Tutorial week 2

1. Hopf Bifurcation

Notebook: Tutorial-w2-1.ipynb

We consider first the two-dimensional dynamical system I

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

with $A_{11} = \lambda$, $A_{12} = 1$, $A_{21} = -1$, $A_{22} = \lambda$, where λ is a real number.
a.
Plot the phase portraits for $\lambda = -1, \lambda = 0$ and $\lambda = 1$.

b.
For which values of λ is the origin a stable fixed point?

Next, we study the dynamical system II

$$\frac{dx}{dt} = \lambda x - y - x(x^2 + y^2)$$

$$\frac{dy}{dt} = \lambda y + x - x(x^2 + y^2)$$

c.
What is the relation between the dynamical systems I and II?

d.
Determine the phase portraits of system II for $\lambda = -1, \lambda = 0$ and $\lambda = 1$ and describe what new dynamical behaviour occurs when λ crosses zero.

2. Back-to-back saddle-node bifurcation

Notebook: Tutorial-w2-2.ipynb

In this exercise, we consider the dynamical system

$$\frac{dx}{dt} = ax^3 + bx + \phi$$

where $a < 0$ and ϕ is considered as a control parameter.

a.
Show that this system has multiple (real valued) equilibria if and only if $b > 0$ and

$$|\phi| < \sqrt{\frac{-4b^3}{27a}}$$

b.
If the conditions under a. are satisfied, determine the positions ϕ of the two saddle-node bifurcations.

c.
Plot the bifurcation diagram versus ϕ for $a = -0.5$ and $b = 0.5$.

Next, let $\phi = \phi_0 + \alpha t$ be time dependent.

d. For $a = -0.5$, $b = 0.5$ and $\phi_0 = -0.5$, study the behaviour of trajectories for several well-chosen values of α. Describe and explain the behaviour found.

3. An example dynamical system

Consider the two-dimensional dynamical system described by the equations

$$\frac{dx}{dt} = x + y - x(x^2 + y^2) \quad \text{and} \quad \frac{dy}{dt} = -(x - y) - y(x^2 + y^2).$$

(a) Identify the fixed points of this dynamical system. Are they stable or unstable?

(b) Show that the system has a limit cycle, given by $x^2 + y^2 = 1$ [Hint: use cylindrical co-ordinates rather than Cartesian co-ordinates]. Use flow lines in the phase space to show how phase space trajectories should converge to the limit cycle.

4. Van der Pol oscillator

Consider the nonlinear second order differential equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0,$$

known as van der Pol equation. The Van der Pol equation was originally proposed by the Dutch electrical engineer and physicist Balthasar van der Pol while he was working at Philips. He was trying to describe oscillations in electrical circuits that employed vacuum tubes.

(a) Convert the equation to a first order differential equation (as discussed in the class), so that you can describe the dynamics in phase space.

(b) Consider the case $\mu = 0$, which renders the equation to that of a simple harmonic oscillator with angular frequency unity. Solve the equation for $x(t)$ and $\dot{x}(t) = dx(t)/dt$, and draw the phase space orbits. Show that the phase space dynamics is conservative.

(c) Find the fixed points of the dynamics. Are these fixed points stable or unstable (consider both cases, $\mu > 0$ and $\mu < 0$)?

(d) Show that for $\mu \neq 0$ the phase space dynamics is non-conservative.

From here on you will assume that $\mu > 0$, and attempt to solve the van der Pol equation for $\mu > 0$. The exercise will demonstrate to you that van der Pol admits a limit cycle solution. In part (b) you have already solved for $x(t)$ and $\dot{x}(t)$ for the case $\mu = 0$. Using that, attempt a form of the solution

$$x(t) = a(t)\cos[\phi(t)], \quad \dot{x}(t) = -a(t)\sin[\phi(t)]$$

for the van der Pol equation. Note that this is a trial solution, there is no guarantee that a solution must be of this form.

(e) First, using the trial solution obtain the equations

$$\dot{a}(t) = -\mu a\left[a^2\cos^2\phi - 1\right]\sin^2\phi$$

and

$$\dot{\phi}(t) = 1 - \mu\left[a^2\cos^2\phi - 1\right]\sin\phi\cos\phi.$$

In order to derive these equations you need to (i) first convert the van der Pol equation into first order differential equations, then (ii) check under what conndition the trial solution can be valid, as well as (ii) use the van der Pol equation itself. Note that the above equations for $\dot{a}(t)$ and $\dot{\phi}(t)$ are still exact. However, they are nonlinear, hence not easy to solve.

(f) Next, assume that $\mu \to 0$, for which we can replace the rhs of the differential equations in (d) by their averages over an entire cycle, i.e., the rhs integrated from 0 to 2π, so you basically replace the rhs by

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\,(\text{rhs}).$$

Show that if you do so, you obtain

$$\dot{a}(t) = \mu a(4 - a^2)/8 \quad \text{and} \quad \dot{\phi}(t) = 1.$$

You need identities like $1 + \cos 2\phi = 2\cos^2\phi$ and $1 - \cos 2\phi = 2\sin^2\phi$.

(g) Solve the equations for $a(t)$ and $\phi(t)$ keeping only the first order terms in μ, and obtain the limit cycle in phase space.

5. One-dimensional tent map

The equation for one-dimensional tent map is given by

$$T(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2-2x & 1/2 < x \leq 1 \end{cases},$$

which maps a point x to $T(x)$. The graph fo $T(x)$ as a function of x simply looks like the triangle below (you can see why it is called a tent map):

(a) Find the formula for $T^(x) = T[T(x)]$ and plot it.

(b) Identify the fixed points of $T(x)$ and $T^2(x)$

(c) What does the graph $T^n(x)$ look like and how many fixed points does it have?

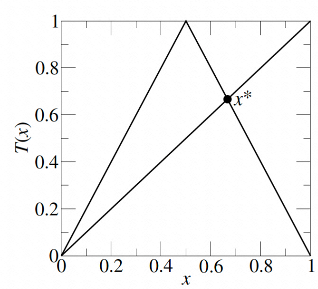

Figure 1: Figure for problem 3.

6. Logistic map

Show that the period 2 orbit for the logistic map $x_{n+1} = rx_n(1-x_n)$ loses stability and undergoes a period-doubling bifurcation at $r = 1+\sqrt{6}$.

7. Period 3 orbits in logistic map

Consider the third iteration of the logistic map $f(f(f(x)))$ where $x_{n+1} = f(x_n) = rx_n(1-x_n)$, and show that third iteration develops fixed points at $r^* = 1+\sqrt{8}$. What happens at the trajectories of the logistic map for $r \geq r^*$ (e.g. $r = r^* + \epsilon$, with $\epsilon = 10^{-3}, 10^{-4}, 10^{-5}$) — i.e., what kind of bifurcation do you observe?