Turbulence in Fluids

Tutorial week 2 : solutions

1. Hopf Bifurcation

See notebook.

2. Back-to-back saddle-node bifurcation

See notebook.

3. An example dynamical system

(a) At a fixed point we have

$$\frac{dx}{dt} = x + y - x(x^2 + y^2) = 0 \tag{1}$$

and

$$\frac{dy}{dt} = -x + y - y(x^2 + y^2) = 0 \tag{2}$$

The only point in $\mathbb{R}^2$ that satisfies these equations is $(0, 0)$. The linearized dynamical system around $(0, 0)$ is described by

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \mathbf{A}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \qquad \text{with } \mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{3}$$

The spectrum of $\mathbf{A}$ (i.e. the set of its eigenvalues) is given by $\sigma(\mathbf{A}) = \{1 + i, 1 - i\}$. Because there is a $\lambda \in \sigma(\mathbf{A})$ with $\mathrm{Re}(\lambda) > 0$, the point $(0, 0)$ is an unstable fixed point. In fact, in this case both eigenvalues have positive real part.

(b) One can use $r(x, y) = \sqrt{x^2 + y^2}$ and $\theta(x, y) = \arctan(\frac{y}{x})$ to transform equations (1) and (2) to

$$\frac{dr}{dt} = r - r^3 \qquad \text{and} \qquad \frac{d\theta}{dt} = -1 \tag{4}$$

Now we again recognize the fixed point $(0, 0)$, but notice that also the circle $r^2 = 1$ is invariant under the dynamics. The angle θ has a constant change in time. Note that $\frac{dr}{dt} > 0$ for $r \in (0, 1)$ and $\frac{dr}{dt} < 0$ for $r \in (1, \infty)$, i.e. flow lines spiral to $r^2 = 1$, which is

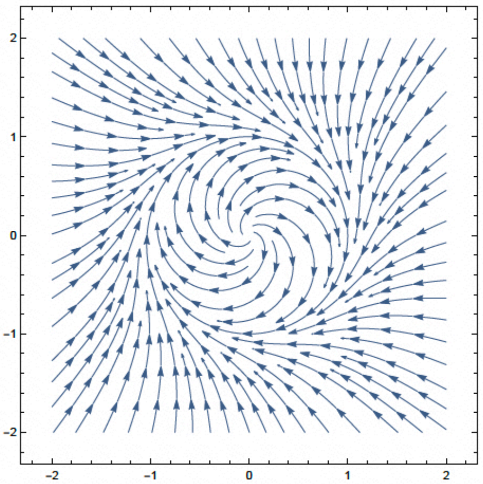

Figure 1: Streamplot of the dynamical system in exercise 1.

the limit cycle of this dynamical system, see figure 1.

4. Van der Pol oscillator

(a) The differential equation can be written as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\mu(x^2 - 1)y - x \end{cases} \tag{5}$$

where $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$.

(b) Let $\mu = 0$. Then the dynamical system in equation (5) can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{6}$$

which implies $\ddot{x} = x$ which has solutions of the form

$$x(t) = A\cos(t) + B\sin(t) \qquad \text{for some } A, B \in \mathbb{R} \tag{7}$$

which implies that

$$y(t) = \dot{x}(t) = A\sin(t) - B\cos(t) \tag{8}$$

Note that the constants $A, B \in \mathbb{R}$ depend on the initial conditions. In phase space, $(x(t), y(t))$ are circles around $(0, 0)$ with different radii for different choices of A and B.

Also note that the Jacobian matrix of $\dot{\mathbf{x}} = (\dot{x}, \dot{y})$ has determinant 1, in other words

$$\det(J(\dot{\mathbf{x}})) = 1 \tag{9}$$

which implies that the dynamical system is conservative.

(c) The only fixed point of the system in equation (5) is $(0,0)$. The linearized system around the origin is given by

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \mathbf{A}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \qquad \text{with } \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \tag{10}$$

The elements in $\sigma(\mathbf{A})$ (i.e. the eigenvalues of $\mathbf{A}$) both have a real part equal to $\frac{1}{2}\mu$, which implies that the fixed point is stable for $\mu < 0$ and unstable for $\mu > 0$.

(d) Let $\mu \neq 0$. Then with equation (5) we see that

$$\det(J(\dot{\mathbf{x}})) = 2\mu xy + 1 \tag{11}$$

which is in general not equal to 1, i.e. the dynamical system is not conservative.

(e) Assume we can write

$$\begin{cases} x(t) = a(t)\cos(\phi(t)) \\ \dot{x}(t) = -a(t)\sin(\phi(t)) \end{cases} \tag{12}$$

then the derivative of the first equation should equal the second, i.e.

$$\dot{a}(t)\cos(\phi(t)) - a(t)\dot{\phi}(t)\sin(\phi(t)) = -a(t)\sin(\phi(t)) \tag{13}$$

which relates $\dot{a}(t)$ and $\dot{\phi}(t)$. Also, the assumption in equation (12) should satisfy the van der Pol equation, which gives the second relation between $\dot{a}(t)$ and $\dot{\phi}(t)$. Solving these two equations gives

$$\begin{cases} \dot{a}(t) = -\mu a(a^2\cos^2\phi - 1)\sin^2\phi \\ \dot{\phi}(t) = 1 - \mu(a^2\cos^2\phi - 1)\sin\phi\cos\phi \end{cases} \tag{14}$$

(f) For ϕ we see that

$$\begin{aligned} \dot{\phi}(t) &= \frac{1}{2\pi}\int_0^{2\pi}(1 - \mu(a^2\cos^2\phi - 1)\sin\phi\cos\phi)d\phi \\ &= 1 - \frac{\mu a^2}{2\pi}\int_0^{2\pi}\cos^3\phi\sin\phi d\phi + \frac{\mu}{2\pi}\int_0^{2\pi}\cos\phi\sin\phi d\phi \\ &= 1 - 0 + 0 = 1 \end{aligned} \tag{15}$$

For determining $\dot{a}(t)$, we need the identities

$$\sin^2\phi = \frac{1}{2} - \frac{1}{2}\cos 2\phi, \qquad \cos^2\phi = \frac{1}{2} + \frac{1}{2}\cos 2\phi \tag{16}$$

and also remember the integrals

$$\int_0^{2\pi} \cos(n\phi)d\phi = 0, \qquad \int_0^{2\pi} \cos^2(n\phi)d\phi = \frac{1}{2} \qquad \forall n \in \mathbb{N} \tag{17}$$

With these ingredients we can calculate

$$\begin{aligned}
\dot{a}(t) &= \frac{1}{2\pi}\int_0^{2\pi} (-\mu a(a^2\cos^2\phi - 1)\sin^2\phi)d\phi \\
&= -\frac{\mu a^3}{2\pi}\int_0^{2\pi} \cos^2\phi\sin^2\phi d\phi + \frac{\mu a}{2\pi}\int_0^{2\pi} \sin^2\phi d\phi \\
&= -\frac{\mu a^3}{2\pi}\int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2\phi\right)\left(\frac{1}{2} - \frac{1}{2}\cos 2\phi\right) d\phi + \frac{\mu a}{2\pi}\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2\phi\right) d\phi \\
&= -\frac{\mu a^3}{2\pi}\int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{4}\cos^2 2\phi\right) d\phi + \frac{\mu a}{2\pi}\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2\phi\right) d\phi \\
&= -\frac{\mu a^3}{4} + \frac{\mu a^3}{8} + \frac{\mu a}{2} - 0 = \frac{\mu a(4-a^2)}{8}
\end{aligned} \tag{18}$$

(g) First note that equation (15) implies a constant change of the function $\phi(t)$. More interesting is the solution of equation (18) which is found by separating variables. Also note that for $a = 2$ we have $\dot{a} = 0$, so there will be our limit cycle. The calculation goes as follows:

$$\frac{da}{dt} = \frac{\mu a(4-a^2)}{8} \qquad \Longrightarrow \qquad \frac{1}{a(4-a^2)}da = \frac{\mu}{8}dt \tag{19}$$

which we can work out to be

$$\begin{aligned}
\frac{1}{a(2-a)(2+a)}da &= \frac{\mu}{8}dt \\
8\left(-\frac{1}{4a} + \frac{1}{8(2-a)} + \frac{1}{8(2+a)}\right) da &= \mu dt \\
(-2\log(a) - \log(2-a) + \log(2+a)) &= \mu(t+t_0) \\
\log\left(\frac{2+a}{a^2(2-a)}\right) &= \mu t \qquad \text{(for } t_0 = 0\text{)}
\end{aligned} \tag{20}$$

A Taylor series of a around $a = 2$ then gives us

$$a(t) \approx 2 - e^{-\mu t} \tag{21}$$

which confirms our limit cycle at $a = 2$.

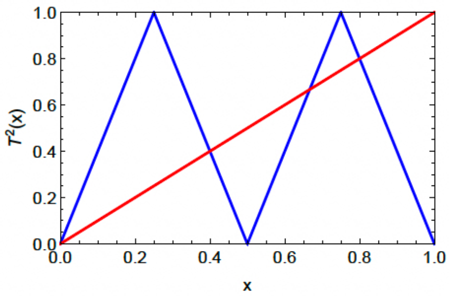

Figure 2: Plot of the function $T^2(x)$

5. One-dimensional tent map

(a) The formula for $T^2(x)$ is given by

$$T^2(x) = \begin{cases} 4x & \text{if } 0 \leq x < 1/4 \\ 2-4x & \text{if } 1/4 \leq x < 1/2 \\ 4x-2 & \text{if } 1/2 \leq x < 3/4 \\ 4-4x & \text{if } 3/4 \leq x \leq 1 \end{cases} \tag{22}$$

which is plotted in figure 2.

(b) The fixed points of $T(x)$ are $\{0, 2/3\}$ and those of $T^2(x)$ are $\{0, 2/5, 2/3, 4/5\}$.

(c) With every iteration, the tent folds itself into the interval $[0, 1]$, which in sloppy terms means that T^n consists of 2^{n-1} 'tents'. The number of fixed points is then 2^n.

6. Logistic map

The logistic map is given by

$$x_{n+1} = f(x_n) \qquad \text{where } f(x_n) = rx_n(1-x_n) \tag{23}$$

with r a parameter. It is easy to see that the fixed points of $f(x_n)$ are $\bar{x}_n = 0$ and $\bar{x}_n = 1 - 1/r$. For a period 2 orbit of the logistic map, we have to look at the fixed points of $f^2(x_n) = f(f(x_n))$. Note that we can write

$$f^2(x_n) = r^2 x_n(1-x_n)(1-rx_n+rx_n^2) \tag{24}$$

and in order to find its fixed points we have to solve $f^2(x_n) = x_n$. That means we have to solve

$$r^2 x_n(1-x_n)(1-rx_n+rx_n^2) - x_n = 0 \tag{25}$$

Note that fixed points of $f(x_n)$ are also fixed points of $f^2(x_n)$, therefore we write equation (25) as

$$x_n(x_n - 1 + 1/r)(r^3 x_n^2 - (r^3 + r^2)x_n + r^2 + r) = 0 \tag{26}$$

Because we know that the points $\bar{x}_n = 0$ and $\bar{x}_n = 1 - 1/r$ are also fixed points of $f(x_n)$, we are interested in the solutions of

$$r^3 x_n^2 - (r^3 + r^2)x_n + r^2 + r = 0 \tag{27}$$

and we call those points p_1 and p_2. In that case, the period 2 orbit is given by $\{p_1, p_2\}$. The points are

$$p_{1,2} = \frac{1}{2} + \frac{1}{2r}\left(1 \pm \sqrt{(r-3)(r+1)}\right) \tag{28}$$

The period 2 orbit $\{p_1, p_2\}$ is stable if $|f'(p_1)f'(p_2)| < 1$. With equations (23) and (28) we find

$$f'(p_1)f'(p_2) = -r^2 + 2r + 4 \tag{29}$$

which equals 1 at $r = 3$ and decreases monotonically to -1 at $r = 1 + \sqrt{6}$. This implies that the period 2 orbit $\{p_1, p_2\}$ is stable for $r \in (3, 1 + \sqrt{6})$ and becomes unstable at $r = 1 + \sqrt{6}$. With similar arguments, one can demonstrate that for $r > 1 + \sqrt{6}$ a period 4 orbit arises.

7. Period 3 orbits in logistic map

We want to show that there exists a period 3 orbit of the logistic map for $r = 1 + \sqrt{8}$. We are going to find this orbit both numerically and analytically. One can use any programming language, the solution is here given for Mathematica.

Numerical solution

Let us define the function `f[x]` in Mathematica, according to equation (23), and we set `r = 1 + Sqrt[8]`. Then we search for fixed points of the third iteration of `f[x]` by solving

```
NSolve[f[f[f[x]]]==x,x]
```
(30)

which gives us the numerical solution for the period 3 orbit, namely $\{0.159929, 0.514355, 0.956318\}$. For $r = 1 + \sqrt{8} + \epsilon$ with ϵ small, we find two period 3 orbits. One can easily show that one of these is stable, whereas the other one is unstable. For ϵ big enough (however, still very small) a period 6 orbit arises and for even larger r orbits of period 12, 24 etc. These are period doubling bifurcations, just as the ones for smaller r (of period 1,2,4,...).

Analytical solution
We want to find the critical value for r where a period 3 orbit arises. Call this critical value r_c. We want to proof that $r_c = 1 + \sqrt{8}$. Points in a period 3 orbit must satisfy $f^3(x) \equiv f(f(f(x))) = x$. Note that fixed points of $f(x)$ itself are automatically fixed points of $f^3(x)$. This means that $x = 0$ and $x = 1 - 1/r$ are solutions of $f^3(x) - x = 0$. Furthermore, note that at $r = r_c$, $f^3(x) - x$ must have three double roots (as the stable and unstable fixed points coincide for this critical value). This means that we can write

$$f^3(x) - x = Ax(x - 1 + 1/r)((x-a)(x-b)(x-c))^2 = 0 \qquad \text{for some } A \in \mathbb{R} \tag{31}$$

where $\{a, b, c\}$ will be the period 3 orbit. Another important observation is that at a, b, c we must have

$$\frac{d(f^3(x))}{dx} = 1 \tag{32}$$

which we can write as

$$\begin{aligned} \frac{d(f^3(x))}{dx} &= \frac{d(f^3(x))}{d(f^2(x))}\frac{d(f^2(x))}{d(f(x))}\frac{d(f(x))}{dx} \\ &= \frac{d(f(x))}{dx}(c)\frac{d(f(x))}{dx}(b)\frac{d(f(x))}{dx}(a) \\ &= r^3(1-2a)(1-2b)(1-2c) \\ &= r^3(1 - 2\alpha + 4\beta - 8\gamma) = 1 \end{aligned} \tag{33}$$

where we define $\alpha = a + b + c$, $\beta = ab + bc + ac$ and $\gamma = abc$. We use these to write equation (31) as

$$\begin{aligned} f^3(x) - x = A(&x^8 - [2\alpha + 1 - 1/r]x^7 + [2\beta + \alpha^2 + 2(1 - 1/r)\alpha]x^6 \\ &- [2\gamma - 2\alpha\beta + 2(1-1/r)\beta + (1-1/r)\alpha^2]x^5 + ...) \end{aligned} \tag{34}$$

On the other hand, we can use the definition in equation (23) to explicitly write $f^3(x)$ as

$$\begin{aligned} f^3(x) - x &= r^3x(1-x)[1 - rx(1-x)][1 - r^2x(1-x)(1 - rx(1-x))] \\ &= -r^7[x^8 - 4x^7 + (6 + 2/r)x^6 - (4 + 6/r)x^5 + ...] \end{aligned} \tag{35}$$

and together with equation (34) we can find $2\alpha = 3 + \frac{1}{r}$, $4\beta = \frac{3}{2} + \frac{5}{r} + \frac{3}{2r^2}$ and $8\gamma = -\frac{1}{2} + \frac{7}{2r} + \frac{5}{2r^2} + \frac{5}{2r^3}$. If we fill this in in equation (33), we get

$$r^2 - 2r - 7 = 0 \tag{36}$$

which has positive root $r_c = 1 + \sqrt{8}$.