Introduction to Response Theory

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MAGYAR AGRÁR- ÉS
ÉLETTUDOMÁNYI EGYETEM

Outline: Part I

- **Linear** in/homogeneous ordinary differential equation (ODE)
  - The "particular solution"
- Linear **system** of in/homogeneous ODEs
- **Non**linear systems
  - Nonchaotic systems
  - Chaotic systems: response via an **ensemble**
    - Snapshot/pullback attractor
    - **Linear response** from Fokker-Planck
- Second-order nonlinear response
- Example: Ornstein-Uhlenbeck process
- Higher-order corrections: Voterra series and its numerical estimation

Linear inhomogeneous ODE

$$\dot{x} = ax + b(t)$$

with inhomogeneity $b(t)$

Daring do(s): *look for* the solution as a sum $x(t) = x_h(t) + x_p(t)$

where one ansatz* is $x_h(t) = A\exp(\lambda t)$

This straight away specifies that $A = x_h(0)$

but, more importantly, substitute it into the homogeneous ODE!

$$\dot{x} = \lambda A \exp(\lambda t) = aA\exp(\lambda t) \quad \longrightarrow \quad \lambda = a$$

*Welcome home!

Linear inhomogeneous ODE

The form of the homogeneous solution implies an ODE for the particular solution

$$\dot{x}_p = a x_p + b(t)$$

This time round, an ansatz $x_p(t) = \exp(\lambda t)\xi(t)$ would lead to an *integrable* ODE (https://en.wikipedia.org/wiki/Linear_time-invariant_system)

$$\dot{\xi}(t) = \exp(-\lambda t) b(t)$$ that is

$$\xi(t) = \int_0^t \mathrm{d}\tau \exp(-\lambda \tau) b(\tau)$$ and so

$$x_p(t) = \int_0^t \mathrm{d}\tau \exp(\lambda(t-\tau)) b(\tau)$$ is a convolution.

Linear inhomogeneous ODE

The inhomogeneity can actually go **back** a *looooooong* way.

Example: $\dot{x} = ax + t$ (Credit: Mickael Chekroun)

General solution: $x(t) = x(t_0)\exp(a(t-t_0)) + \dfrac{\exp(a(t-t_0))(at_0+1) - (at+1)}{a^2}$

Why not take the *limit* “pulling back” to *negative* infinity!

$$\lim_{t_0 \to -\infty} x = -\frac{t}{a} - \frac{1}{a^2}$$

Remember: *a* is *negative* for a physical/stable system, so, the straight “asymptote” is indeed *increasing*.

For any real IC $x(t_0)$, the “asymptote” **attracts** the trajectory, initialised in the “distant” past, by any *finite* time *t*. This is *attraction in the pull-back sense*.

Linear inhomogeneous ODE

Take the Laplace (or Fourier) transform

e.g. $X(s) \stackrel{\text{def}}{=} \mathcal{L}\{x(t)\} = \int_0^\infty \mathrm{d}t x(t) \exp(-st)$

of both sides of the ODE (and use the "rule for differentiation")!

$$s\mathcal{L}\{x(t)\} = \lambda\mathcal{L}\{x(t)\} + \mathcal{L}\{b(t)\}$$

From this, we have that $X(s) = (\lambda - s)^{-1} B(s)$ in which

$(\lambda - s)^{-1} = \mathcal{L}\{\exp(\lambda t)\}$ indeed, in accord with the Wiener-Khinchin theorem: $(h * b)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(t-\tau)b(\tau)\,\mathrm{d}\tau \stackrel{\text{def}}{=} \mathcal{L}^{-1}\{H(s)B(s)\}$

Linear inhomogeneous “mapping”

Discretise in time the ODE using the (forward) Euler numerical integrator scheme!

$$x_{n+1} = x_n + \Delta t f(t_n, x_n) = (1 + a\Delta t)x_n = \varphi x_n$$

It’s easy to *see* that the (homogeneous) solution is $x_n = \varphi^n x_0$

If there is some inhomogeneity, $x_{n+1} = \varphi x_n + b_n$, then we can also simply see that (https://en.wikipedia.org/wiki/Autoregressive_model)

$$x_n = \varphi^N x_{n-N} + \sum_{k=0}^{N-1} \varphi^k b_{t-k}$$

and because $\lim_{N\to\infty} \varphi_N = 0$

$$x_n = \sum_{k=0}^{\infty} \varphi^k b_{n-k} = \sum_{k=0}^{\infty} \varphi^{n-k} b_k$$

(commutativity of conv’n)

Linear inhomogeneous “mapping”

That is, in discrete time, the impulse response function (IRF) is $h[n] = \varphi^n$

The discrete time convolution is also written in this notation as

$$x[n] = (h * b)[n] = \sum_{m=-\infty}^{\infty} h[n-m]b[m]$$ and we have a similar

theorem to Wiener-Khinchin’s $(h * b)[n] = \mathcal{Z}^{-1}\{H(z)B(z)\}$

involving the Z-transform, e.g. $H(z) = \mathcal{Z}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$

which is called the discrete time Fourier transform (DTFT) for $z = \exp(j\omega)$

Technical note

For **finite** length time series, $h[l],\ b[l],\ l = 0, \ldots, L-1$ the *linear* convolution

$$(h * b)[k] = \sum_{l=0}^{L-1} h[k-l]b[l]$$

and the *circular* convolution $\mathrm{DFT}^{-1}\{\mathrm{DFT}\{h[l]\}\mathrm{DFT}\{b[l]\}\}$ are **not**

equal. Here, DFT denotes the discrete Fourier transform, which samples the *continuous* function of the DTFT at **discrete** frequencies. We can **pad** the time series by a number of *L*-1 zeros in **front**, something that we denote by $\tilde{h}[l],\ \tilde{b}[l],\ l = 0, \ldots, 2(L-1)$. Then, the first useful half ($l = 0, \ldots, L-1$) of the circular convolution (very **effectively** computable by FFT)

$$\mathrm{DFT}^{-1}\{\mathrm{DFT}\{\tilde{h}[l]\}\mathrm{DFT}\{\tilde{b}[l]\}\}$$

will be qual with the above linear convolution according to the circ' conv' theor'm.

Check out the Appendices of

Chaos

Can we use linear response theory to assess geoengineering strategies?

Cite as: Chaos **30**, 023124 (2020); https://doi.org/10.1063/1.5122255
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https://doi.org/10.1063/1.5122255

System of linear ODEs

General form of a **system** of linear first-order ODEs: $\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x} + \mathbf{b}(t)$

Assume $\mathbf{A}$ is constant, i.e., time-independent; and $\mathbf{b} = 0$, i.e., we are treating the homogeneous problem. Then, the homogeneous solution takes the form:

$$\mathbf{x}_h = \sum_{i=1}^{n} c_i \mathbf{u}_i \exp(\lambda_i t) = \mathbf{X} \cdot \mathbf{c}$$

since we denote $\mathbf{x}_i = \mathbf{u}_i \exp(\lambda_i t)$ and $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$

where $\mathbf{u}_i$ and λ_i are unique eigenvectors and eigenvalues of $\mathbf{A}$, respectively, satisfying the characteristic equation:

$$\mathbf{A} \cdot \mathbf{u} = \lambda \mathbf{u} \longrightarrow (\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{u} = \mathbf{0} \longrightarrow \det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

System of linear ODEs

When $\mathbf{b} \neq 0$, the particular solution can be shown to take the form:

$$\mathbf{x}_p = \mathbf{X}(t) \cdot \int_0^t \mathbf{X}^{-1}(\tau) \cdot \mathbf{b}(\tau) d\tau$$

with which the general solution is: $\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t)$

The homogeneous solution can also be written in terms of a matrix exponential:

$\mathbf{x}_h(t) = \exp(\mathbf{A}t)\mathbf{x}_0$ where (by a Taylor expansion/series)

$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^3 t^3 + ... + \frac{1}{n!}\mathbf{A}^n t^n + \dots$$

System of linear ODEs

In the presence of inhomogeneity or *forcing*, the particular solution is also sought in a form *analogous* with the 1D ("ordinary") form

$$\mathbf{x}_p(t) = \exp(\mathbf{A}t)\xi(t)$$ with which

$$\mathbf{x}_p(t) = \int_0^\tau \mathrm{d}\tau \exp(\mathbf{A}(t-\tau))\mathbf{b}(\tau)$$

Note #1: This too can of course be defined in a "pullback sense".

Note #2: Linear systems are amenable to analytic solution, the IRF is analytic, and, therefore, the response to any forcing is straightforward to calculate.

Nonlinear dynamical systems

Nonchaotic nonlinear “inhomogeneous”, or *explicitly time dependent*, systems

$$\dot{x} = f(t, x)$$

also have – stable – “pullback” particular solutions. At worst, they can be found *numerically* only. It is *feasible by forward simulation*, because the system “forgets” the past forcing *exponentially fast*. In other words, the particular solution attracts trajectories exponentially fast. Otherwise: no *mechanic* way of getting the response!!

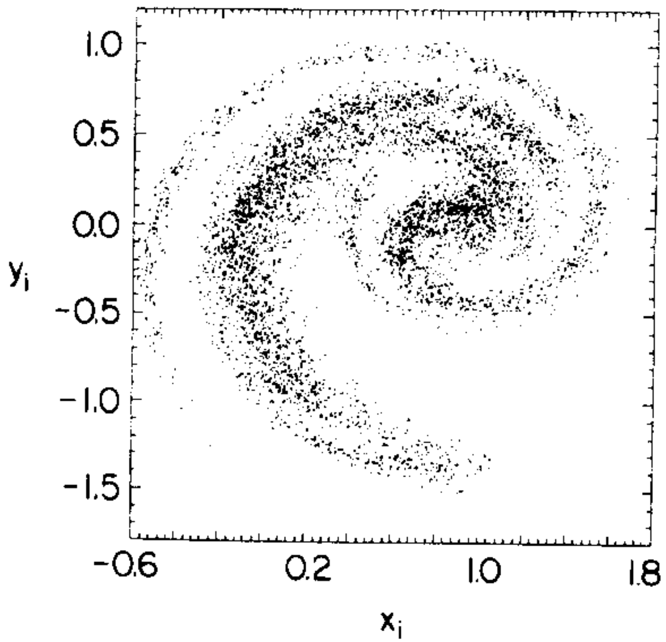
In the case of explicitly time dependent **chaotic** systems, there is *no particular solution*. One can follow an **ensemble** of trajectories, though. These trajectories are also attracted to an object in a pullback sense, which is called a pullback or *snapshot attractor*. The attracting property makes it *unique*, i.e., the way of initialisating it in the distant past is forgotten. The snapshot attractor *evolves* in time *completely determined by the forcing* – just like a particular solution.

Chaotic dynamical systems

- Dynamical system theory is well developed for **autonomous** systems. Such a theory is in its infancy for ***non*****autonomous** systems.
- The snapshot attractor would be a corner stone for such a theory.
- The *same* driving acts on all trajectories (ensemble members, realisations) locally
- After a while a fractal pattern might emerge in the ensemble **Romeiras, Grebogi, Ott, Phys. Rev. A, 1990.**
- Problems: trajectories can be determined only numerically AND now it's a whole ensemble.

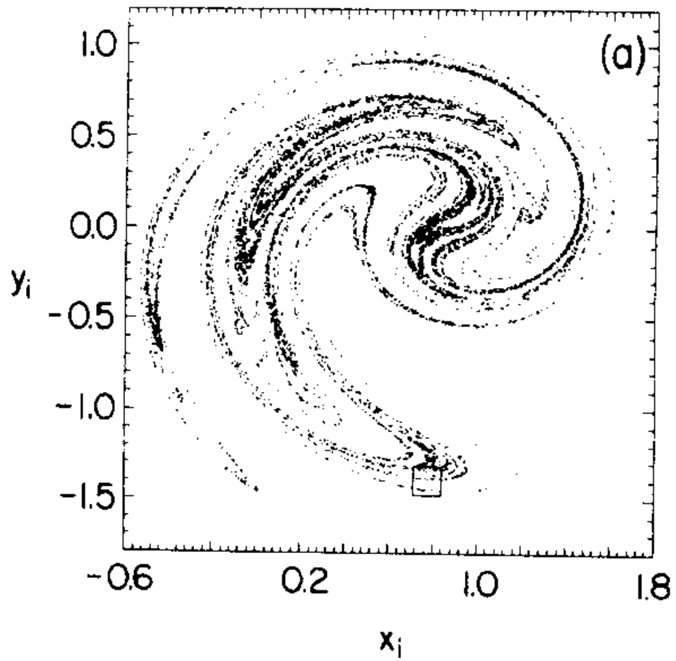
Example I.

For a “noisy” map:



A single trajectory noisy attractor

Snapshot attractor: clear fractality

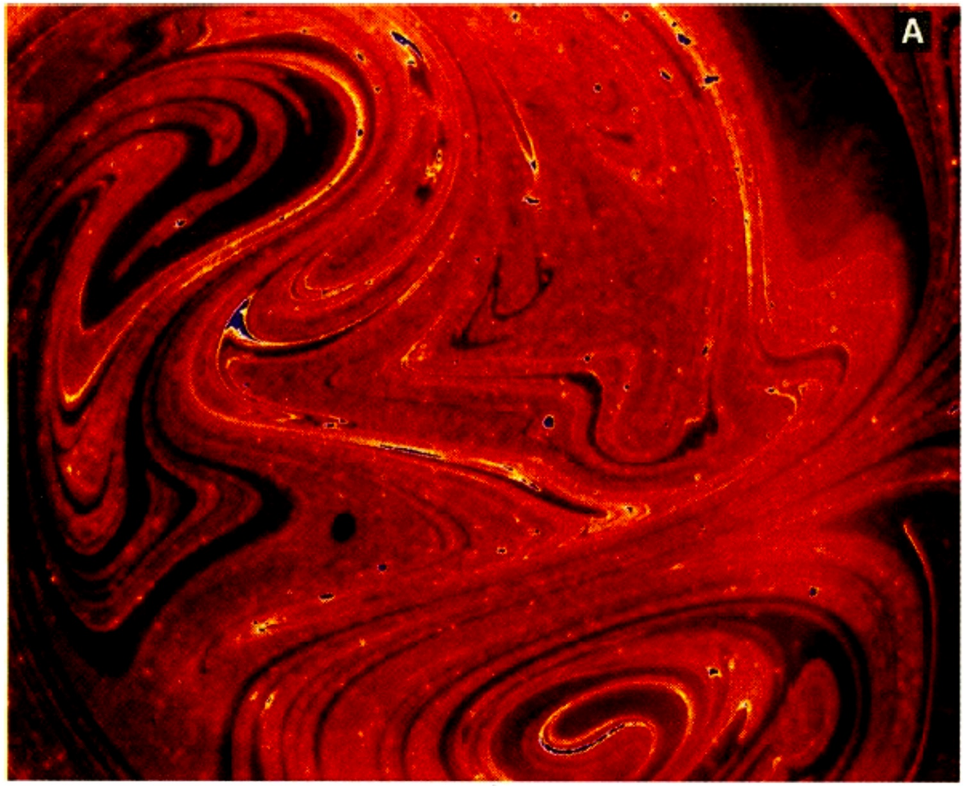


Ergodicity does **not** hold. See also: Drótos, Bódai, Tél, PRE, 2016 and Bódai et al. Frontiers, 2021 and Bódai et al., GRL, 2022

Example II.

Experimental realisation: floaters on the surface of a fluid

Sommerer & Ott, Science, 1993



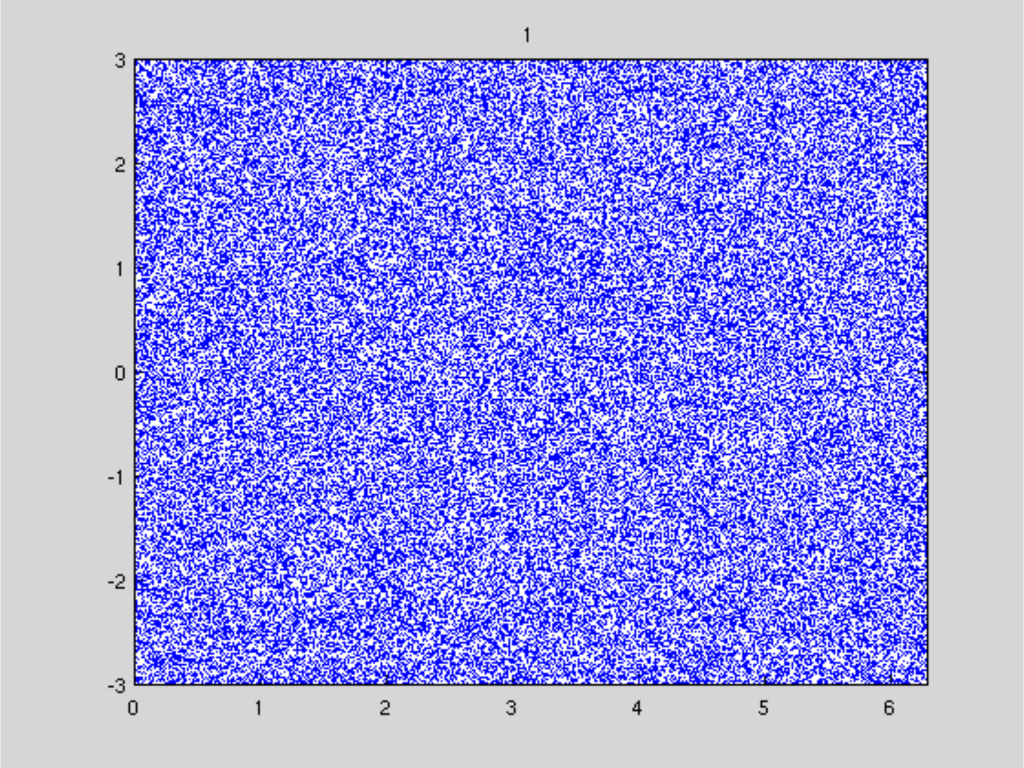
Example II.

Mathematical model:

Namenson, Ott, Antonsen, Phys. Rev. E, 1996

$$x_{n+1} = [x_n + y_n(1 - e^{-\alpha})/\alpha] \quad \mathrm{mod}\ 2\pi$$

$$y_{n+1} = \kappa \sin(x_{n+1} + c_n) + e^{-\alpha} y_n$$



Broken video

Example III.: Lorenz84 & *aperiodic* forcing

$$\dot{x} = -y^2 - z^2 - x/4 \quad + F/4,$$
$$\dot{y} = xy - 4xz - y \quad + 1,$$
$$\dot{z} = xz + 4xy - z$$

Nonlinear conservative
Linear dissipative
Asymmetric forcing

x : mean wind speed of the Westerlies
y, z : amplitudes of two principal modes of cyclonic activity

1 [MTU] = 5 [days]

Set *F* = 8: Perpetual winter

Driving includes **seasonality**…

$$F(t) = F_0(t) + A \sin(\omega t)$$

$$A = 2$$

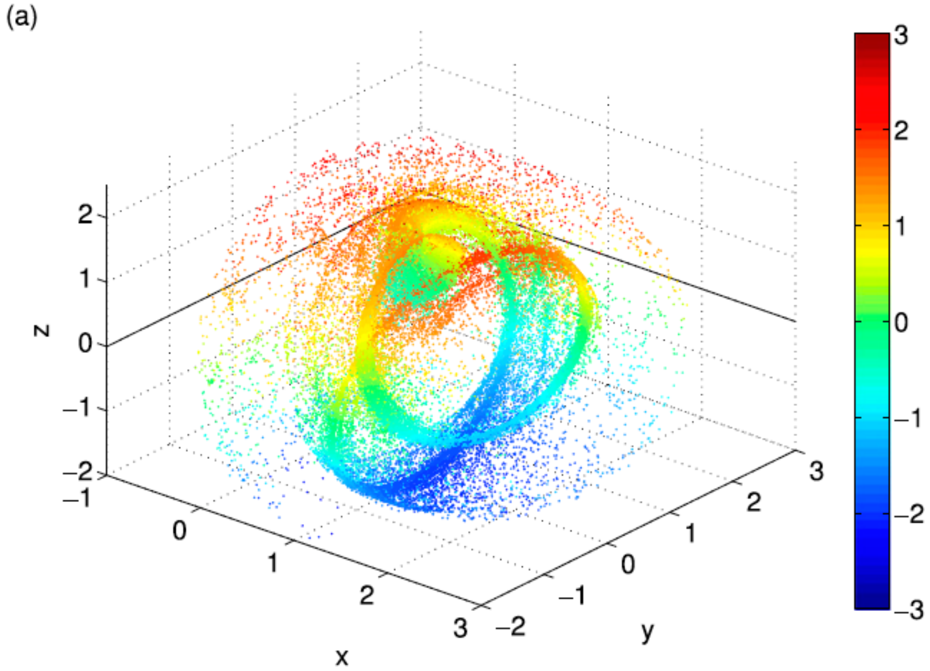
and a **parameter shift**, like climate change

$$F_0(t) = \begin{cases} 9.5 & \text{if} \quad t < t_0 \\ 9.5 - \frac{2}{t_0}(t - t_0) & \text{if} \quad t_0 < t \end{cases}$$

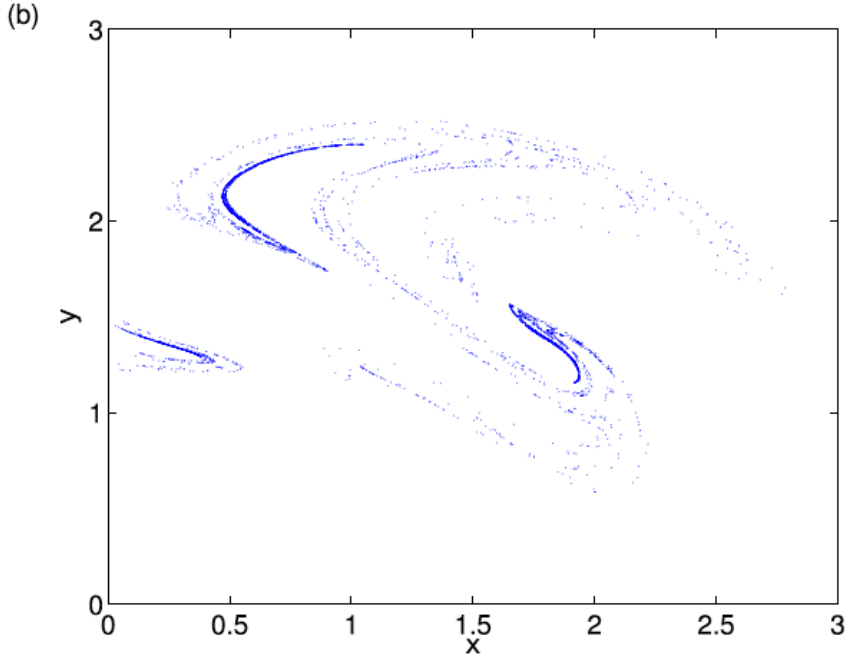
$$t_0 = 100 \text{ yr}$$

Stationary climate: $F_0(t) = 9.5$

Full phase space

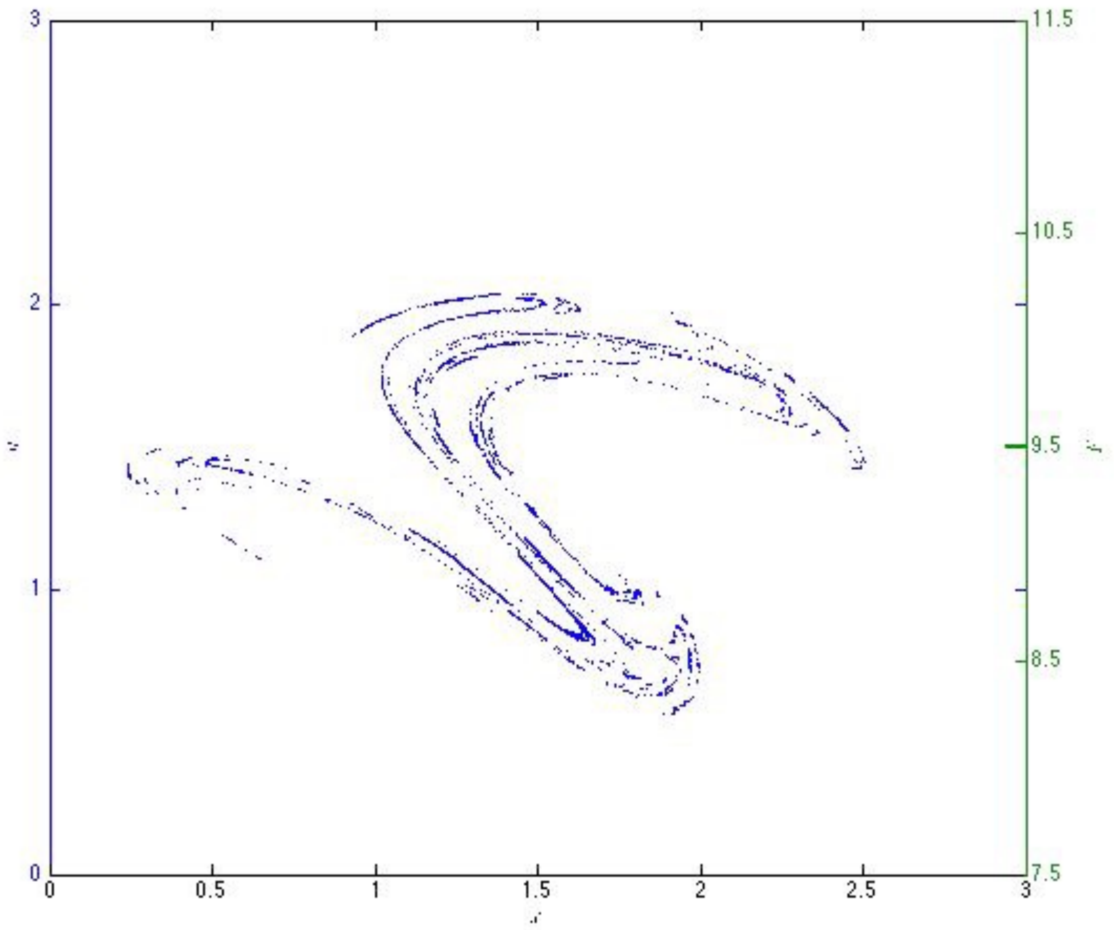


Poincaré section: $z = 0$



Stroboscopic view at midwinter: $t \mod T = T/4$

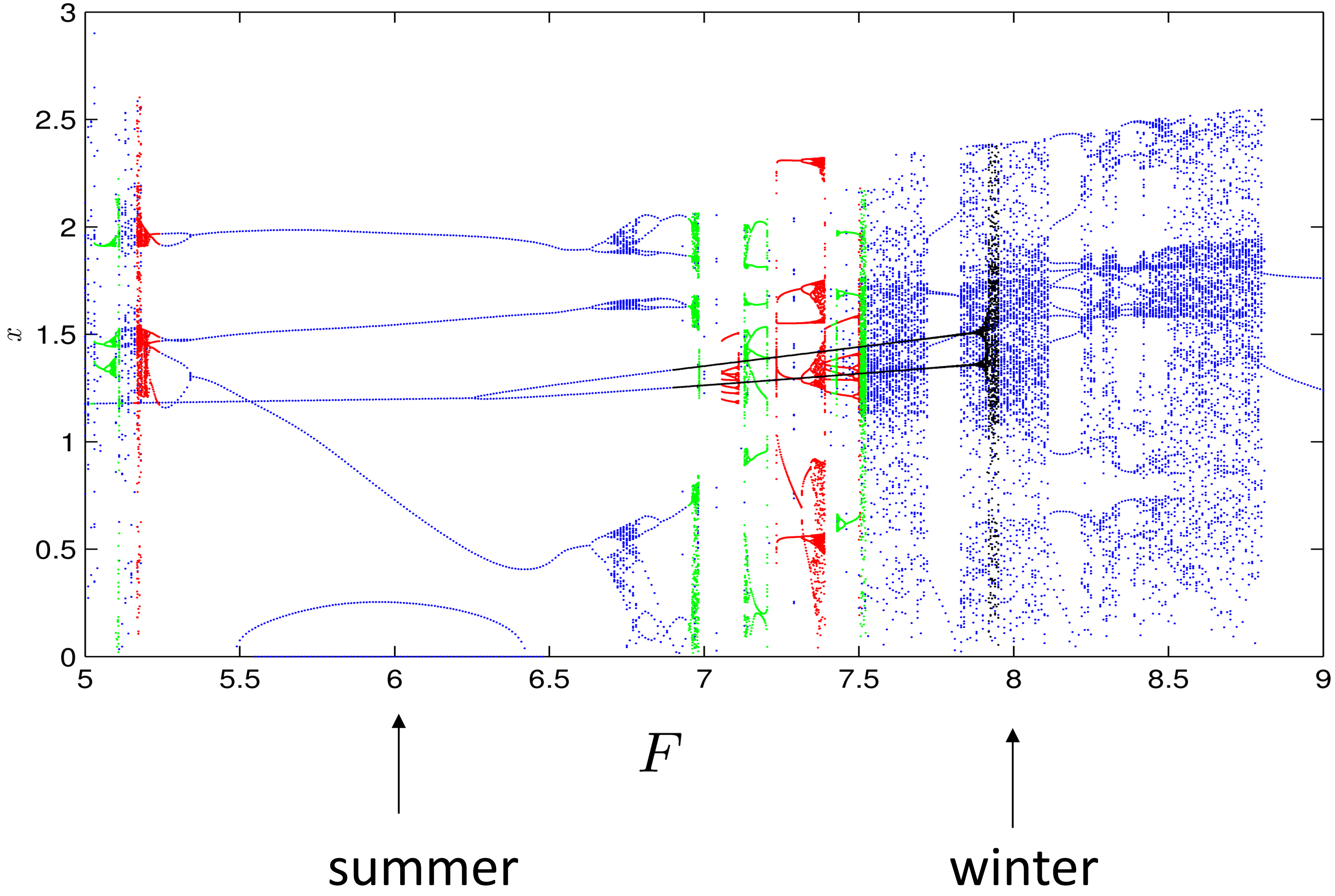
A swipe through the seasons



Video online

Bódai & Tél, Chaos, 2012

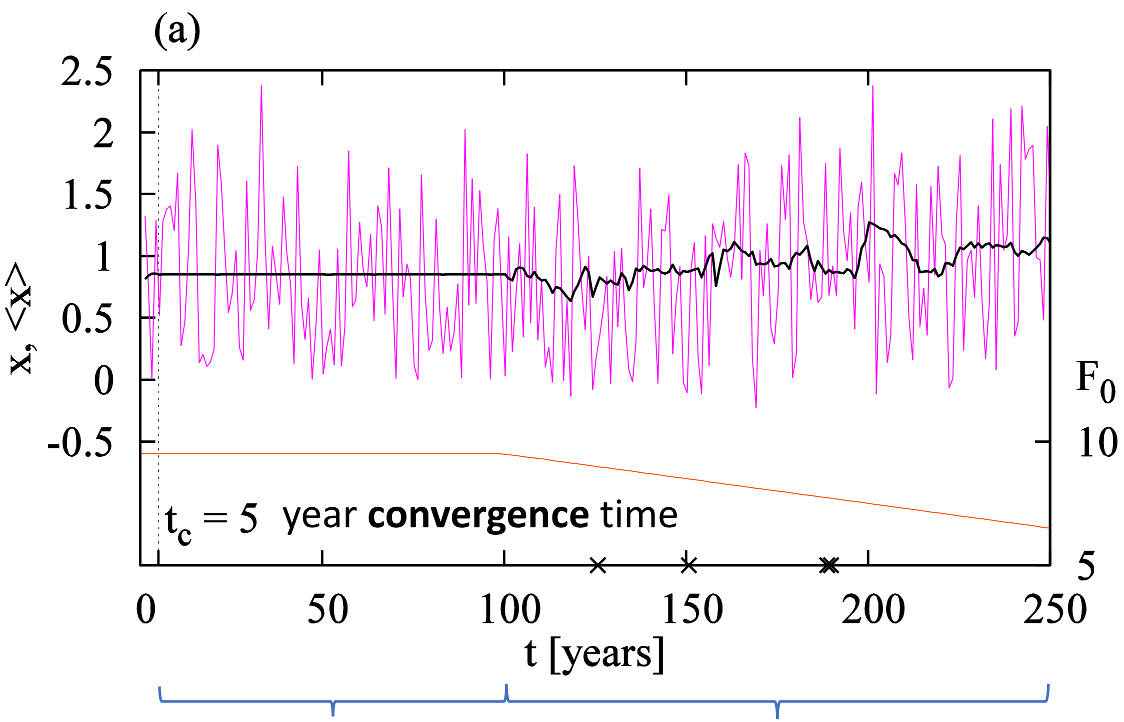
Bifurcation diagram with constant driving



Temporal driving washes the periodic windows into permanent chaos!

‘Expectations’ should be taken wrt. the natural distribution of the snapshot attractor at any time

Midwinter x values



periodic driving
stationary climate
ergodic

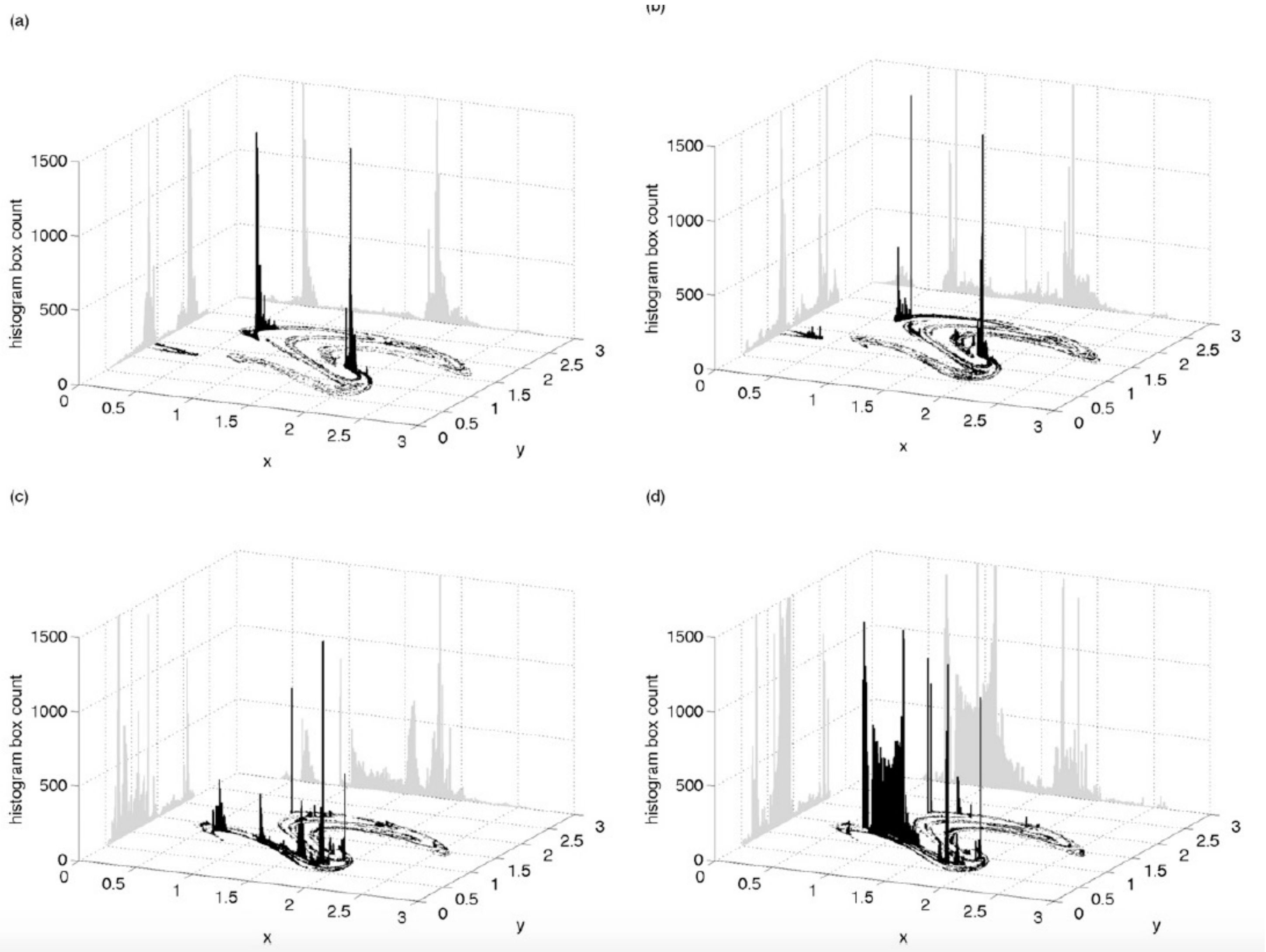
aperiodic driving
changing climate
nonergodic

Magenta: single realization

Black: snapshot average
$N=10^6$

Drótos, Bódai, Tél, J. Clim., 2015

Midwinter measures at (a) 25, (b) 50, (c) 88 and (d) 89 years after climate change begins



Snapshots at (a)-(b) many years ((c)-(d) a single year) apart can be very similar (different)

The seasonal driving does *not* achieve a very *smooth* result of washing things out.

These are clearly some kind of “bifurcations on the fly”. Lot to discover here…

Q. Is brute force the only way to determine the forced response of chaotic systems?

A. Not necessarily, but there are conditions.

Response theory

The forced nonautonomous explicitly time dependent dynamical system

$$\dot{x} = F(x) + \epsilon g(x,t)$$

The Fokker-Planck eq. (or Liouville eq. for the deterministic dynamics):

$$\partial_t \rho = \dot{\rho} = -\mathrm{div}[(F(x) + \epsilon g(x,t))\rho] = (L_F(x) + \epsilon L_g(x,t))[\rho] = L(x,t)[\rho]$$

Operator notation

Perturbative approach: series expansion of the measure ρ wrt. the small parameter ϵ

$$\rho(x,t) = \rho_0(x) + \sum_{i=1}^{\infty} \epsilon^i \rho^{(i)}(x,t)$$

Stationary solution: $\dot{\rho}_0 = 0 = L_F[\rho_0]$

Consider only second-order response

$$\begin{aligned} \epsilon\dot{\rho}^{(1)} + \epsilon^2\dot{\rho}^{(2)} &= L_F[\qquad \epsilon\rho^{(1)} + \epsilon^2\rho^{(2)}] \\ &\quad + \epsilon L_g[\rho_0 + \epsilon\rho^{(1)} + \epsilon^2\rho^{(2)}] \end{aligned}$$

(observing the stationary solution)

Response theory

Obtain eq. for $\rho^{(1)}$ by multiplying by ϵ^{-1} and taking the limit of $\epsilon \to 0$:

$$\dot{\rho}^{(1)} = L_F \rho^{(1)} + L_g \rho_0$$

whose formal solution is:

$$\rho^{(1)} = \int_{-\infty}^{t} \mathrm{d}\tau \exp((t-\tau)L_F) L_g(x,\tau)\rho_0$$

The response of observable Ψ as an ensemble mean:

$$\langle \Psi \rangle^{(1)}(t) = \int \mathrm{d}x \Psi(x) \rho^{(1)} = \int \mathrm{d}x \Psi(x) \int_{-\infty}^{t} \mathrm{d}\tau \exp((t-\tau)L_F) L_g(x,\tau)\rho_0$$

For a multiplicative forcing $g(x,t) = g(x)f(t)$ we have

$$\langle \Psi \rangle^{(1)}(t) = \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}x \Psi(x) \exp((t-\tau)L_F) L_g(x)\rho_0 f(\tau) = \int_{-\infty}^{t} \mathrm{d}\tau G_{\Psi}^{(1)}(t-\tau) f(\tau)$$

That is, we can identify a linear Green's function as:

$$G_{\Psi}^{(1)}(t) = \int \mathrm{d}x \Psi(x) \exp(t L_F) L_g(x) \rho_0$$

Response theory

- Followed Chapter 7 of “Risken, The Fokker-Planck equation, Springer”
- Conditions:
  - Weak forcing – we embraced a **perturbative** approach and have gone only to first/linear order
  - Even so, there’s a radius of convergence for the series expansion
  - No “bifurcations on the fly”
  - Add a bit of noise to avoid mathematical problems
- Formal analogy with the solution of linear systems
- But abstarction is up a notch (operators instead of functions)
- Anyhow, how can we evaluate the linear Green’s functtion?
- Rethoric Q: Not analytically in general – numerically super easily! Lucarini, Ragone, Lunkeit, JOSP, 2017

Nonlinear (quadratic) response

Obtain eq. for $\rho^{(2)}$ by multiplying' by ϵ^{-2} and taking the limit of $\epsilon \to 0$:

$$\dot{\rho}^{(2)} = L_F \rho^{(2)} + L_g \rho^{(1)} + \epsilon^{-1} \underbrace{(L_F \rho^{(1)} + L_g \rho_0 - \dot{\rho}^{(1)})}_{0}$$

whose formal solution is:

$$\rho^{(2)} = \int_{-\infty}^{t} \mathrm{d}\tau_2 \exp((t-\tau_2)L_F) L_g(x,\tau_2)\rho^{(1)}$$

$$= \int_{-\infty}^{t} \mathrm{d}\tau_2 \int_{-\infty}^{\tau_2} \mathrm{d}\tau_1 \exp((t-\tau_2)L_F) L_g(x,\tau_2) \exp((\tau_2-\tau_1)L_F) L_g(x,\tau_1)\rho_0$$

Therefore, for a multiplicative forcing we can identify a second-order Green's function as:

$$G_{\Psi}^{(2)}(t_1,t_2) = \int \mathrm{d}x \Psi(x) \exp(t_2 L_F) L_g(x) \exp(t_1 L_F) L_g(x) \rho_0(x)$$

with which

$$\langle \Psi \rangle^{(2)}(t) = \int_{-\infty}^{t} \mathrm{d}\tau_2 \int_{-\infty}^{\tau_2} \mathrm{d}\tau_1 G_{\Psi}^{(2)}(\tau_2-\tau_1, t-\tau_2) f(\tau_1) f(\tau_2)$$

Question: What sort of experimental procedure can identify $G_{\Psi}^{(2)}(t_1,t_2)$?

Nonlinear (quadratic) response: example

Consider the SDE of the forced OU process:

$$\mathrm{d}x = (ax + Gf(t))\mathrm{d}t + \sqrt{2b^{-1}}\mathrm{d}W \tag{1}$$

whose Fokker-Planck eq. is:

$$\frac{\partial \rho}{\partial t} = \mathcal{L}^*[\rho] + \mathcal{L}_G[\rho], \tag{2}$$

$$\mathcal{L}^*[h(x)] = -\frac{\partial}{\partial x}[axh(x)] + b^{-1}\frac{\partial^2}{\partial x^2}[h(x)], \tag{3}$$

$$\mathcal{L}_G[h(x)] = -\frac{\partial}{\partial x}[Gh(x)] \tag{4}$$

The stationary solution satisfying $\mathcal{L}^*[\rho_0(x)] = 0$ is:

$$\rho_0(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{x^2}{2\sigma_0^2}}, \ \sigma_0^2 = -\frac{1}{ab}.$$

Pavliotis, Stochastic Processes and Applications, Springer

Eigenfunctions and eigenvalues of the generator $\mathcal{L}$

$$\mathcal{L}[h(x)] = ax\frac{\partial}{\partial x}[h(x)] + b^{-1}\frac{\partial^2}{\partial x^2}[h(x)]$$

of (1), solutions of $\mathcal{L}[f_n(x)] = \lambda_n f_n(x)$, are:

$$f_n(x) = \frac{1}{\sqrt{n!}}H_n(\sqrt{-ab}x),\ \lambda_n = -an,\ n = 0, 1, 2, \ldots, \tag{5}$$

where $H_n(\cdot)$ are the Hermite polynomials. E.g.:

$$H_0(x) = 1,\ f_0(x) = 1, \tag{6}$$

$$H_1(x) = x,\ f_1(x) = \sqrt{b}x, \tag{7}$$

$$H_2(x) = x^2 - 1,\ f_2(x) = \frac{ab}{\sqrt{2}}x^2 - \frac{1}{\sqrt{b}}. \tag{8}$$

We can use the identities $H_n'(x) = nH_{n-1}(x)$ and $H_{n+1} = xH_n(x) - H_n'(x)$ to verify this. Eigenvalues of $\mathcal{L}^*$ are the same as those of $\mathcal{L}$, and eigenfunctions are those of $\mathcal{L}$ multiplied by the stationary density:

$$\mathcal{L}^*[f_n(x)\rho_0(x)] = \lambda_n f_n(x)\rho_0(x).$$

This implies that

$$\mathcal{L}^{*m}[f_n(x)\rho_0(x)] = \lambda_n^m f_n(x)\rho_0(x),$$

and so

$$e^{\tau\mathcal{L}^*}[f_n(x)\rho_0(x)] = e^{\tau\lambda_n} f_n(x)\rho_0(x).$$

We can use some of these formulae to detail the second-order contribution to the solution for (2). Assuming $\rho^{(2)}(t=0)=0$ and $f(t<0)=0$, the generic formula of that can be obtained (I haven't tried to do this using the Duhamel-Dyson identity yet) as:

$$\rho^{(2)}(x,t) = \int_0^t \mathrm{d}\tau_2 \int_0^{\tau_2} \mathrm{d}\tau_1 f(\tau_1) f(\tau_2) e^{(t-\tau_2)\mathcal{L}^*} \mathcal{L}_G e^{(\tau_2-\tau_1)\mathcal{L}^*} \mathcal{L}_G \rho_0(x). \qquad (9)$$

Step-by-step detailing of the actions of the operators:

$$\mathcal{L}_G\rho_0(x) = -Ga\sqrt{b} f_1\rho_0, \qquad (10)$$

$$e^{(\tau_2-\tau_1)\mathcal{L}^*}\mathcal{L}_G\rho_0(x) = -Ga\sqrt{b}e^{-(\tau_2-\tau_1)a} f_1\rho_0, \qquad (11)$$

$$\mathcal{L}_G e^{(\tau_2-\tau_1)\mathcal{L}^*}\mathcal{L}_G\rho_0(x) = -G^2 abe^{-(\tau_2-\tau_1)a}\sqrt{2} f_2\rho_0, \qquad (12)$$

$$e^{(t-\tau_2)\mathcal{L}^*}\mathcal{L}_G e^{(\tau_2-\tau_1)\mathcal{L}^*}\mathcal{L}_G\rho_0(x) = -G^2 ab\sqrt{2}e^{-(\tau_2-\tau_1)a} e^{-(t-\tau_2)2a} f_2\rho_0 \qquad (13)$$

$$= -G^2 ab\sqrt{2} f_2\rho_0 e^{-a(2t-\tau_2-\tau_1)}. \qquad (14)$$

Nonlinear response

The complete series is the Volterra series:

$$\Psi(t) = h_0 + \lim_{N\to\infty} \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{j=1}^{n} f(t-\tau_j)\, d\tau_j = h_0 + \lim_{N\to\infty} \sum_{n=1}^{N} H_n f(t)$$

Or in discrete time: $g(\mathbf{f}) = \sum_{n=0}^{p} H_n \mathbf{f} = \sum_{n=0}^{p} \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} h^{(n)}_{i_1 \ldots i_n} f_{i_1} \cdots f_{i_n}$

Entries of the Volterra kernels can be found by ordinary regression minimizing the sum of squared errors for a data set of *N* pairs of $(\mathbf{f}_j, \Psi_j)$

$$SS = \sum_{j=1}^{N} (g(\mathbf{f}_j) - \Psi_j)^2$$

which involves the inversion of *M* x *M* matrices using SVD where *M* ~ m^n.

Nonlinear response

- This is unfeasible for many practical applications.
- Luckily, it can be shown (Franz & Schoelkopf, Neural Comp., 2006) that "all finite discrete Volterra series of degree *p* that are solutions of a linear regression problem can be generated by an expansion in **polynomial kernel functions**"

$$g(\mathbf{f}) = \sum_{n=0}^{p} H_n \mathbf{f} = \sum_{j=1}^{N} \alpha_j (1 + \mathbf{x}_j^T \mathbf{x}_j)^p$$

which greatly reduces the number of parameters (*N*) to be estimated.

- Matlab and Python codes by Matthias O Franz and co are available on github: https://github.com/mof2/volterra

Part II

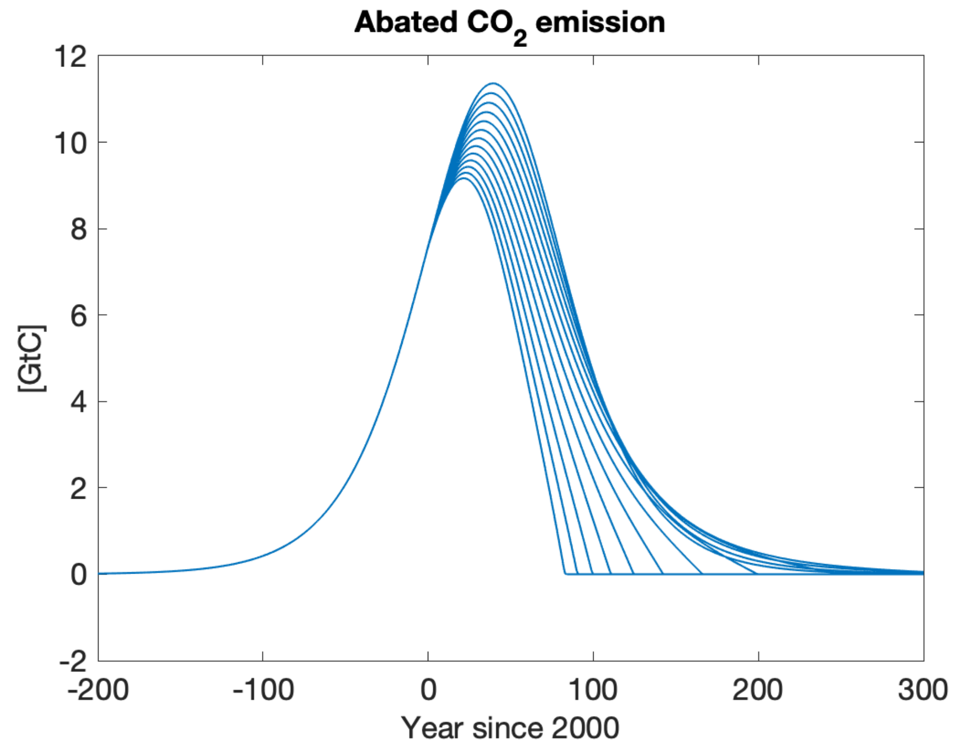
Application to the carbon cycle

Say, we model cum. anthropogenic **CO_2 emissions** "business as usual" (BAU) by a logistic eq.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right)$$

whose solution is the logistic curve:

$$P(t) = \frac{KP_0e^{rt}}{K + P_0\left(e^{rt} - 1\right)} = \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right)e^{-rt}}$$



We then model CO_2 abatement by reducing the carbon budget like this:

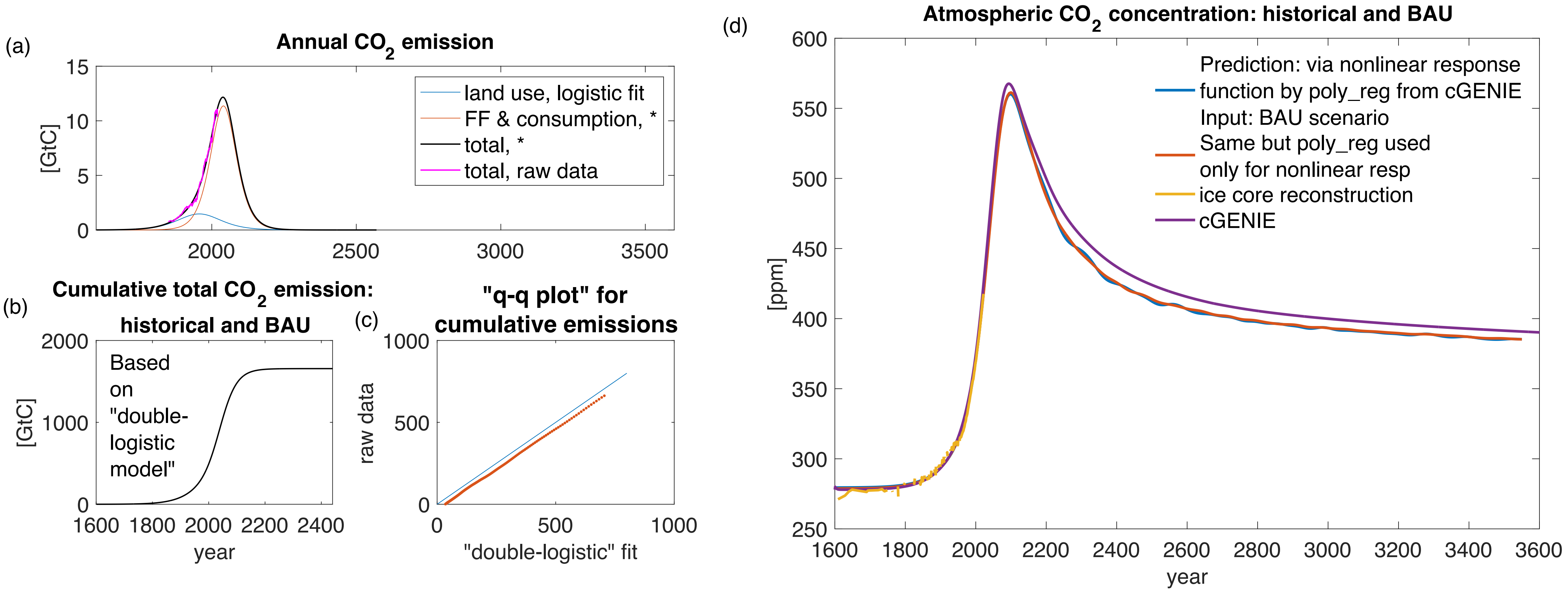
$$K(t) = K_{BAU} + (P(t) - K_{BAU})f(t), \quad f(t) = \min(r_a t, 1)$$

scenarios:

$r_a = k\text{e-}3$ [1/yr], $k = 0, \ldots, 12$

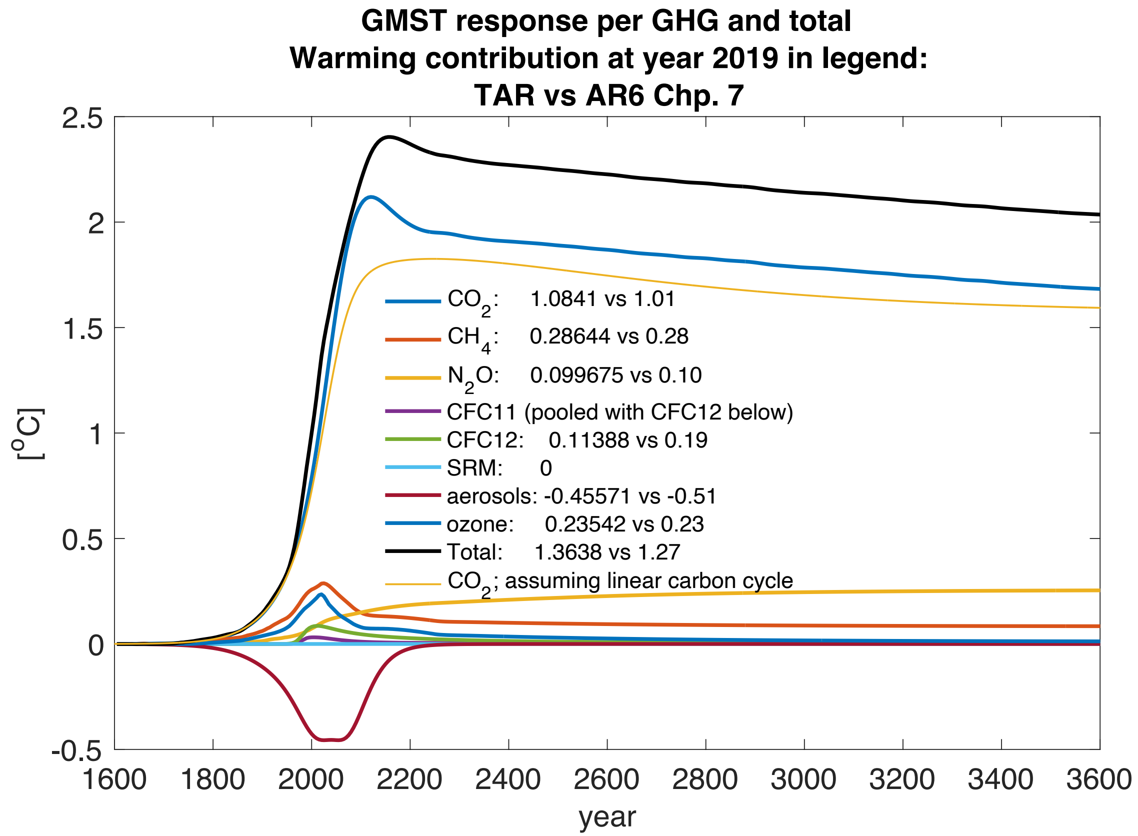
BAU

Carbon cycle model: [cGENIE](#)
by Andy Ridgwell

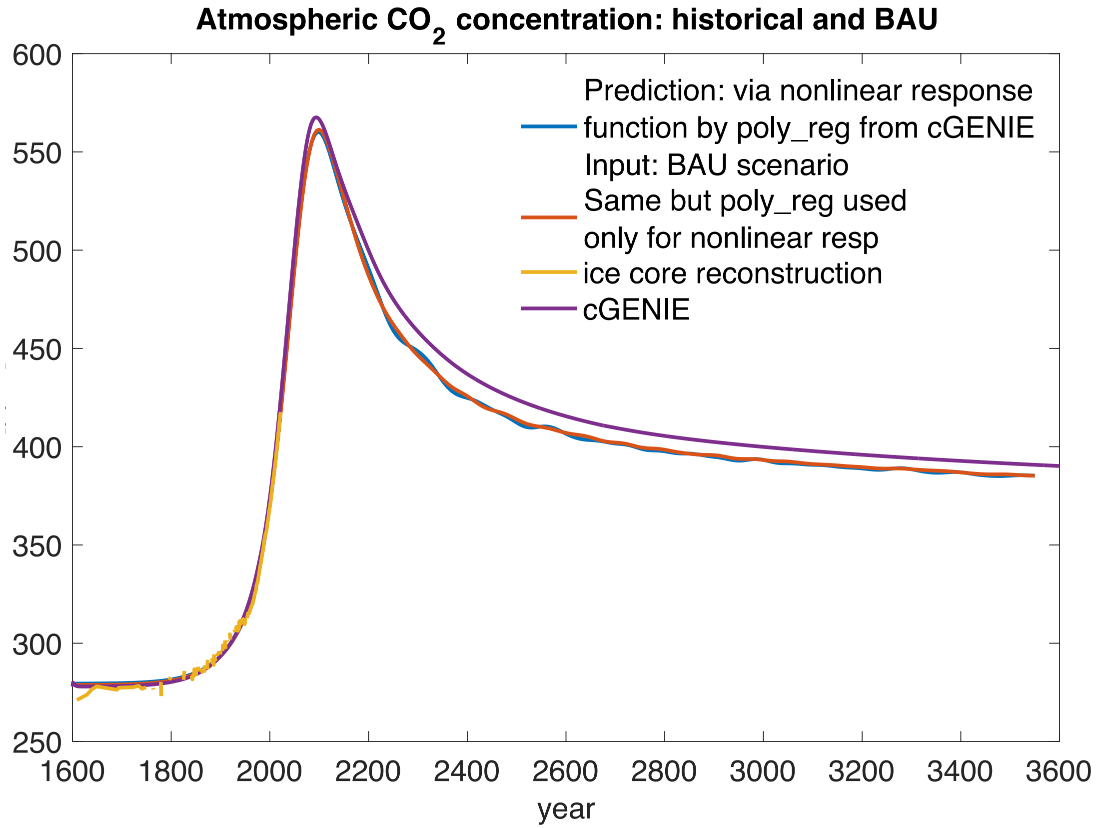


BAU

(e)



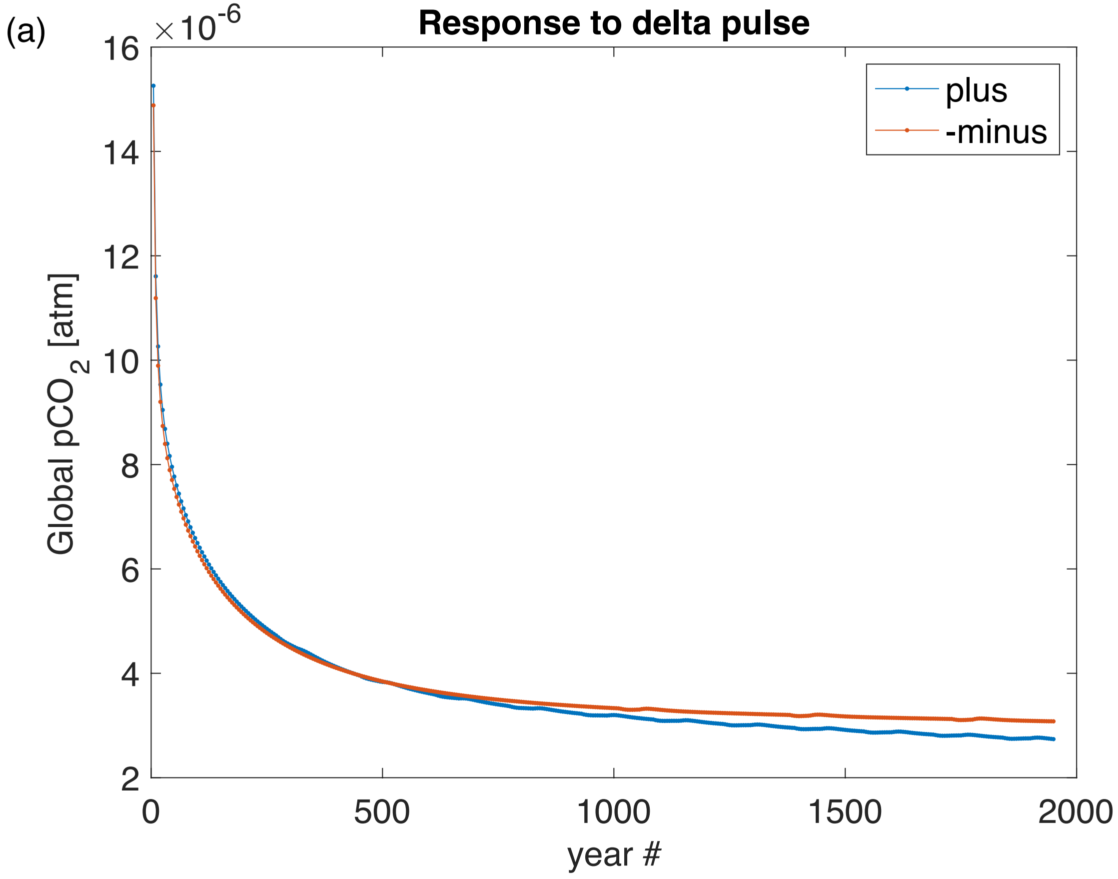
GMST response per GHG and total
Warming contribution at year 2019 in legend:
TAR vs AR6 Chp. 7
[°C]
year
CO2: 1.0841 vs 1.01
CH4: 0.28644 vs 0.28
N2O: 0.099675 vs 0.10
CFC11 (pooled with CFC12 below)
CFC12: 0.11388 vs 0.19
SRM: 0
aerosols: -0.45571 vs -0.51
ozone: 0.23542 vs 0.23
Total: 1.3638 vs 1.27
CO2; assuming linear carbon cycle

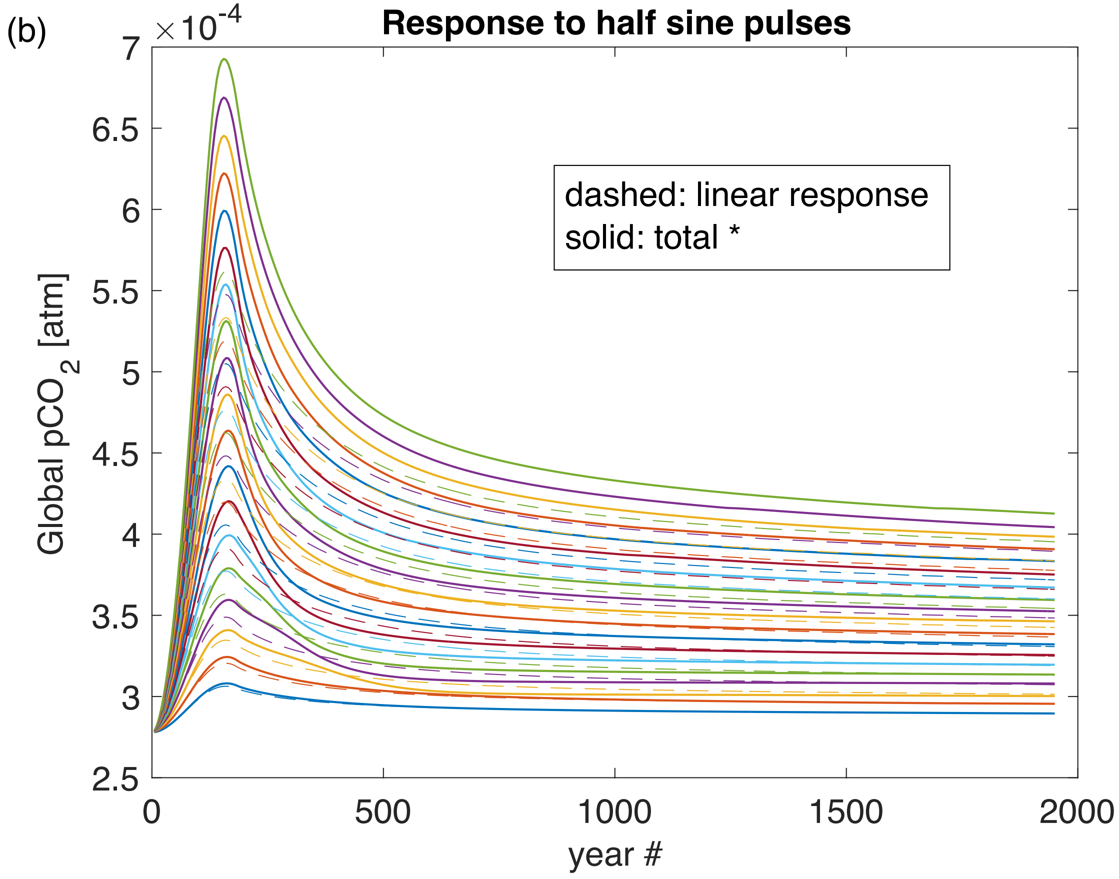


Atmospheric CO2 concentration: historical and BAU
year
Prediction: via nonlinear response
function by poly_reg from cGENIE
Input: BAU scenario
Same but poly_reg used
only for nonlinear resp
ice core reconstruction
cGENIE

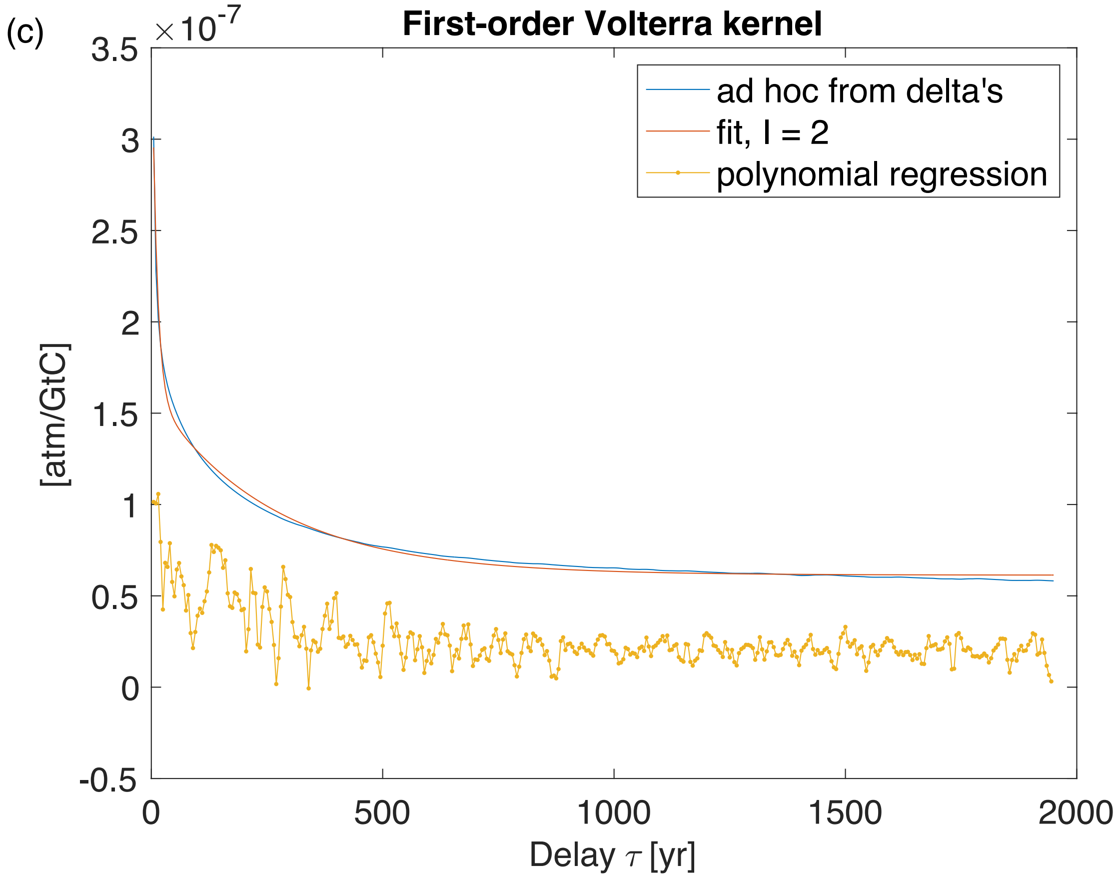
Application to the carbon cycle

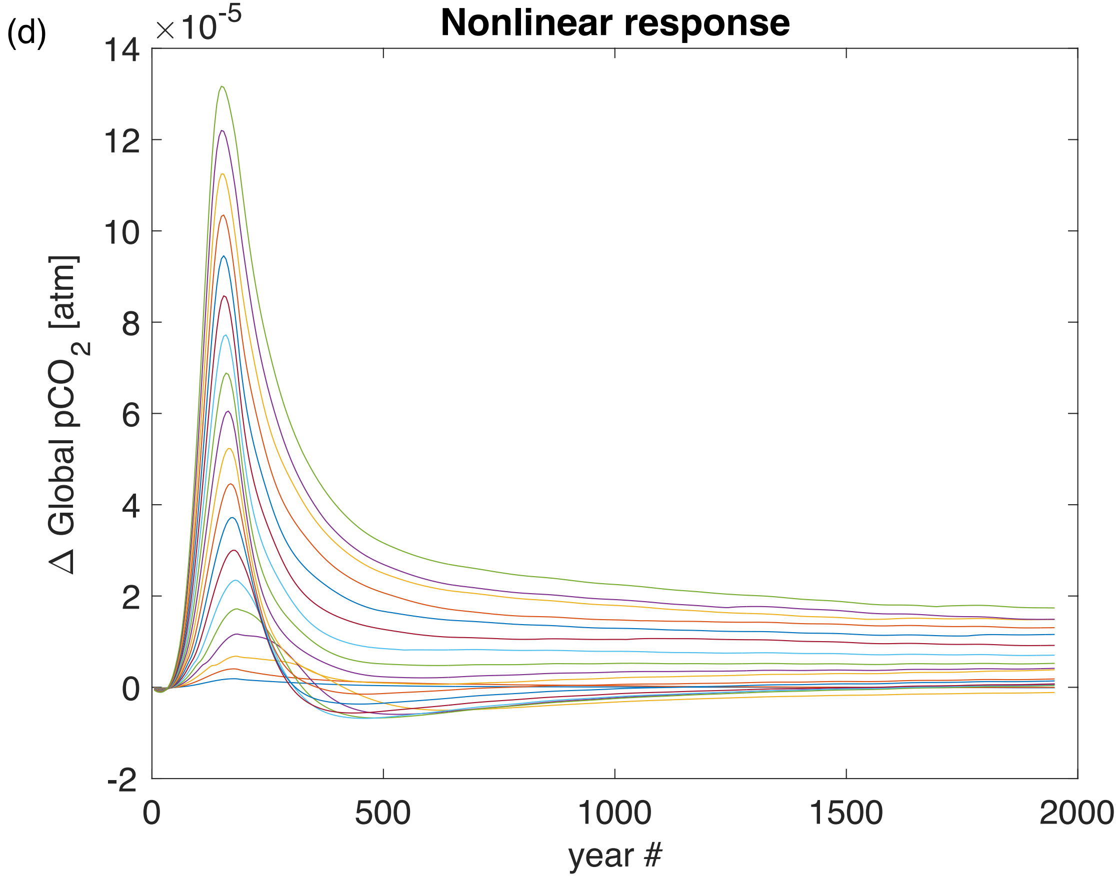
System identification via **external** forcing



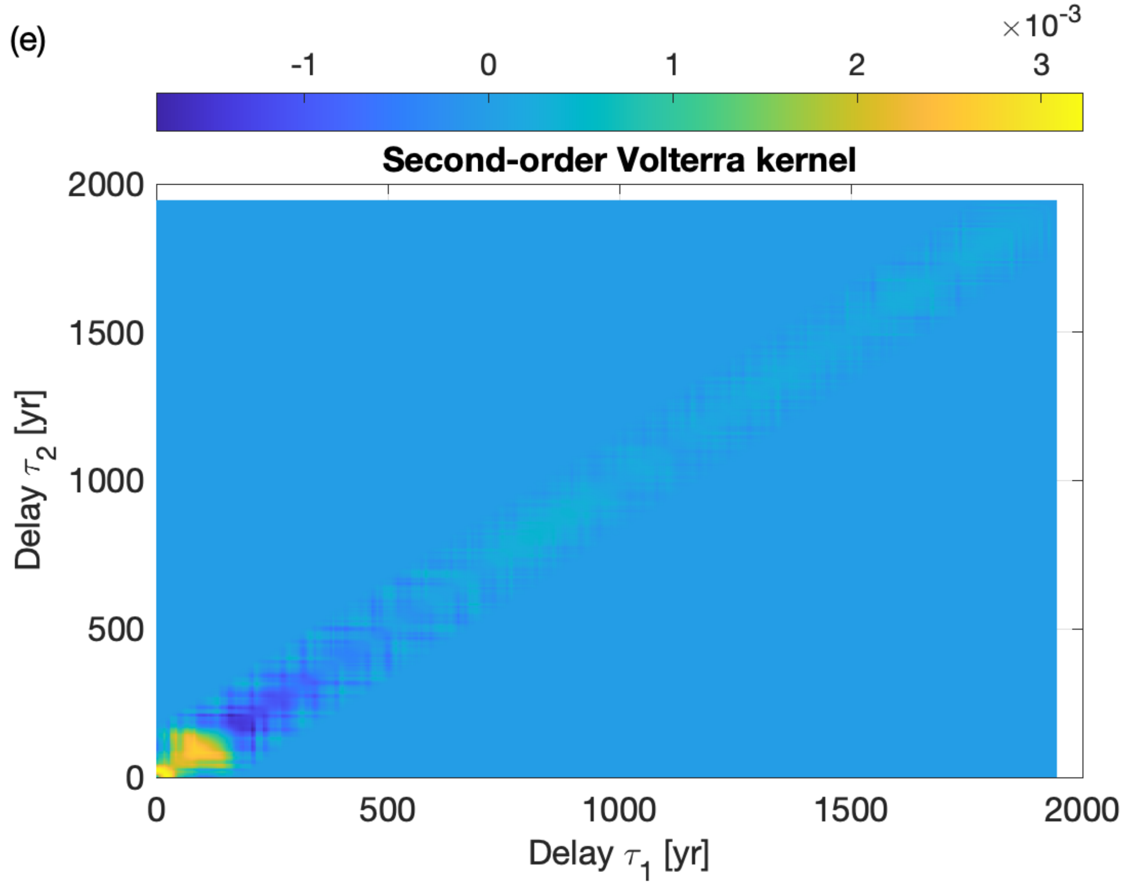


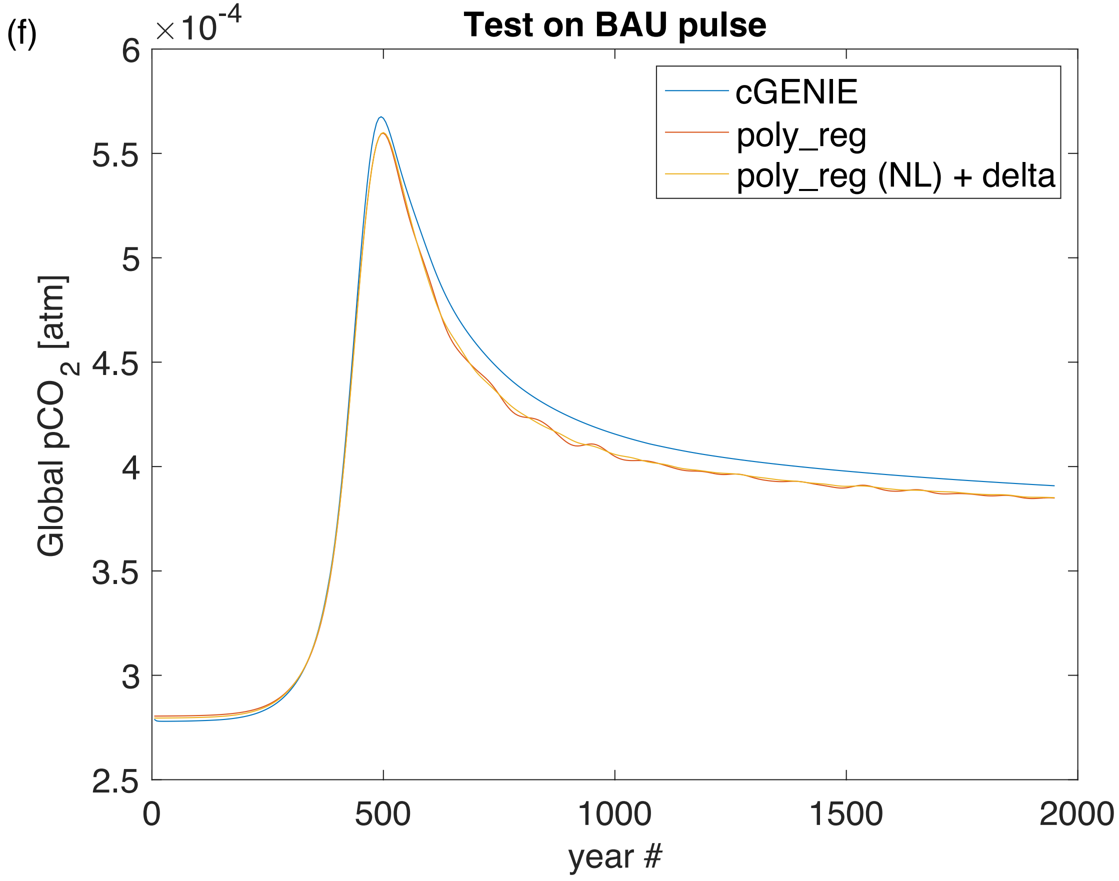
Application to the carbon cycle





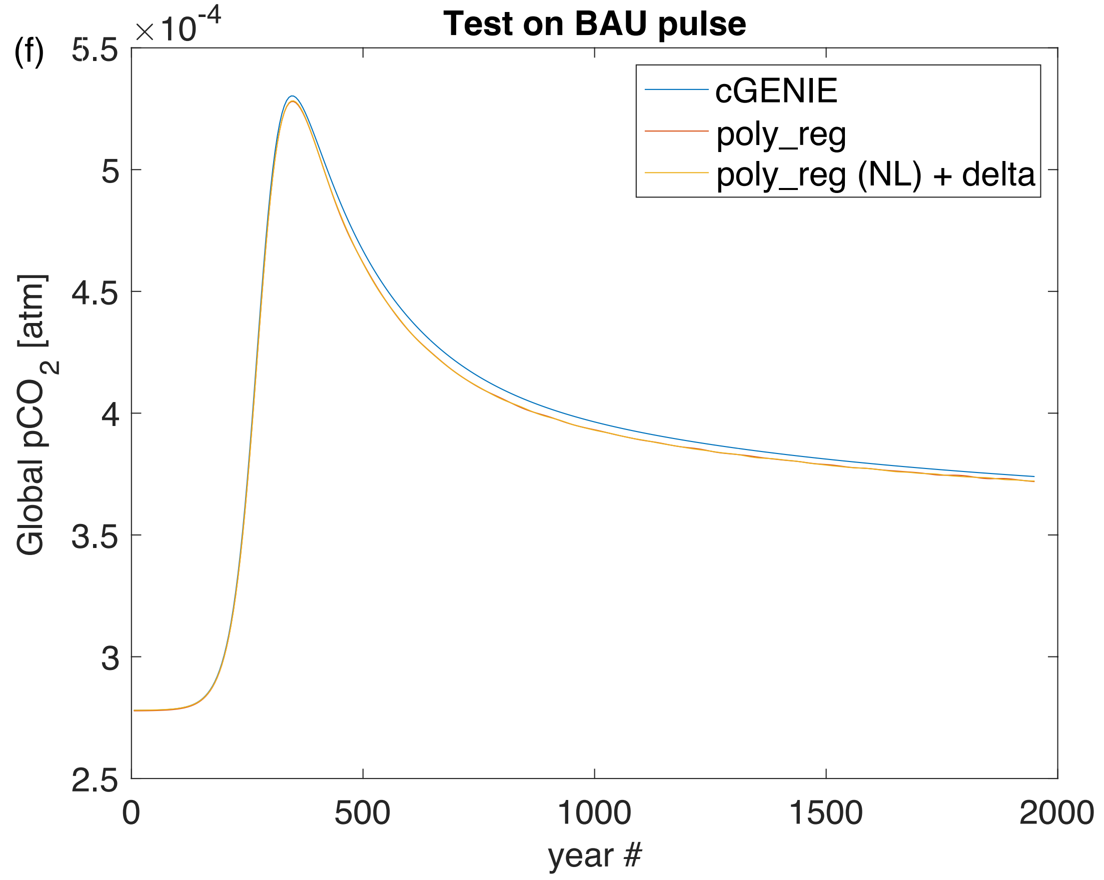
Application to the carbon cycle

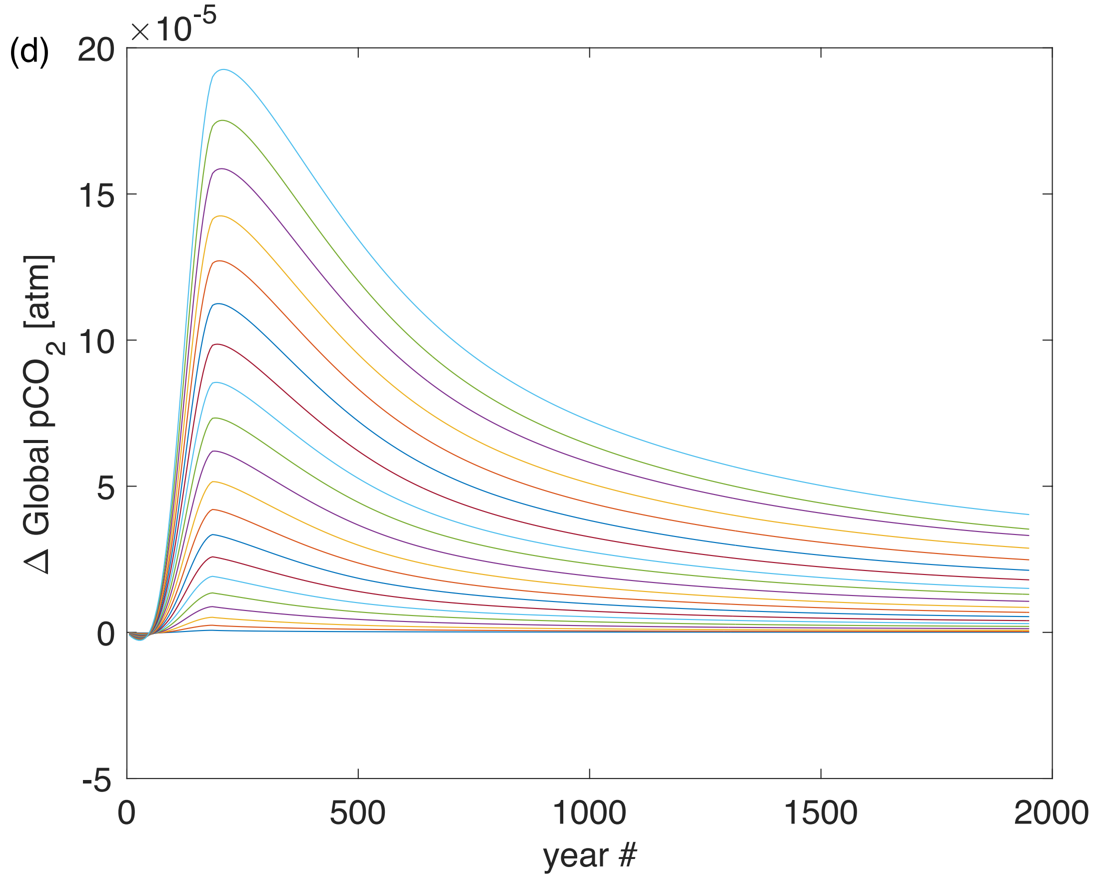




No vegetation in cGENIE

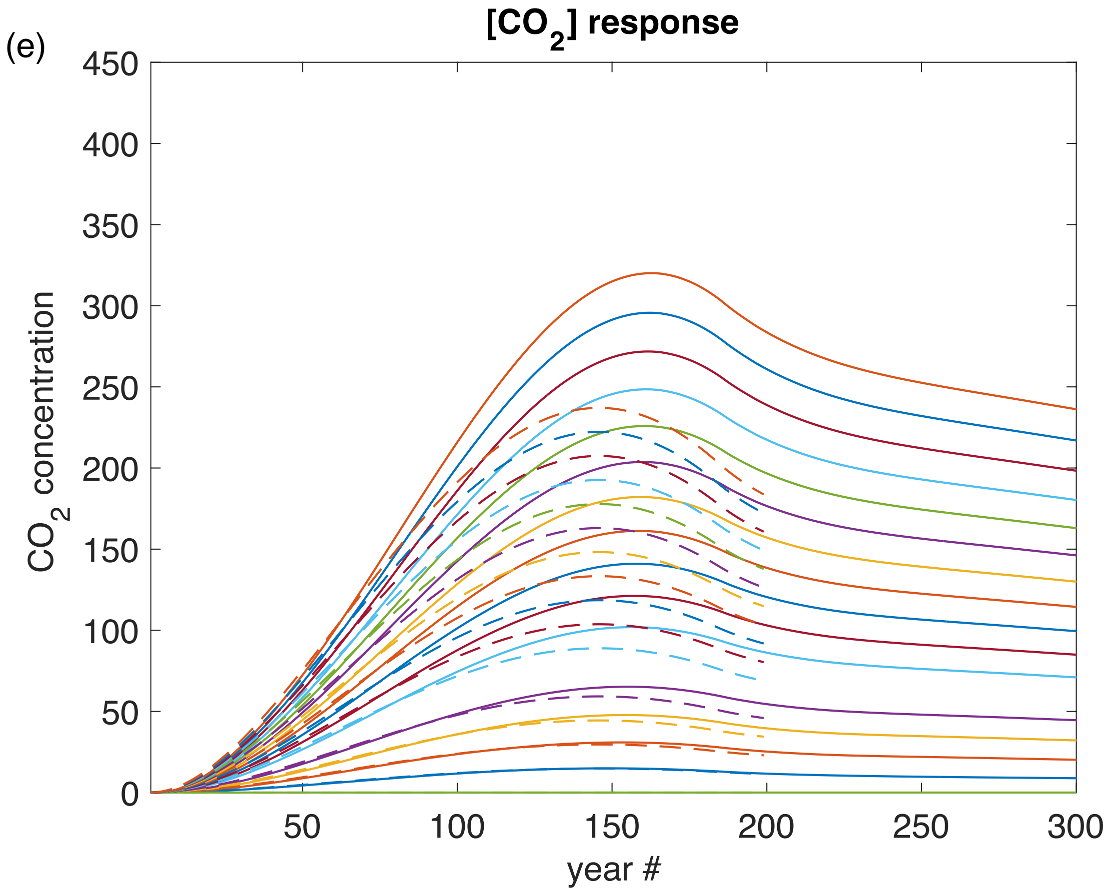
Stronger nonlinearity but better beahved



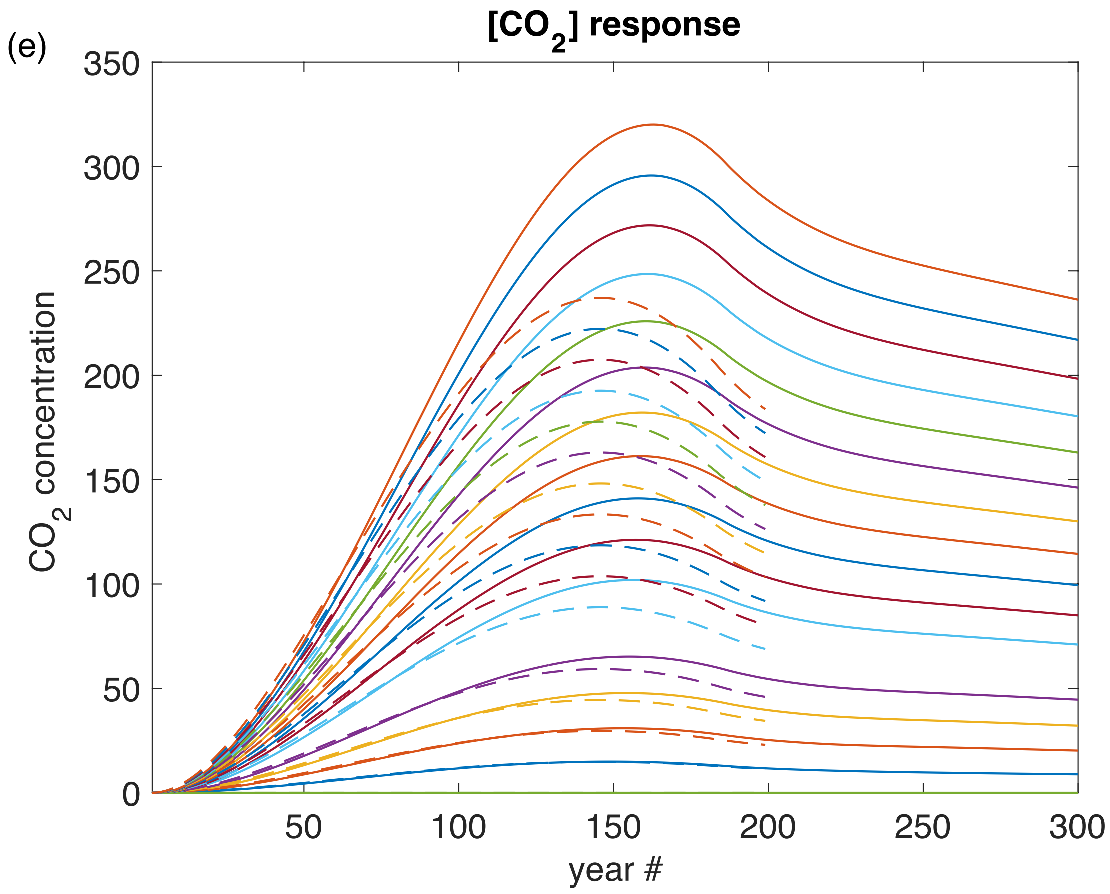


Other model: HECTOR (web app)

No vegetation



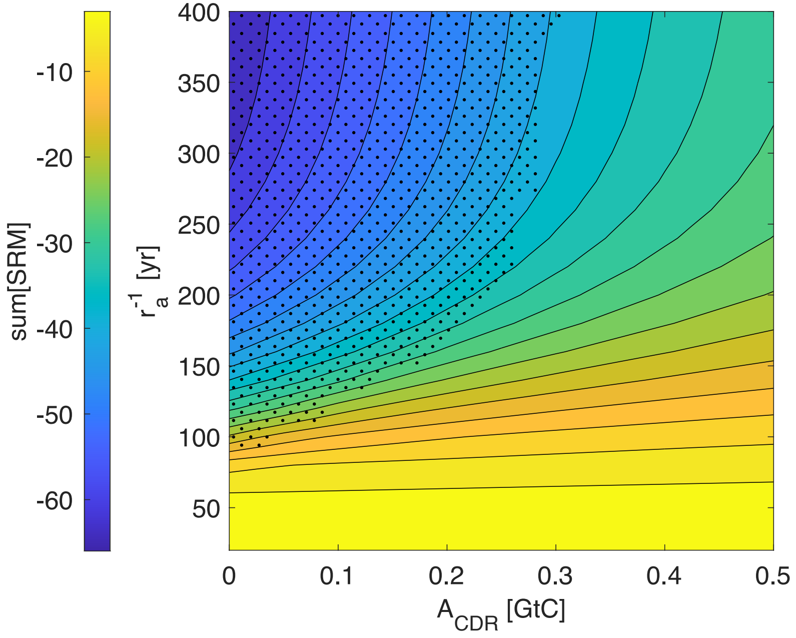
With vegetation



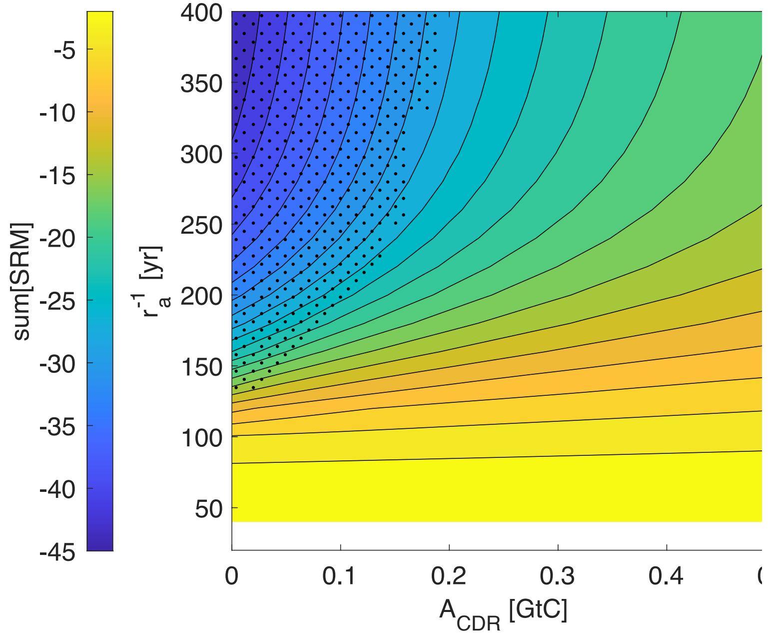
Minimal SRM geoengineering needed…

…in various scenarios by 1. temperature limits (Paris 2015 agreement), 2. abatement effort (r_a) and 3. CDR ability (A_{CDR}) ([Bodai et al. preprint]())

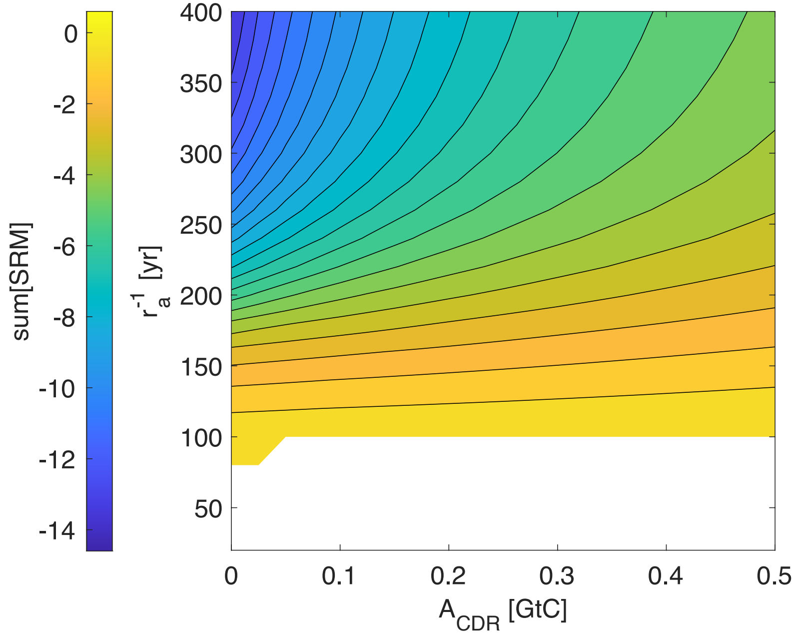
1.5 °C



1.7 °C



2.0 °C

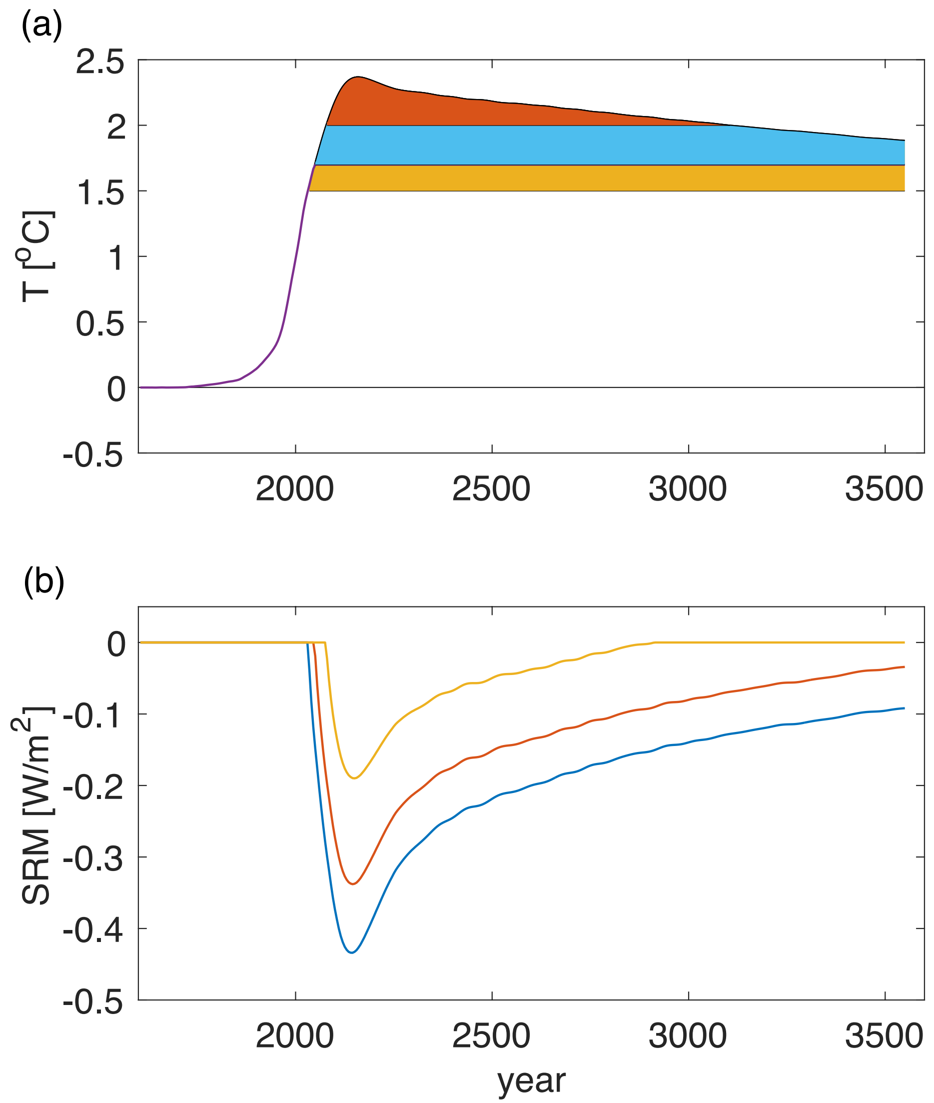


Minimal SRM geoengineering needed…

…in various scenarios by temperature limits (and e.g. r_a = 1e-3 1/yr, A_{CDR} = 0.1 GtC per annum)

Inverse problem for the SRM forcing time series: Bodai et al., Chaos, 2020

Solving inverse problems need a numerically lightweight emulator

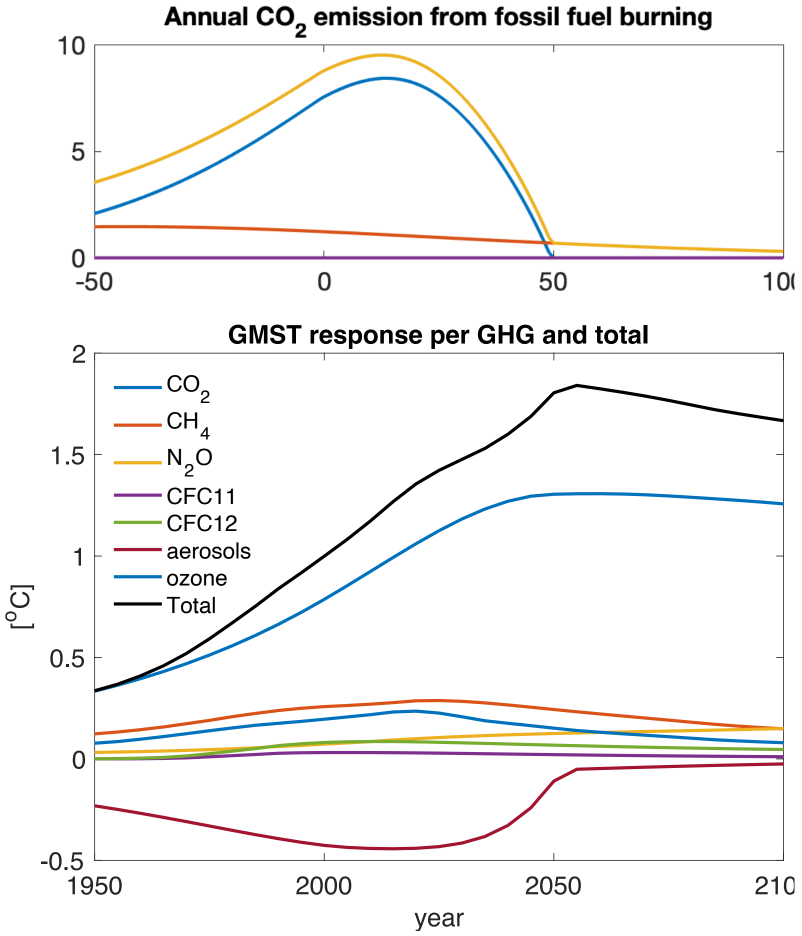


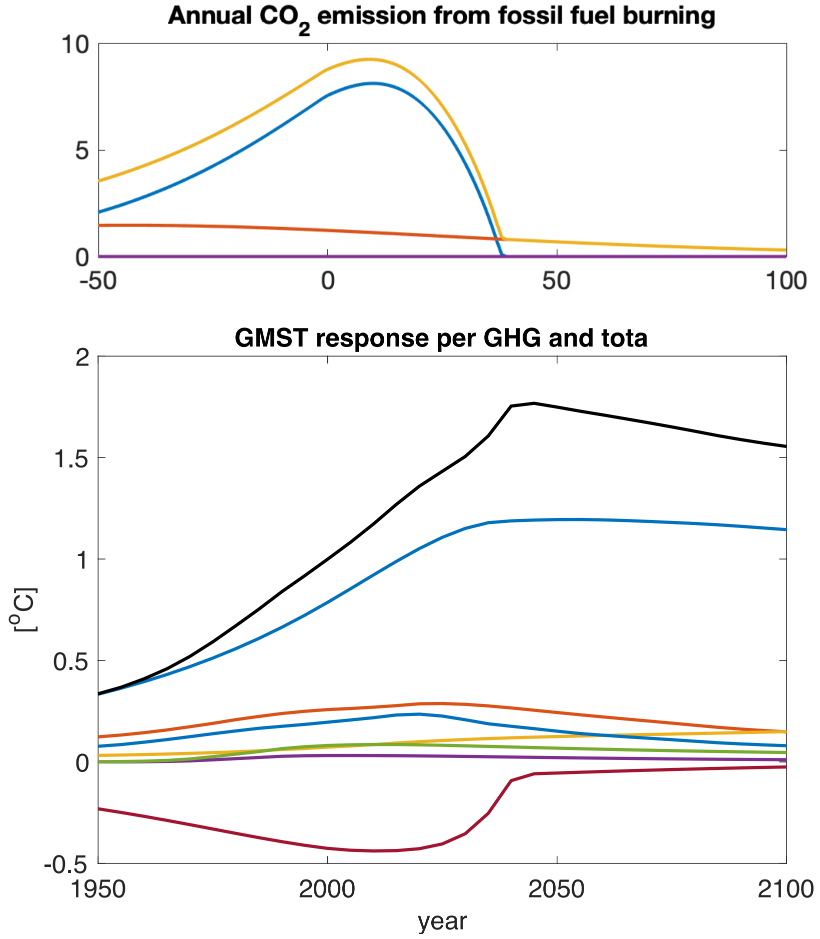
Problem with the IPCC special report on 1.5 °C warming??

Quote: “In order to achieve the 1.5 °C target, CO_2 emissions must decline by 45% (relative to 2010 levels) by 2030, reaching net zero by around 2050. […]

Under the pledges of the countries entering the Paris Accord, a sharp rise of 3.1 to 3.7 °C is still expected to occur by 2100.”

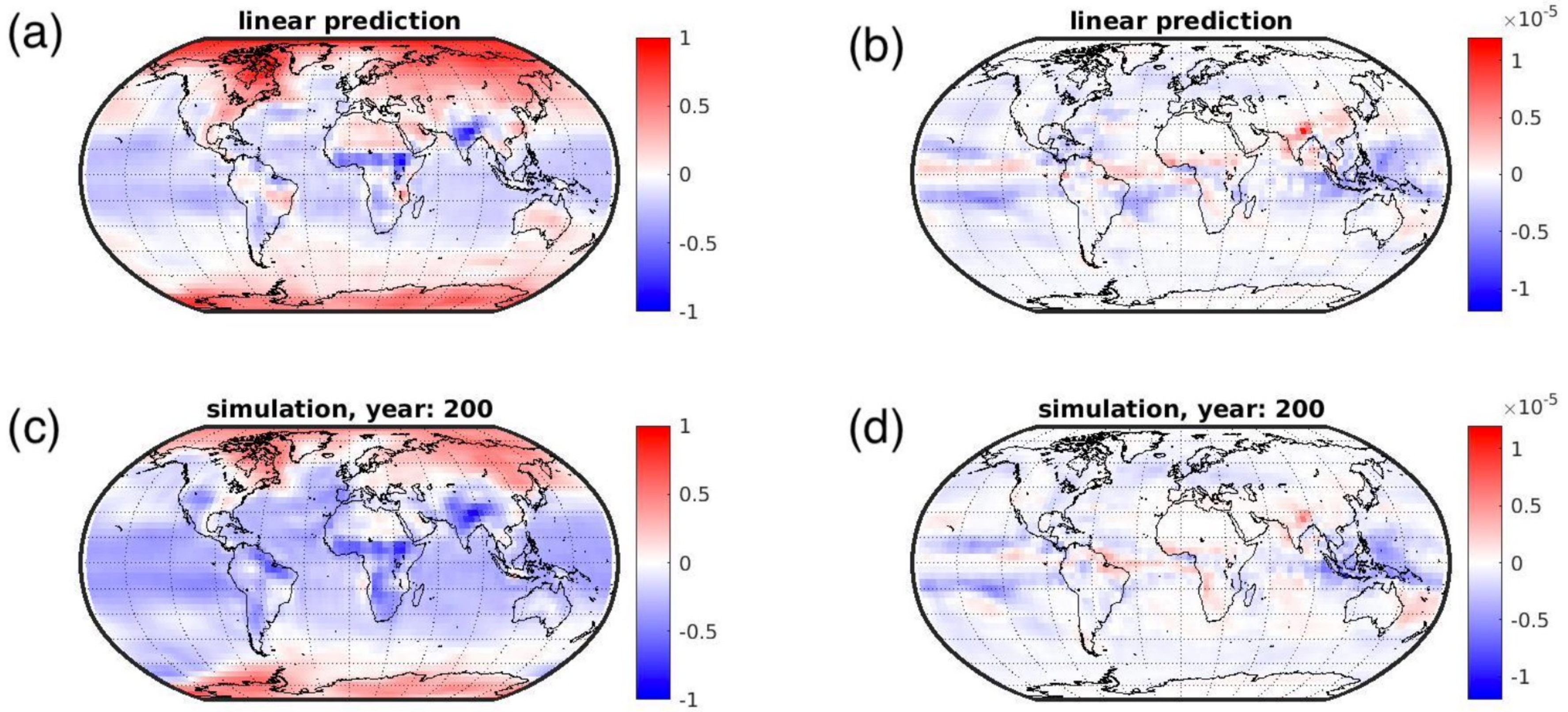
My calculations



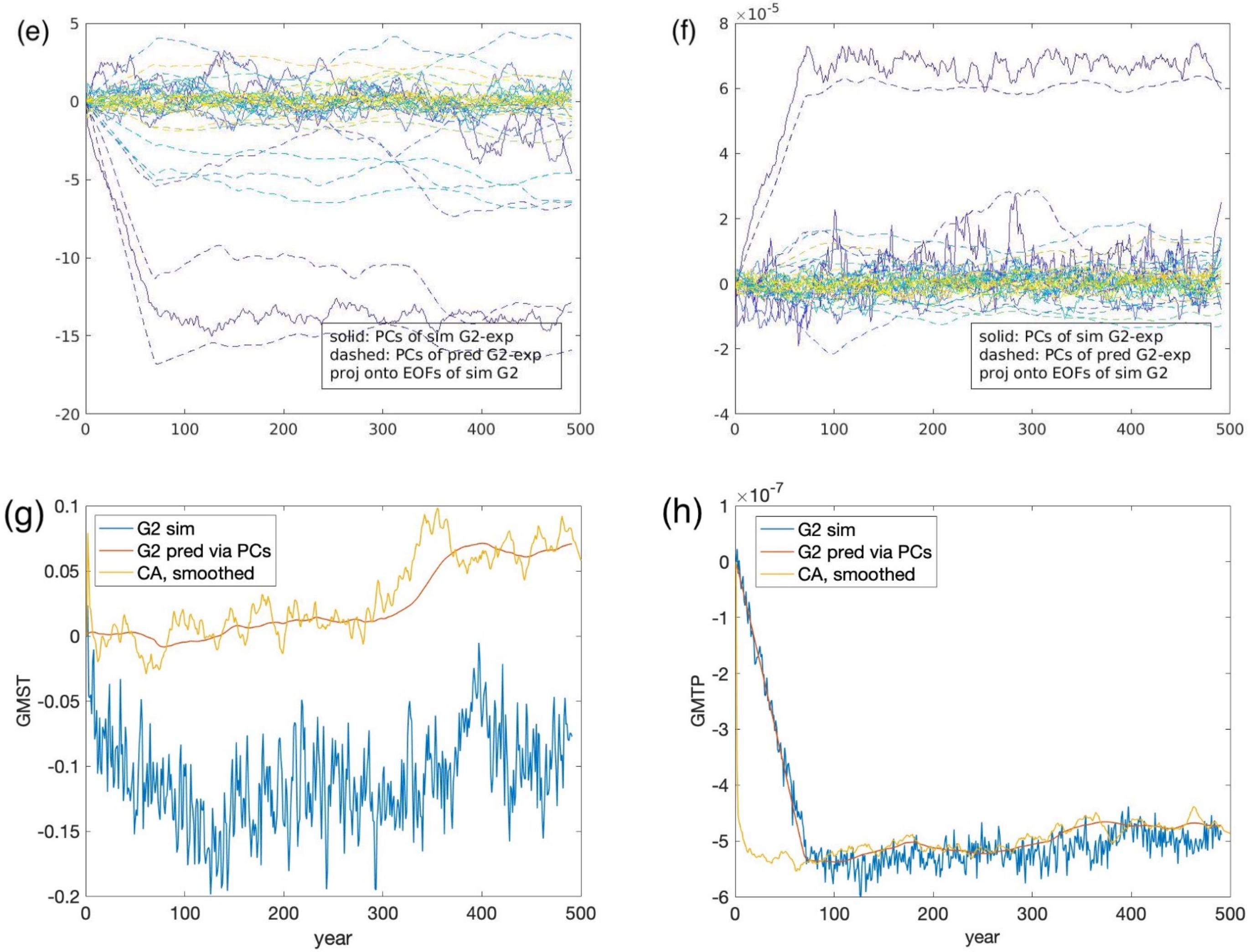


Spatial features/prediction…

…in the GeoMIP G2-type cancellation experiment



Spatial features/predictio – nonlinearity!



References

- **Preprint #1: The minimal geoengineering problem concerning the Paris 2015 agreement https://www.researchsquare.com/article/rs-3302963/v4**
- **Preprint #2: Modest geoengineering side effects predicted by an emulator https://www.researchsquare.com/article/rs-3308863/v2**
- **Emulator etc. code and data**
  - **#1: https://zenodo.org/records/10572748**
  - **#2: https://zenodo.org/records/8406742**
- **Web apps: http://bodaimatlab.zapto.org:9988**

THANK YOU!