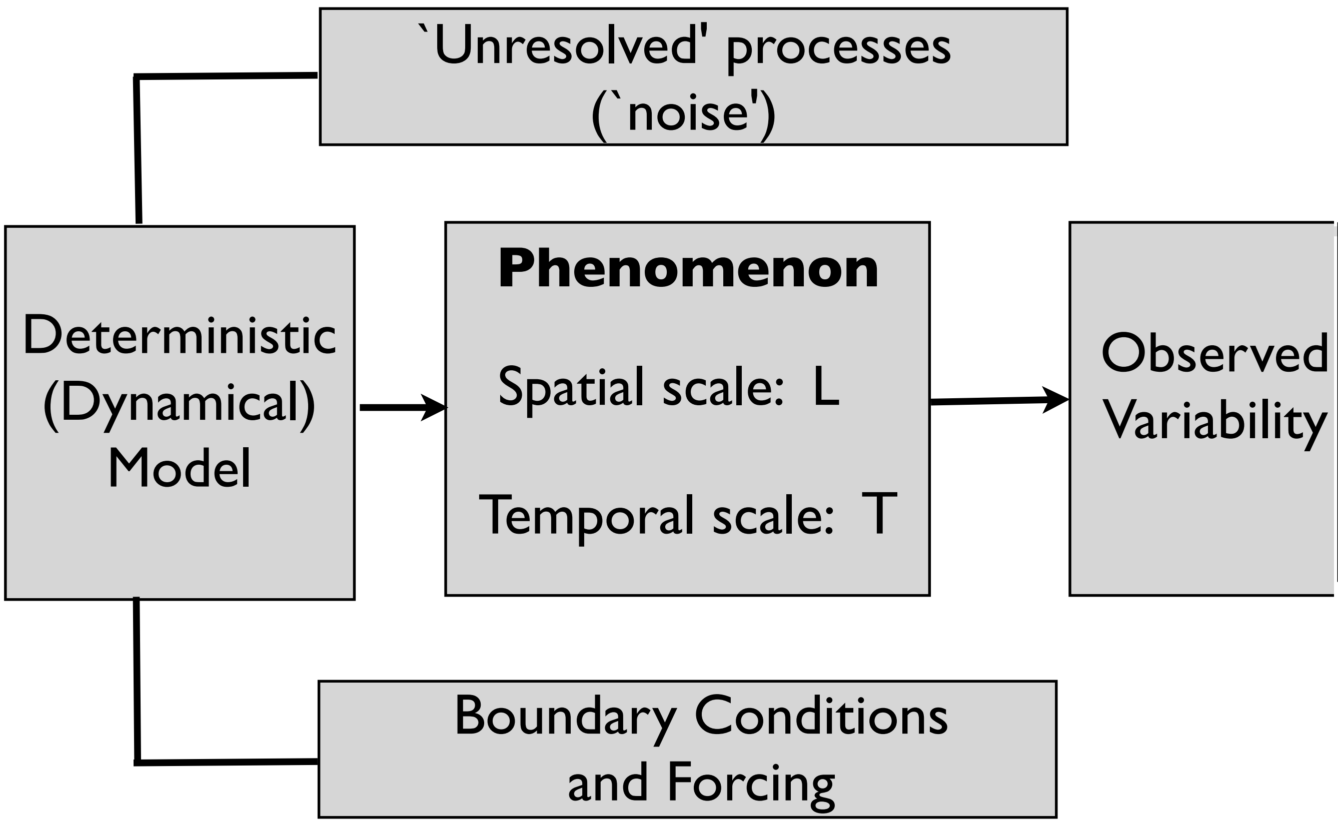
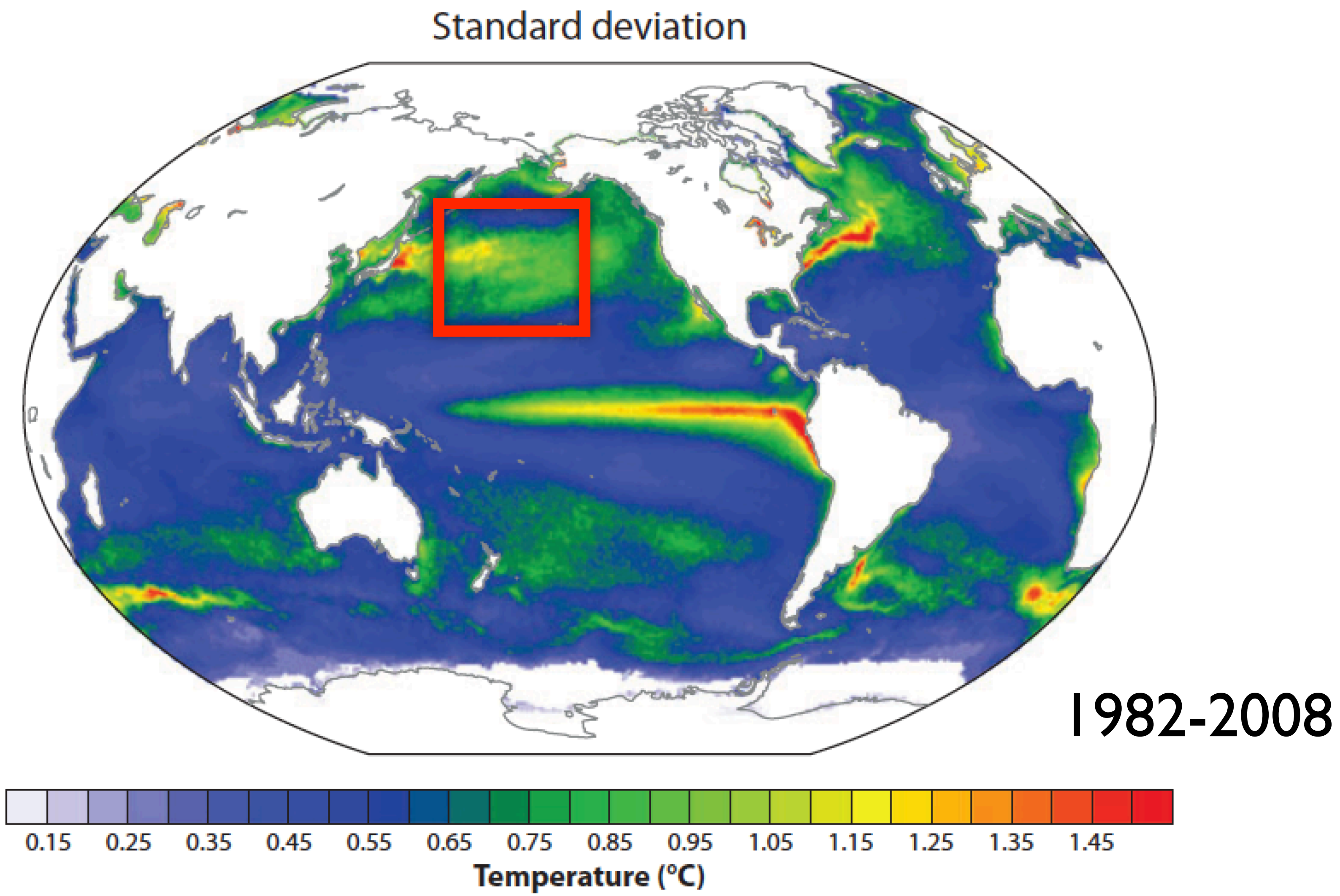
Stochastic Dynamical Modeling
UniTN: 27/1/2025 - 7/2/2025

Stochastic Dynamical Systems Framework



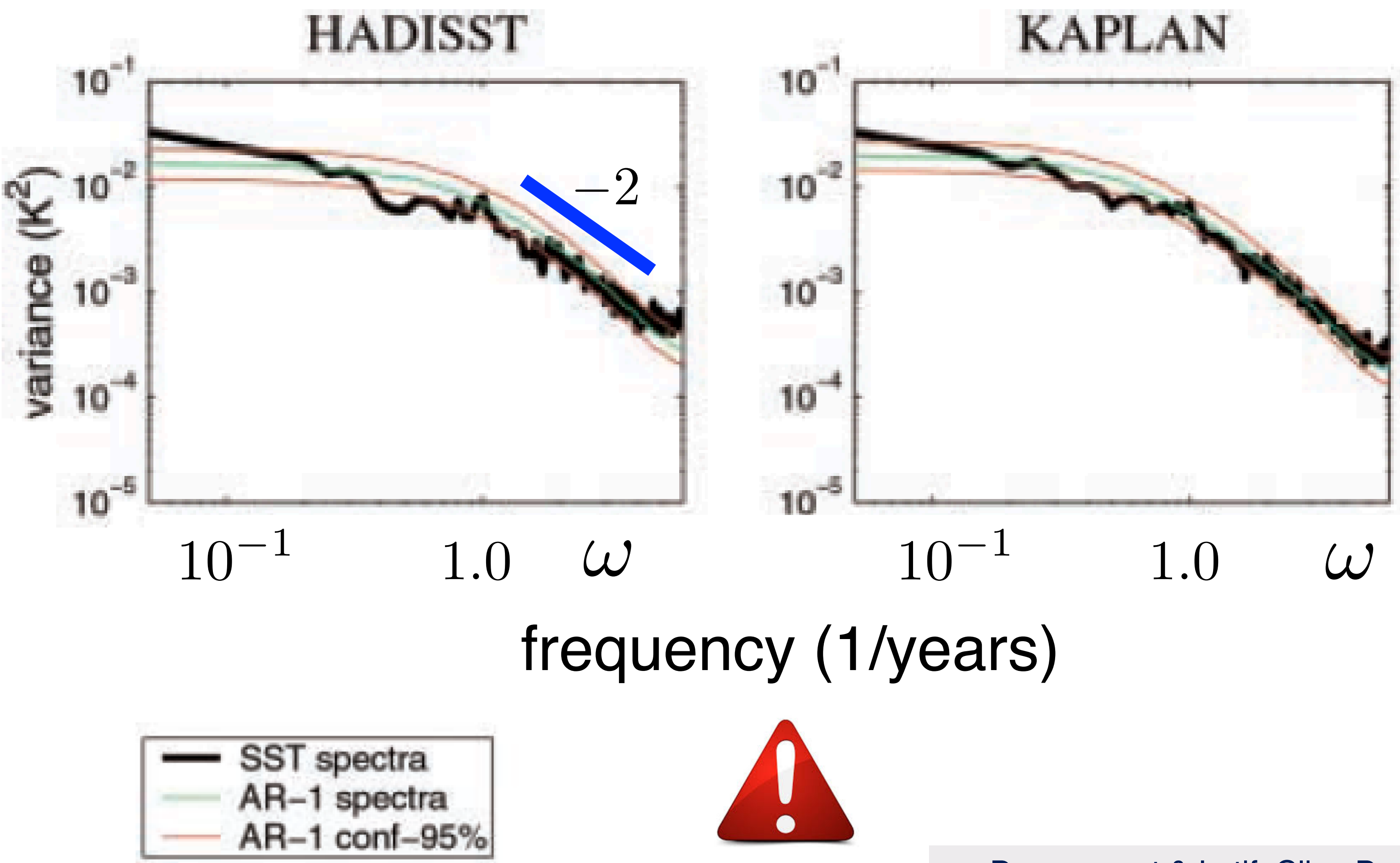
Sea surface temperature variability



Deser et al., Ann. Rev. Mar. Sc. (2010)

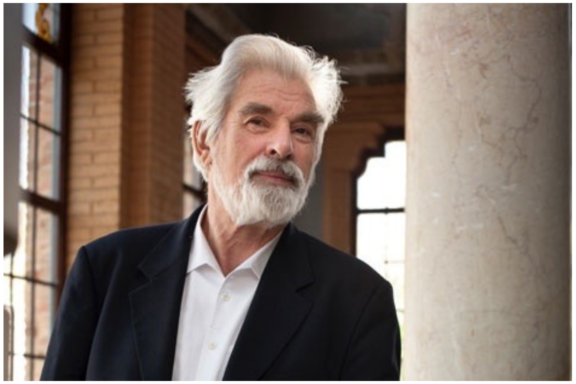
Midlatitude SST Spectra

25N – 50N Pacific (1903-1994)



Dommenget & Latif, Clim. Dyn. (2002)

The Hasselmann (1976) stochastic climate model



Hasselmann K. (1976), "Stochastic climate models, Part 1: Theory", *Tellus*, 28: 473-485.

$$\boxed{\frac{d\tilde{T}}{dt} = -\gamma\tilde{T} + \sigma\xi} \qquad \gamma = \frac{\alpha}{\rho C_p h}$$

white noise

$$E[\xi(t)] = 0$$

$$E[\xi(t)\xi(s)] = \delta(t-s)$$

"The choice between a deterministic and a stochastic formulation of the equations …. [is] dictated by convenience" Lorenz (1987)

Stochastic Differential Equations (SDEs)

from $$\boxed{\frac{d\tilde{T}}{dt} = -\gamma\tilde{T} + \sigma\xi}$$

Stochastic process: $X_t = \tilde{T}$

Wiener process: W_t $N(0, t)$ distributed

$$E[(dW_t)^2] = dt$$

to $$\boxed{dX_t = -\gamma X_t dt + \sigma dW_t}$$

or $$X_t = X_0 + \int_0^t (-\gamma X_s) ds + \int_0^t \sigma dW_s$$

Statistics Ornstein-Uhlenbeck process

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

Solution: $$X_t = e^{-\gamma t}(X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$$

Wanted: $$E[X_t X_{t+s}], s > 0$$

Result: $$E[X_t X_{t+s}] = e^{-\gamma(2t+s)}(X_0^2 + \sigma^2 \frac{e^{2\gamma t} - 1}{2\gamma})$$

$$E[X_t X_{t+s}] \to \frac{\sigma^2}{2\gamma} e^{-\gamma s}, \quad t \to \infty$$

Spectrum: $$S(\omega) = \frac{\sigma^2}{2\gamma} \mathcal{F}(e^{-\gamma s}) = \frac{\sigma^2}{\omega^2 + \gamma^2}$$

Numerical solution of SDEs

$$X(t) = X(0) + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s)$$

$$\tau_j = j\Delta t, j = 0, \ldots, n \text{ on } [0, T]$$

$$\Delta t = T/n$$

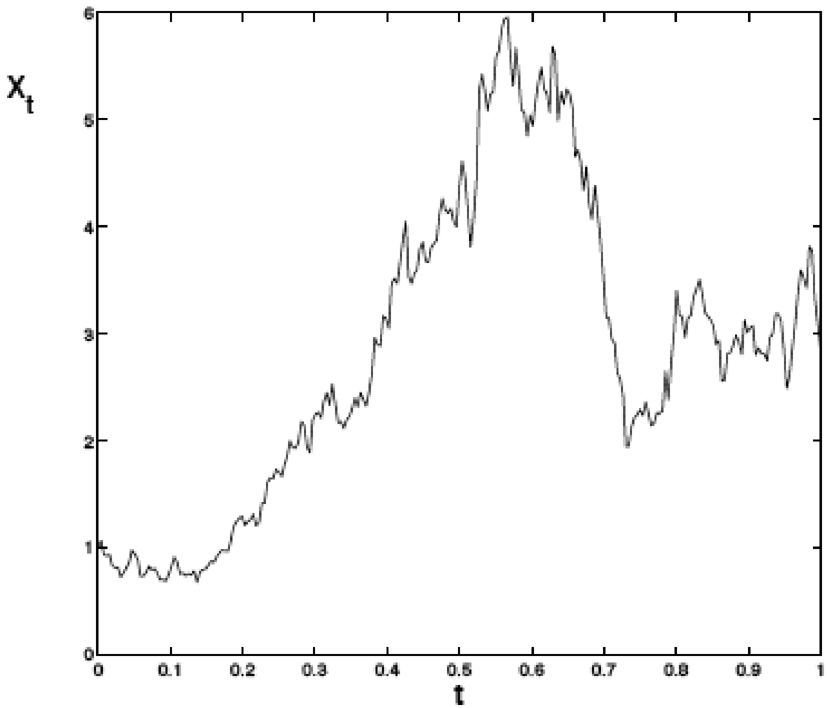
Gisiro Maruyama (1916-1986)

Euler-Maruyama scheme:

$$X_j - X_{j-1} = f(X_{j-1})\Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1}))$$

Geometric Brownian Motion

Equation: $$X_t = X_0 + \int_0^t \lambda X_x dx + \int_0^t \mu X_x dW_x$$



Solution:

$$X_t = X_0 e^{(\lambda - \mu^2/2)t + \mu W(t)}$$

$$X_0 = 1\ ;\ \lambda = 2\ ;\ \mu = 1$$

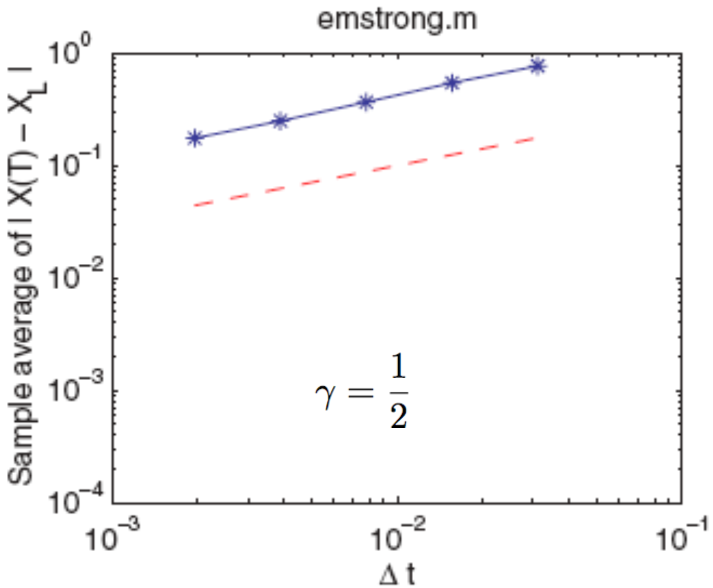
Convergence of EM-method

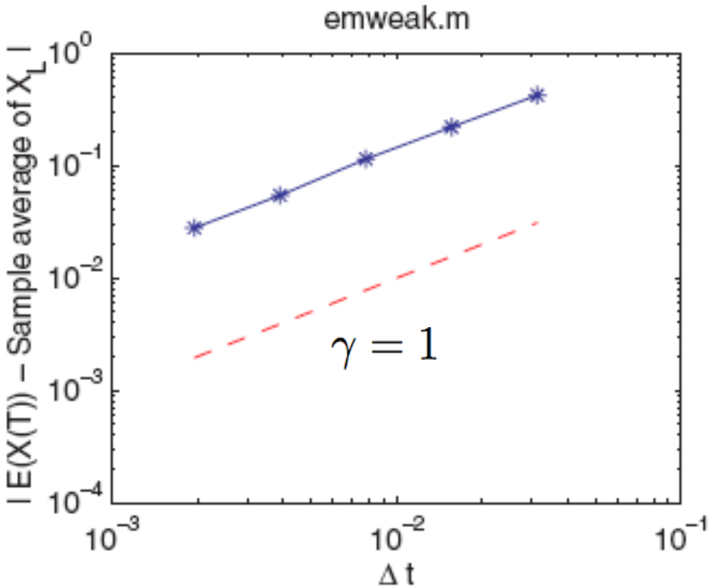
Strong convergence:

$$E[|X_k - X(\tau_k)|] \leq (\Delta t)^\eta$$

Weak convergence:

$$|E[X_k] - E[X(\tau_k)]| \leq (\Delta t)^\eta$$





Probability density

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$

$$E[f(X_t)] = \int f(x)p(x,t)dx$$

Probability Density Function (PDF)

Fokker-Planck Equation

$$\frac{\partial p}{\partial t} = -\frac{\partial (ap)}{\partial x} + \frac{1}{2}\frac{\partial^2 (b^2 p)}{\partial x^2}$$

+ Bc's + IC

Example

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

$$a = -\gamma x \; ; \; b = \sigma$$

$$\frac{\partial p}{\partial t} = -\frac{\partial (ap)}{\partial x} + \frac{1}{2}\frac{\partial^2 (b^2 p)}{\partial x^2} = 0$$

equilibrium

$$p_e(x) = Ce^{-\frac{\gamma x^2}{\sigma^2}}$$

Result:

$$\int_{-\infty}^{\infty} p_e(x) = 1 \rightarrow C = \frac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}$$

Summary: Ornstein-Uhlenbeck process

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

Leonard Ornstein
(1880-1941)

George Uhlenbeck
(1900-1988)

Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = \frac{\partial(\gamma x p)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}$$

$$x \to \pm\infty : p \to 0$$

Stationary density:

$$p_{stat} = \sqrt{\frac{\gamma}{\pi}}\frac{1}{\sigma}e^{\frac{-\gamma x^2}{\sigma^2}}$$

Stationary autocorrelation:

$$E[X_t X_s] = \frac{\sigma^2}{2\gamma}e^{-\gamma|\tau|}, \tau = t - s$$

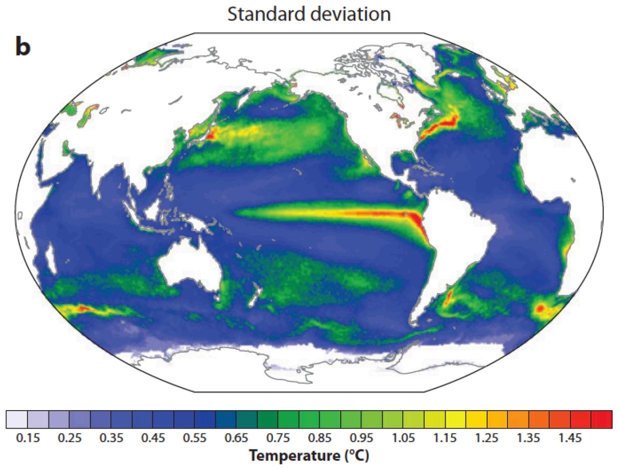
Stationary spectrum:

$$S(\omega) = \frac{\sigma^2}{\gamma^2 + \omega^2}$$

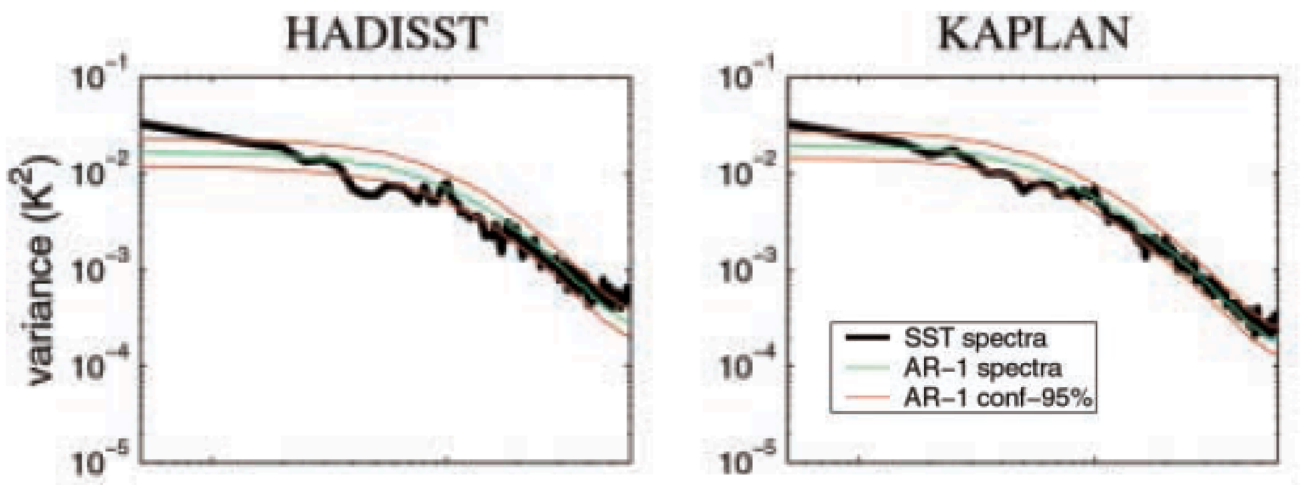
Summary: SST variability

Ocean mixed layer temperature: $$dX_t = -\gamma X_t dt + \sigma dW_t$$

Wiener process



Continuous: Ornstein-Uhlenbeck process
Discrete: red noise or AR(1) process



The red noise spectrum serves as a null-hypothesis for climate variability!