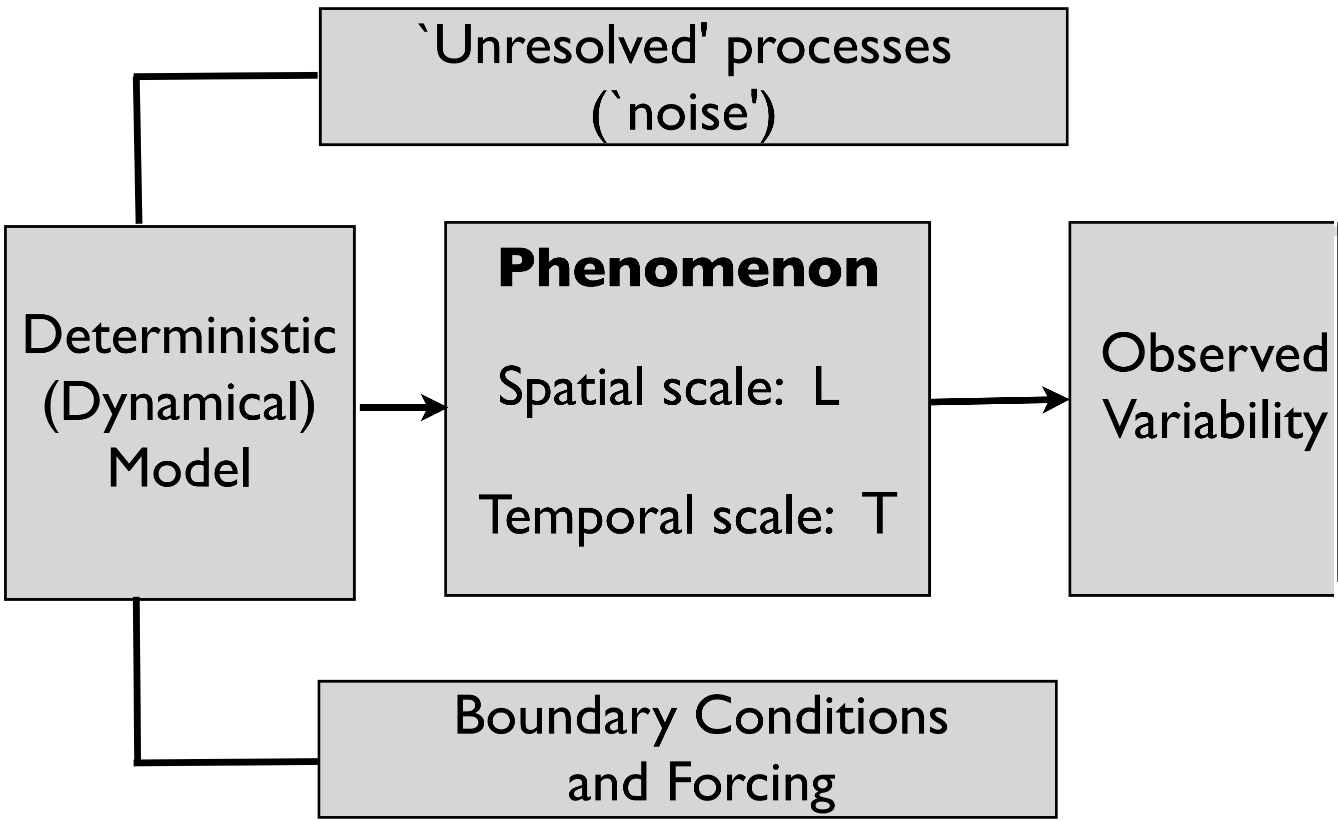
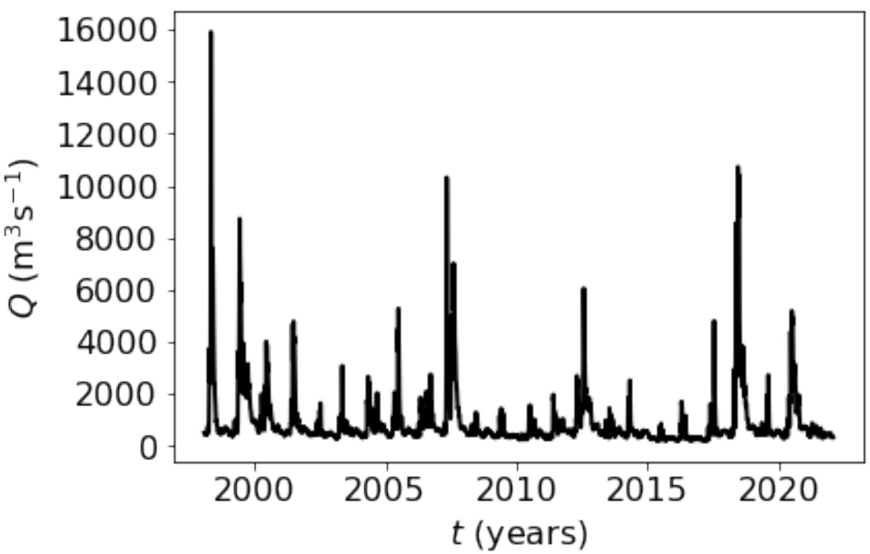
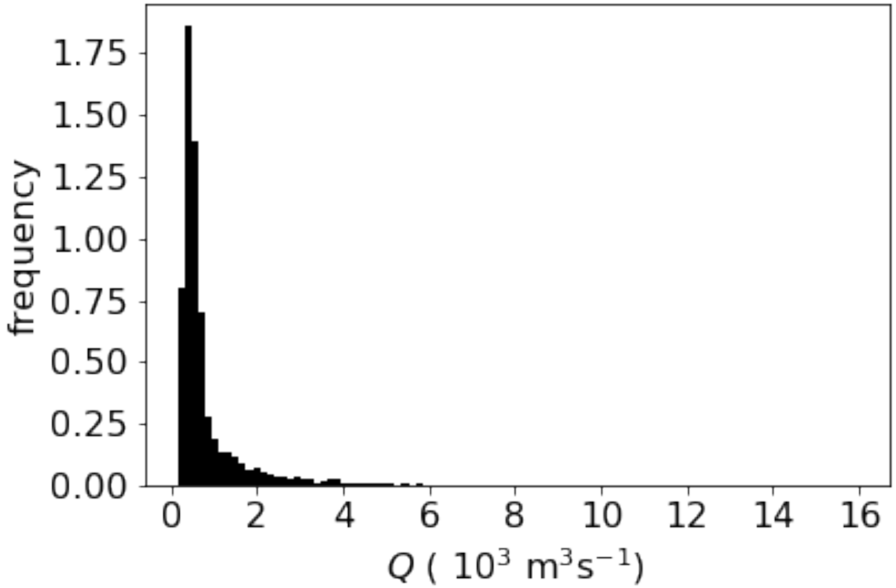
Stochastic Dynamical Modeling

Stochastic Dynamical Systems Framework



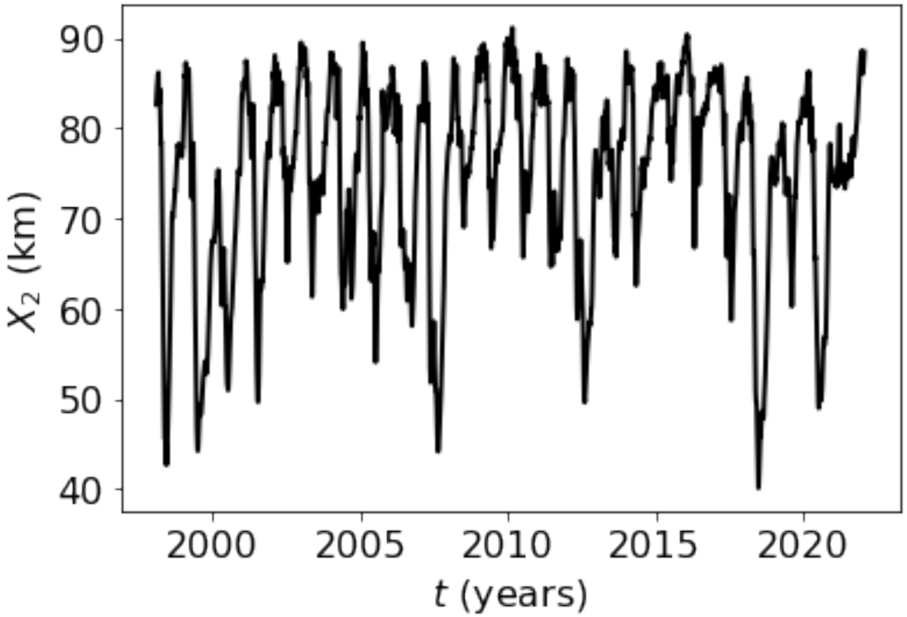
Example: river discharge (San Francisco Bay)

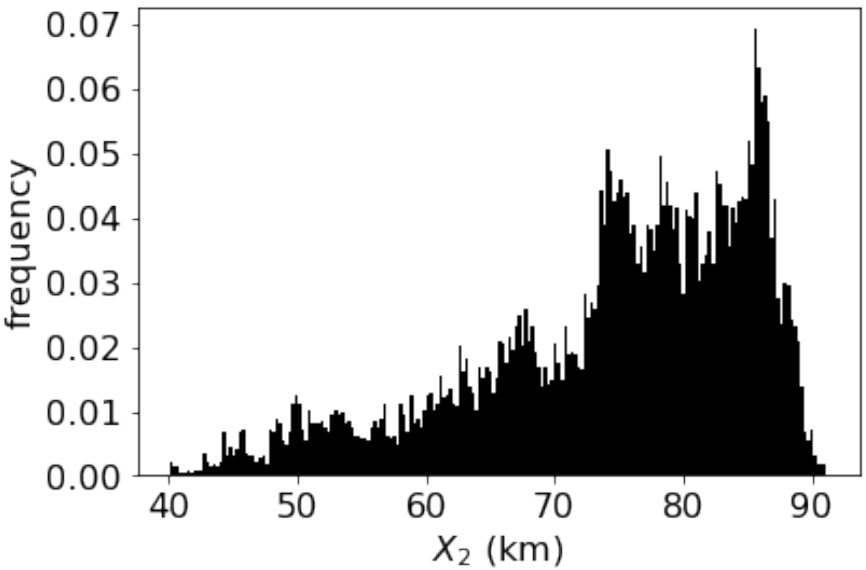




Example: salt intrusion length (San Francisco Bay)

X_2: position of S = 2 psu contour to mouth





Deterministic model

$$\frac{1}{2}\frac{dX}{dt} = \frac{C_3}{X^3} + \frac{Q}{A}(-1 + \frac{C_2}{X^2}) + \frac{C_0}{X} + (\frac{Q}{A})^2\frac{C_1}{X},$$

$$X_2 = (1 - 2/s_{ocn})X \sim 0.94X$$

Q: river discharge given
A: cross section

General stochastic model

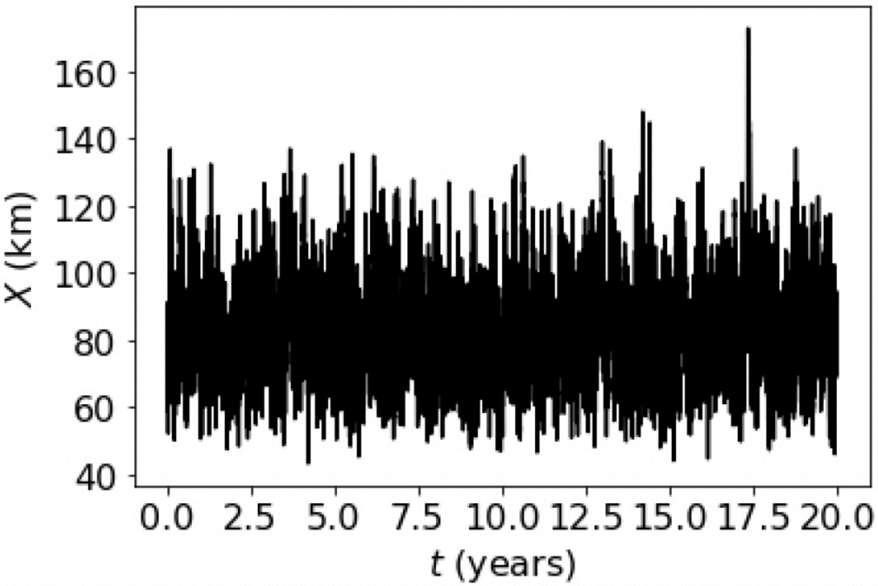
$$
\begin{aligned}
dX_t &= 2\left(\frac{C_3}{X_t^3} + \frac{Q_t}{A}\left(-1 + \frac{C_2}{X_t^2}\right) + \frac{C_0}{X_t} + \left(\frac{Q_t}{A}\right)^2 \frac{C_1}{X_t}\right)dt + \sigma_X(t, X_t)dW_t^{(1)} \\
dQ_t &= \mu_Q(t, Q_t)dt + \sigma_Q(t, Q_t)dW_t^{(2)}.
\end{aligned}
$$

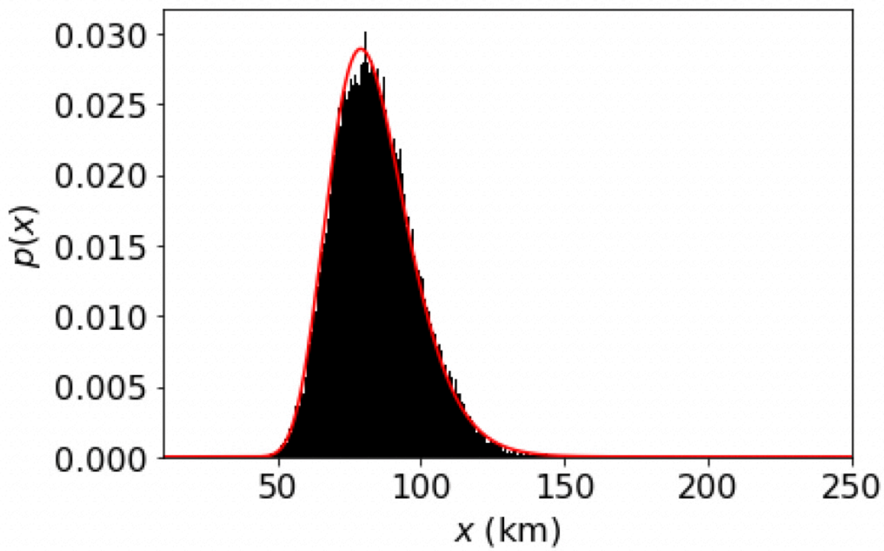
Noise X

Drift Q

Noise Q

Case 1 : constant Q



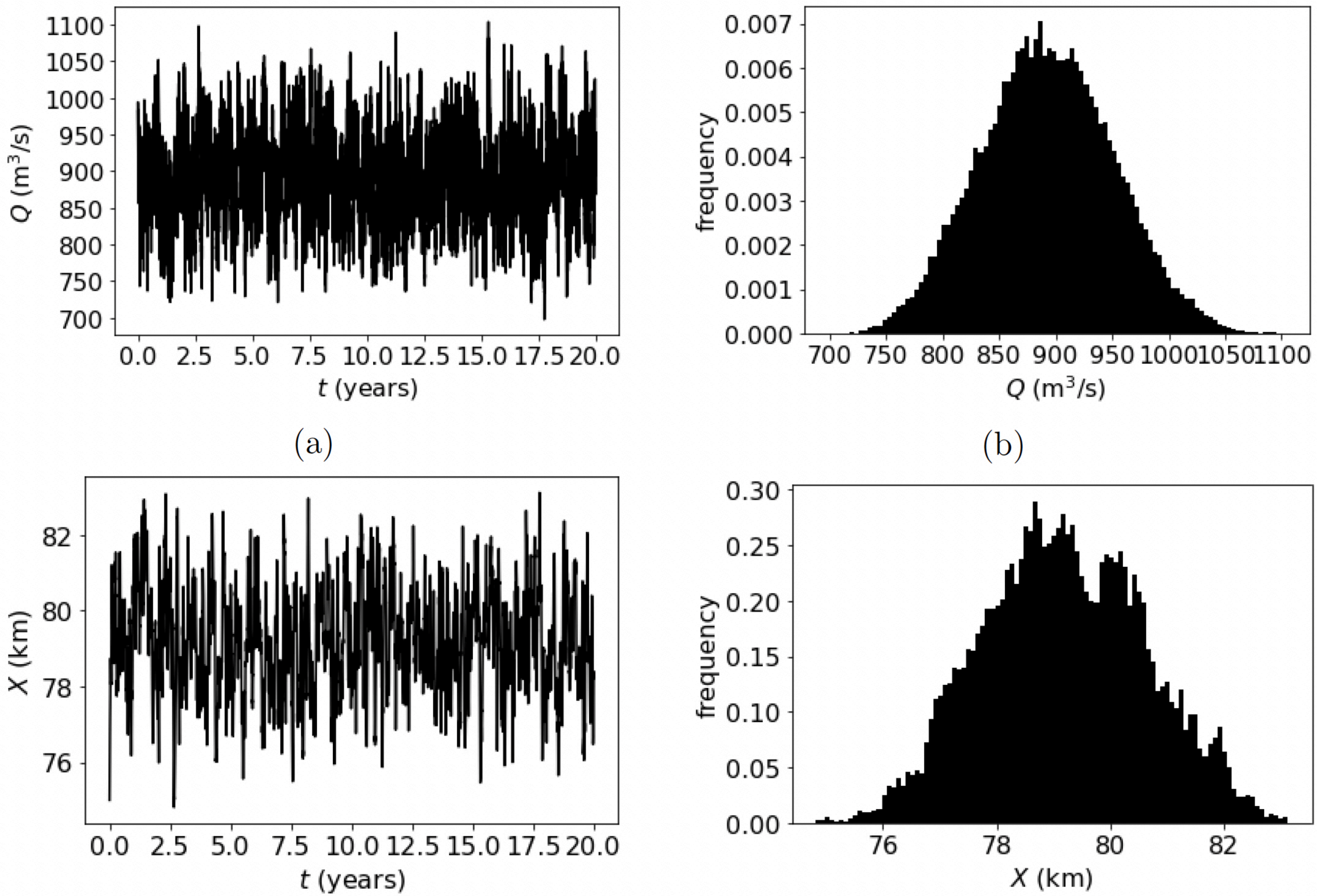


Case 1 : red noise Q

$$\dot{Y_t} = (Q_t - \bar{Q})/\bar{Q},$$

$$\begin{aligned}
dX_t &= \begin{cases} 2(\frac{C_0}{X_t} - \frac{\bar{Q}(1+Y_t)}{A})dt + \sigma_X dW_t^{(1)} & \text{(diffusive limit)}, \\ 2(\frac{C_3}{X_t^3} - \frac{\bar{Q}(1+Y_t)}{A})dt + \sigma_X dW_t^{(1)} & \text{(exchange limit)}, \end{cases} \\
dY_t &= -\frac{Y_t}{\tau}dt + \sigma dW_t^{(2)}.
\end{aligned}$$

Case 1 : red noise Q



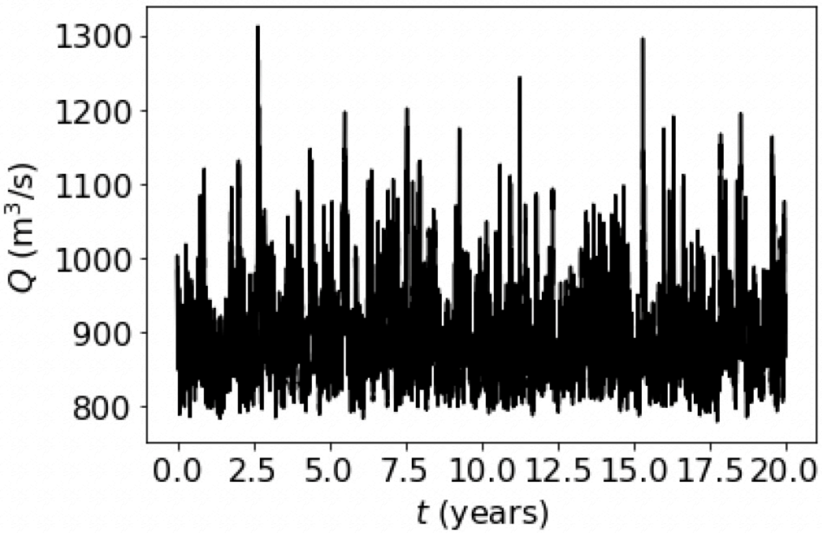
Case 2 : CAM noise Q

$$\dot{Y_t} = (Q_t - \bar{Q})/\bar{Q},$$

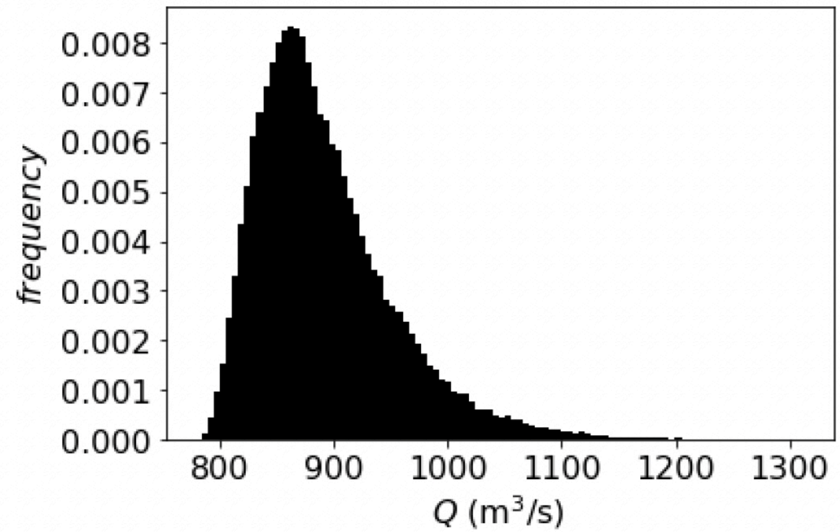
$$dX_t = \begin{cases} 2(\frac{C_0}{X_t} - \frac{\bar{Q}(1+Y_t)}{A})dt + \sigma_X dW_t^{(1)} & \text{(diffusive limit)}, \\ 2(\frac{C_3}{X_t^3} - \frac{\bar{Q}(1+Y_t)}{A})dt + \sigma_X dW_t^{(1)} & \text{(exchange limit)}, \end{cases}$$

$$dY_t = -\mu Y_t dt + (\sigma_A + \sigma_M Y_t) dW_t^{(2)},$$

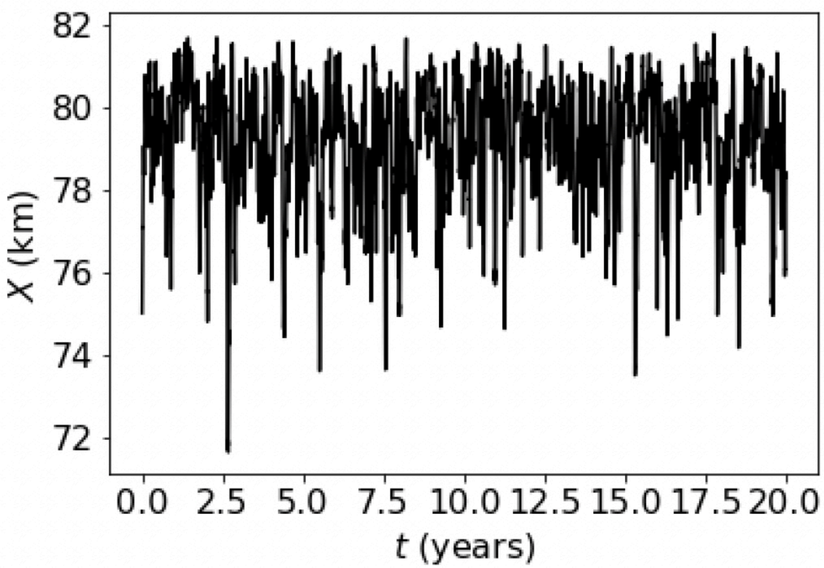
Case 2 : CAM noise Q

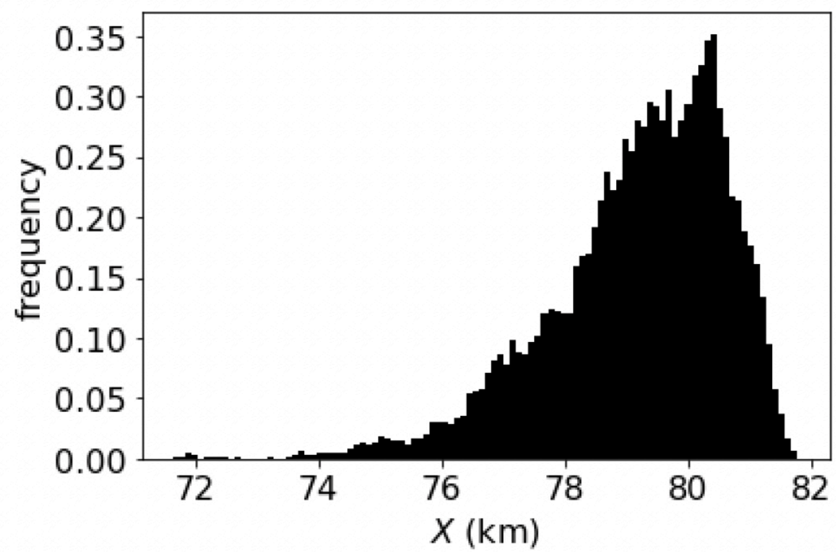


(a)



(b)





Fit of CAM noise model from observations

$$
\begin{aligned}
dX_t &= 2\left(\frac{C_3}{X_t^3} + \frac{Q_t}{A}\left(-1 + \frac{C_2}{X_t^2}\right) + \frac{C_0}{X_t} + \left(\frac{Q_t}{A}\right)^2 \frac{C_1}{X_t}\right)dt + \sigma_X dW_t^{(1)}, \\
dY_t &= (-\mu Y_t + S_Y(t))dt + (\sigma_A + \sigma_M Y_t)dW_t^{(2)},
\end{aligned}
$$

Seasonal Cycle

Fit of CAM noise model from observations

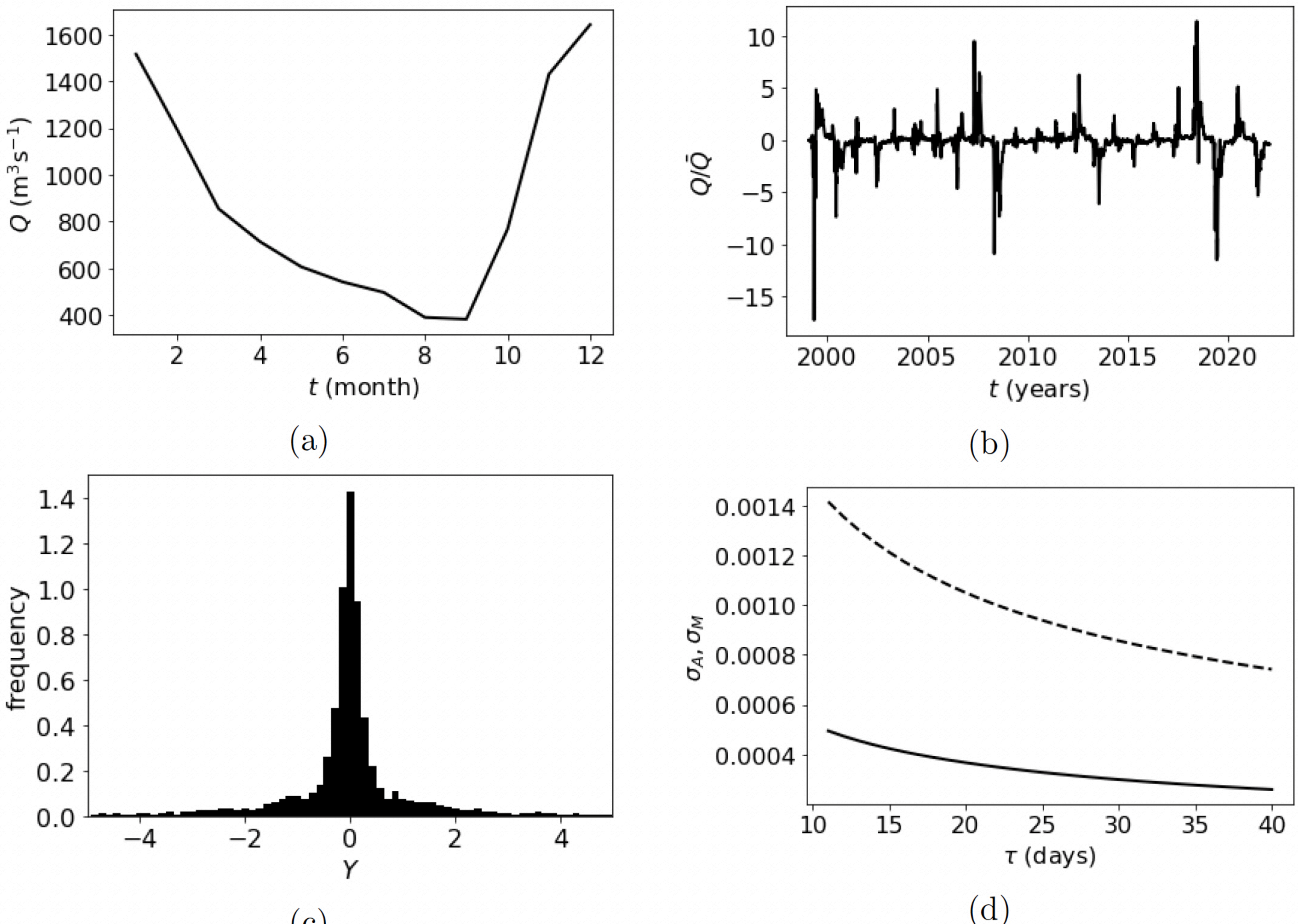
$$dY_t = -\mu Y_t dt + (\sigma_A + \sigma_M Y_t) dW_t^{(2)},$$

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y}(-\mu y p) + \frac{1}{2}\frac{\partial^2}{\partial y^2}[(\sigma_A + \sigma_M y)^2 p]$$

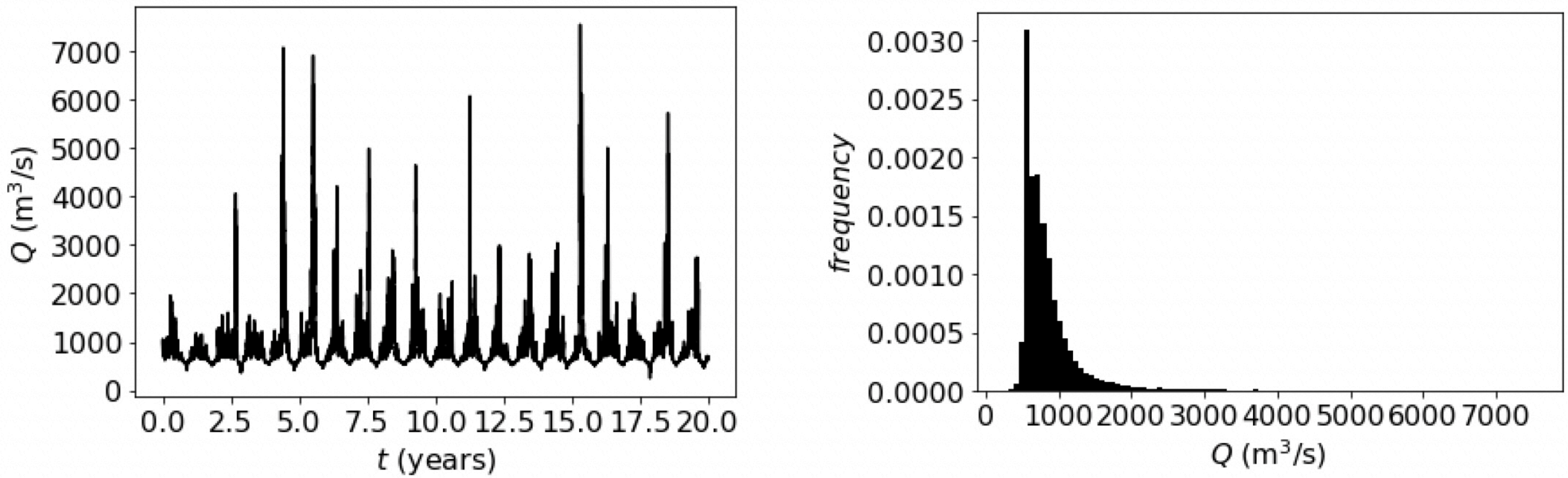
$$p_e(y) = N \exp\left\{ \frac{-2\mu}{\sigma_M^2}\left[\left(1 + \frac{\sigma_A^2}{\mu}\right) \ln(|\sigma_M y + \sigma_A|) + \frac{\sigma_A}{\sigma_M y + \sigma_A} \right] \right\}$$

$$M_1 = 0 \ ; \ M_2 = -\frac{\sigma_A^2}{2\mu + \sigma_M^2} \ ; \ M_3 = \frac{3\sigma_A^4(\mu - 3\sigma_M^2)}{(2\mu + \sigma_M^2)(\mu + \sigma_M^2)}$$

Fit of CAM noise model from observations



Final results



(a) (b)