ECS 132, Spring 2024, Midterm 1 Topics

1. Counting and Sampling
    - Permutations
    - Combinations
    - Product rule
    - Sum rule
    - Sampling:
        - with replacement; without replacement; order matters; order does not matter
        - Multinomial coefficient
    - Applications
        - Simple poker and bridge hands
        - Lottery outcomes
        - The Birthday paradox
        - HW1 #1,2,3,4
        - HW2 #2, 3, 4

2. The basics of probability
    - Normalization: $P(A^C) = 1 - P(A)$
    - Law of total probability:

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A^C) \\ &= P(B|A)P(A) + P(B|A^C)P(A^C) \end{aligned}$$

    - Independent events: $P(A \cap B) = P(A)P(B)$ and $P(A|B) = P(A)$
    - Disjoint events: $P(A \cup B) = P(A) + P(B)$ and $P(A|B) = 0$
    - Inclusion-exclusion principle for non-disjoint events
    - Applications:
        - Summing probabilities for servers
        - Sensor networks active in timeslots (lecture example, HW2 #5, HW3 #3)
        - The matching problem
        - HW1 # 5,6
        - HW2 # 2,4,5,6

3. Conditional probability
   - Bayes' Rule:
   $$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$
   - Set theory perspective
   $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
   - Theorem 1:
   $$\begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ &= P(B|A)P(A) \end{aligned}$$
   - Conditioning on event B collapses the sample space to set B. Normalization means that:
   $$P(A^C|B) = 1 - P(A|B)$$
   - From written English to mathematics:
     - P("A and B") = $P(A \cap B)$
     - P("A given B") = $P(A|B)$
   - Applications:
     - Naive approach of collapsing the sample space to B (e.g., HW2 #1)
     - Direct application of Bayes's Rule (e.g., HW3 #1)
     - Accounting for false positives/negatives (e.g., HW3 #2, rare-disease example from lecture)
     - Prosecutor's fallacy ($P(A|B) \neq P(B|A)$). (Monty Hall is beyond the scope of exam.)

4. Discrete random variable, X
   - Sample space $S = \{x_1, x_2, ...\}$ where $x_j \in \mathbb{Z}$
   - $p_j = P(X = x_j)$ and $\sum_j p_j = 1$.
   - $E(X) = \sum_j p_j x_j$
   - $Var(X) = E(X^2) - \big(E(X)^2\big)$
   - $F_X(x) = P(X \leq x)$
   - $P(a < X \leq b) = F_X(b) - F_X(a)$

5. Common parametric distributions for a discrete random variable
   - Bernoulli; $X \sim Bern(p)$
   - Binomial; $X \sim Binom(n, p)$
   - Geometric; $X \sim Geom(p)$
   - Geometric*; $X \sim \text{Geom}^*(p)$
   - R functions (e.g., dbinom, pbinom)
   - Applications
     - Lecture example: MAC protocol with nodes sending packets in time slots.
     - HW3 #3,4,5