ECS 132 Spring 2024: Assignment 1
Solution

1 Problem

Consider the Manhattan grid network shown in Figure 1. Suppose that starting at the point labelled A, you can go one step up (denoted u) or one step to the right (denoted r) at each move. This procedure is is continued until the point labelled B is reached. For instance, the path along the upper left corner would be denoted (u u u r r r r). How many different paths from A to B are possible?

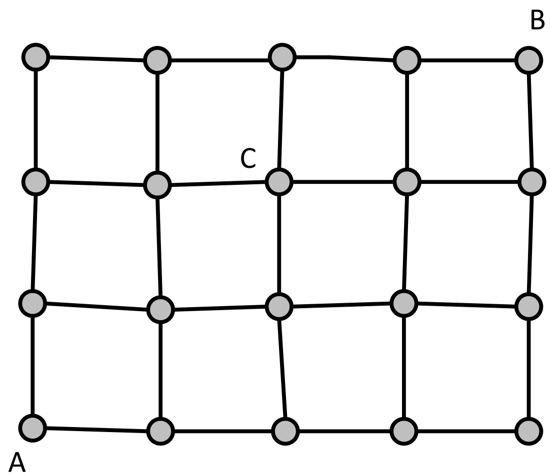

Figure 1: The Manhattan grid network.

Answer

Each path from A to B is a permutation of 7 steps consisting of 4 r's (rights) and 3 u's (ups). So there are 7 positions and we choose 4 of them to be r's and the remaining 3 to be u's. Once the r's are selected, the u's are automatically selected. The total number of ways is $\binom{7}{4}$ = $\binom{7}{3}$. Note that order is not relevant because the r's are indistinguishable from one another as are the u's. For example, if the r's are labelled r_1, r_2, r_3, r_4, and the u's are labeled u_1, u_2, u_3 then $(r_1 u_1 r_2 u_2 r_3 u_3 r_4) = (r_2 u_1 r_3 u_3 r_4 u_2 r_1)$. Also, note that $\binom{n}{k}$ is the same as $\binom{n}{n-k}$.

2 Problem

Consider a pool of six I/O buffers. Assume that any buffer is just as likely to be available (or occupied) as any other. Compute the probabilities associated with the following events

1. A = "A least two but no more than five buffers are occupied"
2. B = "At least one buffer is occupied"

Answer

There are 6 I/O buffers and each o f them can be in one of two states available (0) and occupied (1). The state space S consist of 2^6 possible outcomes of the I/O buffer pool. For example $(0,0,0,0,0,0)$ corresponds to the outcome that all 6 I/O buffers are available. Similarly, $(0,1,0,0,0,1)$ correspond to the outcome that 2 I/O buffers (specifically 2 and 6 in this case) are occupied.

1. For event A we are looking for the outcomes in which 2, 3, 4, and 5 I/O buffers are occupied. The number of outcomes that have i I/O buffers occupied is $\binom{6}{i}$. Thus,

$$P(A) = \frac{\binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5}}{2^6} \tag{1}$$

2. For the event B, we can follow the same logic as before and

$$P(B) = \frac{\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + +\binom{6}{6}}{2^6} \tag{2}$$

The other way think about is to consider the complement of event B denoted as $B^{\complement}$ which is the event that 0 I/O buffers are available:

$$\begin{aligned} P(B) &= 1 - P(B^{\complement}) && (3) \\ &= 1 - \frac{\binom{6}{0}}{2^6} && (4) \\ &= 1 - \frac{1}{2^6} && (5) \end{aligned}$$

3 Problem

Suppose your MP3 player contains 100 songs. 10 of those songs are by your favorite group (say Cake, which is a band from Sacramento). Suppose the shuffle feature is used to play the songs in random order. Note that after a shuffle all the 100 songs are played before the next shuffle.

1. What is the probability that the first Cake song heard is the 5th song played?
2. Similar to the **simulation** of the Birthday paradox that we discussed in class, write an R code to simulate the problem and determine the probability that the first Cake song heard is the 5th song played.

Answer

This is a sampling problem. Always ask yourself these two questions: (i) does order matter or not matter? (ii) is sampling with our without replacement. Here the answers are (i) yes order matters, (ii) sampling is without replacement. So we will use a permutation ${}^{n}P_{k}$.

1. What is the probability that the first Cake song heard is the 5th song played? The number of ways in which the first 5 songs can be played is $100 \times 99 \times 98 \times 97 \times 96 = {}^{100}P_5$. The number of ways in which the first 4 songs are non-Cake songs is $90 \times 89 \times 88 \times 87 = {}^{90}P_4$. And the number of ways in which the 5th song is a Cake song is 10. Thus, P(5th song is Cake) is given by

$$\begin{aligned} P(\text{5th song is Cake}) &= \frac{90 \times 89 \times 88 \times 87 \times 10}{100 \times 99 \times 98 \times 97 \times 96} \\ &= \frac{10 \times {}^{90}P_4}{{}^{100}P_5} \\ &= .0679 \end{aligned}$$

2.
```
# m is the number of trials
# n is total number of songs.
# We can assume that songs 1 through 10 are Cake songs,
# without loss of generality (WLOG).
# vector x records the outcomes.
# x(i)=1 if during trial i the 5th song is the 1st Cake song

 m = 100000
 n = 100
 x = numeric(m)
  for (i in 1:m)
 {
 b = sample(1:100, n, repl=F) # a random shuffle of 100 songs

 if ((b[1] > 10) & (b[2] > 10) & (b[3] > 10) & (b[4] >10) & (b[5] <= 10)) {
     x[i] = 1
     }
 else {
     x[i] = 0
     }
```

```
}
pmean = mean(x == 1)
print(pmean) # We take advantage of the mean function as a shorthand
             # to calculate number of times x(i)=1 divided by m.
```

4 Problem

Suppose that you want to backup the 100 songs on your MP3 player onto 4 different external drives each of different size. The first drive can store 20 songs, the second can store 30 songs, the third can store 40 songs, and the forth can store 10 songs. How many different ways can you arrange the songs on the four external drives.

Answer

This is an example of a multinomial coefficient. A set of n distinct objects is to be divided into r distinct groups of sizes $n_1, n_2, \ldots n_r$ such that $\sum_{i=1}^{r} n_i = n$. The total number of ways is given by $T = \frac{n!}{n_1!n_2!\ldots n_r!}$. Applying this here:

$$T = \frac{100!}{20!\,30!\,40!\,10!}$$

5 Problem

Consider two specific memory locations that are within the address space of a computer program when it executes. With probability 0.5, the program will access the first location; with probability 0.4 it will access the second location and with probability 0.3 it will access both locations. What is the probability that the program will access neither location?

Answer

Let B_i denote the event that the computer program access location $i, i = 1, 2$. Then the probability that the program accesses at least one of the locations is given by

$$\begin{aligned} P(B_1 \cup B_2) &= P(B_1) + P(B_2) - P(B_1 B_2) && (6) \\ &= 0.5 + 0.4 - 0.3 && (7) \\ &= 0.6 && (8) \end{aligned}$$

The event that the computer program will access neither location is

$$\begin{aligned} P(B_1^{\complement} \cap B_2^{\complement}) &= P((B_1 \cup B_2)^{\complement}) && (9) \\ &= 1 - P(B_1 \cup B_2) && (10) \\ &= 1 - 0.6 && (11) \\ &= 0.4 && (12) \end{aligned}$$

6 Problem

John is 31 years old, single, outspoken, and has multiple talents. He majored in Computer Science. As a student, he was deeply concerned with issues of discrimination and social justice, and environment. Which of the following scenarios is more probable? **A**) John is a computer programmer. **B**) John is a computer programmer and is an activist in the environmental movement. Give a reason for your answer.

Answer

The event that John is both a computer programmer and environmental activist is a subset of the event that John is a computer programmer. Thus, it cannot be more likely than the event that John is a computer programmer.

If event X is a subset of event Y then

$$P(X) \leq P(Y)$$

Here, $P(X) < P(Y)$ since there is some non-zero probability that John is a computer programmer but not active in the environmental movement.

A similar problem was made famous by Amos Tversky and Daniel Kahneman, whose 1983 study showed that 85% of respondents erroneously thought it was more likely that John is a computer programmer and is an activist in the environmental movement. These are related to mental/implicit biases that can/should be avoided and probability provides a framework for analyzing and avoiding them `brilliant.org`. Note Kahneman won the 2002 Nobel Prize in Economics, in part for his work establishing Prospect Theory and he is well known for his book ”Thinking fast and slow” published in 2011. Wikipedia has quite a lot of information on this.