &= \sum_{i=1}^4 P(H_i) \\ &= 4 * \frac{1}{\binom{52}{13}} \\ &= \sim 6.3 \times 10^{-12} \end{aligned}$$

5 Problem

In class we looked at an example of wireless sensor networks where each sensor can be active or asleep during each time slot of a timeframe. Here we will consider 2 sensors and 3 time slots. Each sensor is active with probability 1/2 in a given slot, and the sensors are independent of one another. Let A_i denote the event that both sensors are active in slot i.

1. What is $p(A_i)$, the probability that both sensors are active during the i-th time slot?
2. What is $p(A_i \cap A_j)$, for $i \neq j$? This is the probability that both sensors are active during two time slots.
3. What is $p(A_1 \cap A_2 \cap A_3)$, the probability that both sensors are active during all three time slots?
4. What is $p(A_1 \cup A_2 \cup A_3)$, the probability that they are both active in at least one time slot? Give the mathematical expression and then the numerical answer.

Solution

1. For a given slot i there are two bits present: one for the first sensor, and one for the second sensor. The state space $S = \{(0,0),(0,1),(1,0),(1,1)\}$. There is only one state with both sensors on so $P(A_i) = 1/4$.
2. Since events A_i and A_j are independent for $i \neq j$, then $p(A_i \cap A_j) = p(A_i)\, p(A_j) = (1/4)^2 = 1/16$. (Note we say $i \neq j$ for mathematical correctness since if $i = j$, then $A_i \cap A_i = A_i$.)
3. Since events A_1, A_2, A_3 are independent, $p(A_1 \cap A_2 \cap A_3) = p(A_1)\, p(A_2)\, p(A_3) = (1/4)^3 = 1/64$.
4. The events are **independent so they are not disjoint** and we use the inclusion-exclusion principle:

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) && (1) \\ & -P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) && (2) \\ & +P(A_1 \cap A_2 \cap A_3) && (3) \\ &= 3\left(\frac{1}{4}\right) - 3\left(\frac{1}{16}\right) + \left(\frac{1}{64}\right) = 0.578. && (4) \end{aligned}$$

6 Problem

Write a simulation of the (hat) matching problem discussed in class. The simulation should calculate the probability of at least one match for different values of n (the number of people). For each value of n do 1000 experiments to calculate the probability. Plot the probability as a function of n. In the same plot, add a line showing the asymptotic value, $1 - \frac{1}{e}$. Notice how quickly the asymptotic value is reached.

Solution

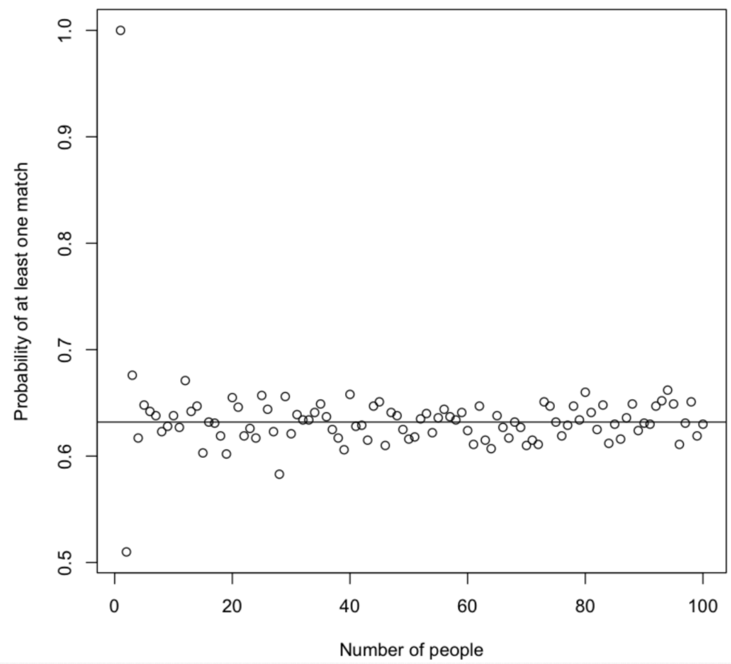
```
# m is number of experiments
# n is number of people in room
# x is the vector of outcomes

 m = 1000
 n = 100
 x = numeric(m)
 p = numeric(n)

 for (k in 1:n)
     {
         for (i in 1:m)
         {
             matches = 0                  # initialize the match count
             b = sample(1:k, k, repl=F)   # n random permutations of 1 to k
             for (j in 1:k)
             {
                 if (b[j] == j) {
                  matches = matches +  1
                 }
             }
             if (matches > 0)
             {
                 x[i] = 1
             }

         }
         p[k] = mean(x==1)      # probability of at least 2 match
         x = numeric(m)
     }

 plot(p, ylab = "Probability of at least one match", xlab = "Number of people")
 abline(h = 0.632)
```


Probability of at least one match
1.0
0.9
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Number of people