Notes
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Land-based Ice Sheet

Plastic ice, yield stress τ_0

Stress balance (Nye, 1952)

On $[0, \frac{L}{2}]$: $\tau_0 = \rho_i g h \frac{\partial h}{\partial x}$

$$\rightarrow \quad \frac{\partial h^2}{\partial x} = 2h \frac{\partial h}{\partial x} = \frac{2\tau_0}{\rho_i g} \equiv \sigma$$

$$\rightarrow \quad h^2(x,t) = \sigma x + c(t)$$

$$h(0,t) = 0 \quad \cdots$$

$$\rightarrow \quad h^2(t) = \sigma x$$

On $\left[\frac{L}{2}, L\right]$: $\tau_0 = -\rho g h \frac{\partial h}{\partial x}$

$$\longrightarrow \begin{cases} h^2(x,t) = D(t) - x \\ h(L,t) = 0 \end{cases}$$

$$\longrightarrow \quad h^2(x,t) = \sigma\left(L(t) - x\right)$$

<u>Use</u>

$$\frac{L}{2} - \left|x - \frac{L}{2}\right| = \begin{cases} x & x < \frac{L}{2} \\ L - x & x > \frac{L}{2} \end{cases}$$

On $[0, L]$

$$\rightarrow \quad h^2(x,t) = \sigma\left(\frac{L(t)}{2} - \left|x - \frac{L(t)}{2}\right|\right)$$

$$\longrightarrow \quad h^2\left(\frac{L}{2}, t\right) = \sigma \frac{L(t)}{2} = H^2(t)$$

max height: $h_{max} = H(t)$

$$= \sqrt{\frac{\sigma L}{2}}$$

Mass balance

$$\rho_i \frac{\partial h}{\partial t} = \rho_i \beta \left(h - \gamma \left(x - x_p \right) \right)$$

$$\int_{L/2}^{L} \frac{\partial h}{\partial t} \, dx = \beta \int_{L/2}^{L} \left(\sqrt{\sigma (L - x)} - \gamma \left(x - x_p \right) \right) dx$$

$$= \beta \left[\sqrt{\sigma} \left(L - x \right)^{3/2} \left(-\frac{2}{3} \right) \Big|_{L/2}^{L} - \gamma \left(\frac{1}{2} x^2 - x_p x \right) \Big|_{L/2}^{L} \right]$$

$$= \beta \left[\frac{2}{3} \sqrt{\sigma} \left(\frac{L}{2} \right)^{3/2} - \gamma \left(\frac{1}{2} L^2 - x_p L - \frac{1}{2} \left(\frac{L}{2} \right)^2 + x_p \frac{L}{2} \right) \right]$$

$$= \beta \left[\frac{1}{3} \sqrt{\frac{\sigma}{2}} \, L \sqrt{L} - \gamma \left[\frac{L^2}{4} - x_p \frac{L}{2} \right] \right]$$

On the other hand :

$$\int_{L/2}^{L} \frac{\partial h}{\partial t}\, dx = \frac{\partial}{\partial t} \int_{L/2}^{L} h\, dx$$

$$- \underbrace{h(L)}_{=0} \frac{dL}{dt} + h\left(\frac{L}{2}\right) \frac{d(L/2)}{dt}$$

$$= \frac{1}{2} H(t) \frac{dL}{dt} + \frac{\partial}{\partial t} \int_{L/2}^{L} h\, dx$$

use $\displaystyle\int_{L/2}^{L} h\, dx = \frac{1}{3}\sqrt{\frac{5}{2}}\, L\sqrt{L}$

$$= \frac{H}{2} \frac{dL}{dt} + \frac{1}{3}\sqrt{\frac{5}{2}}\, \frac{d(L\sqrt{L})}{dt}$$

$$= \frac{H}{2} \frac{dL}{dt} + \frac{3}{2}\,\frac{1}{3}\sqrt{\frac{5}{2}}\,\sqrt{L}\, \frac{dL}{dt} = H \frac{dL}{dt}$$

Finally

$$H \frac{dL}{dt} = \beta\left(\frac{1}{3}HL - \gamma\left(\frac{L^2}{4} - X_p\frac{L}{2}\right)\right)$$

$$\frac{dL}{dt} = \frac{\beta}{3}L - \frac{\beta\gamma}{4}\frac{L^2}{H} + \beta\gamma X_p \frac{L}{2H}$$

use $H = \sqrt{\frac{\sigma L}{2}}$

$$\frac{\beta\gamma}{4}\frac{L^2}{H} = -\frac{\beta\gamma}{4}\sqrt{\frac{2}{\sigma}}\,L\sqrt{L} \qquad \beta\gamma X_p\frac{L}{2H} = \frac{\beta\gamma X_p}{\sqrt{2\sigma}}\sqrt{L}$$

$$\longrightarrow \quad F_1 = \frac{\beta\gamma X_p}{\sqrt{2\sigma}}$$

$$F_2 = \beta/3$$

$$F_3 = -\frac{\beta\gamma}{4}\sqrt{\frac{2}{\sigma}}$$

$$\frac{dL}{dt} = F_1 \sqrt{L} + F_2 L + F_3 L\sqrt{L}$$

<u>fixed</u> points $\quad \ell = \sqrt{L}$

$$\rightarrow \quad F_1 \ell + F_2 \ell^2 + F_3 \ell^3 = 0$$

$\ell = 0$: fixed point

$$F_1 + F_2 \ell + F_3 \ell^2 = 0$$

$$\ell_1, \ell_2 = \frac{-F_2 \pm \sqrt{F_2^2 - 4F_1F_3}}{2F_3}$$

two solutions, when

$$F_2^2 - 4F_1F_3 > 0$$

Saddle node when: $F_2^2 = 4F_1F_3$

only for $x_p < 0$

Stochastic Diff Eq.

$$dX_t = -\gamma X_t\, dt + \sigma\, dW_t$$

Deterministic

$$X(t) = X_0 + \int_0^t -\gamma X(s)\, ds$$

<u>solution</u> $X(t) = X_0 e^{-\gamma t}$

Stochastic

$$X(t) = X_0 + \int_0^t -\gamma X_s\, ds + \underbrace{\int_0^t \sigma\, dW_s}_{\text{stochastic integral}}$$

Consider smooth function $f(t,x)$

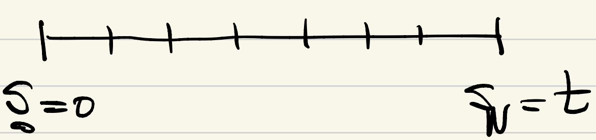
$$f_1 = \frac{\partial f}{\partial t} \quad f_2 = \frac{\partial f}{\partial x}, \quad f_{22} = \frac{\partial^2 f}{\partial x^2}$$

$$f(t+dt, W_t + dW_t) - f(t, W_t)$$

$$= f_1\, dt + f_2\, dW_t + \frac{1}{2}\Big(f_{11}\, dt^2 + 2 f_{12}\, dt\, dW_t + f_{22} \underbrace{(dW_t)^2}_{= dt} \Big) + \ldots$$

$$\sim f_1\, dt + f_2\, dW_t + \frac{1}{2} f_{22}\, dt$$

Interval



$$\sum_{j=0}^{N-1} \left\{ f\left(s_{j+1}, W_{j+1}\right) - f\left(s_j, W_j\right) \right\}$$

$$\cong \int_0^t \left(f_1 + \frac{1}{2} f_{22} \right) ds + \int_0^t f_2 \, dW_s$$

$$= f\left(t, W_t\right) - f\left(0, W_0\right)$$

<u>Now</u> $\quad dX_t = A_t \, dt + B_t \, dW_t$

see slide

$$f\left(t, X_t\right) - f\left(0, X_0\right) =$$

$$\int_0^t f_1 ds + f_2 \left(A_s \, ds + B_s \, dW_s \right) +$$

$$\frac{1}{2} f_{22} \left(A_s^2 \, ds^2 + 2 A_s B_s \, ds \, dW_s + B_s^2 \underbrace{\left(dW_s\right)^2}_{= ds} \right)$$

Finally

$$f(t, X_t) - f(0, X_0) =$$

$$\int_0^t \left(f_1 + \frac{1}{2} B_s^2 f_{22} + f_2 A_s \right) ds$$

$$+ \int_0^t f_2 B_s \, dW_s$$

Example : $A_s = -\gamma X_s$, $B_s = \sigma$

take $f(t, x) = x e^{\gamma t}$

$f_1 = \gamma f$, $f_2 = e^{\gamma t}$, $f_{22} = 0$

$$\rightarrow X_t e^{\gamma t} - X_0 = \int_0^t \left(\gamma X_s e^{\gamma s} + e^{\gamma s}(-\gamma X_s \text{[illegible]} \right) ds$$

$$+ \int_0^t e^{\gamma s} \sigma \, dW_s$$

hence

$$X_t = e^{-\gamma t}\left(X_0 + \sigma \int_0^t e^{\gamma s}\, dW_s\right)$$

Ornstein - Uhlenbeck

process

Autocorrelation :

$$\mathrm{Cov}\left[X_t, X_{t+s}\right] = E\left[X_t X_{t+s}\right]$$

$$= E\left[\left(e^{-\gamma t} X_0 + \sigma e^{-\gamma t} \int_0^t e^{\gamma q}\, dW_q\right) * \right.$$

$$\left. * \left(-e^{\gamma(s+t)} X_0 + \sigma e^{-\gamma(s+t)} \int_0^{t+s} e^{\gamma p}\, dW_p\right)\right]$$

$$= X_0^2 e^{-\gamma(2t+s)} + \sigma^2 e^{-\gamma(2t+s)} *$$

$$\underbrace{E\left[\int_0^t dW_q \int_0^{t+s} e^{\gamma(p+q)}\, dp\right]}_{= I}$$

note that $E[dW] = 0$ so cross products are zero.

<u>write</u> $\quad dW_q = \eta(q)\, dq$

$\quad dW_p = \eta(p)\, dp$

η : white noise

then

$$I = E\left[\int_0^t dq \int_0^{t+s} e^{\gamma(p+q)} \eta(p)\eta(q)\, dp \right]$$

use $E[\eta(t')\eta(t'')] = \delta(t'-t'')$

$$I = \int_0^t dq \int_0^{t+s} e^{\gamma(p+q)} \delta(p-q)\, dp$$

$$= \int_0^t dq\, e^{2\gamma q} = \frac{1}{2\gamma}\left(e^{2\gamma t} - 1\right)$$

Finally

$$E[X_t X_{t+s}] = e^{-\gamma(2t+s)} \left(X_0^2 + \sigma^2 \frac{e^{2\gamma t} - 1}{2\gamma} \right)$$

s fixed , $t \to \infty$

$$E[X_t X_{t+s}] = \frac{\sigma^2}{2r} e^{-rs}$$

<u>Variance</u> $E[X_t^2] = \frac{\sigma^2}{2r}$

From time series :

$$X_{t_{n+1}} = e^{-rt_{n+1}}\left(X_0 + \sigma\int_0^{t_{n+1}} e^{rs}\,dW_s\right)$$

$$e^{-r\Delta t}\left[X_{t_n} = e^{-rt_n}\left(X_0 + \sigma\int_0^{t_n} e^{rs}\,dW_s\right)\right]$$

$$t_{n+1} = t_n + \Delta t$$

<u>subtract</u> :

$$\alpha \in (0,1)$$

$$X_{t_{n+1}} - \overbrace{e^{-r\Delta t}}^{\alpha} X_{t_n} =$$

$$= \underbrace{\sigma\left\{\int_0^{t_{n+1}} e^{rs}\,dW_s - \int_0^{t_n} e^{r(s-\Delta t)}\,dW_s\right\}}_{Z_{n+1}}$$

$$\rightarrow \quad X_{n+1} - \alpha X_n = Z_{n+1}$$

$$Z \sim N(0, \sigma^2) \qquad Var[Z] = \sigma^2$$

AR(1) process :

$$X_i = \alpha X_{i-1} + Z_i$$

$$= \alpha \left(\alpha X_{i-2} + Z_{i-1} \right) + Z_i$$

$$= Z_i + \alpha Z_{i-1} + \dots + \alpha^i X_0$$

$E[X_i] = 0$ when $E[X_0] = 0$

<u>Use</u> $\mathrm{Var}[\alpha X + \beta Y] =$

$$\alpha^2 \mathrm{Var}[X] + 2\alpha\beta \mathrm{Cov}[X, Y] + \beta^2 \mathrm{Var}[Y]$$

$$\rightarrow \mathrm{Var}[X_i] = \mathrm{Var}[Z_i] + \alpha^2 \mathrm{Var}[Z_{i-1}] + \dots + \alpha^{2i} \underbrace{\mathrm{Var}[X_0]}_{=0}$$

$$= \sigma^2 \left(1 + \alpha^2 + \alpha^4 + \dots \right)$$

$$= \frac{\sigma^2}{1 - \alpha^2}$$

$$E[X_i X_{i+k}] =$$

$$E\left[\left(Z_i + \alpha Z_{i-1} + \dots + \alpha^{i-1} Z_1\right) * \right.$$

$$\left. * \left(Z_{i+k} + \alpha Z_{i+k-1} + \dots + \alpha^{k+i-1} Z_1\right)\right]$$

$$= \alpha^k \sigma^2 \left(1 + \alpha^2 + \alpha^4 + \dots\right)$$

$$= \alpha^k \sigma^2 / (1 - \alpha^2)$$

<u>Hence</u> $$\rho_k = \frac{E[X_i X_{i+k}]}{Var[X_i]}$$

$$= \alpha^k$$