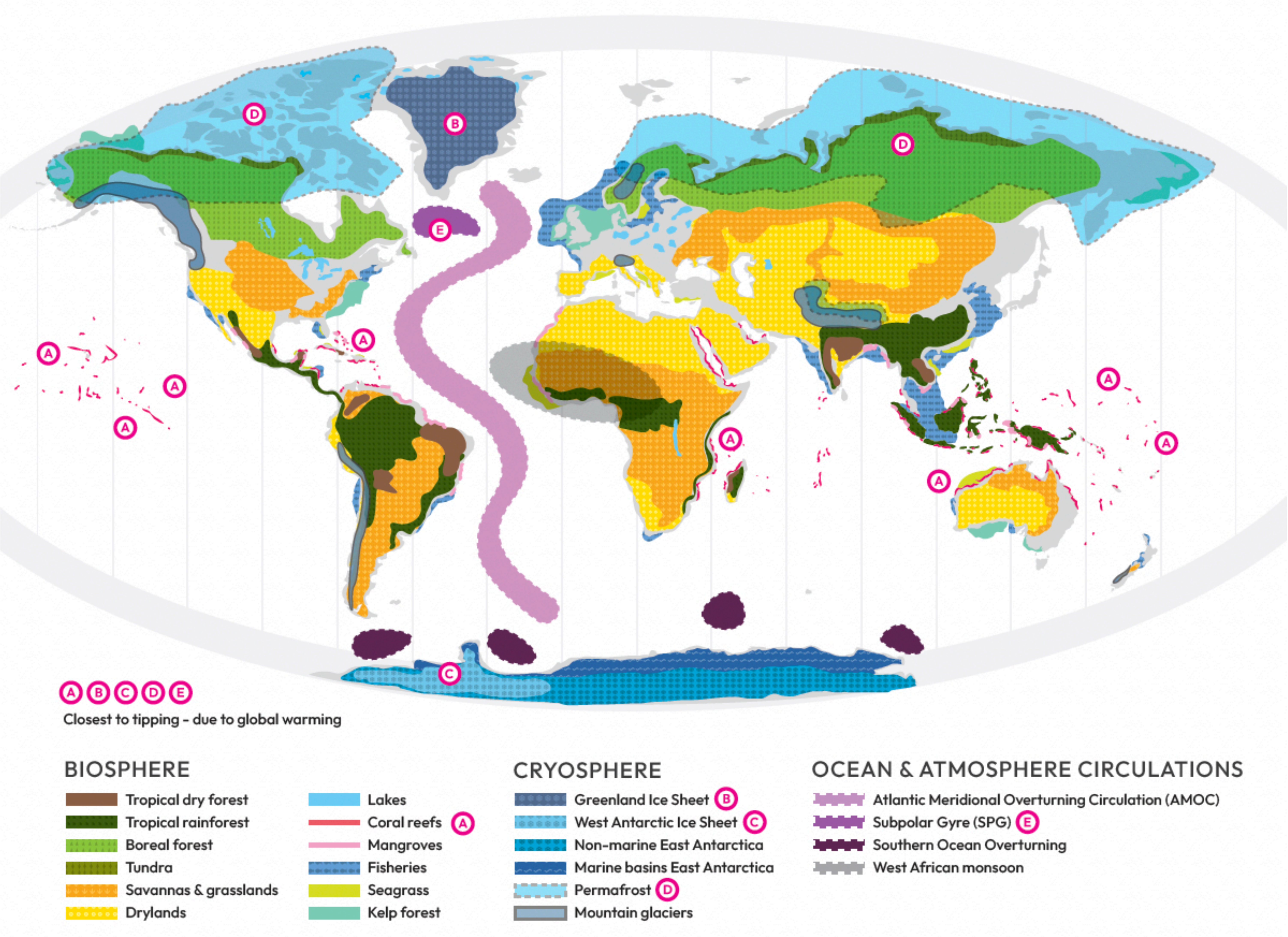
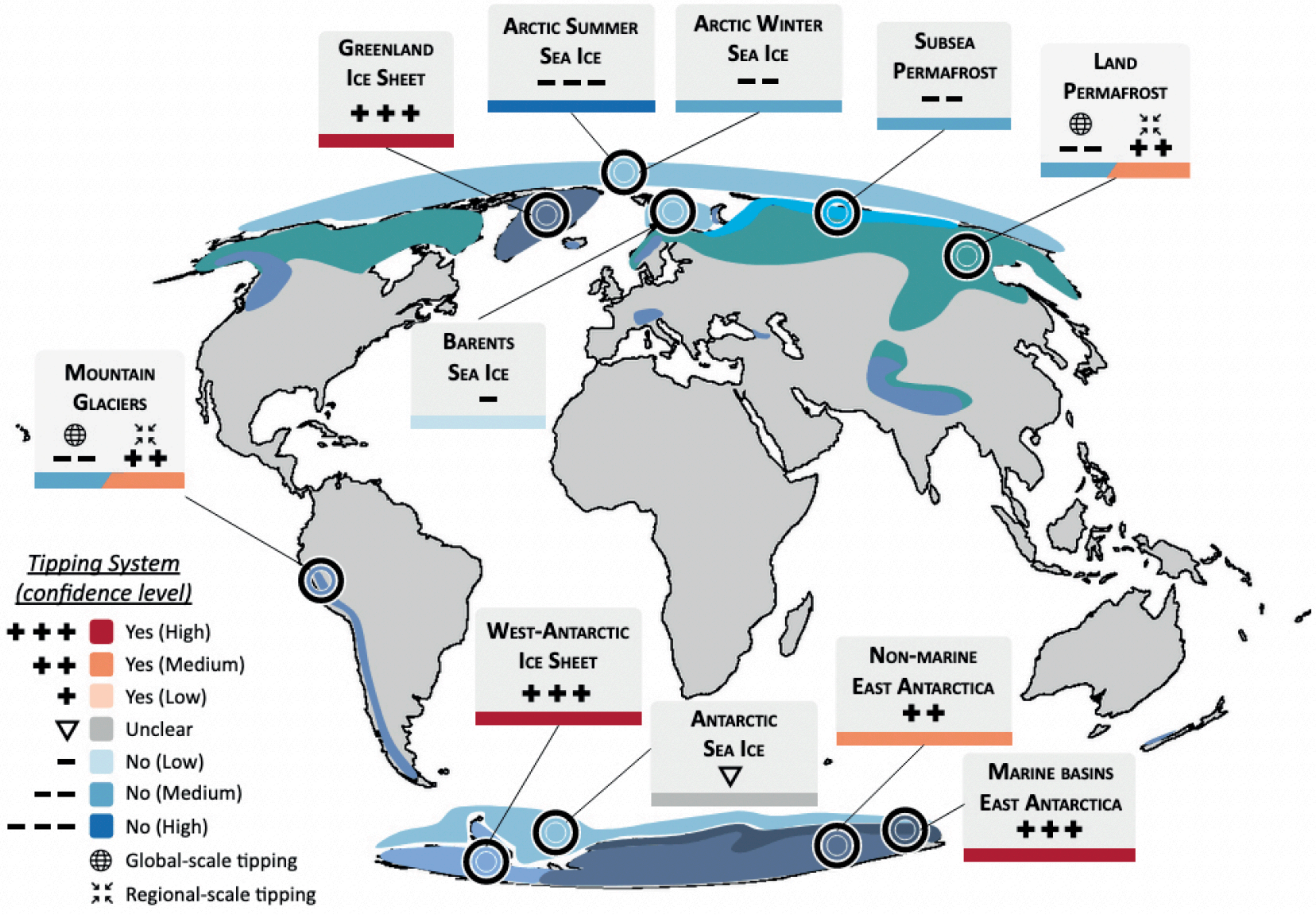
Tipping Behavior in the Climate System

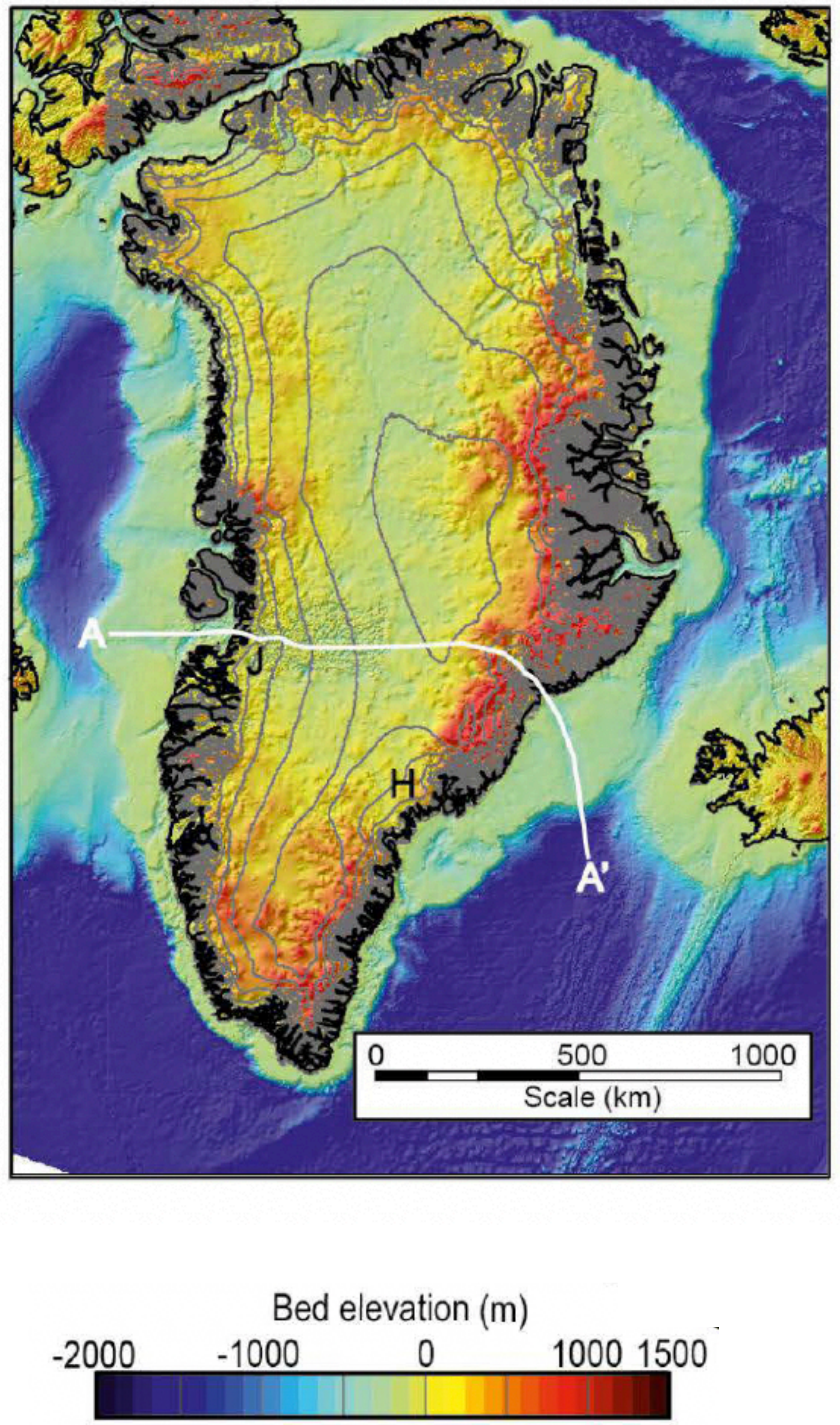


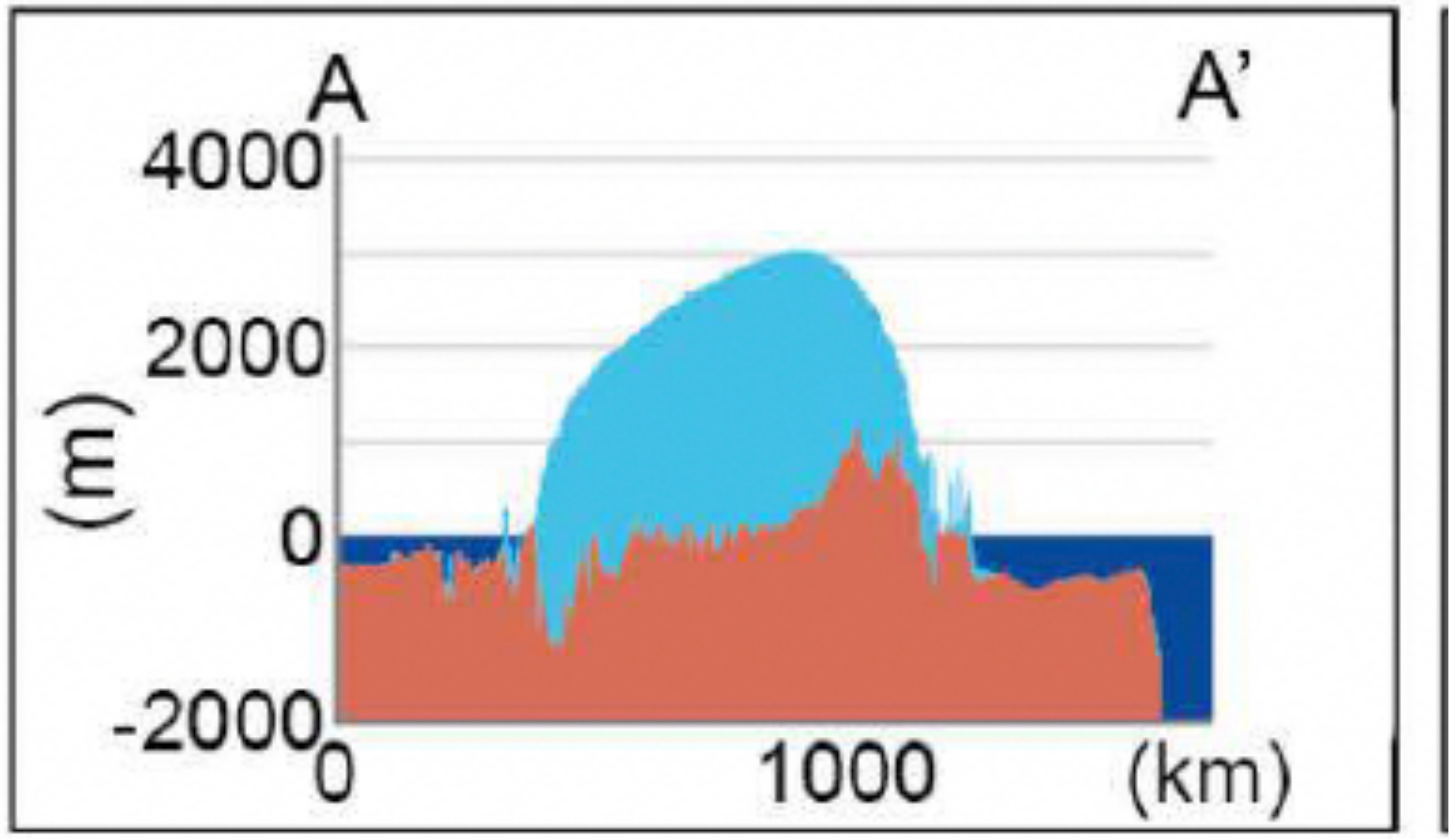
Henk Dijkstra,
Department of Physics,
Utrecht University, NL
&
DICAM,
University of Trento, IT

Tipping Elements: Cryosphere



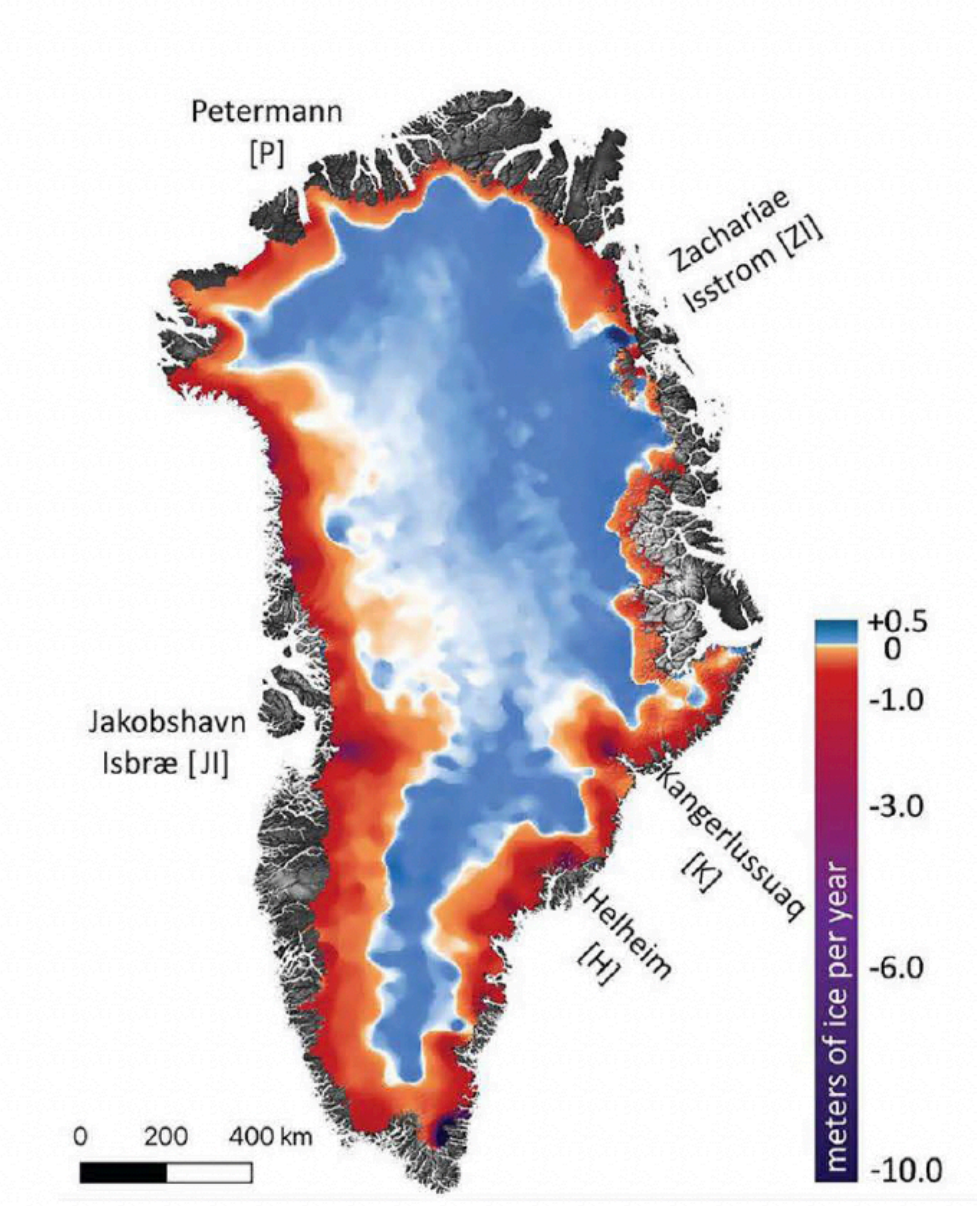
Tipping Element: Greenland Ice Sheet





Land-based ice sheet

Mass Loss: Greenland Ice Sheet



Period: 2003-2019

Model

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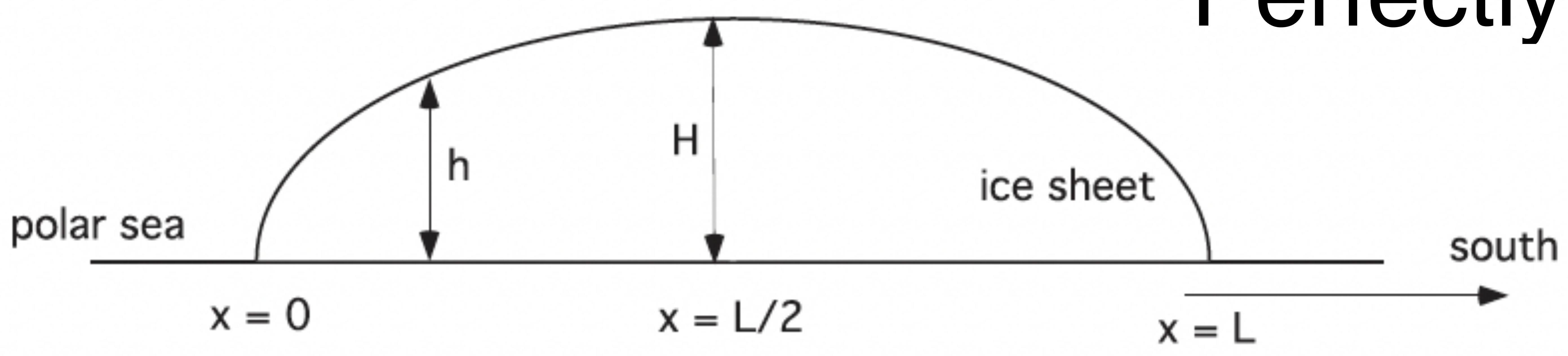
Perfectly plastic ice



On [0,L/2]

$$\tau_0 = \rho_i g h \frac{\partial h}{\partial x}$$

$$2h\frac{\partial h}{\partial x} = \frac{\partial h^2}{\partial x} = \frac{2\tau_0}{\rho_i g} \equiv \sigma \to h^2(x,t) = \sigma x + C(t).$$

On [L/2,L]:

$$\tau_0 = -\rho_i g h \frac{\partial h}{\partial x}$$

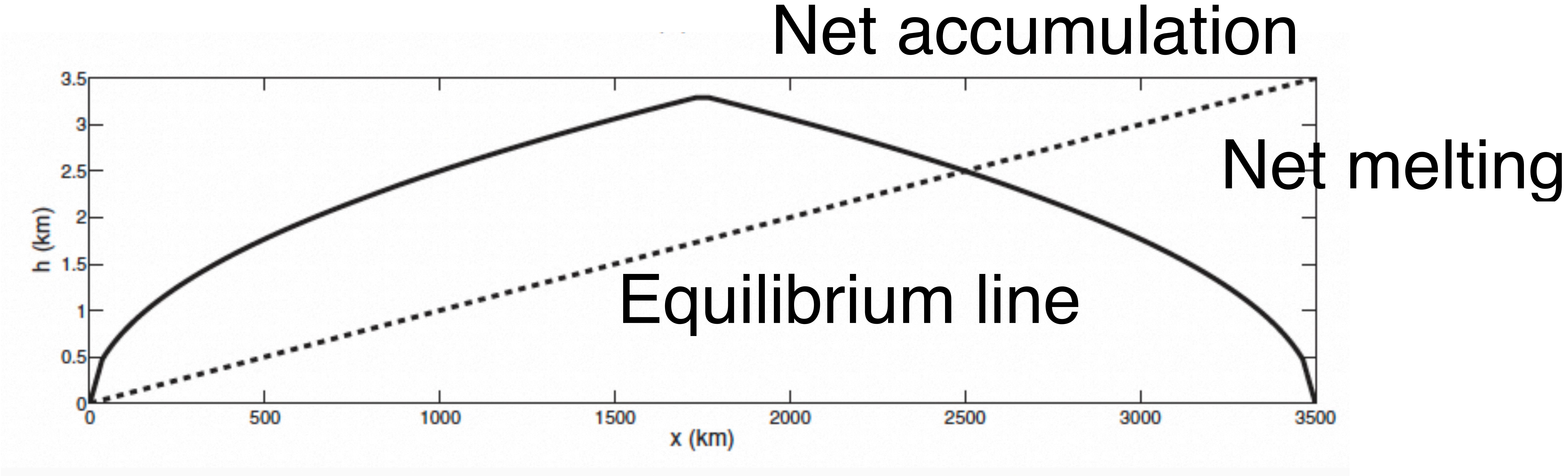
$$h^2(x,t) = D(t) - \sigma x$$

Boundary conditions

$$h(x,t) = \sqrt{\sigma}\left(\frac{L(t)}{2} - |x - \frac{L(t)}{2}|\right)^{\frac{1}{2}}$$

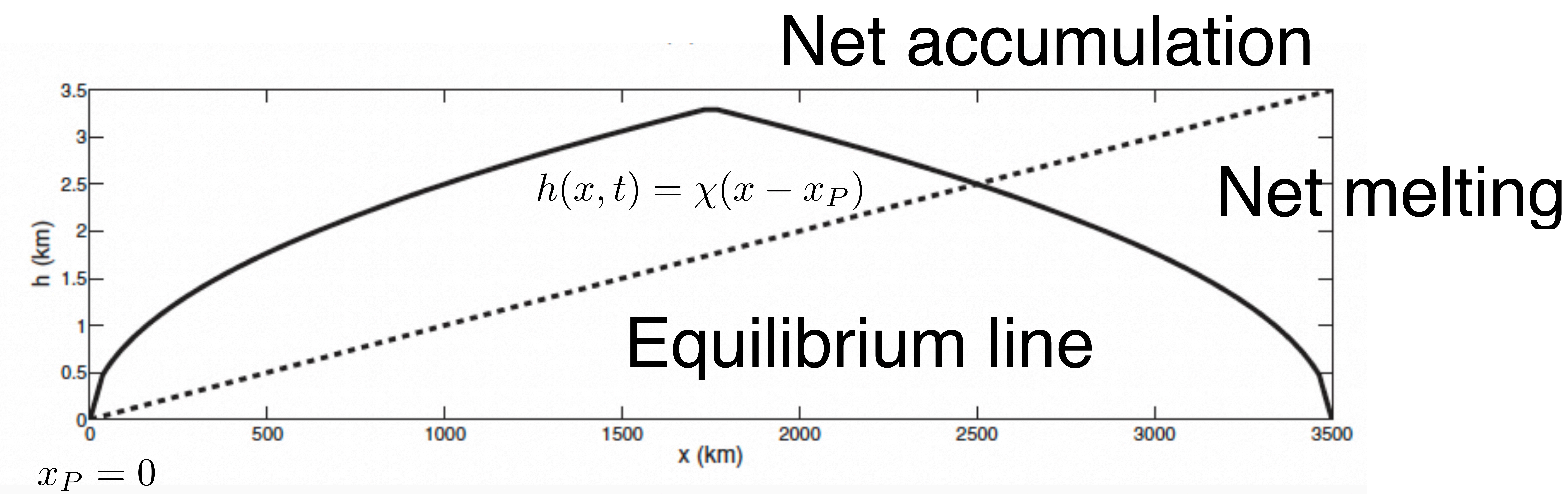
Solution

$$h(x,t) = \sqrt{\sigma}\left(\frac{L(t)}{2} - |x - \frac{L(t)}{2}|\right)^{\frac{1}{2}}$$



Parameter	Meaning	Value	Unit
ρ_i	ice density	0.9	kg m^{-3}
L_0	ice cap length	3,500	km
σ	yield stress parameter	6.25	m

Mass balance

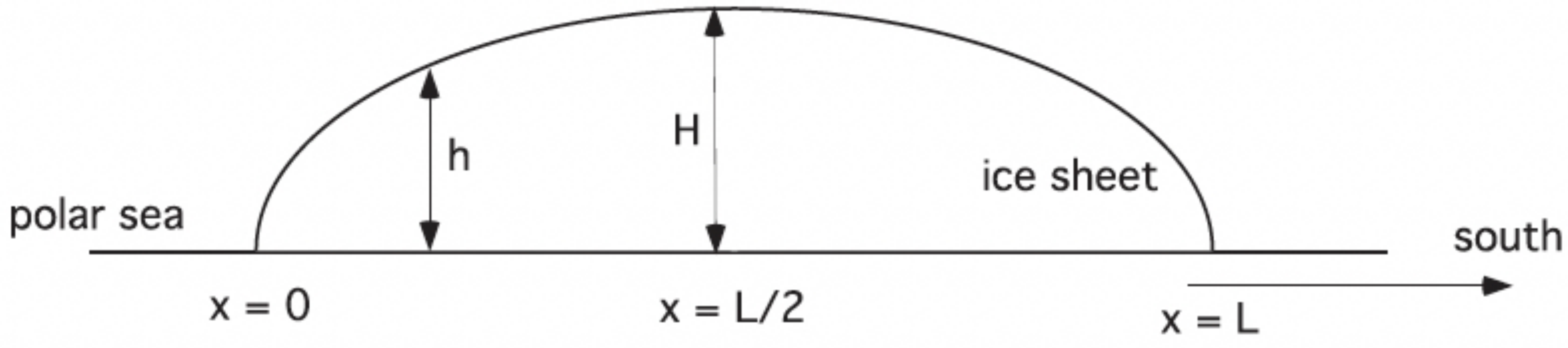


$$\rho_i \frac{\partial h}{\partial t} = P_i - M,$$

$$P_i(x,t) - M(x,t) = G(x,t) = \rho_i \beta(h(x,t) - \chi(x - x_P)),$$

Final reduced model

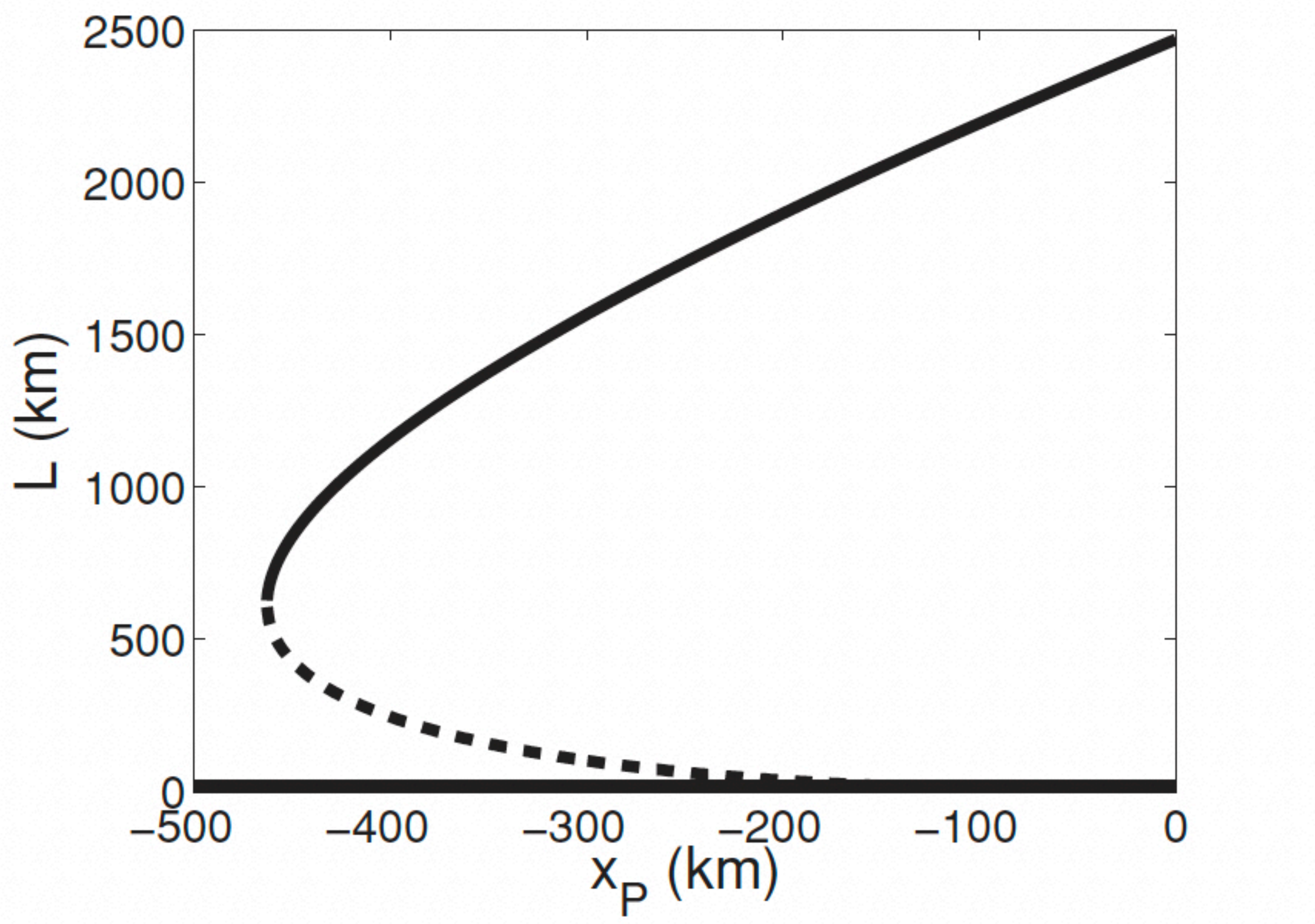
$$\int_{L/2}^{L} \frac{\partial h}{\partial t}\, dx = H \frac{dL}{dt} = \int_{L/2}^{L} \left[\beta(\sqrt{\sigma(L-x)} - \chi(x - x_P)) \right] dx,$$

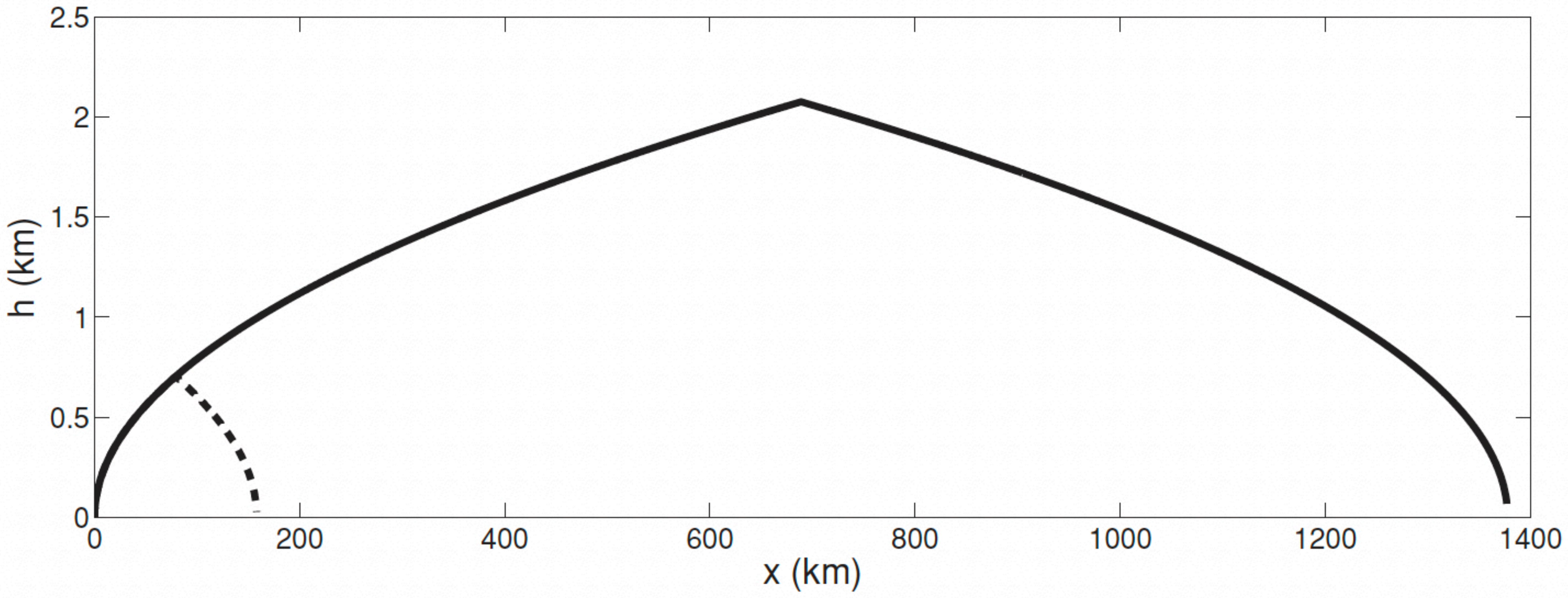


$$\frac{dL}{dt} = F_1\sqrt{L} + F_2 L + F_3 L\sqrt{L},$$

Parameter	Meaning	Value	Unit
ρ_i	ice density	0.9	kg m^{-3}
L_0	ice cap length	3,500	km
σ	yield stress parameter	6.25	m
β	coefficient in the linear mass balance	10^{-3}	yr^{-1}
χ	coefficient in the linear mass balance	10^{-3}	–

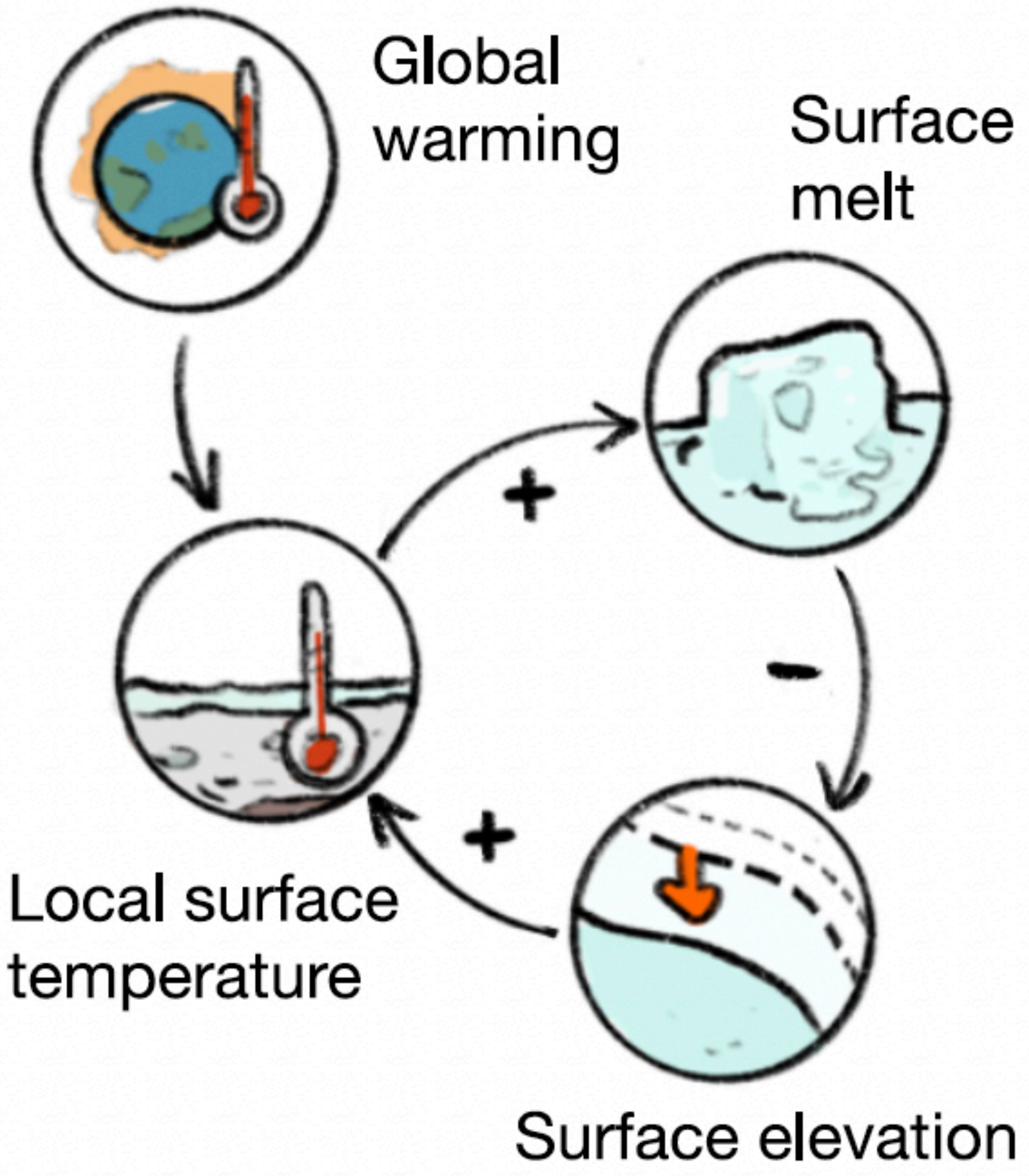
Bifurcation diagram

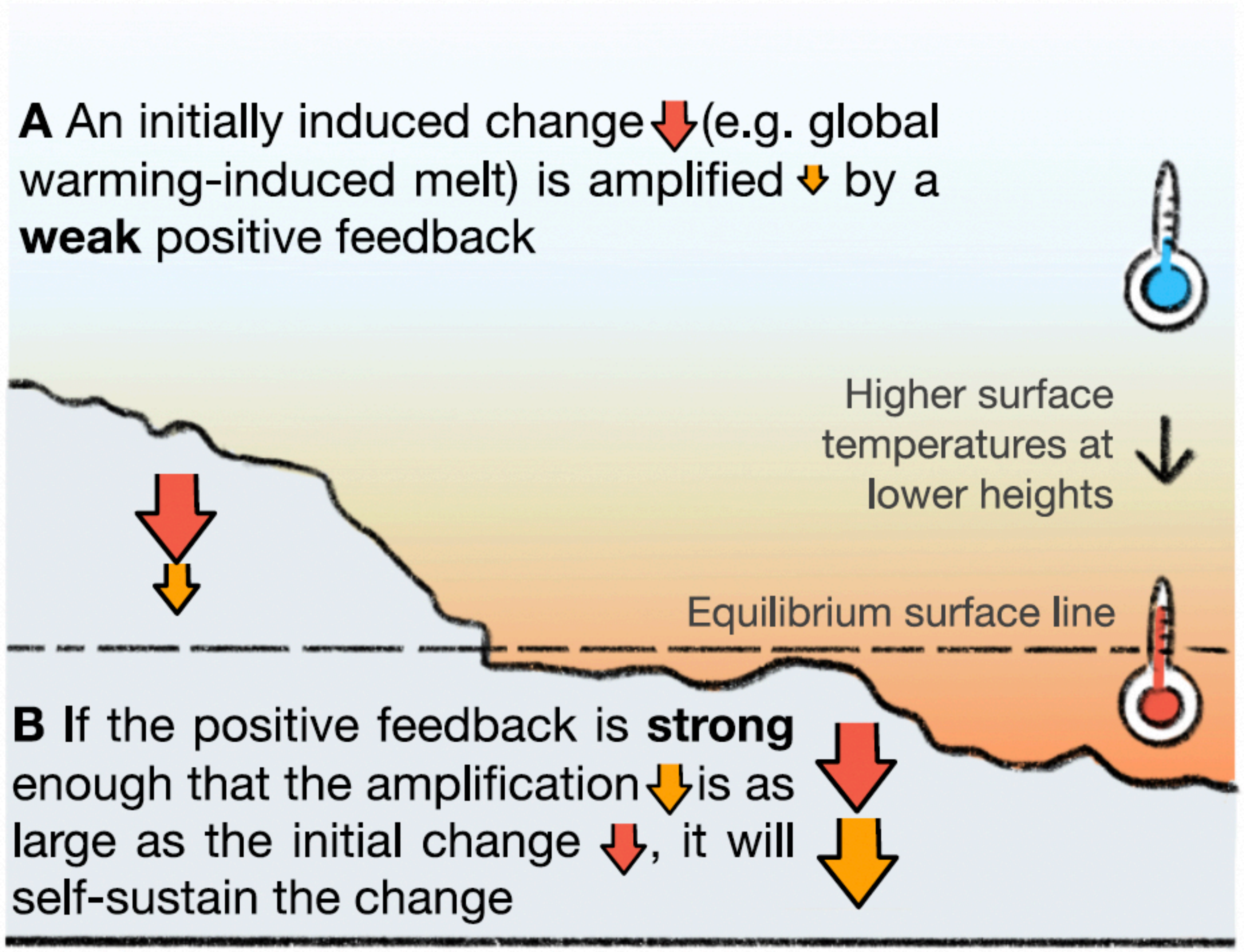




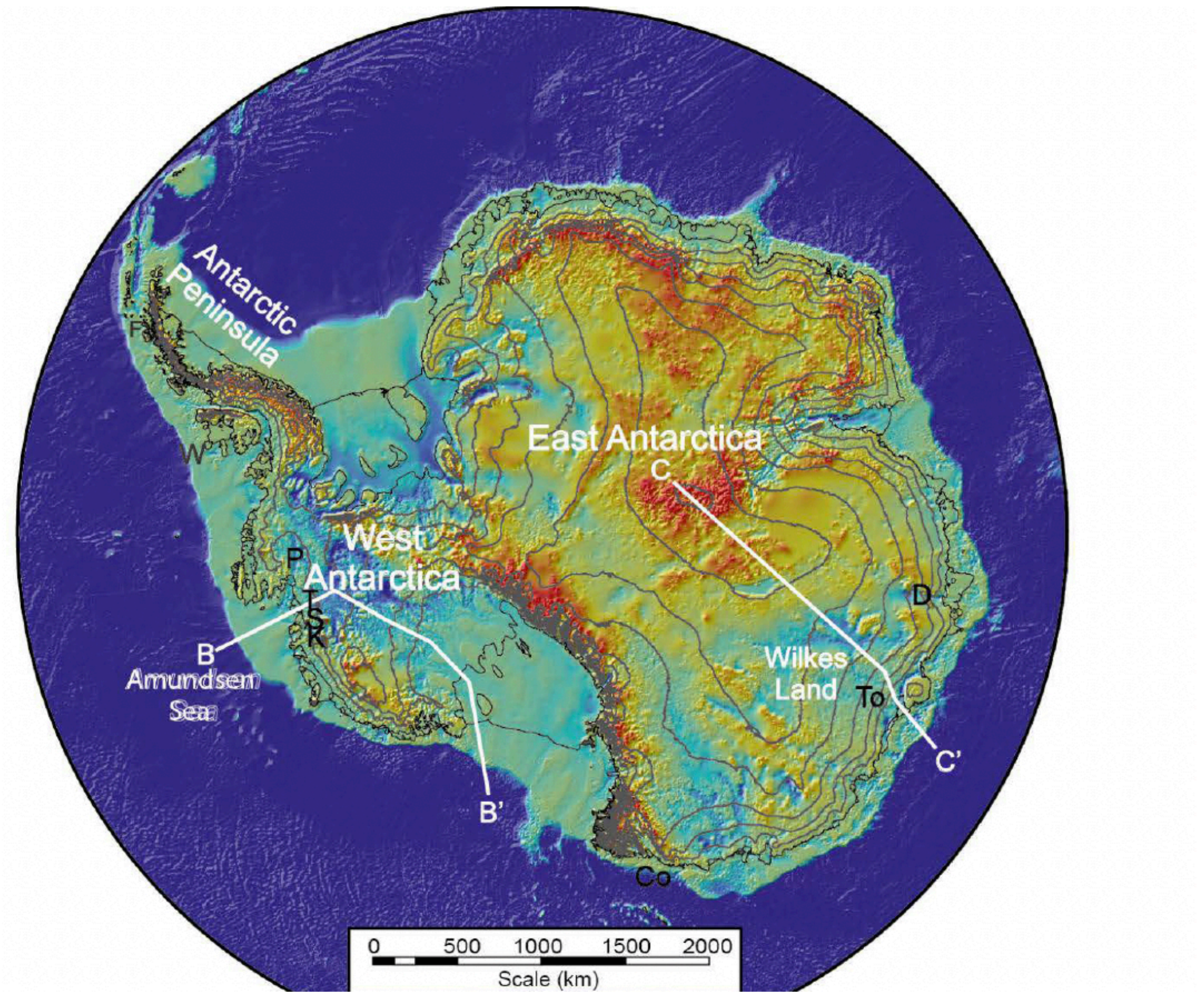
$$x_P = -250\ km$$

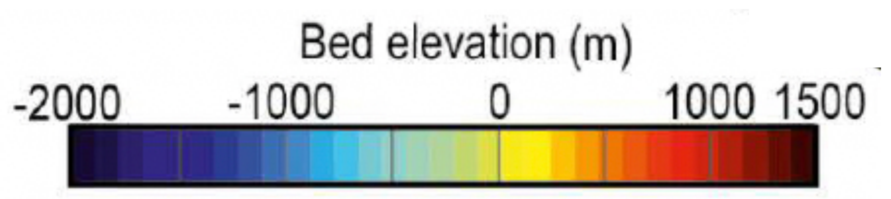
Melt-elevation Feedback

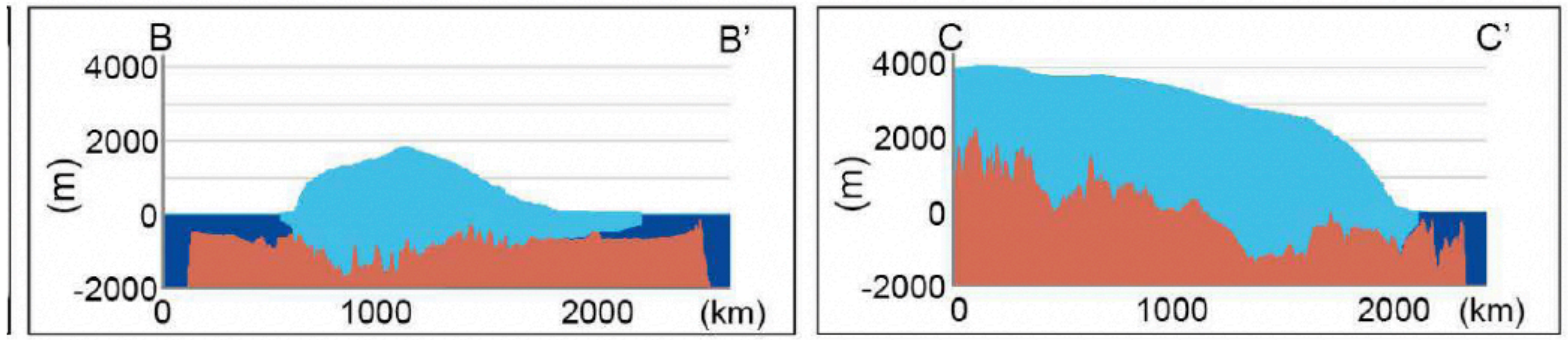




Tipping Element: Antarctic Ice Sheet



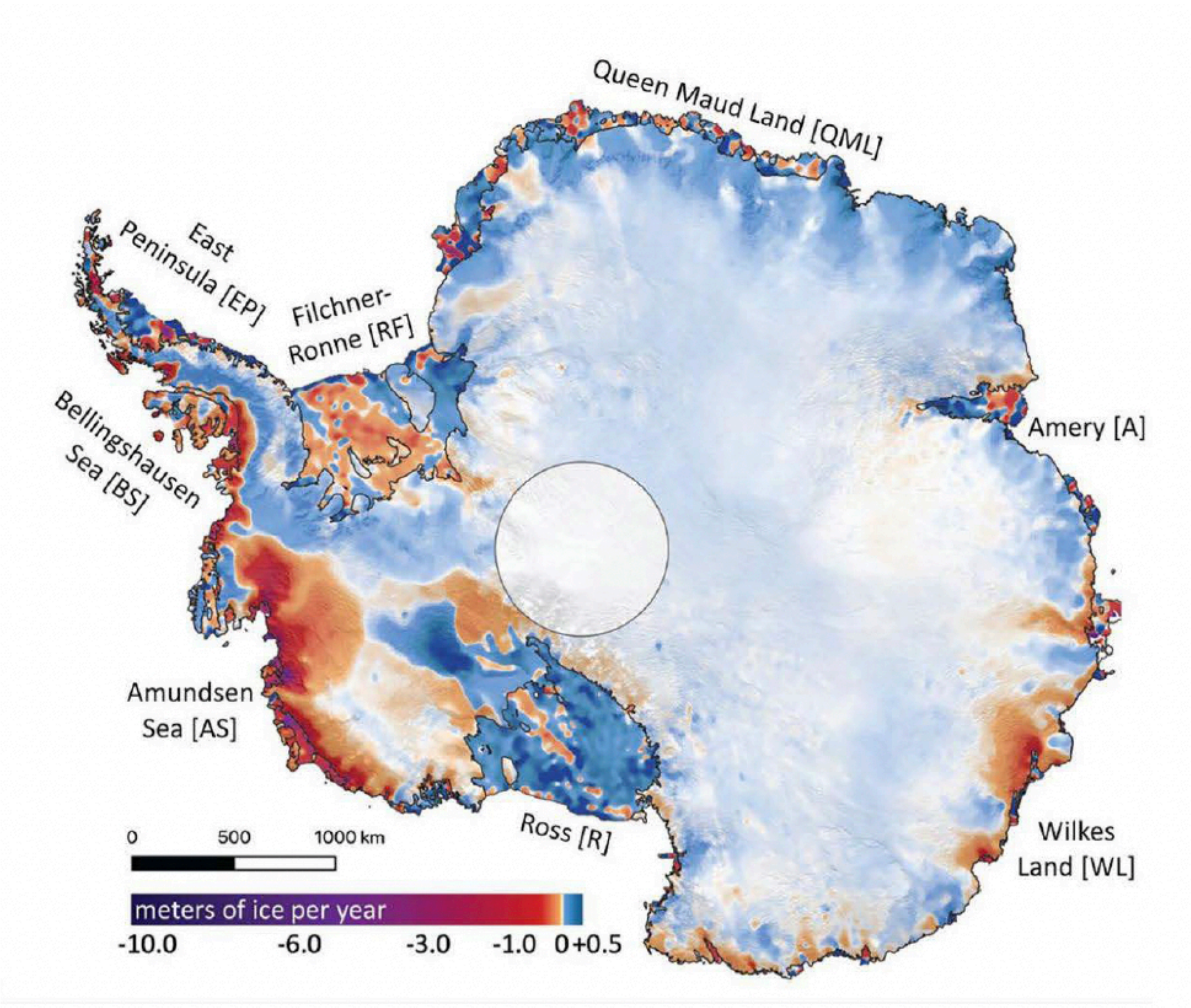




Marine ice sheet

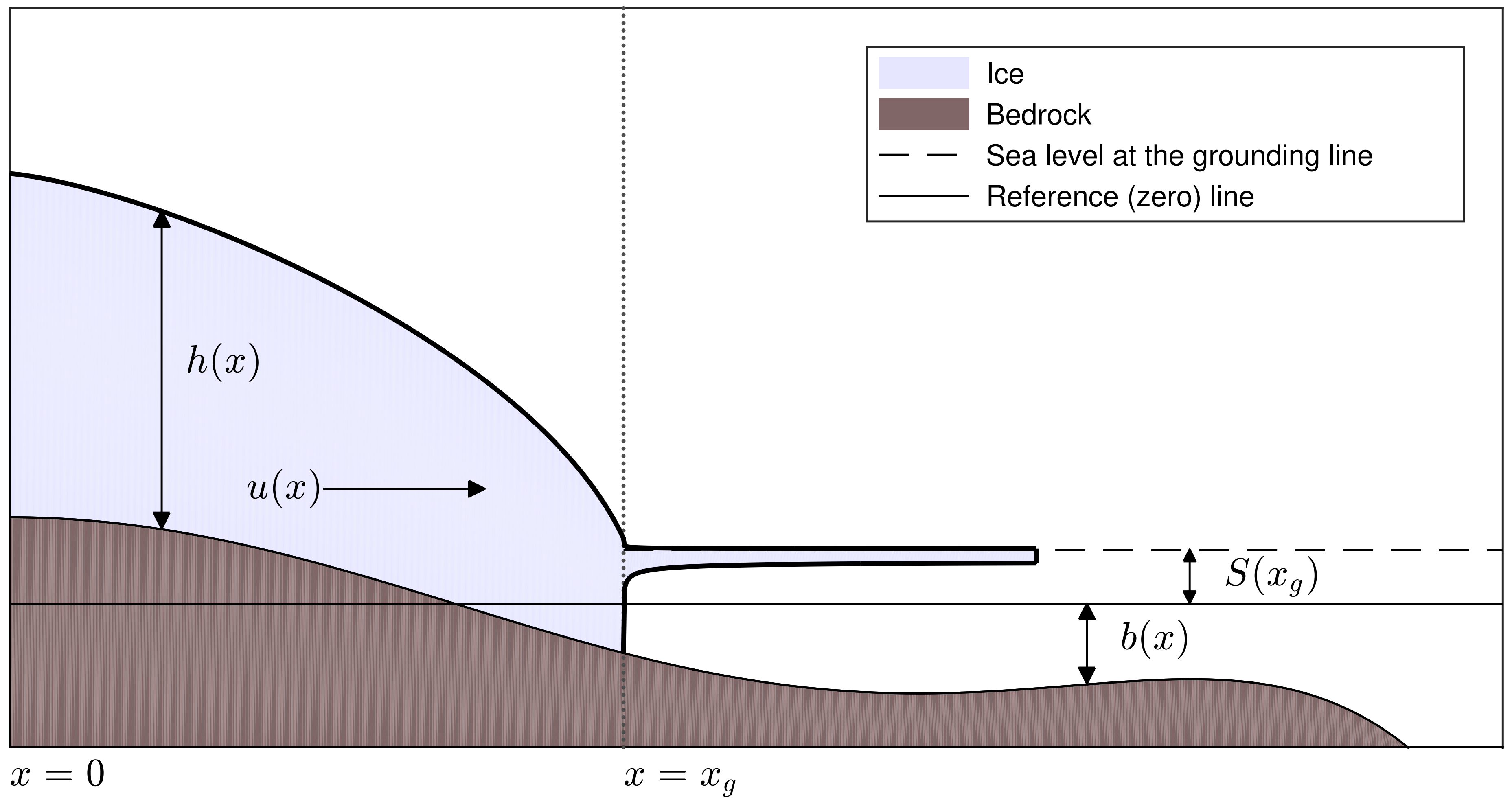
Utrecht University

Mass Loss: Antartic Ice Sheet



Period: 2003-2019

Model



Conservation of mass:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = a,$$

a: accumulation rate.

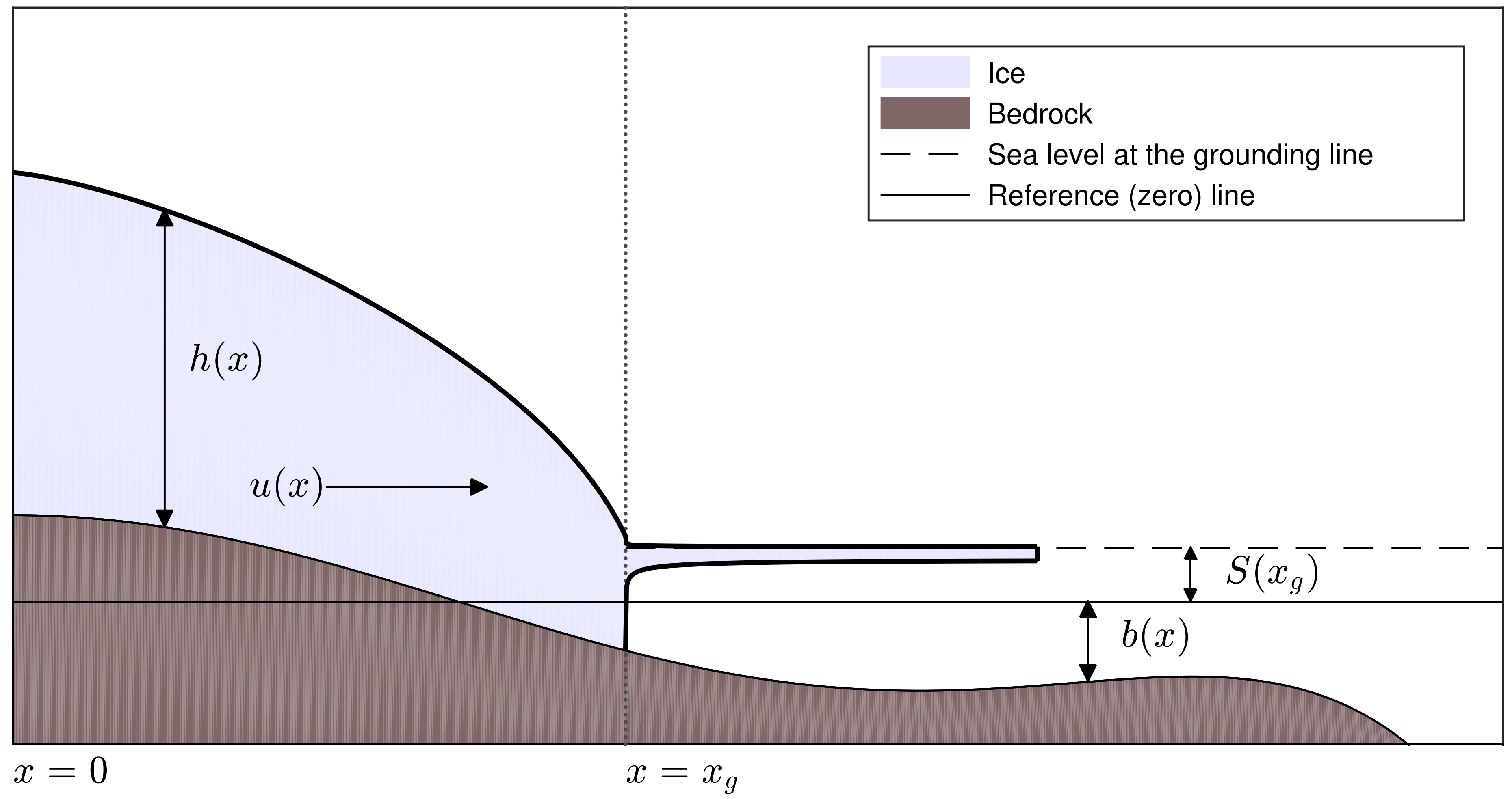
Conservation of momentum:

$$\frac{\partial}{\partial x}\left[2A^{-\frac{1}{n}} h \left|\frac{\partial u}{\partial x}\right|^{\frac{1}{n}-1} \frac{\partial u}{\partial x}\right] - C\,|u|^{(m-1)}\,u - \rho_i g h \frac{\partial (h-b)}{\partial x} = 0,$$

A and n are coefficients of Glen's flow law, a constitutive relation describing the rheology of ice (typically $n = 3$).

The parameters C and m determine the sliding of the ice

Boundary conditions



No flow at the ice divide:

$$\frac{\partial(h-b)}{\partial x} = 0, \quad u = 0 \quad \text{at } x = 0.$$

There are two boundary conditions at the grounding line $x = x_g$. One condition results from an integration of the shelf flow [Schoof,2007a] as:

$$2A^{-\frac{1}{n}} \left| \frac{\partial u}{\partial x} \right|^{\frac{1}{n}-1} \frac{\partial u}{\partial x} = \frac{1}{2} \left(1 - \frac{\rho_i}{\rho_w}\right) \rho_i g h \quad \text{at } x = x_g.$$

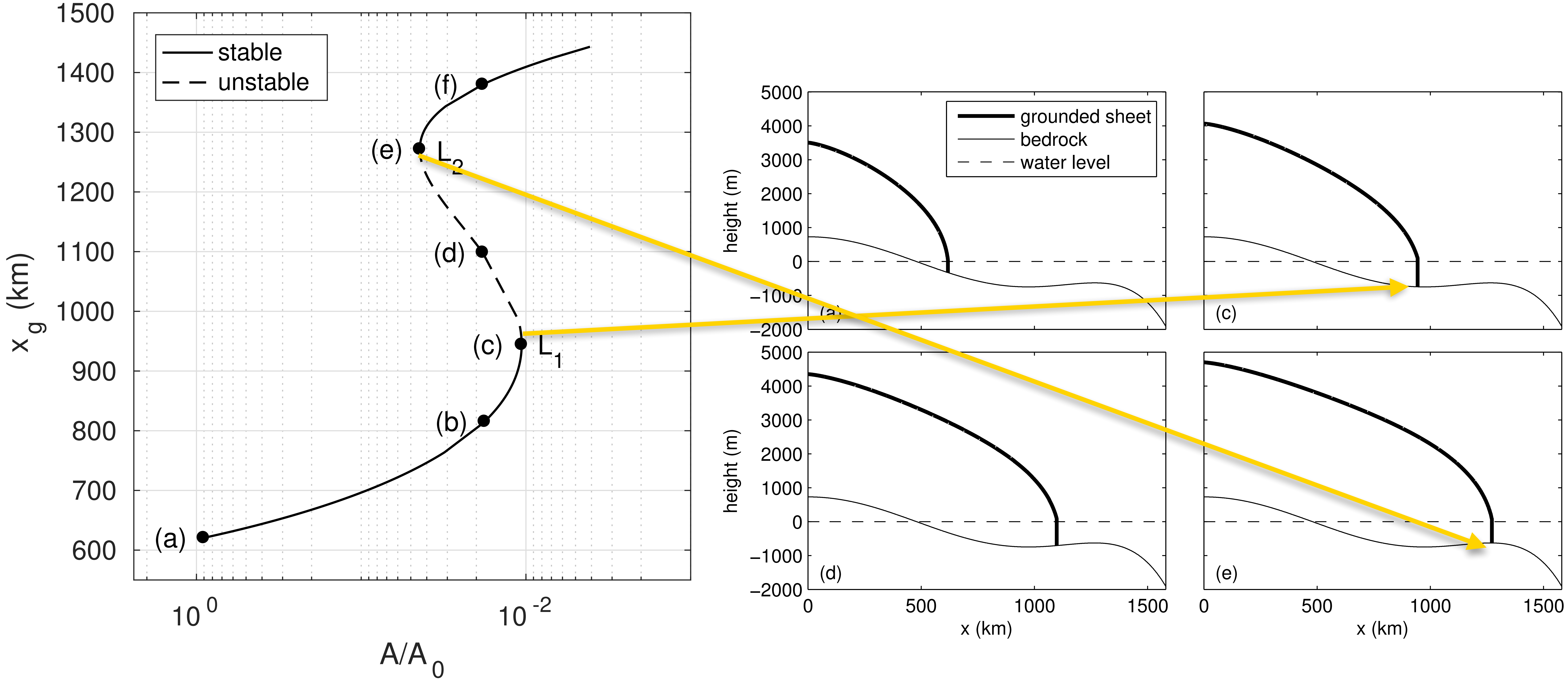
The other boundary condition is the flotation requirement:

$$\rho_i h = \rho_w \left(b + S\right) \quad \text{at } x = x_g,$$

where $S(x_g)$ is the sea level at the grounding line position.

Utrecht University

Bifurcation diagram



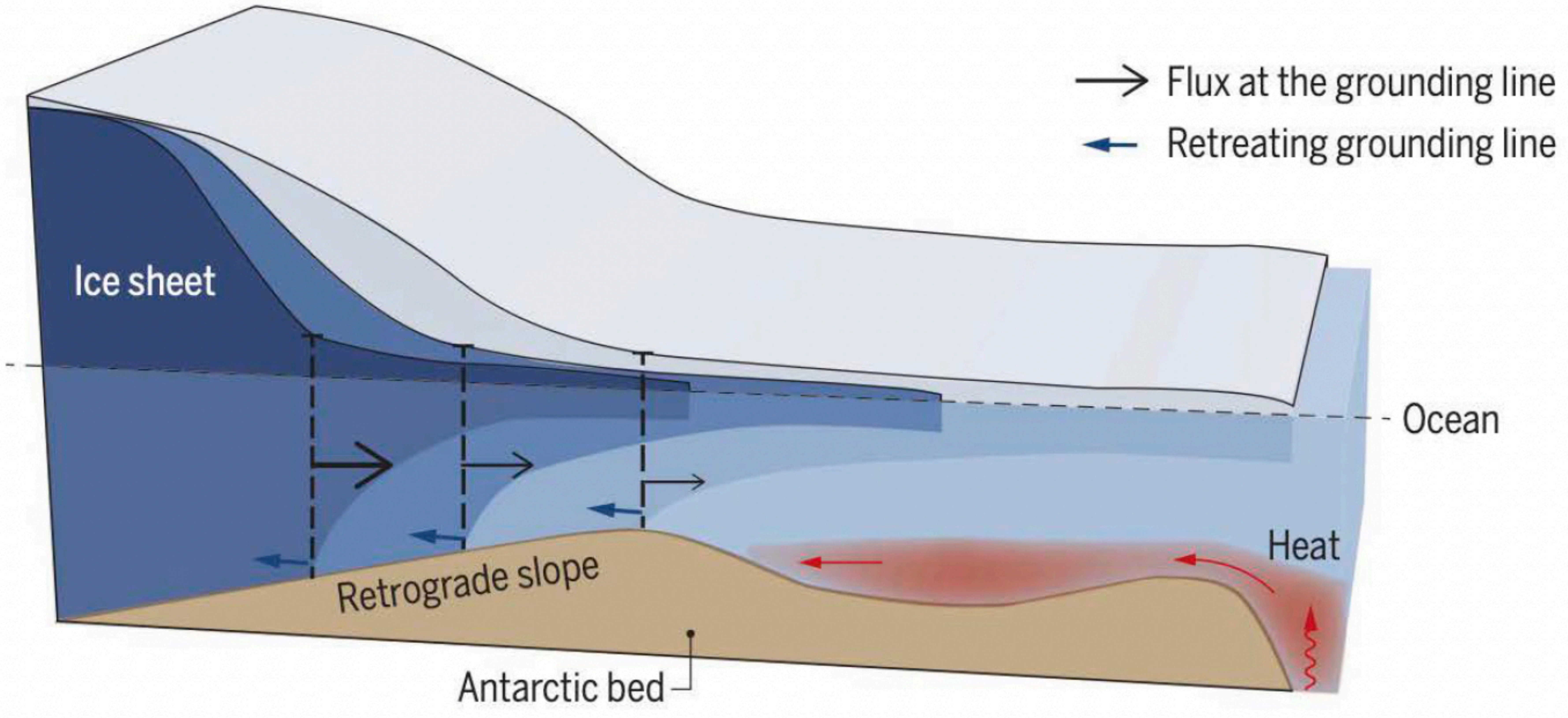
Unstable: upward slope of bottom topography in the ice flow direction

J. Fluid Mech. (2018), *vol.* 843, *pp.* 748–777. © Cambridge University Press 2018
doi:10.1017/jfm.2018.148 748

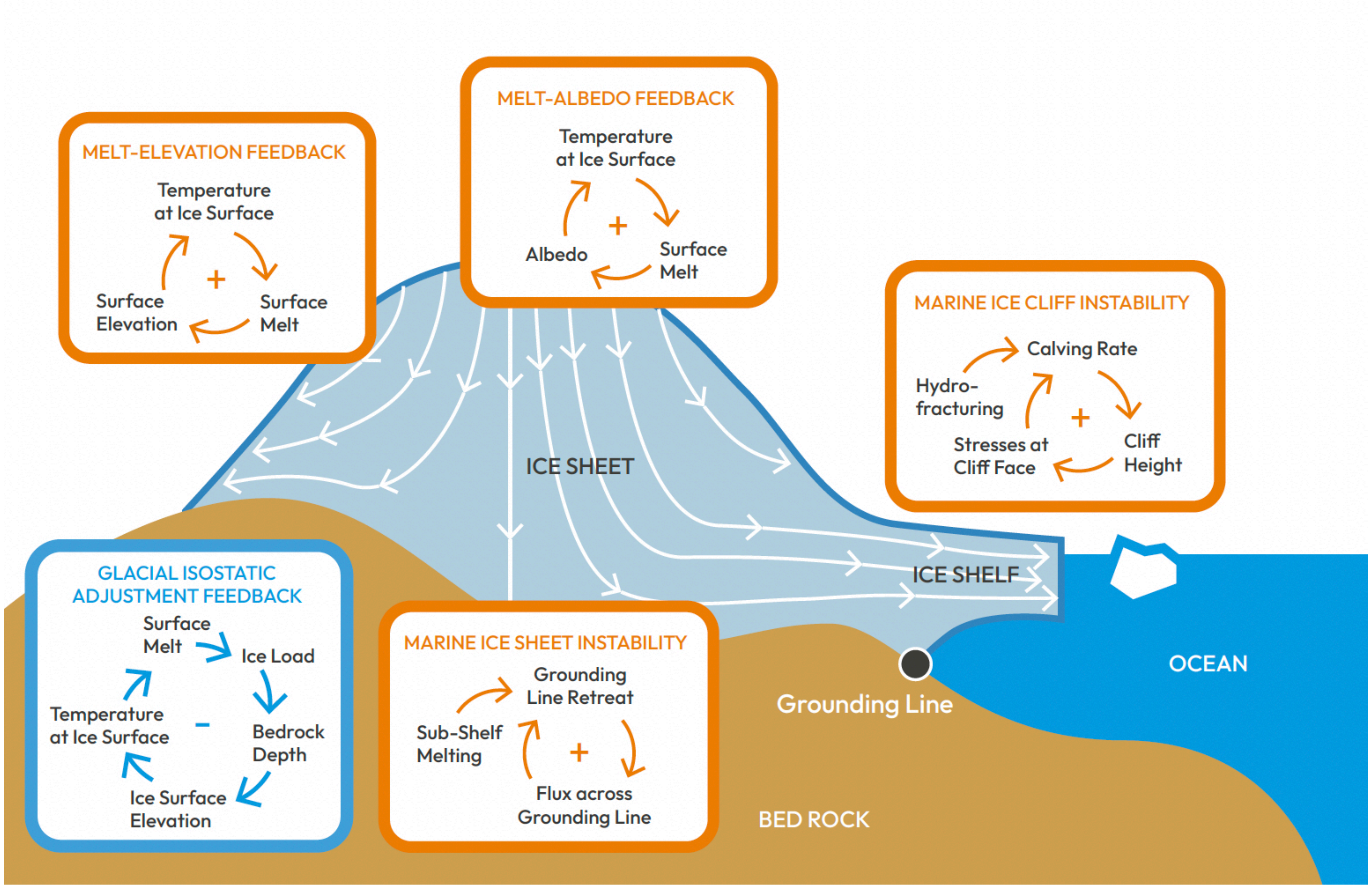
Stochastic marine ice sheet variability

T. E. Mulder[1,†], S. Baars[2], F. W. Wubs[2] and H. A. Dijkstra[1]

Mechanism
Marine Ice Sheet Instability



Other mechanisms



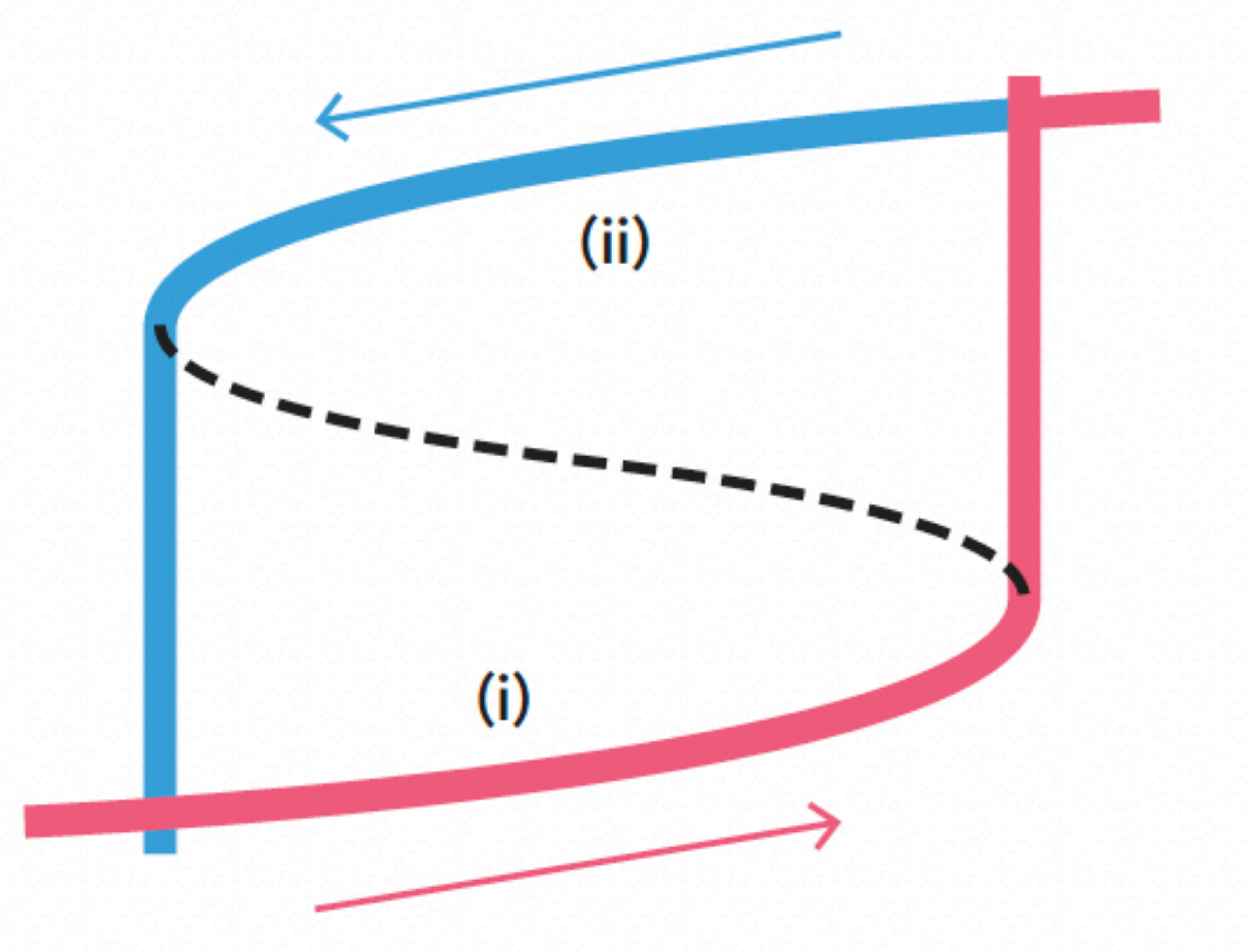
Early Warning Signals of TB

$$\boxed{\begin{aligned} dX_t &= f(X_t)dt + \sigma dW_t \\ X_0 &= X(0) \end{aligned}}$$

Locally near steady state:

$$f(X_t) \sim -\gamma X_t$$

At saddle node: $\gamma = 0$



Stochastic Integrals

Kiyoshi $It\hat{o}$ (1915-2008)

$$\int_0^T h(t)\, dW_t = \lim_{N\to\infty} \sum_{j=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

\`left end point'

Ruslan Stratonovich (1930-1997)

\`mid point'

$$\int_0^T h(t) \circ dW_t = \lim_{N\to\infty} \sum_{j=0}^{N-1} h(\frac{t_j + t_{j+1}}{2})(W(t_{j+1}) - W(t_j))$$

both in the mean-square sense

$$\lim_{N\to\infty} E[(I - I_N)^2] \to 0$$

Evaluation of integrals

Problem: Evaluate $\int_0^T W_t dW_t$

$It\hat{o}$

$$f(W_T) - f(W_0) = \int_0^T f'(W_t) dW_t + \int_0^T \frac{1}{2} f''(W_t) dt$$

Take: $f(t) = t^2$

Answer: $$\int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T)$$

Solution of SDEs

$$f(t+dt, X_t+dX_t) - f(t, X_t) = f_1 dt + f_2 dX_t + \frac{1}{2}(f_{11}(dt)^2 + 2f_{12}dtdX_t + f_{22}(dX_t)^2) + \dots$$

$Itô$

$$\boxed{dX_t = A_t dt + B_t dW_t}$$

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2) ds + \int_0^t f_2 B_s dW_s$$

Local stochastic process near steady state

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

$$A_t = -\gamma X_t \; ; \; B_t = \sigma$$

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2) ds + \int_0^t f_2 B_s dW_s$$

Take: $f(t, x) = x e^{\gamma t}$

Result: $X_t = e^{-\gamma t}(X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$

Ornstein-Uhlenbeck process

Statistics OU process

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

Solution: $$X_t = e^{-\gamma t}(X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$$

Autocorrelation: $$E[X_t X_{t+s}], s > 0$$

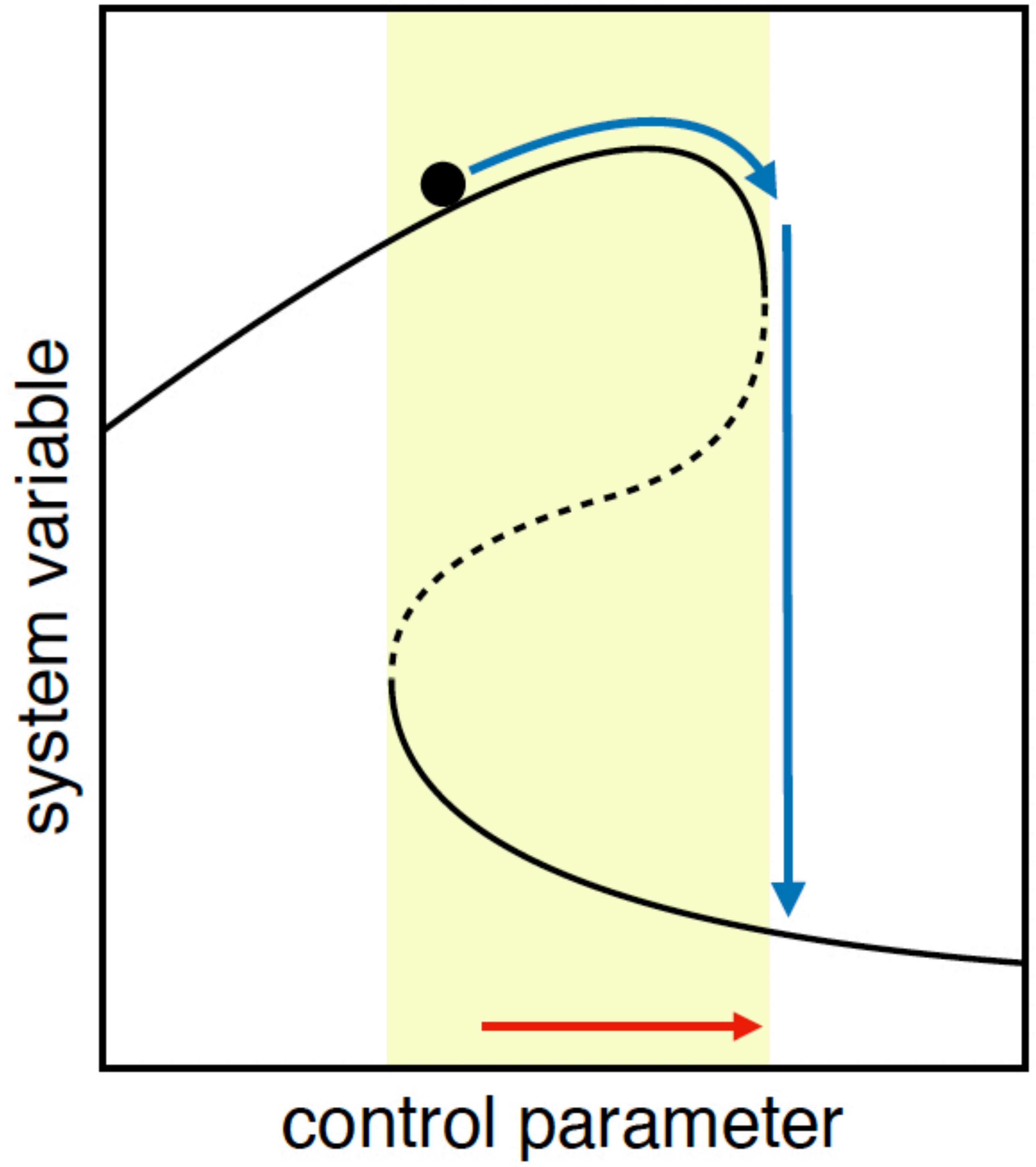
Result: $$E[X_t X_{t+s}] = e^{-\gamma(2t+s)}(X_0^2 + \sigma^2 \frac{e^{2\gamma t} - 1}{2\gamma})$$

$$E[X_t X_{t+s}] \to \frac{\sigma^2}{2\gamma} e^{-\gamma s}, \quad t \to \infty$$

Spectrum: $$S(\omega) = \frac{\sigma^2}{2\gamma} \mathcal{F}(e^{-\gamma s}) = \frac{\sigma^2}{\omega^2 + \gamma^2}$$

Early Warning Signals

$$dX_t = -\gamma X_t dt + \sigma dW_t \quad \longrightarrow \quad X_t = e^{-\gamma t}\left(X_0 + \sigma \int_0^t e^{\gamma s} dW_s\right)$$



AR(1):

$$X_{n+1} = \alpha X_n + Z_{n+1}$$

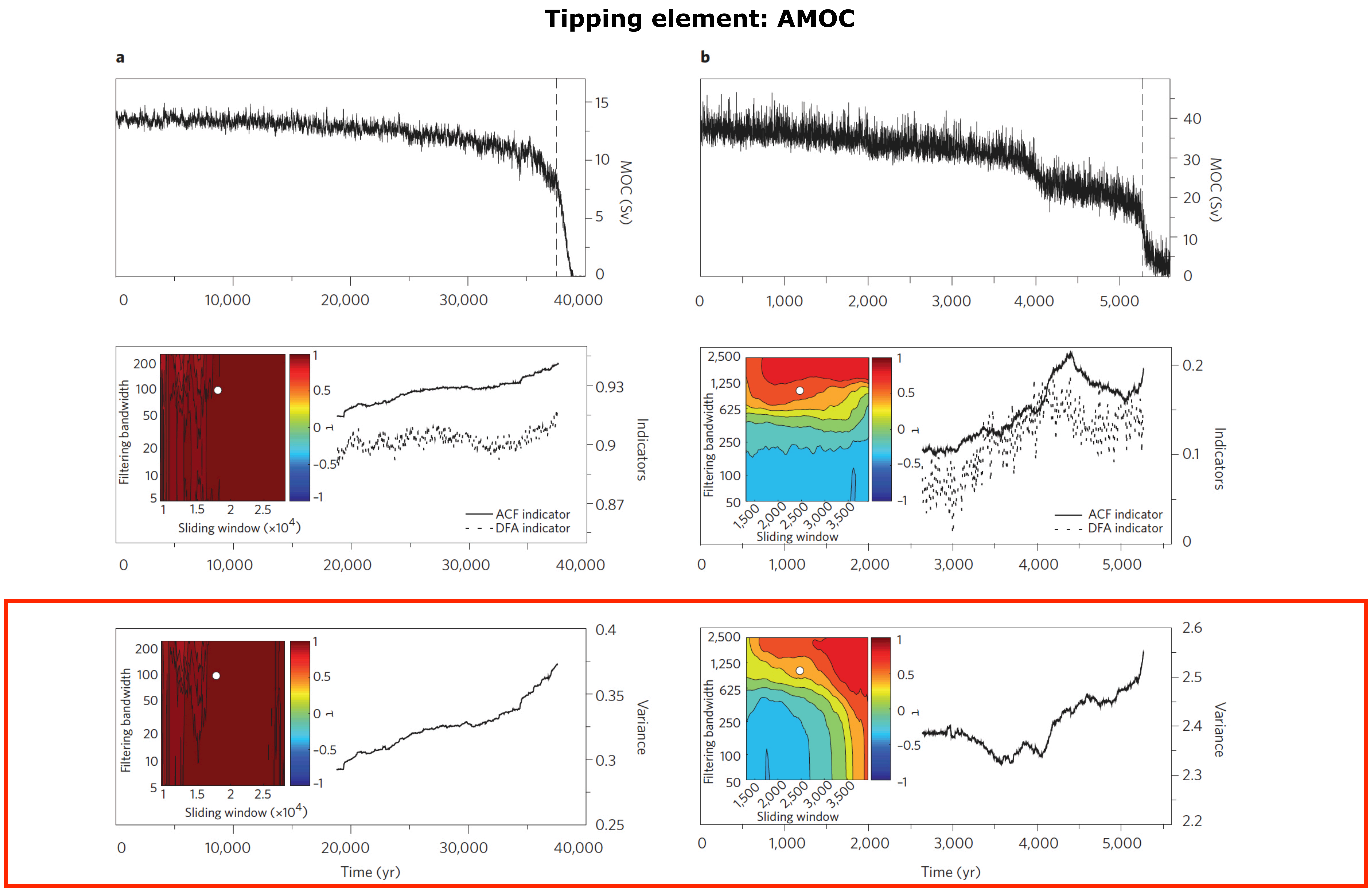
$$\alpha = e^{-\gamma \Delta t}$$

$$Var[X] = \frac{\sigma^2}{1-\alpha^2}$$

$$\rho_n = \alpha^n$$

At saddle node: $\gamma \to 0 \quad \longrightarrow \quad Var[X] \to \infty \;;\; \rho_n \to 1$

Example: AMOC



Lenton (2011)

Exercises!