Notes

5/2/24

After discretization, any model can be written as

$$\text{(DS1)} \qquad \frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, \lambda, t)$$

$\underline{x}$: state vector

$\underline{f}$: vector field

λ : parameter

t : time

$\underline{x} \in \mathbb{R}^n$, n: dimension

Two classes:

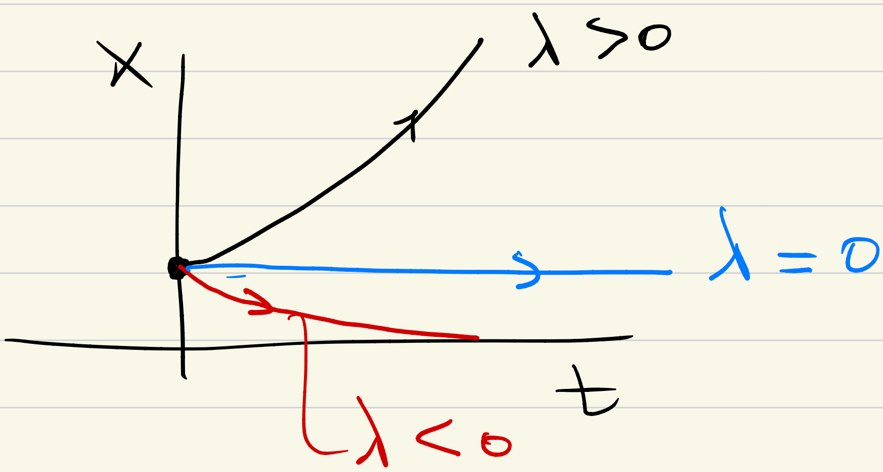
autonomous $\underline{f}(\underline{x}, \lambda)$

non-autonomous $\underline{f}(\underline{x}, \lambda, t)$

Example: $\dot{x} = \lambda x$

$n = 1$

Solution: $x(t) = x_0 e^{\lambda t}$

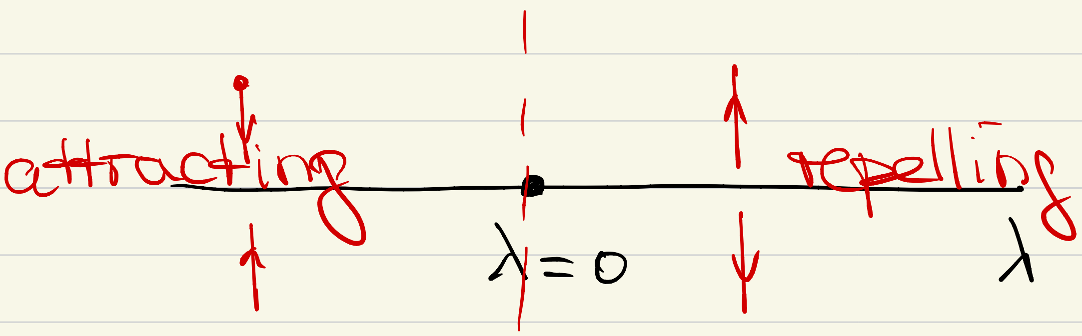


Concepts: $x(t)$ trajectory

x_0 initial condition

Fixed points: $\dot{x} = 0$

$\longrightarrow \bar{x} = 0 \quad \forall \lambda \in \mathbb{R}$



$x = 0$

Example $(n = 2)$

$$\dot{x} = -2x - 3y$$
$$\dot{y} = 3x - 2y$$

Vector plot $\longrightarrow$ Phase portrait

- stable focus

General :

$$\dot{\underline{x}} = A\,\underline{x}$$

with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det A \neq 0$$

$$\underline{\tilde{x}} = T\,\underline{x} \longrightarrow \dot{\underline{\tilde{x}}} = T\,\dot{\underline{x}}$$
$$= T A \underline{x}$$
$$= T A T^{-1} \underline{\tilde{x}}$$

$$\dot{\tilde{\underline{x}}} = B\,\tilde{\underline{x}} \qquad B = TAT^{-1}$$

where B has the same eigenvalue σ as A. $\underline{x} = 0$ fixed point.

Case 1. $\sigma_1 \neq \sigma_2$ real

Case 2. $\sigma_{1,2} = \alpha \pm i\beta$

other special cases

1. $\sigma_1 < 0 < \sigma_2$

$$B = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$\begin{cases} \tilde{x}(t) = e^{\sigma_1 t} c_1 \\ \tilde{y}(t) = e^{\sigma_2 t} c_2 \end{cases}$$

Example

$$\dot{x} = x + 2y$$
$$\dot{y} = 2x + y$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

eigenvalues: $0 = \begin{vmatrix} 1-\sigma & 2 \\ 2 & 1-\sigma \end{vmatrix}$

$$\longrightarrow \quad (1-\sigma)^2 - 4 = 0$$

$$\sigma_1 = -1 \,, \quad \sigma_2 = 3$$

$\longrightarrow$ saddle

Summary

- trajectory
- fixed point
- attractor | repeller
- phase portrait

Still linear models $n=1,2$.

Bifurcations of fixed points

Scalar equation, autonomous
(n=1)

$$\dot{x} = f(x, \lambda)$$

Fixed points $\bar{x}$

$$f(\bar{x}, \lambda) = 0$$

Small perturbations $\tilde{x}$,
with $|\tilde{x}| \ll \bar{x}$ $\longrightarrow$

$$\dot{\tilde{x}} + \underset{\overset{\parallel}{0}}{\dot{\bar{x}}} = f(\bar{x} + \tilde{x}, \lambda)$$

f smooth $\longrightarrow$

$$\dot{\tilde{x}} = f(\bar{x}, \lambda) + \tilde{x} f_x(\bar{x}, \lambda) + \Theta(|\tilde{x}|^2)$$

Let $\sigma = f_x(\bar{x}, \lambda)$

then $\dot{\tilde{x}} = \sigma \tilde{x}$

$\rightarrow \tilde{x}(t) = \tilde{x}_0 e^{\sigma t}$

$\sigma < 0$: $\bar{x}$ stable ; $\sigma > 0$: $\bar{x}$ unstable

Changes in perturbation behavior occur at $\sigma = 0$, i.e. at values of λ_c for which $f_x(\bar{x}, \lambda_c) = 0$; λ_c are called bifurcation points.

Classification of bifurcation points based on Taylor series of f. For a single parameter λ, there are three cases:

(i) saddle-node

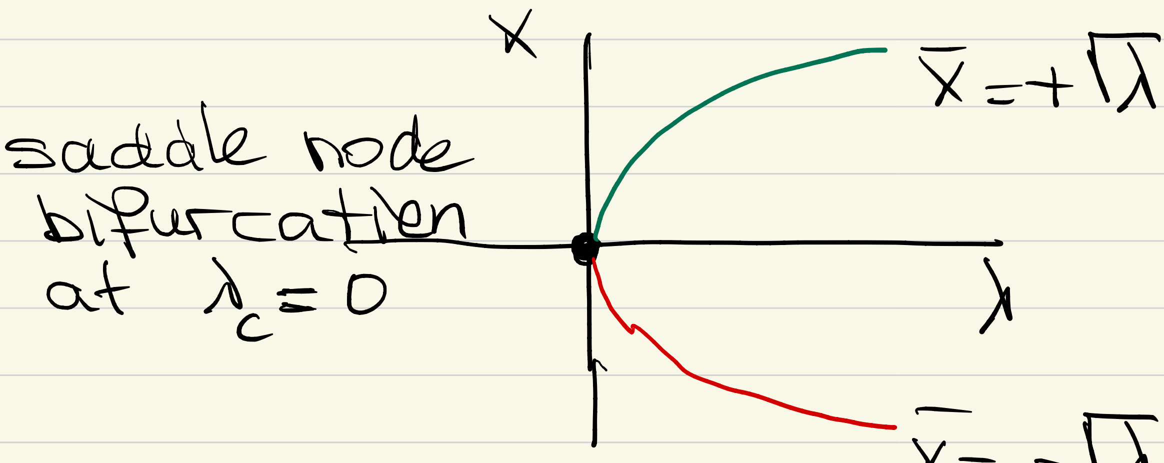
normal form : $\dot{x} = \lambda - x^2$

fixed points : $\bar{x} = \pm\sqrt{\lambda}, \lambda \geq 0$

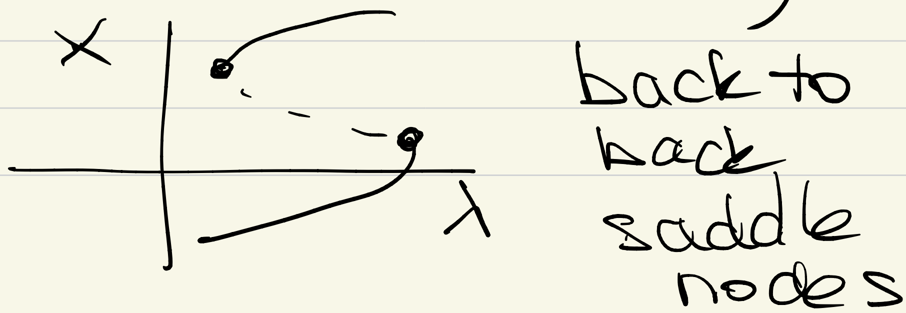
stability : $f_x(\bar{x}, \lambda) = -2\bar{x} = \sigma$

<u>hence</u> : $\bar{x} = +\sqrt{\lambda} \rightarrow \sigma < 0$

$\bar{x} = -\sqrt{\lambda} \rightarrow \sigma > 0$



• physical systems
(bounded solutions)



(ii) pitchfork bifurcation

normal form : $\dot{x} = x(\lambda - x^2)$

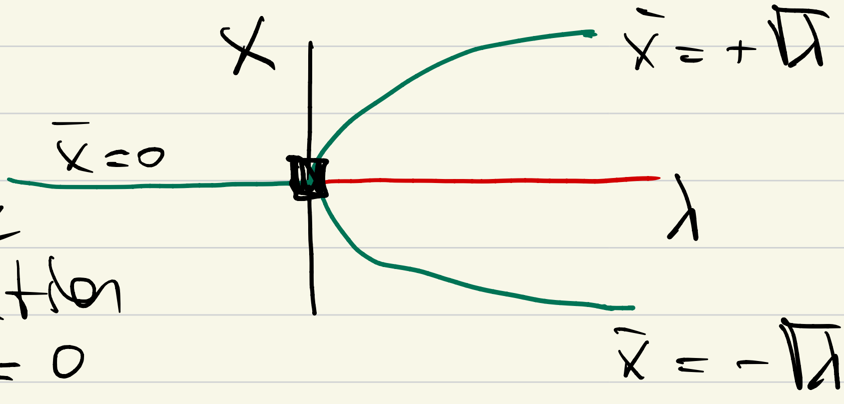
fixed points : $\bar{x} = 0$, $\bar{x} = \pm\sqrt{\lambda}, \lambda > 0$

stability : $\sigma = f_x(\bar{x}, \lambda) = \lambda - 3\bar{x}^2$

$\bar{x} = 0 \rightarrow \sigma = \lambda$ $\quad$ $\sigma < 0$ $\quad$ $\sigma > 0$ $\quad$ 0 $\quad$ λ

$\bar{x} = +\sqrt{\lambda} \rightarrow \sigma = -2\lambda \quad < 0$

$\bar{x} = -\sqrt{\lambda} \rightarrow \sigma = -2\lambda \quad < 0$



pitchfork bifurcation at $\lambda_c = 0$

- physical systems

symmetry breaking

(iii) transcritical bifurcation

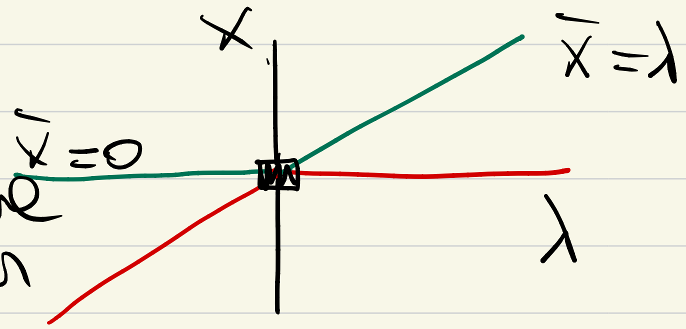
normal form : $\dot{x} = \lambda x - x^2$

fixed points : $\bar{x} = 0, \ \bar{x} = \lambda$

stability : $\sigma = f_x(\bar{x}, \lambda) = \lambda - 2\bar{x}$

$\bar{x} = 0 \longrightarrow \sigma = \lambda$ $\quad \sigma < 0, \ \sigma > 0$ (on the λ axis, either side of 0)

$\bar{x} = \lambda \longrightarrow \sigma = -\lambda$ $\quad \sigma > 0, \ \sigma < 0$ (on the λ axis, either side of 0)



transcritical bifurcation at $\lambda_c = 0$

- physics
  - no symmetry, (trivial) solution for all values of λ.

Two-dimensional case

$$\begin{cases} \dot{x} = f(x, y, \lambda) \\ \dot{y} = g(x, y, \lambda) \end{cases}$$

fixed points : $f(\bar{x}, \bar{y}, \lambda) = 0$

$g(\bar{x}, \bar{y}, \lambda) = 0$

stability : $\begin{cases} x = \bar{x} + \tilde{x} \\ y = \bar{y} + \tilde{y} \end{cases}$

$$\dot{\bar{x}} + \dot{\tilde{x}} = f(\bar{x} + \tilde{x}, \bar{y} + \tilde{y}, \lambda)$$
$$\dot{\bar{y}} + \dot{\tilde{y}} = g(\bar{x} + \tilde{x}, \bar{y} + \tilde{y}, \lambda)$$

$$\rightarrow \begin{cases} \dot{\tilde{x}} = f_x \tilde{x} + f_y \tilde{y} + \dots \\ \dot{\tilde{y}} = g_x \tilde{x} + g_y \tilde{y} + \dots \end{cases}$$

Jacobian matrix

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = J \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

<u>Solution</u>: $\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = e^{tJ} \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix}$

$$J = T \Sigma T^{-1}$$

$\Sigma \rightarrow$ eigenvalues of J

In the 2-dimensional case, there is one additional bifurcation

(iv) Hopf bifurcation

normal form
$$\begin{cases} \dot{x} = \lambda x - \omega y - x(x^2+y^2) \\ \dot{y} = \omega x + \lambda y - y(x^2+y^2) \end{cases}$$

ω fixed ; λ varying

fixed point : $\bar{x} = \bar{y} = 0$

stability :

$$J(\bar{x}=\bar{y}=0) = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}$$

eigenvalues : $\det(J - \lambda I) = 0$

$$\begin{vmatrix} \lambda-\sigma & -\omega \\ \omega & \lambda-\sigma \end{vmatrix} = 0$$

$$\rightarrow \quad (\lambda-\sigma)^2 + \omega^2 = 0$$

$$\sigma_{1,2} = \lambda \pm i\omega$$

$\lambda > 0 \quad : \mathrm{Re}(\sigma) > 0$

$\lambda < 0 \quad : \mathrm{Re}(\sigma) < 0$

Solutions :

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \tilde{x}_0 \cos\omega t + \tilde{y}_0 \sin\omega t \\ -\tilde{x}_0 \sin\omega t + \tilde{y}_0 \cos\omega t \end{pmatrix}$$

Polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - \dot{\theta} r \sin\theta \\ \dot{y} = \dot{r}\sin\theta + \dot{\theta} r \cos\theta \\ x^2 + y^2 = r \end{cases}$$

$\rightarrow$

$$\dot{r}\cos\theta - \dot{\theta} r\sin\theta = \lambda r\cos\theta - \omega r\sin\theta - r^3\cos\theta \qquad (1)$$

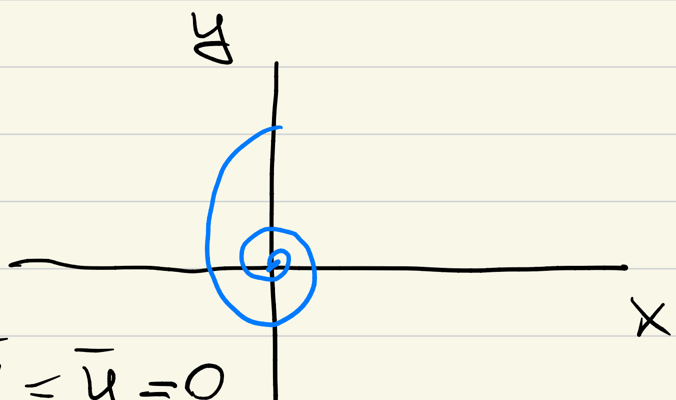
$$\dot{r}\sin\theta + \dot{\theta} r\cos\theta = \omega r\sin\theta + \lambda r\sin\theta - r^3\sin\theta \qquad (2)$$

$\cos\theta\,(1) + \sin\theta\,(2) \Rightarrow$

$$\dot{r} = \lambda r - r^3 \qquad (\text{'pitchfork'})$$

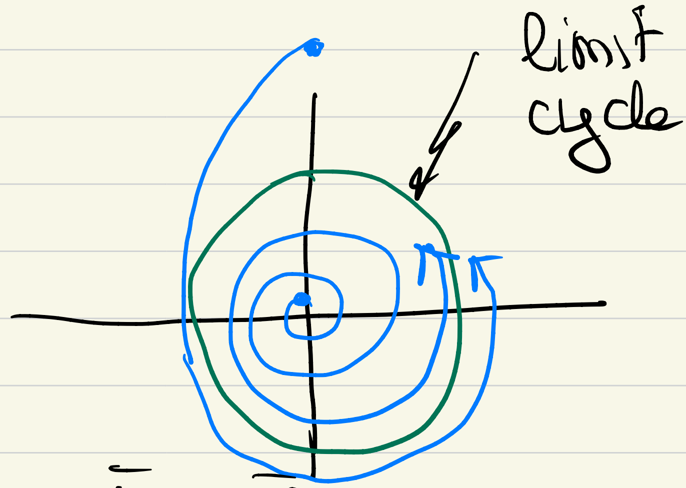
$-\sin\theta\,(1) + \cos\theta\,(2) \Rightarrow$

$$r^2\dot{\theta} = -\omega r^2 \longrightarrow \dot{\theta} = \omega$$



$\bar{x} = \bar{y} = 0$
stable focus

$\lambda < 0$



$\bar{x} = \bar{y} = 0$

unstable focus