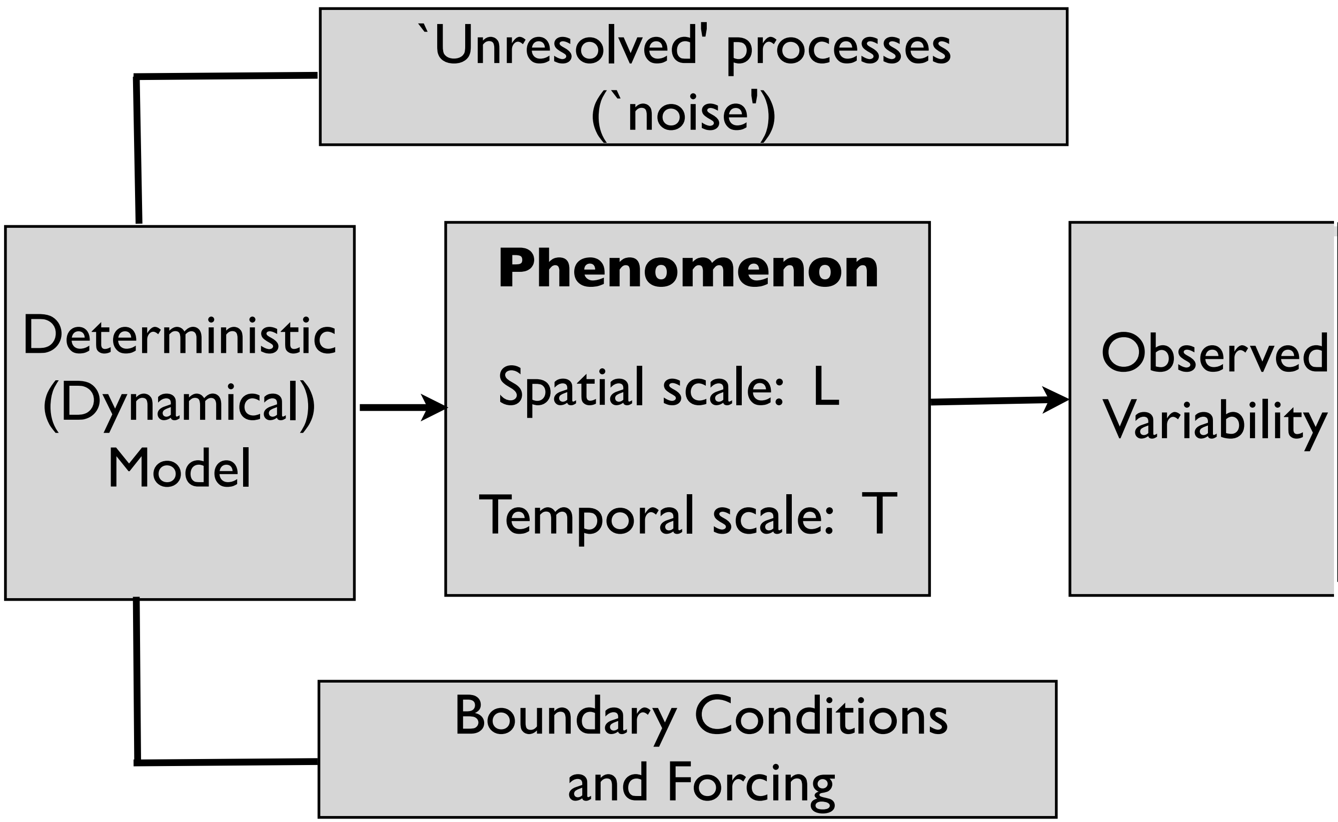
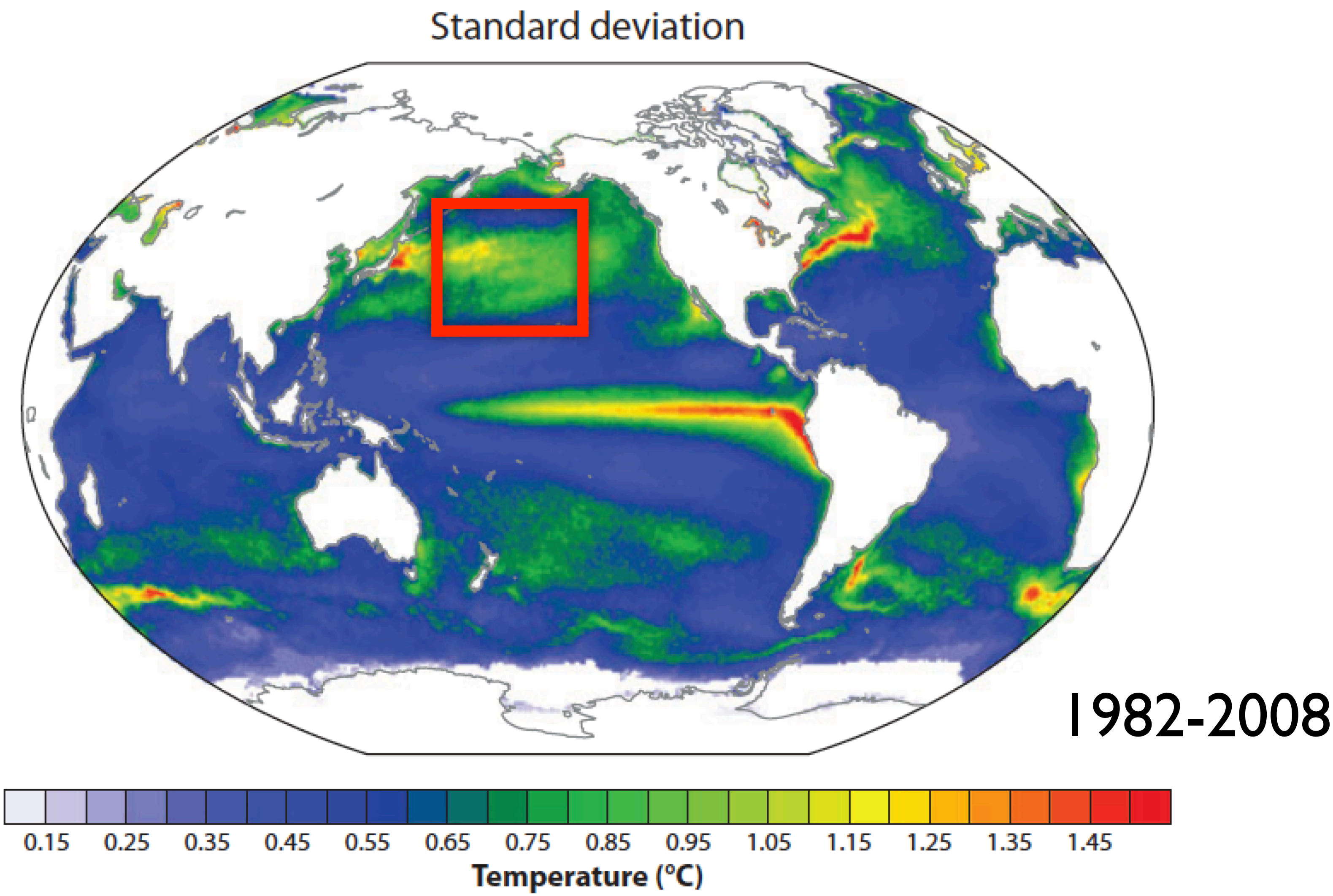
Stochastic Dynamical Modeling
UniTN: 27/1/2025 - 7/2/2025

Stochastic Dynamical Systems Framework



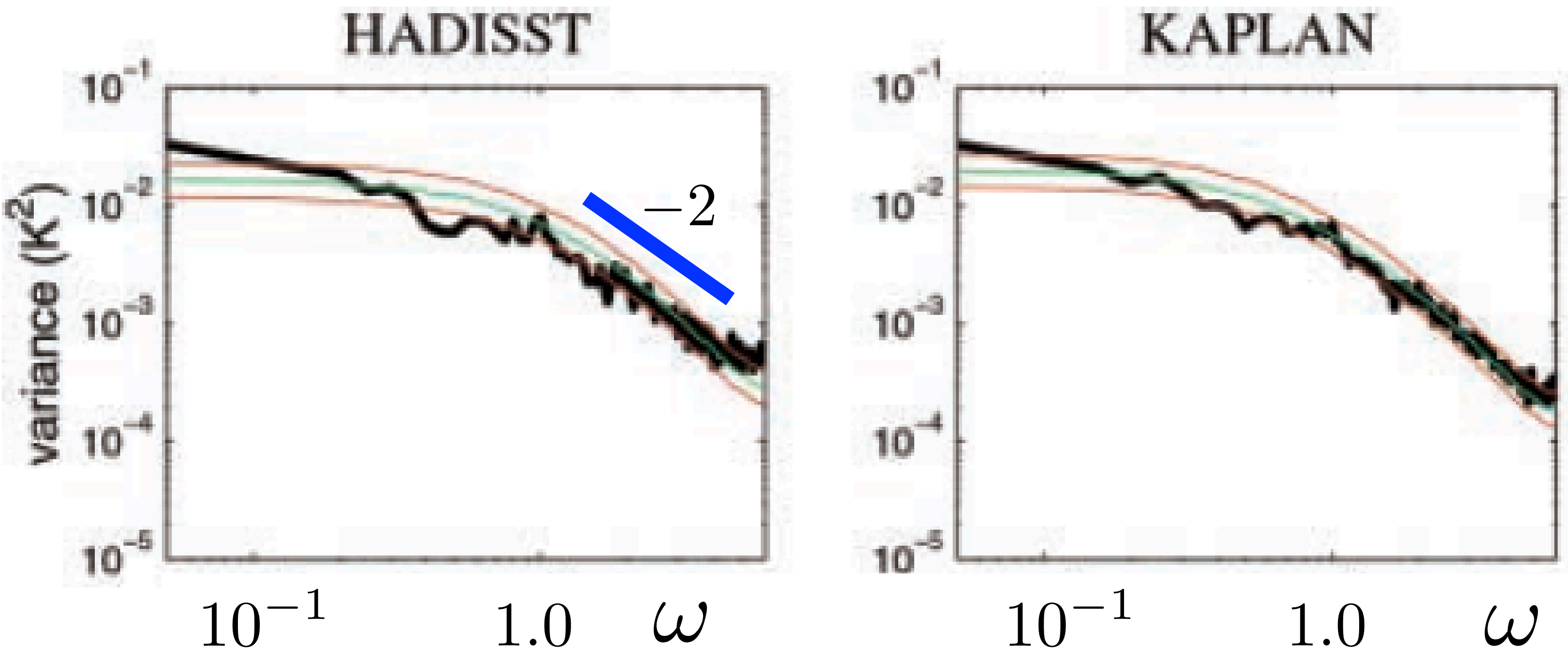
Sea surface temperature variability



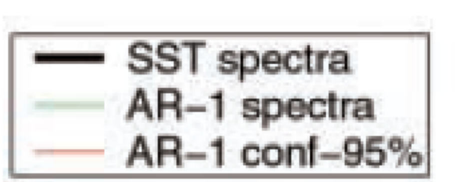
Deser et al., Ann. Rev. Mar. Sc. (2010)

Midlatitude SST Spectra

25N – 50N Pacific (1903-1994)

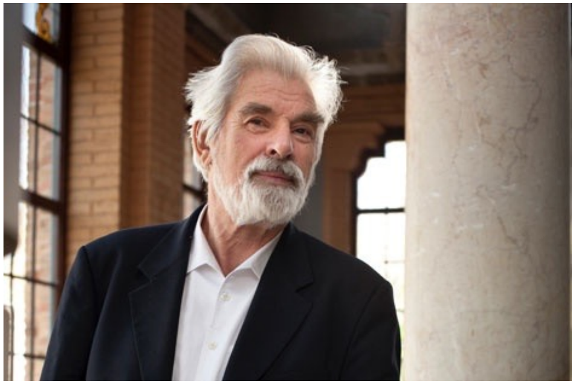


frequency (1/years)



Dommenget & Latif, Clim. Dyn. (2002)

The Hasselmann (1976) stochastic climate model



Hasselmann K. (1976), "Stochastic climate models, Part 1: Theory", *Tellus*, 28: 473-485.

$$\boxed{\frac{d\tilde{T}}{dt} = -\gamma\tilde{T} + \sigma\xi} \qquad \gamma = \frac{\alpha}{\rho C_p h}$$

white noise

$$E[\xi(t)] = 0$$

$$E[\xi(t)\xi(s)] = \delta(t - s)$$

"The choice between a deterministic and a stochastic formulation of the equations …. [is] dictated by convenience" Lorenz (1987)

Stochastic Differential Equations (SDEs)

from $$\boxed{\frac{d\tilde{T}}{dt} = -\gamma\tilde{T} + \sigma\xi}$$

Stochastic process: $X_t = \tilde{T}$

Wiener process: W_t $\quad N(0,t)$ distributed

$$E[(dW_t)^2] = dt$$

to $$\boxed{dX_t = -\gamma X_t dt + \sigma dW_t}$$

or $$X_t = X_0 + \int_0^t (-\gamma X_s)ds + \int_0^t \sigma dW_s$$

Stochastic integrals

Kiyoshi $It\hat{o}$ (1915-2008)

$$\int_0^T h(t)\, dW_t = \lim_{N\to\infty} \sum_{j=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

\`left end point'

Ruslan Stratonovich (1930-1997)

\`mid point'

$$\int_0^T h(t) \circ dW_t = \lim_{N\to\infty} \sum_{j=0}^{N-1} h(\frac{t_j + t_{j+1}}{2})(W(t_{j+1}) - W(t_j))$$

both in the mean-square sense

$$\lim_{N\to\infty} E[(I - I_N)^2] \to 0$$

Stochastic integrals: evaluation

Problem: Evaluate $\int_0^T W_t dW_t$

$It\hat{o}$

$$f(W_T) - f(W_0) = \int_0^T f'(W_t)dW_t + \int_0^T \frac{1}{2} f''(W_t)dt$$

Take: $f(t) = t^2$

Answer: $$\int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T)$$

Solution of SDEs

$$f(t+dt, X_t+dX_t) - f(t, X_t) = f_1 dt + f_2 dX_t + \frac{1}{2}(f_{11}(dt)^2 + 2f_{12}dtdX_t + f_{22}(dX_t)^2) + \ldots$$

$It\hat{o}$

$$\boxed{dX_t = A_t dt + B_t dW_t}$$

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2) ds + \int_0^t f_2 B_s dW_s$$

Example: Hasselmann

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

$$A_t = -\gamma X_t \ ; \ B_t = \sigma$$

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2) ds + \int_0^t f_2 B_s dW_s$$

Take: $f(t, x) = x e^{\gamma t}$

Result: $X_t = e^{-\gamma t}(X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$

Ornstein-Uhlenbeck process

Statistics Ornstein-Uhlenbeck process

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

Solution: $$X_t = e^{-\gamma t}(X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$$

Wanted: $$E[X_t X_{t+s}], s > 0$$

Result: $$E[X_t X_{t+s}] = e^{-\gamma(2t+s)}(X_0^2 + \sigma^2 \frac{e^{2\gamma t} - 1}{2\gamma})$$

$$E[X_t X_{t+s}] \to \frac{\sigma^2}{2\gamma} e^{-\gamma s}, \quad t \to \infty$$

Spectrum: $$S(\omega) = \frac{\sigma^2}{2\gamma} \mathcal{F}(e^{-\gamma s}) = \frac{\sigma^2}{\omega^2 + \gamma^2}$$

The Red Noise (AR(1)) process

$$X_t = e^{-\gamma t}(X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$$

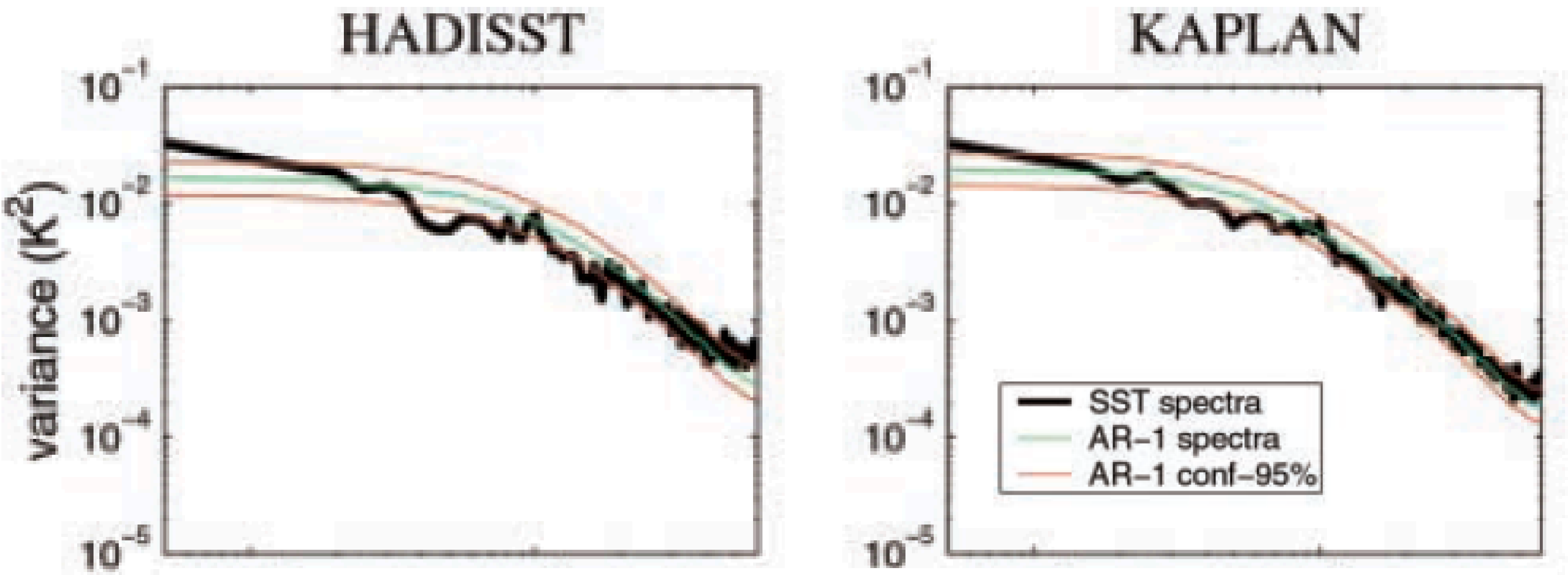
Discrete time: $$t_n = n\Delta t \; ; \; X_n = X_{t_n}$$

$$X_{n+1} = \alpha X_n + Z_{n+1}$$

$$\alpha = e^{-\gamma \Delta t}, Z_n = \sigma dW_n$$

α can be estimated from the autocorrelation of the time series

Red noise representation



The red noise spectrum serves as a null-hypothesis for climate variability!