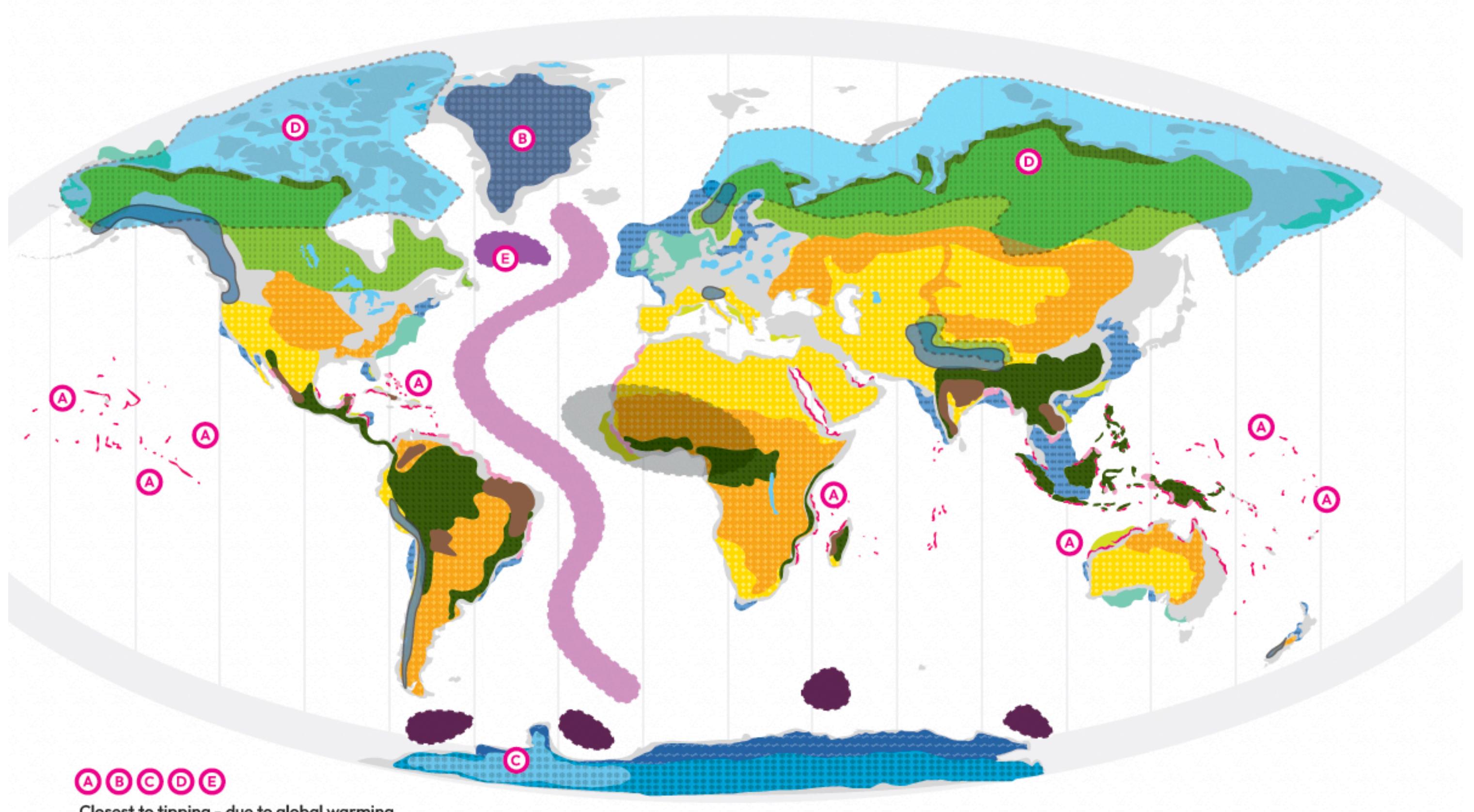


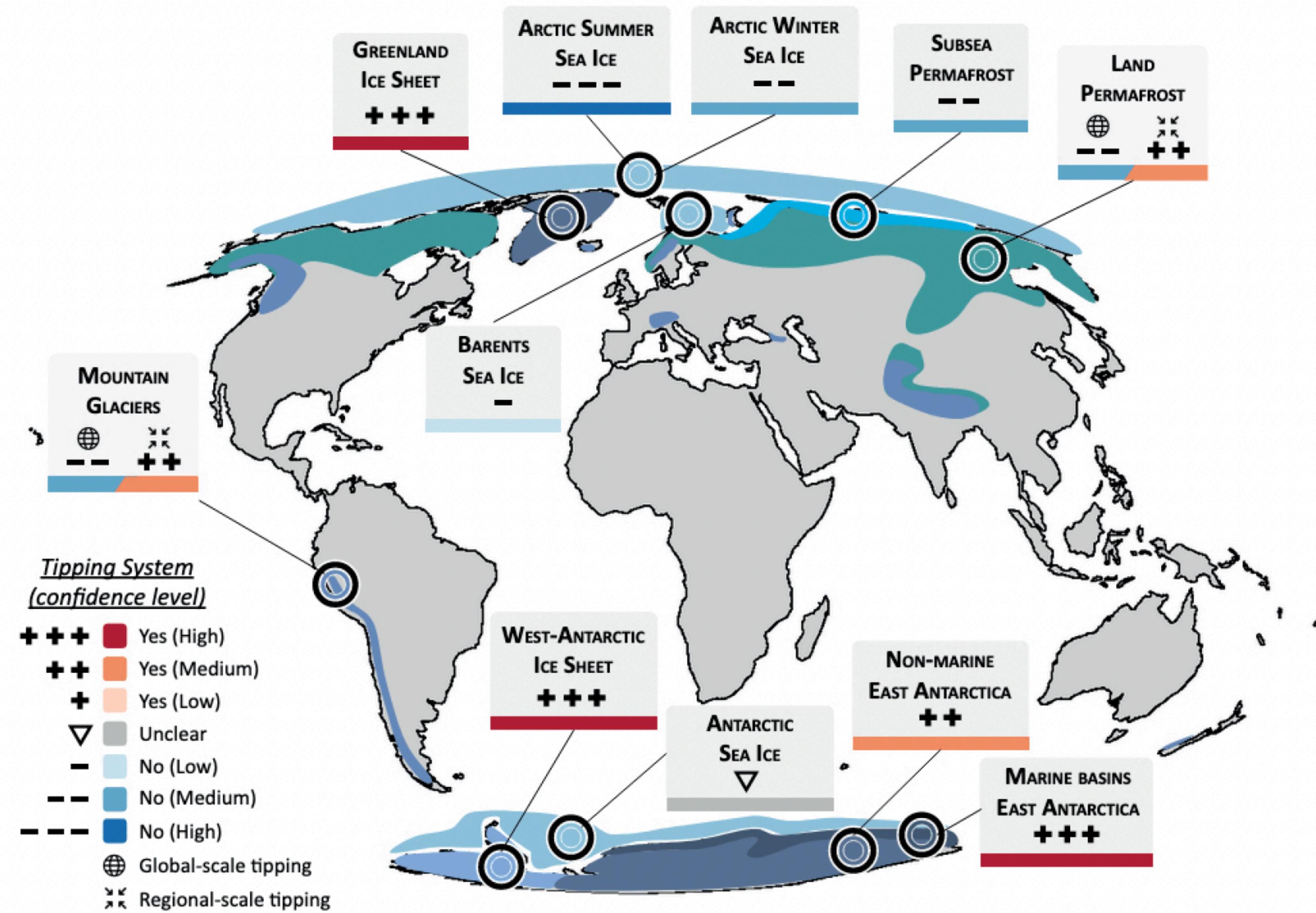
Tipping Behavior in the Climate System



Henk Dijkstra,
Department of Physics,
Utrecht University, NL
&
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University of Trento, IT

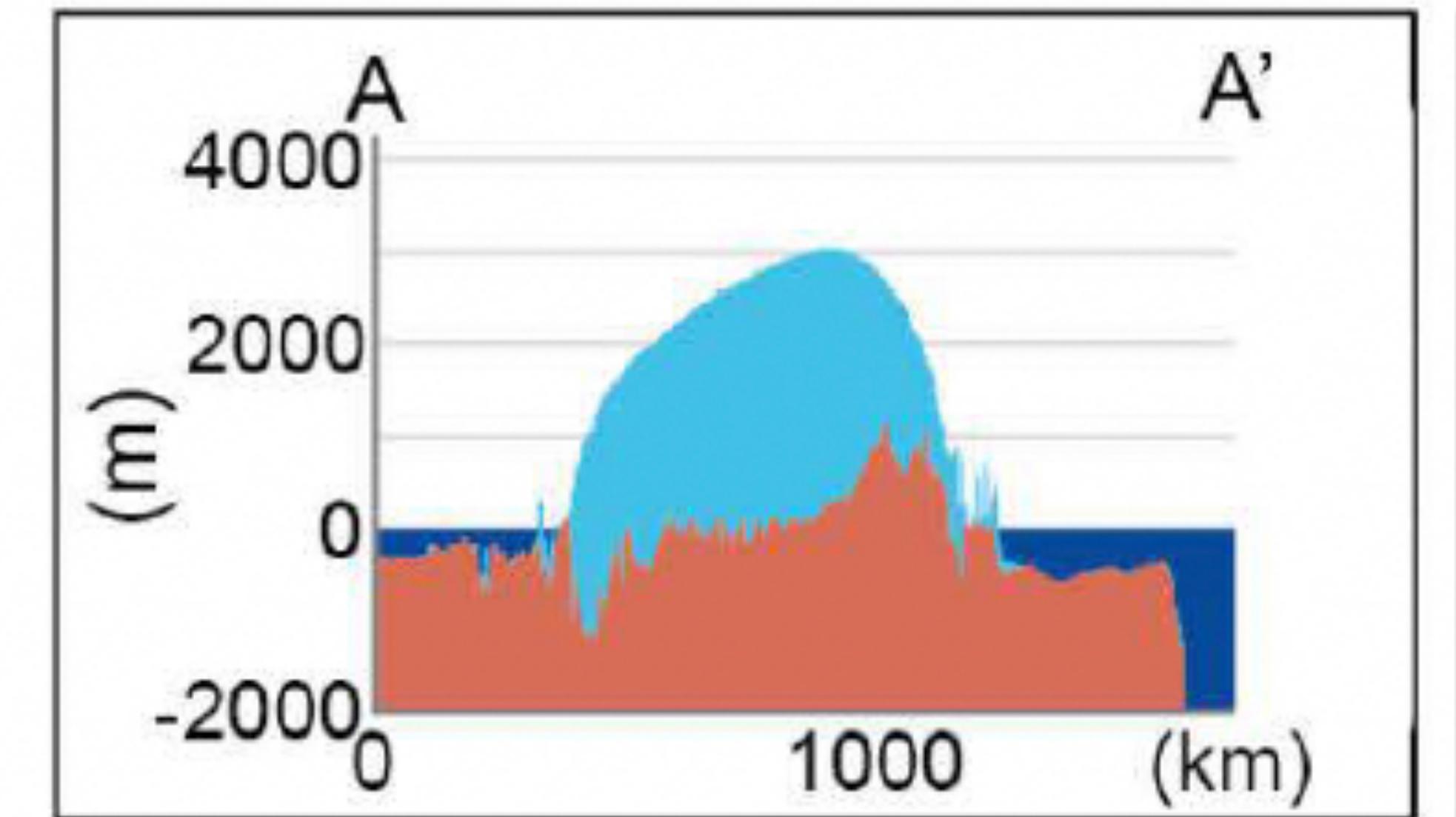
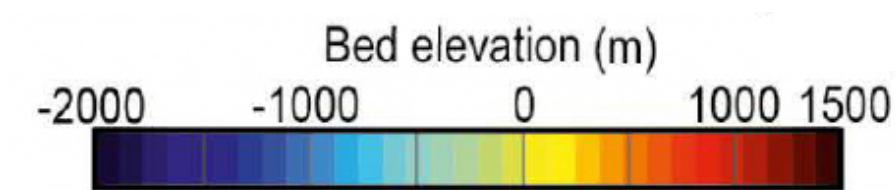
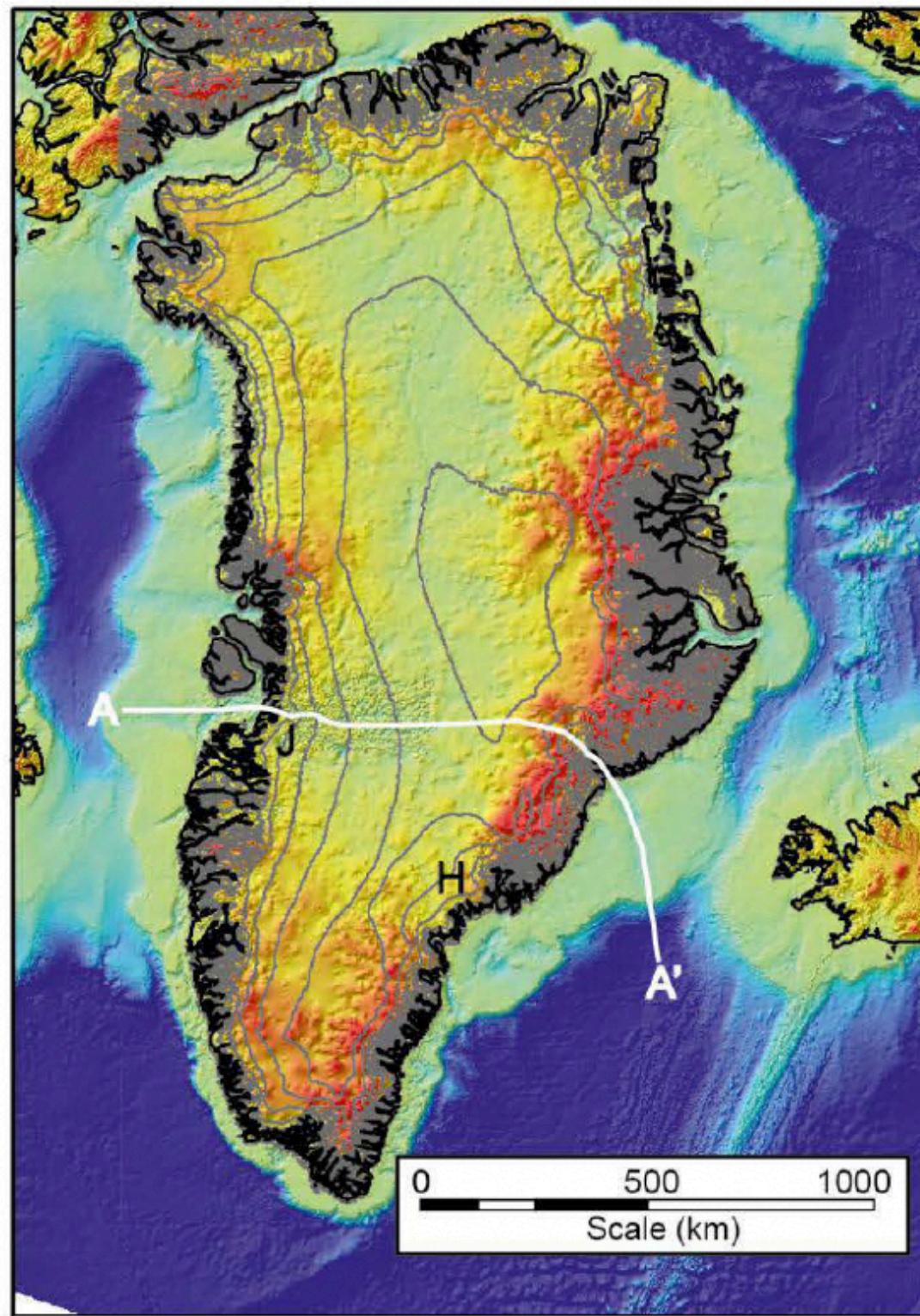


Tipping Elements: Cryosphere





Tipping Element: Greenland Ice Sheet

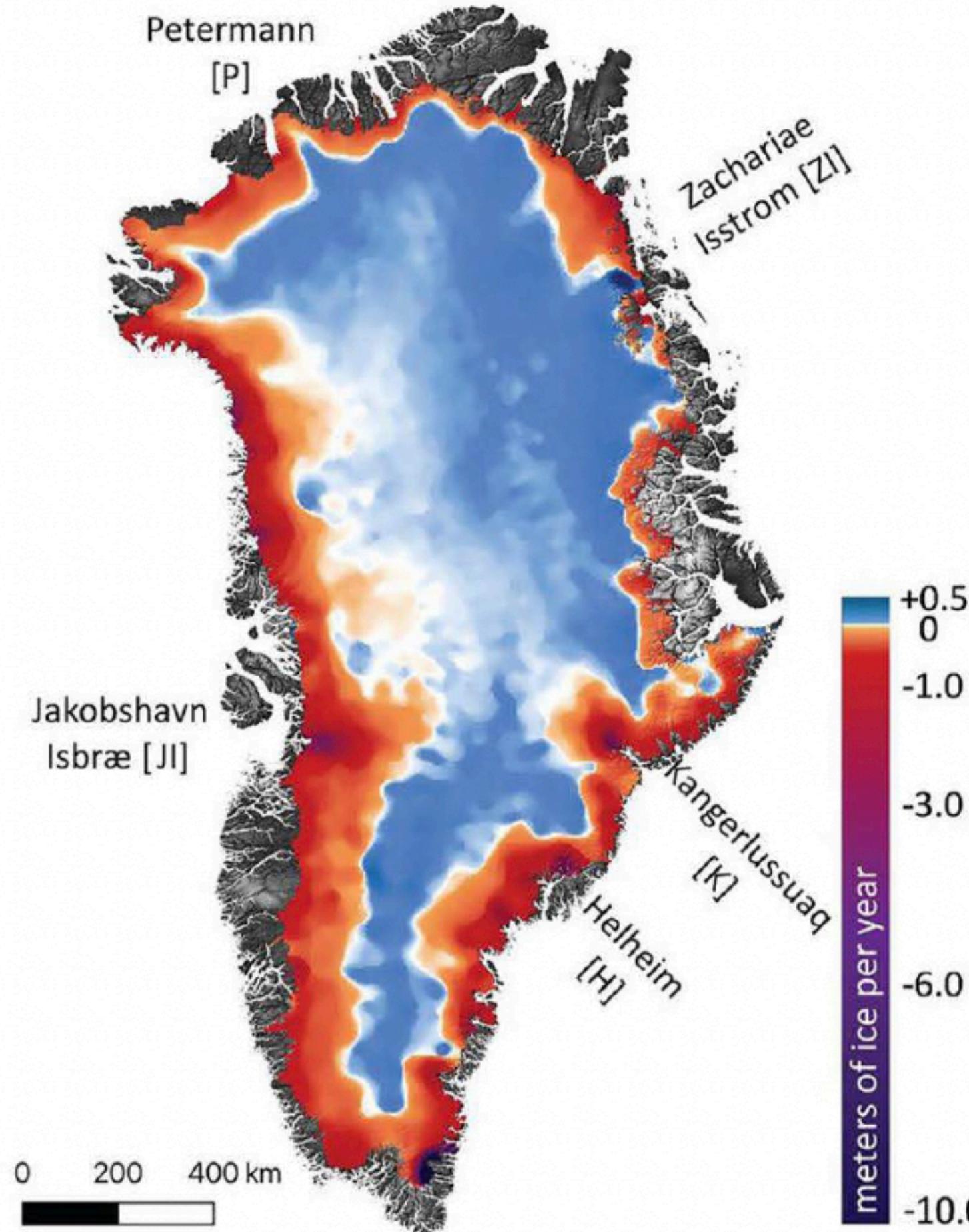


Land-based ice sheet



Utrecht
University

Mass Loss: *Greenland Ice Sheet*

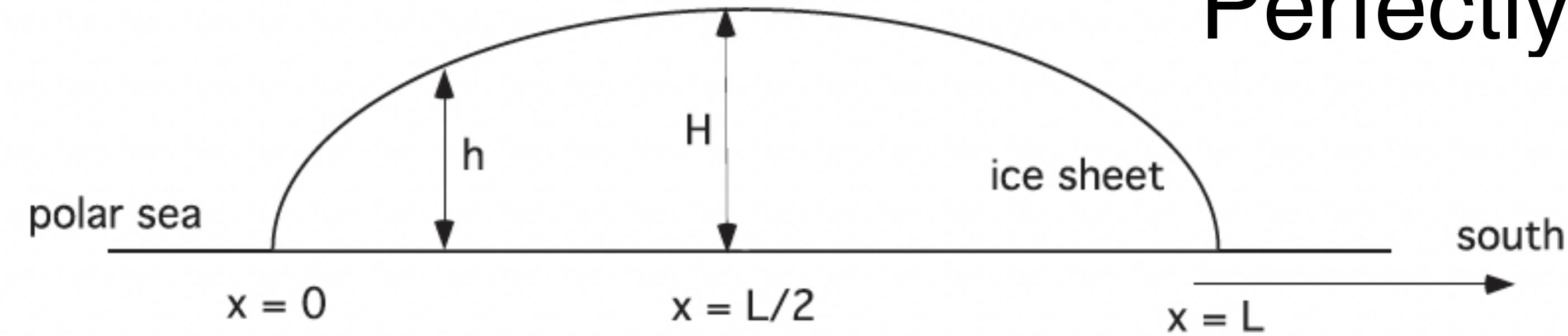


Period: 2003-2019



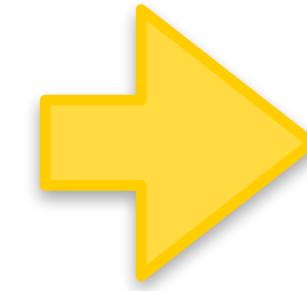
Model

Perfectly plastic ice



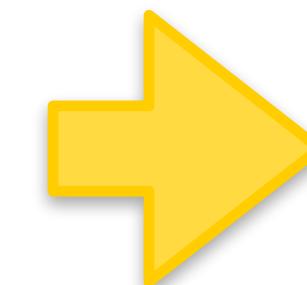
Yield stress

$$\tau_0 = \rho_i g h \frac{\partial h}{\partial x},$$



$$2h \frac{\partial h}{\partial x} = \frac{\partial h^2}{\partial x} = \frac{2\tau_0}{\rho_i g} \equiv \sigma \rightarrow h^2(x, t) = \sigma x + C(t).$$

Boundary conditions

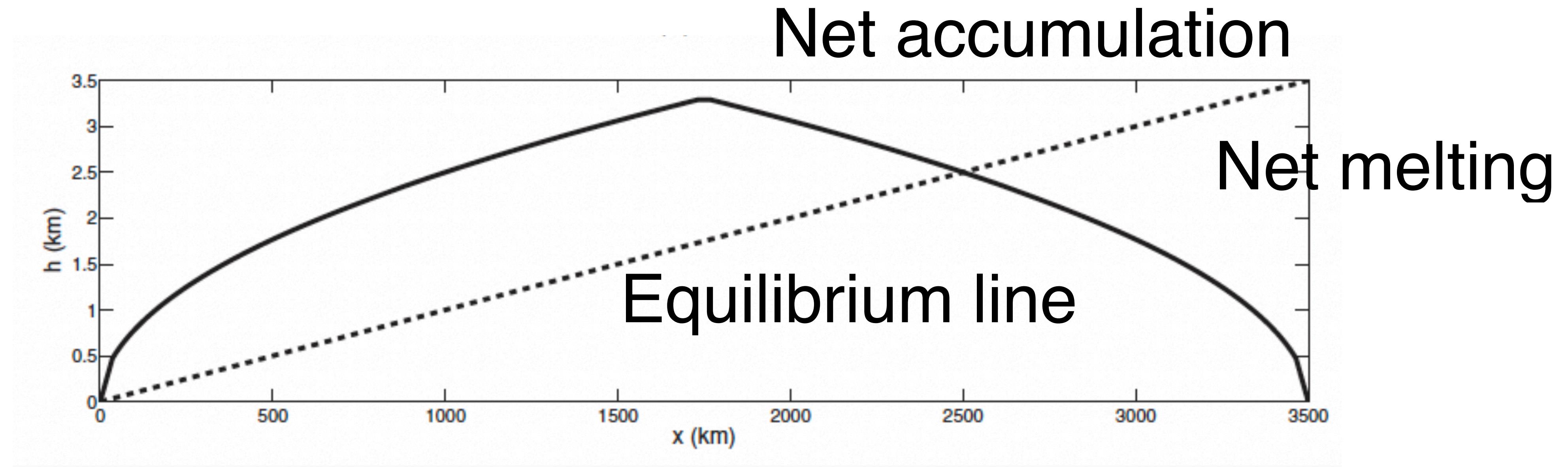


$$h(x, t) = \sqrt{\sigma} \left(\frac{L(t)}{2} - |x - \frac{L(t)}{2}| \right)^{\frac{1}{2}}$$



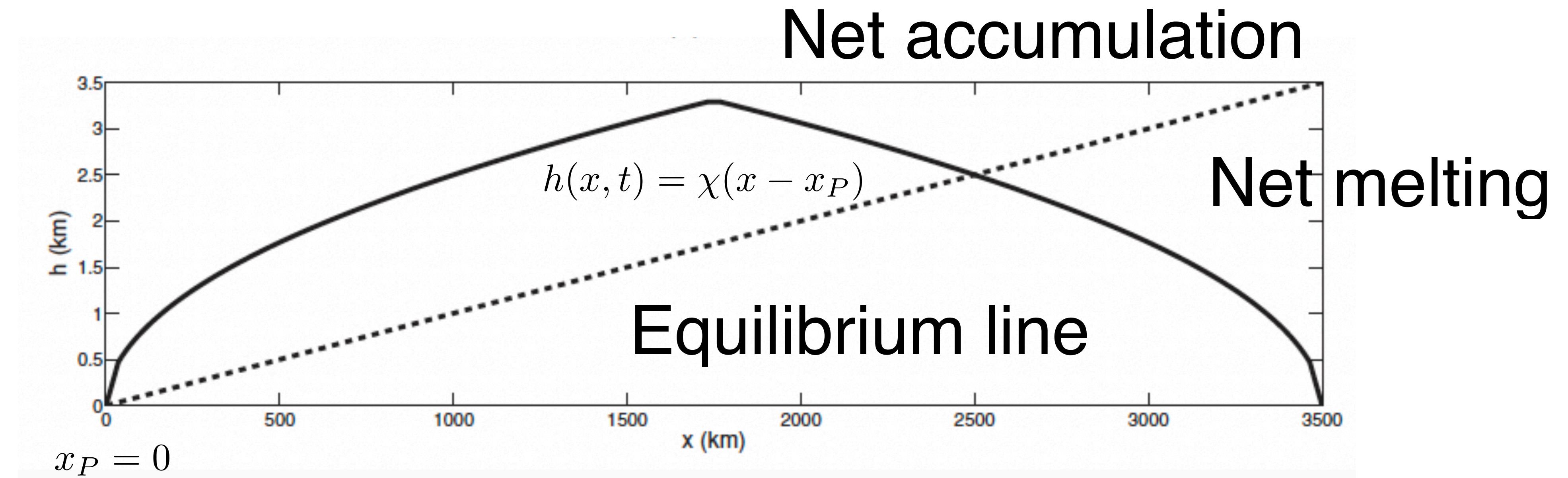
Solution

$$h(x, t) = \sqrt{\sigma} \left(\frac{L(t)}{2} - \left| x - \frac{L(t)}{2} \right| \right)^{\frac{1}{2}}$$



Parameter	Meaning	Value	Unit
ρ_i	ice density	0.9	kg m^{-3}
L_0	ice cap length	3,500	km
σ	yield stress parameter	6.25	m

Mass balance

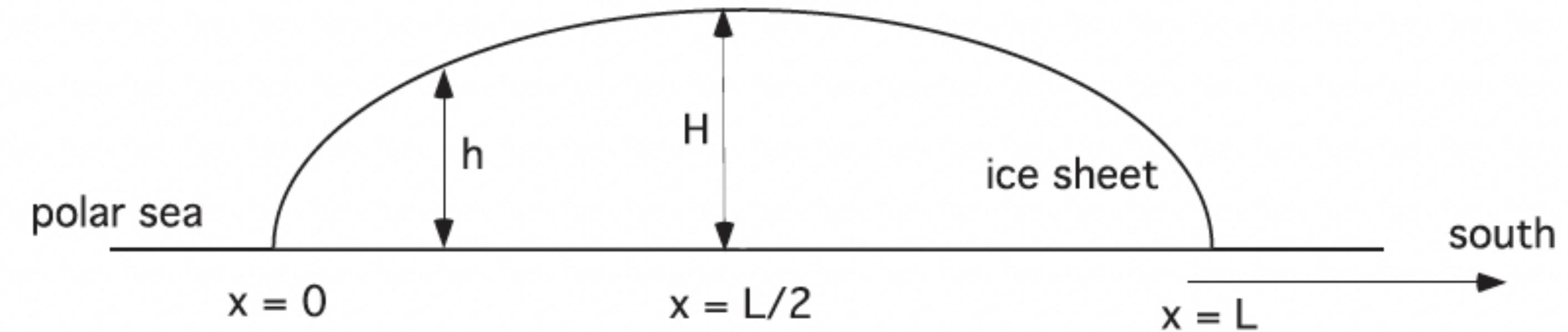
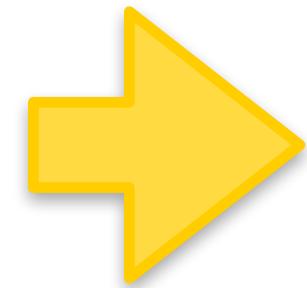


$$\rho_i \frac{\partial h}{\partial t} = P_i - M,$$

$$P_i(x, t) - M(x, t) = G(x, t) = \rho_i \beta(h(x, t) - \chi(x - x_P)),$$

Final reduced model

$$\int_{L/2}^L \frac{\partial h}{\partial t} dx = H \frac{dL}{dt} = \int_{L/2}^L \left[\beta(\sqrt{\sigma(L-x)} - \chi(x-x_P)) \right] dx,$$

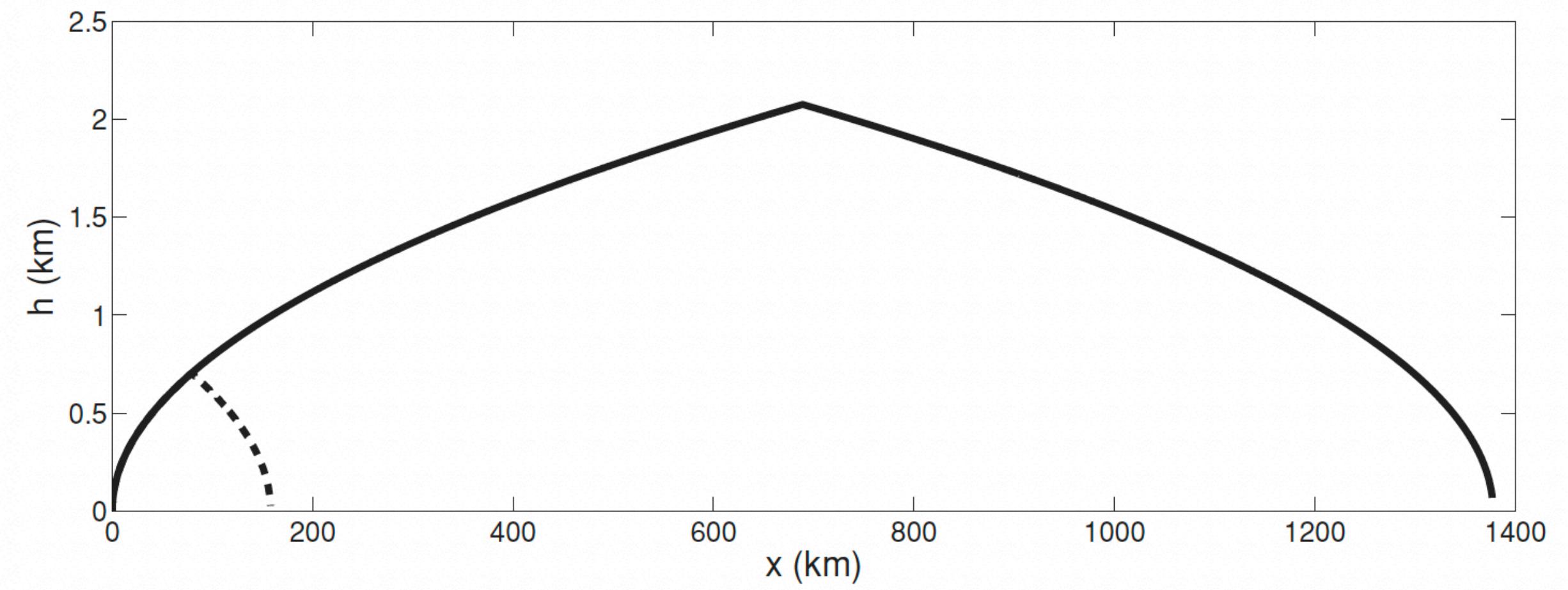
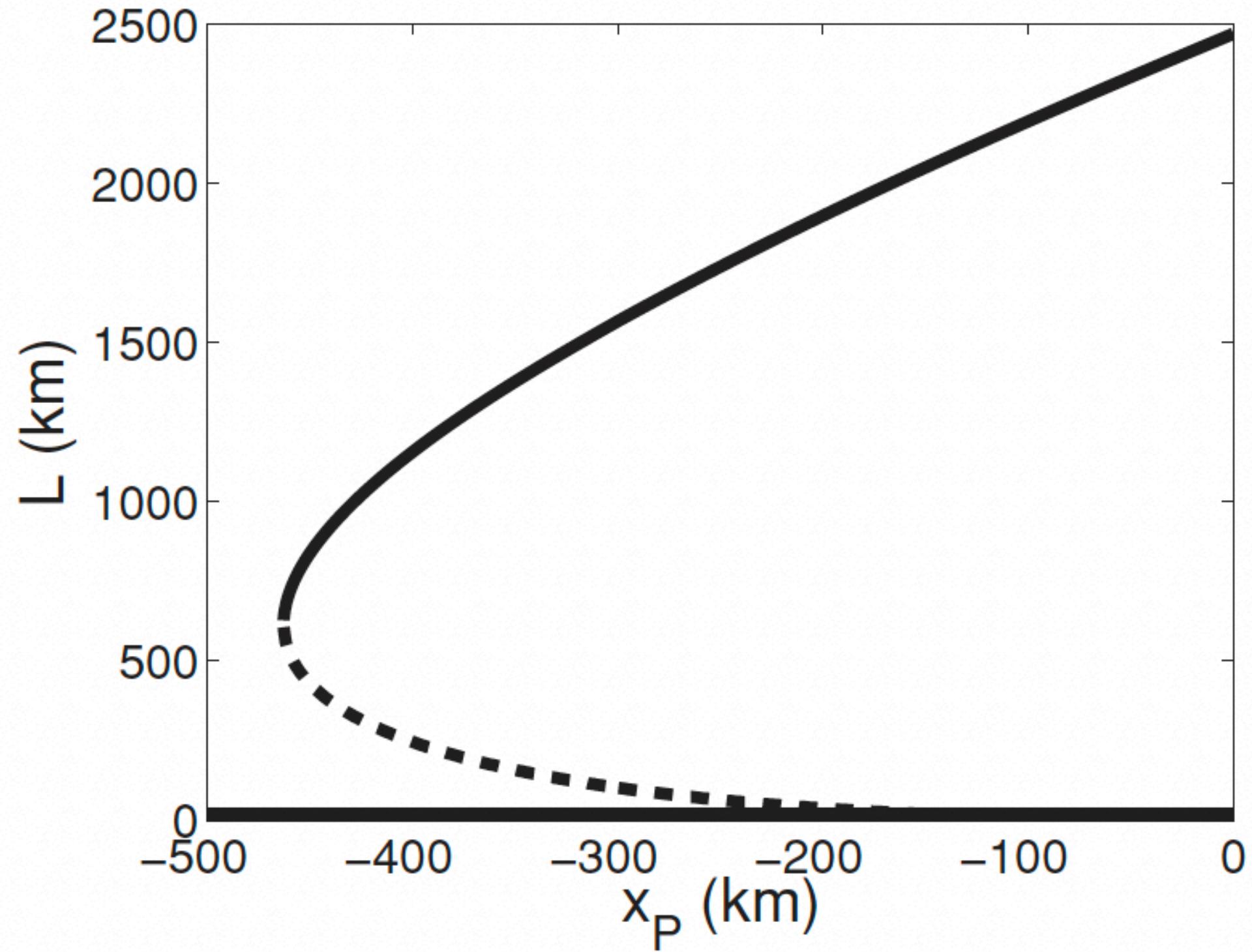


$$\frac{dL}{dt} = F_1 \sqrt{L} + F_2 L + F_3 L \sqrt{L},$$

Parameter	Meaning	Value	Unit
ρ_i	ice density	0.9	kg m^{-3}
L_0	ice cap length	3,500	km
σ	yield stress parameter	6.25	m
β	coefficient in the linear mass balance	10^{-3}	yr^{-1}
χ	coefficient in the linear mass balance	10^{-3}	-



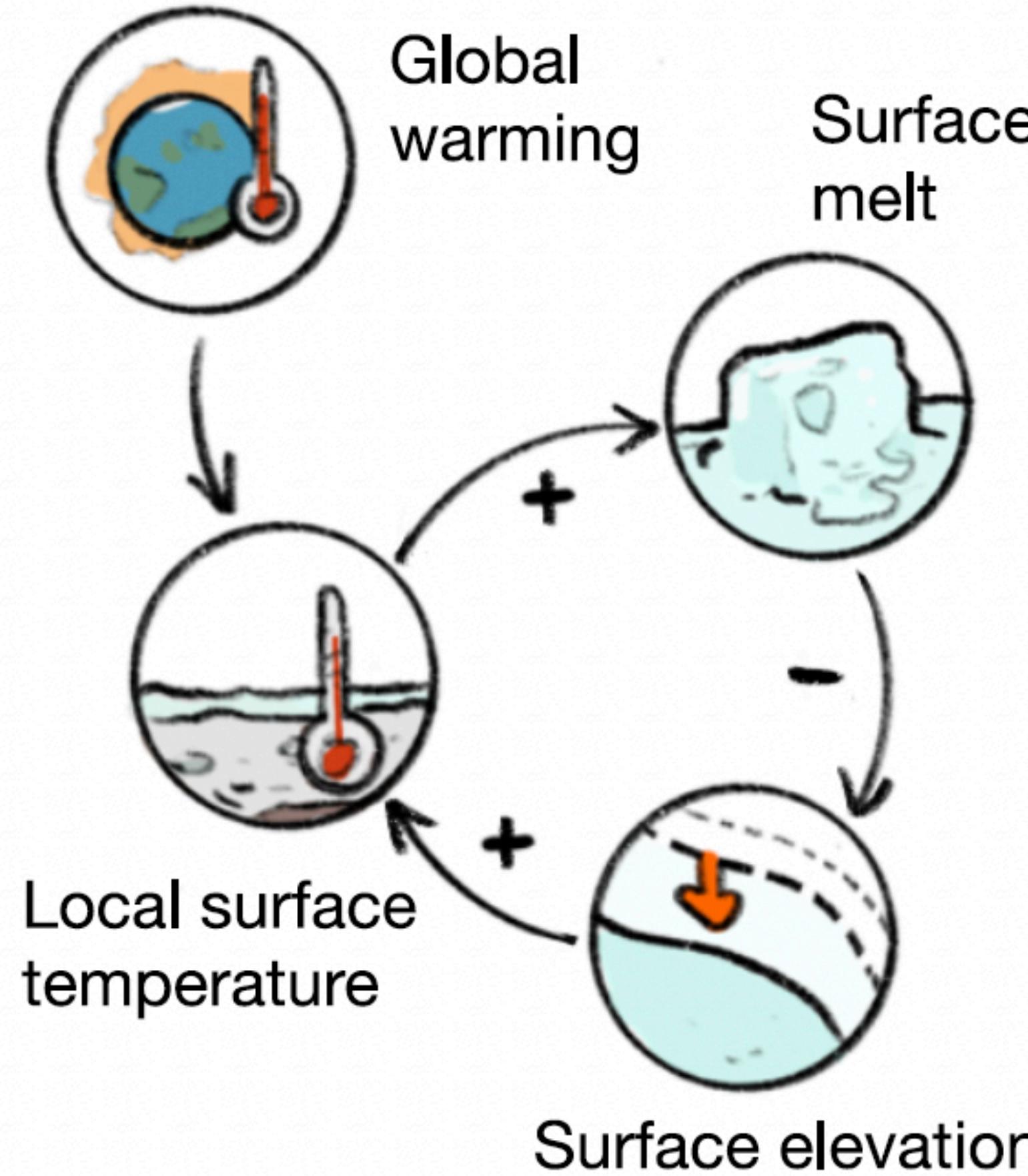
Bifurcation diagram



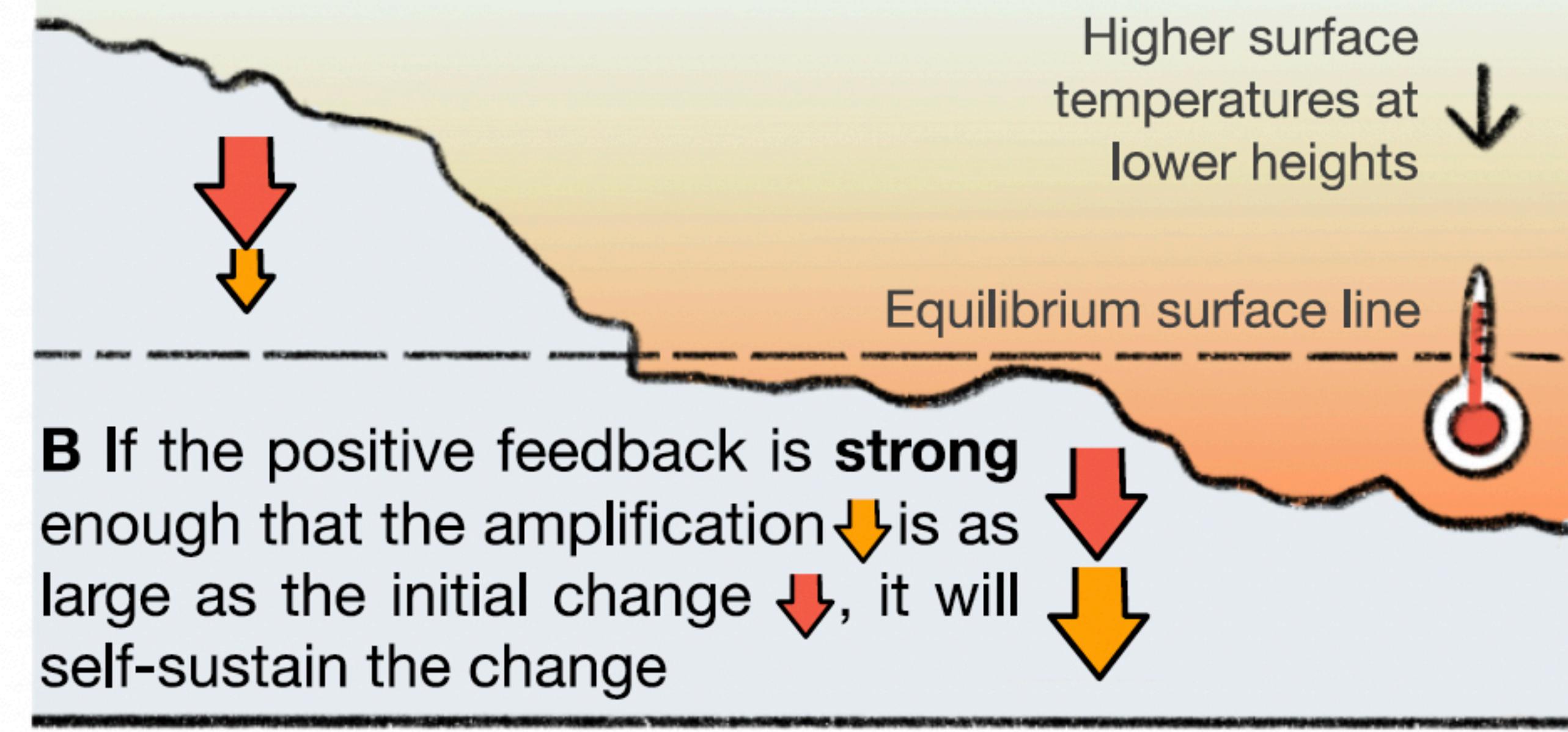
$$x_P = -250 \text{ km}$$



Melt-elevation Feedback

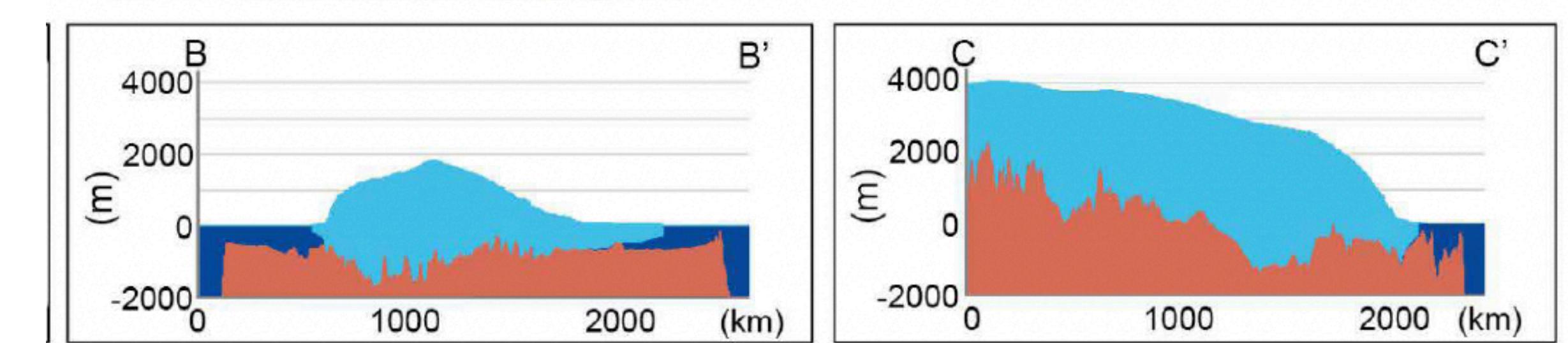
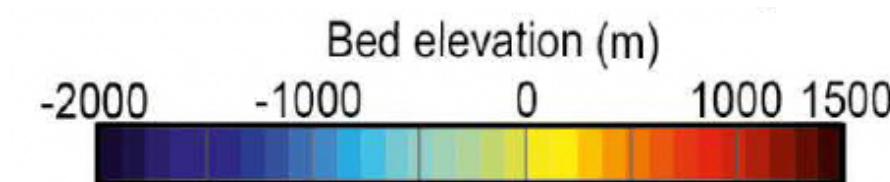
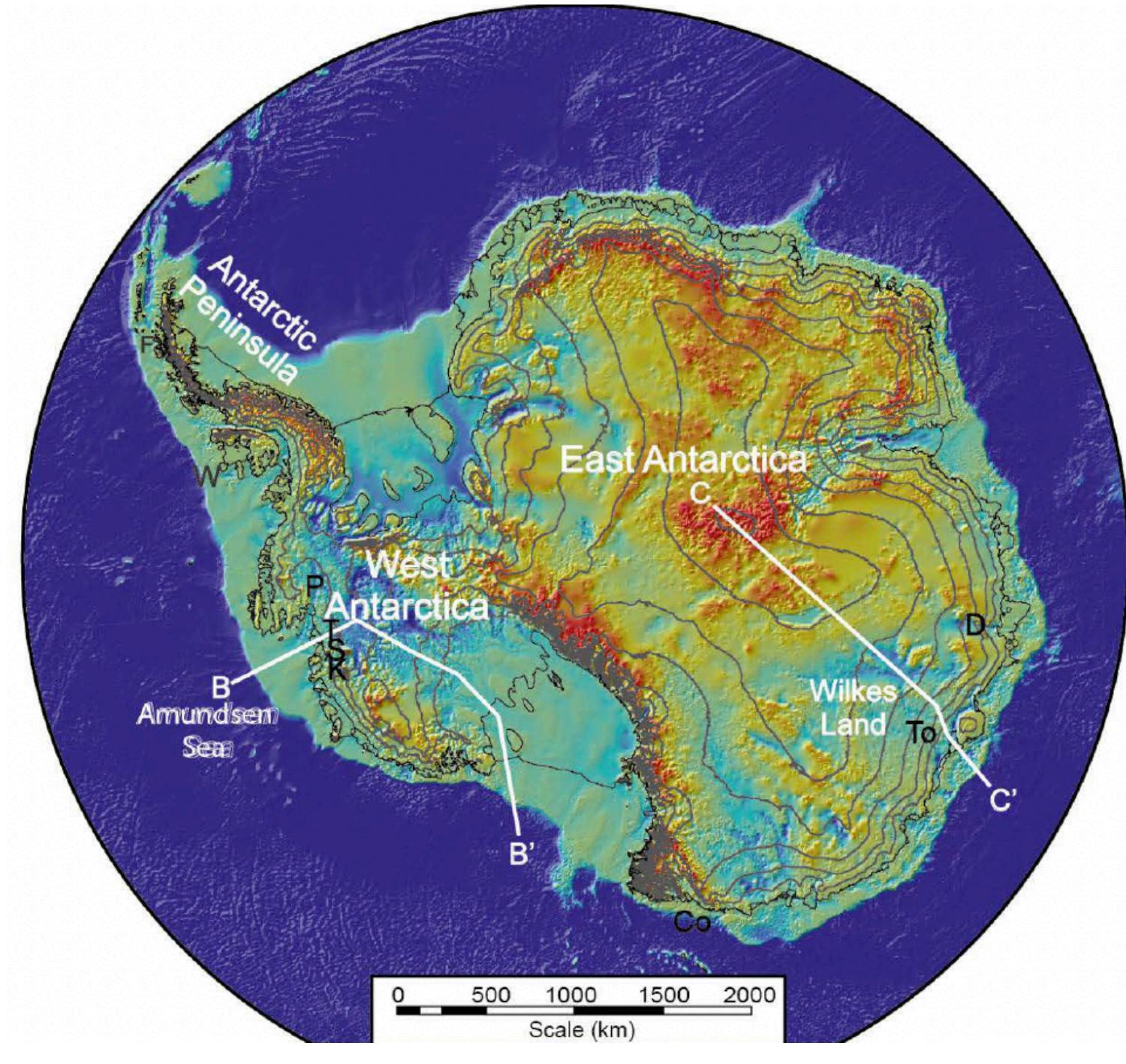


A An initially induced change (e.g. global warming-induced melt) is amplified by a **weak** positive feedback





Tipping Element: Antarctic Ice Sheet

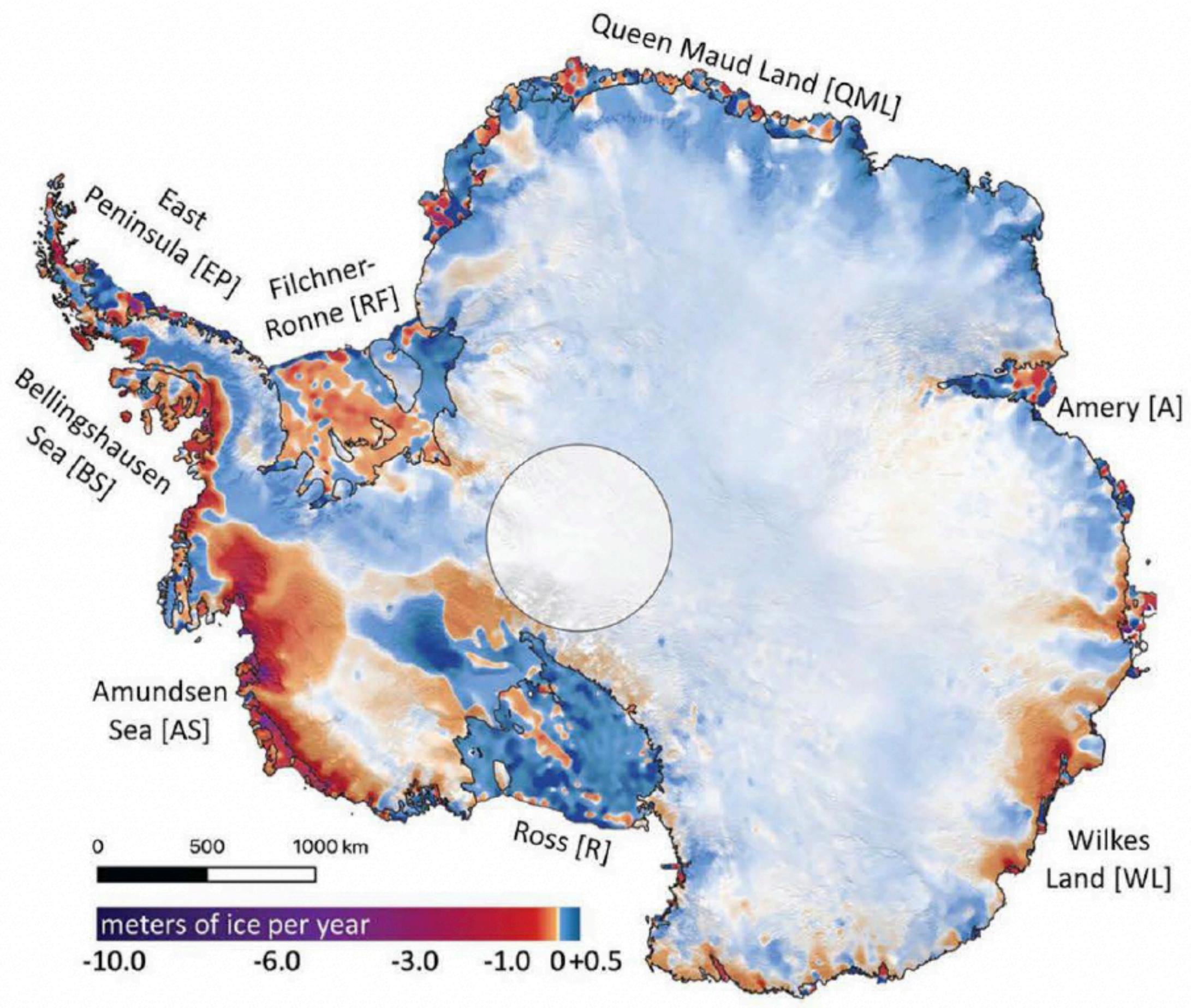


Marine ice sheet



Utrecht
University

Mass Loss: *Antarctic Ice Sheet*



Period: 2003-2019



Model

Conservation of mass:

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = a,$$

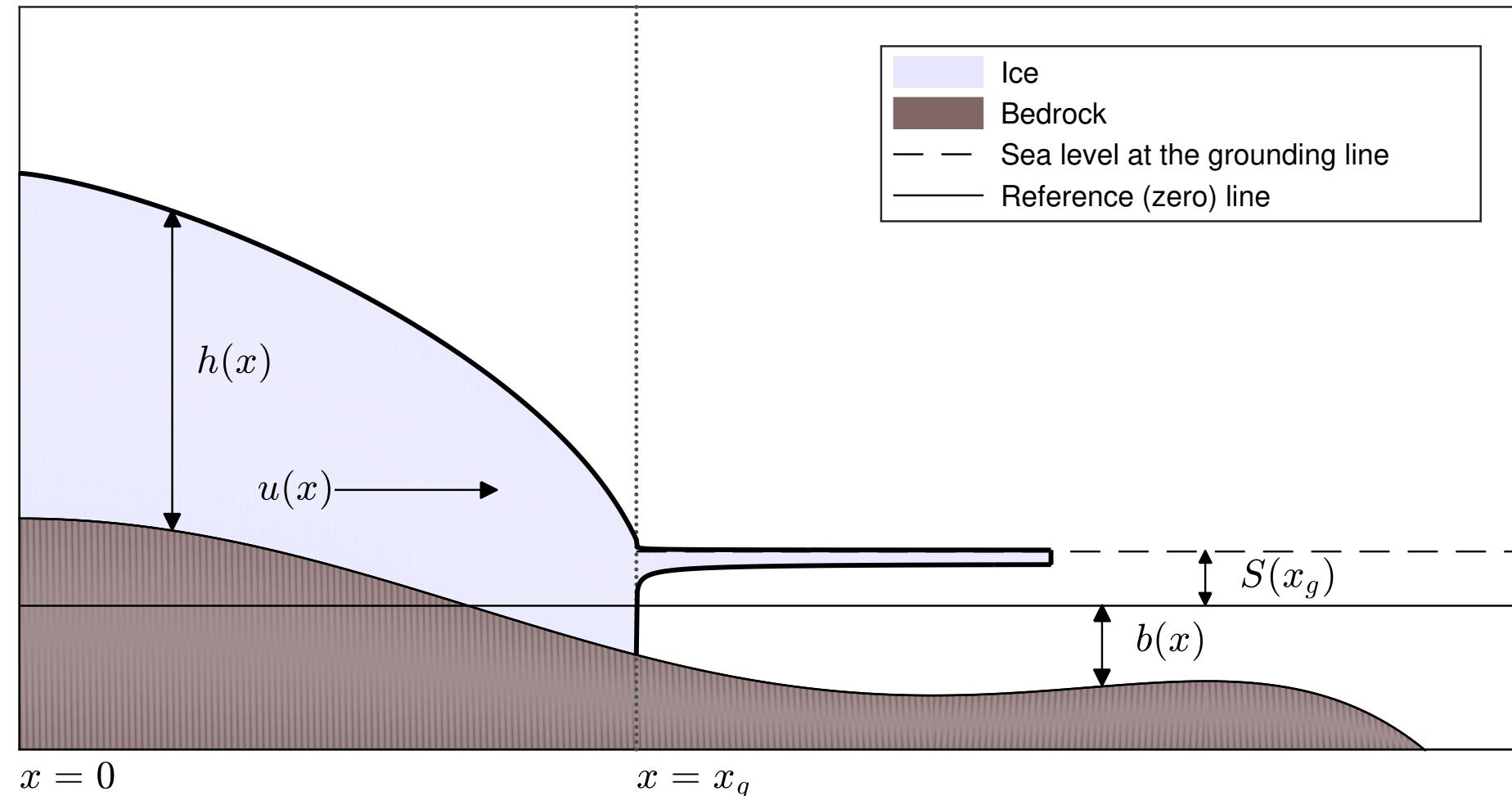
a : accumulation rate.

Conservation of momentum:

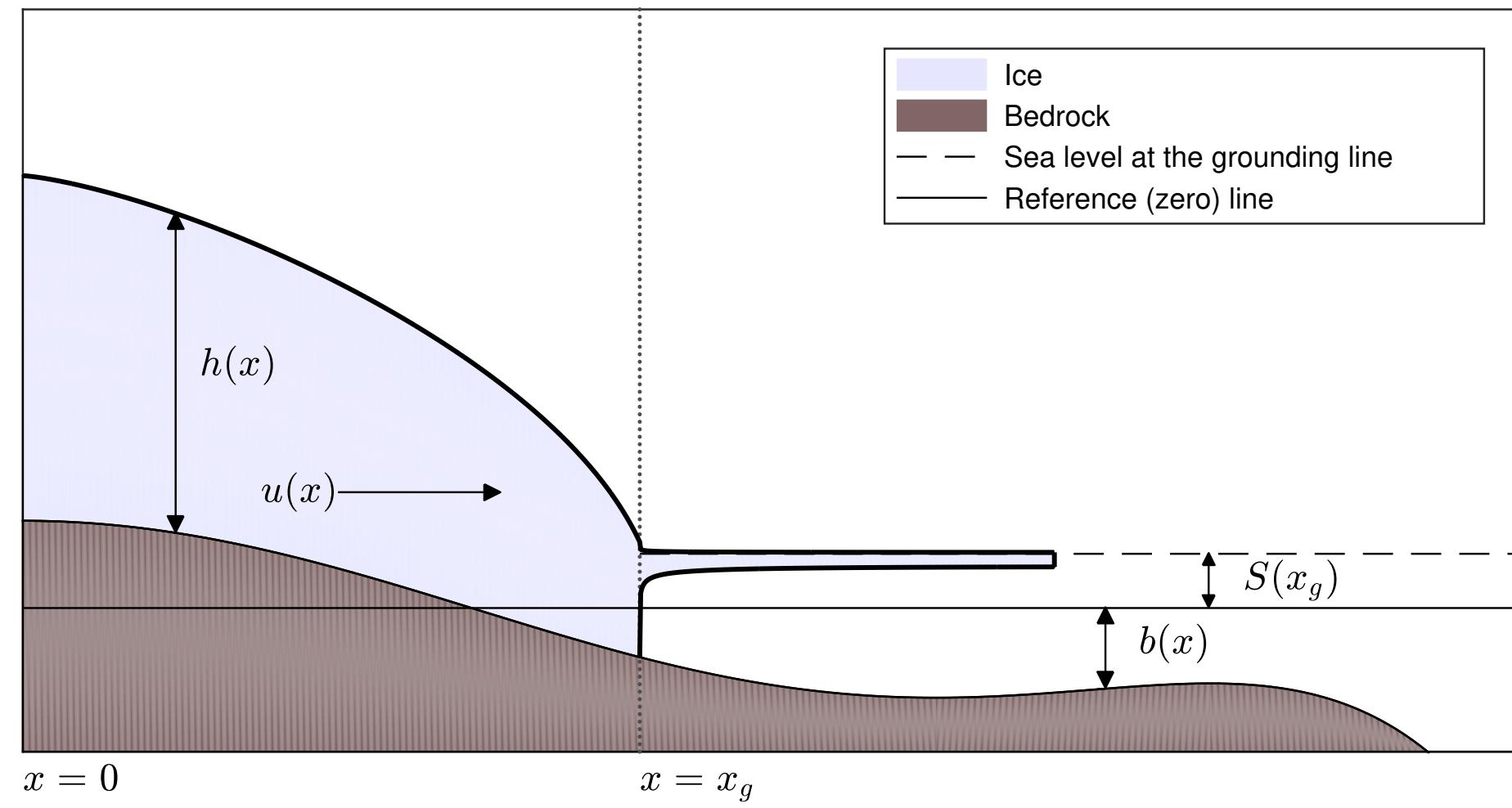
$$\frac{\partial}{\partial x} \left[2A^{-\frac{1}{n}} h \left| \frac{\partial u}{\partial x} \right|^{\frac{1}{n}-1} \frac{\partial u}{\partial x} \right] - C |u|^{(m-1)} u - \rho_i g h \frac{\partial(h-b)}{\partial x} = 0,$$

A and n are coefficients of Glen's flow law, a constitutive relation describing the rheology of ice (typically $n = 3$).

The parameters C and m determine the sliding of the ice



Boundary conditions



No flow at the ice divide:

$$\frac{\partial(h - b)}{\partial x} = 0, \quad u = 0 \quad \text{at } x = 0.$$

There are two boundary conditions at the grounding line $x = x_g$. One condition results from an integration of the shelf flow [Schoof,2007a] as:

$$2A^{-\frac{1}{n}} \left| \frac{\partial u}{\partial x} \right|^{\frac{1}{n}-1} \frac{\partial u}{\partial x} = \frac{1}{2} \left(1 - \frac{\rho_i}{\rho_w} \right) \rho_i g h \quad \text{at } x = x_g.$$

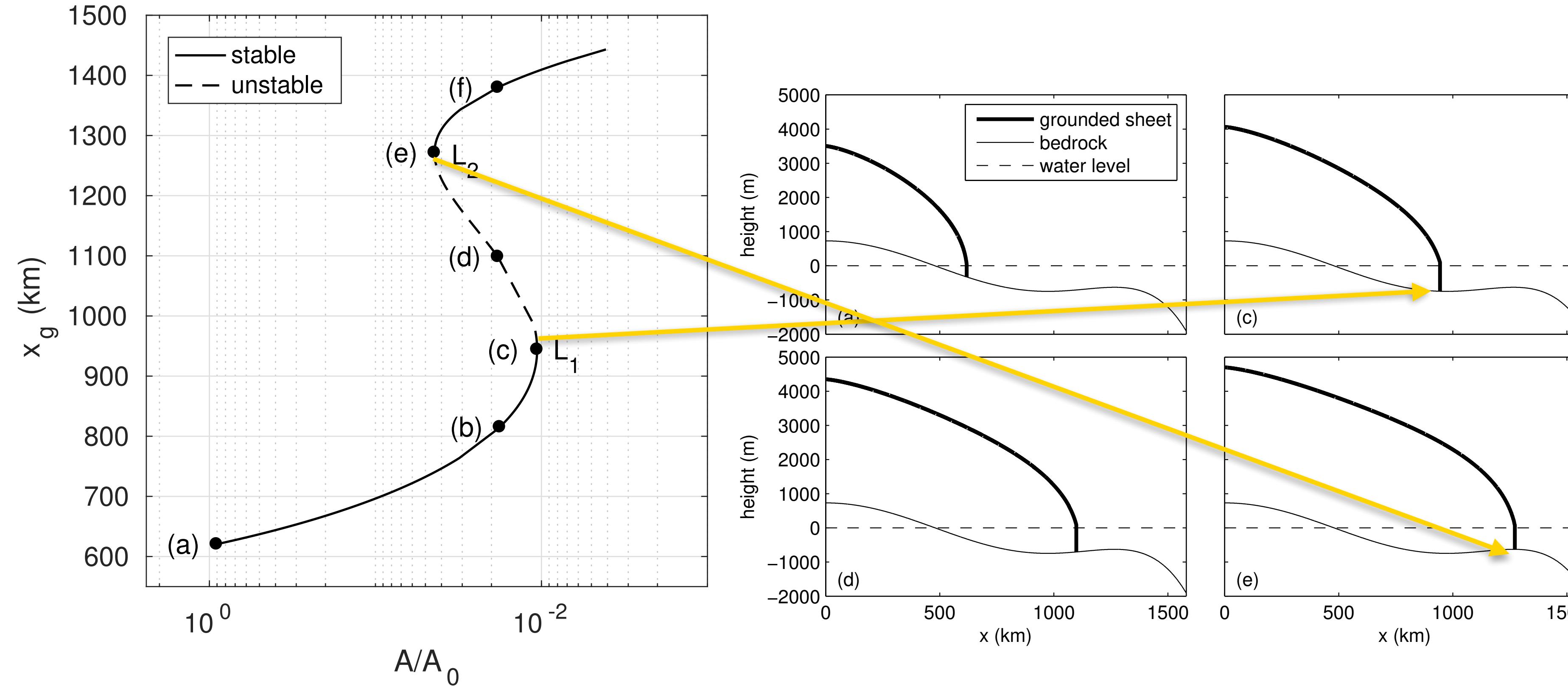
The other boundary condition is the flotation requirement:

$$\rho_i h = \rho_w (b + S) \quad \text{at } x = x_g,$$

where $S(x_g)$ is the sea level at the grounding line position.



Bifurcation diagram



Unstable: upward slope of bottom topography
in the ice flow direction

J. Fluid Mech. (2018), vol. 843, pp. 748–777. © Cambridge University Press 2018
doi:10.1017/jfm.2018.148

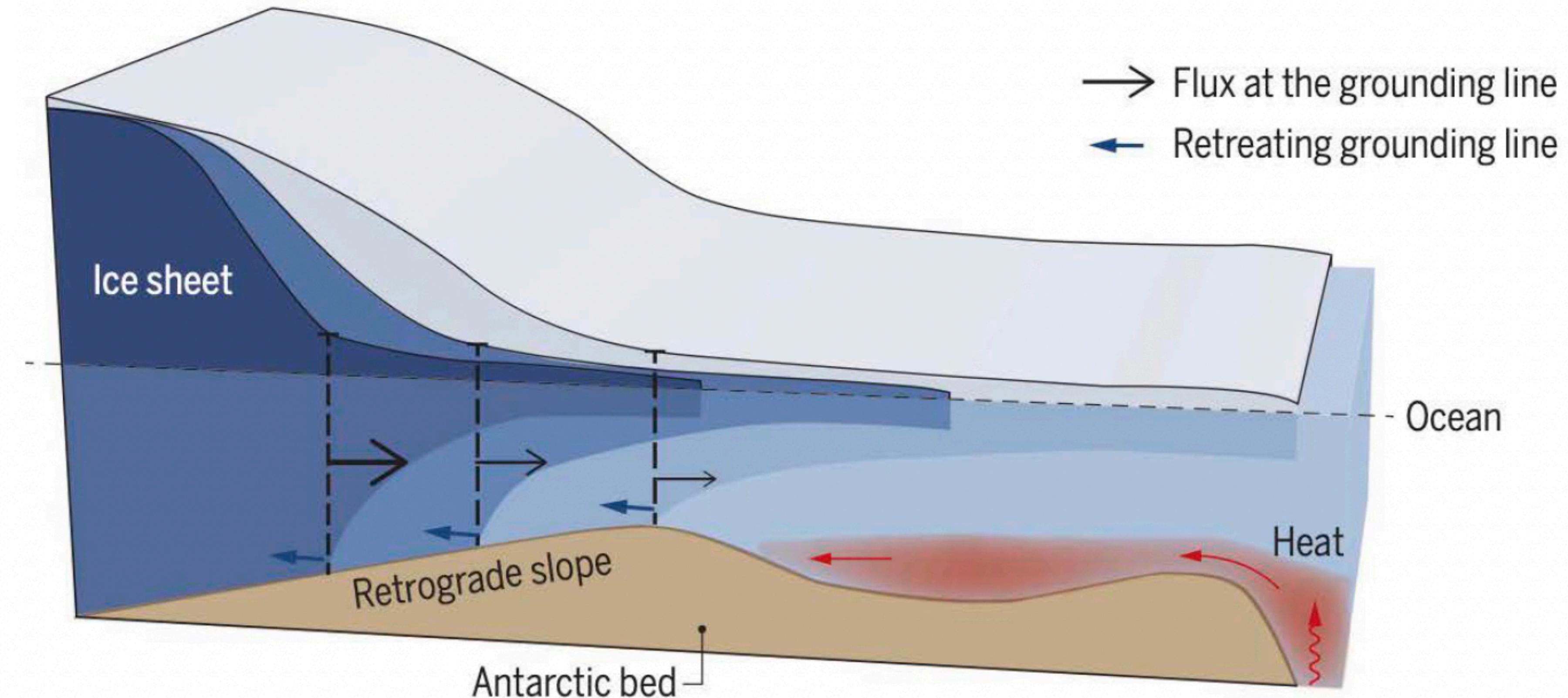
748

Stochastic marine ice sheet variability

T. E. Mulder^{1,†}, S. Baars², F. W. Wubs² and H. A. Dijkstra¹

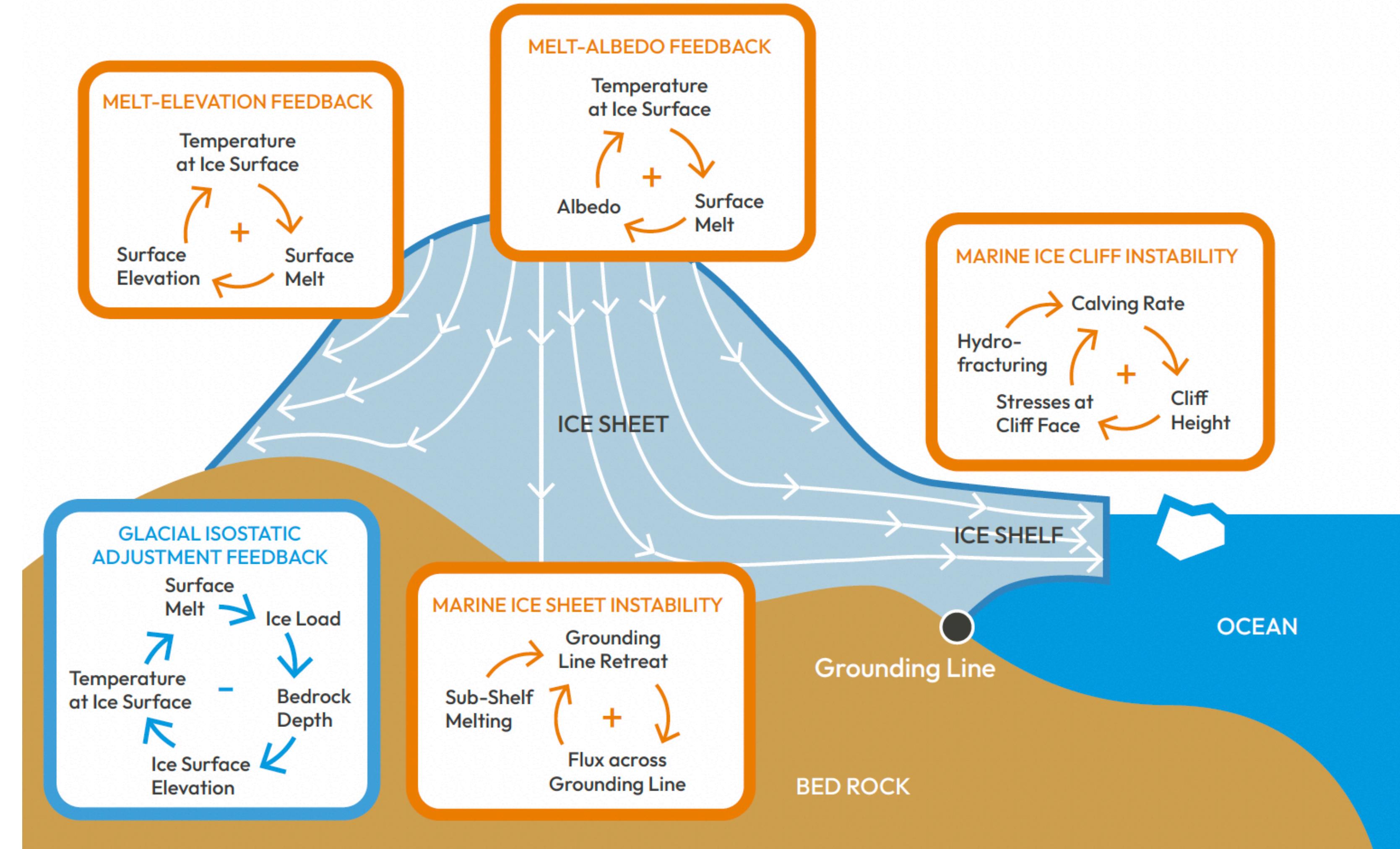


Mechanism Marine Ice Sheet Instability





Other mechanisms



Early Warning Signals of TB

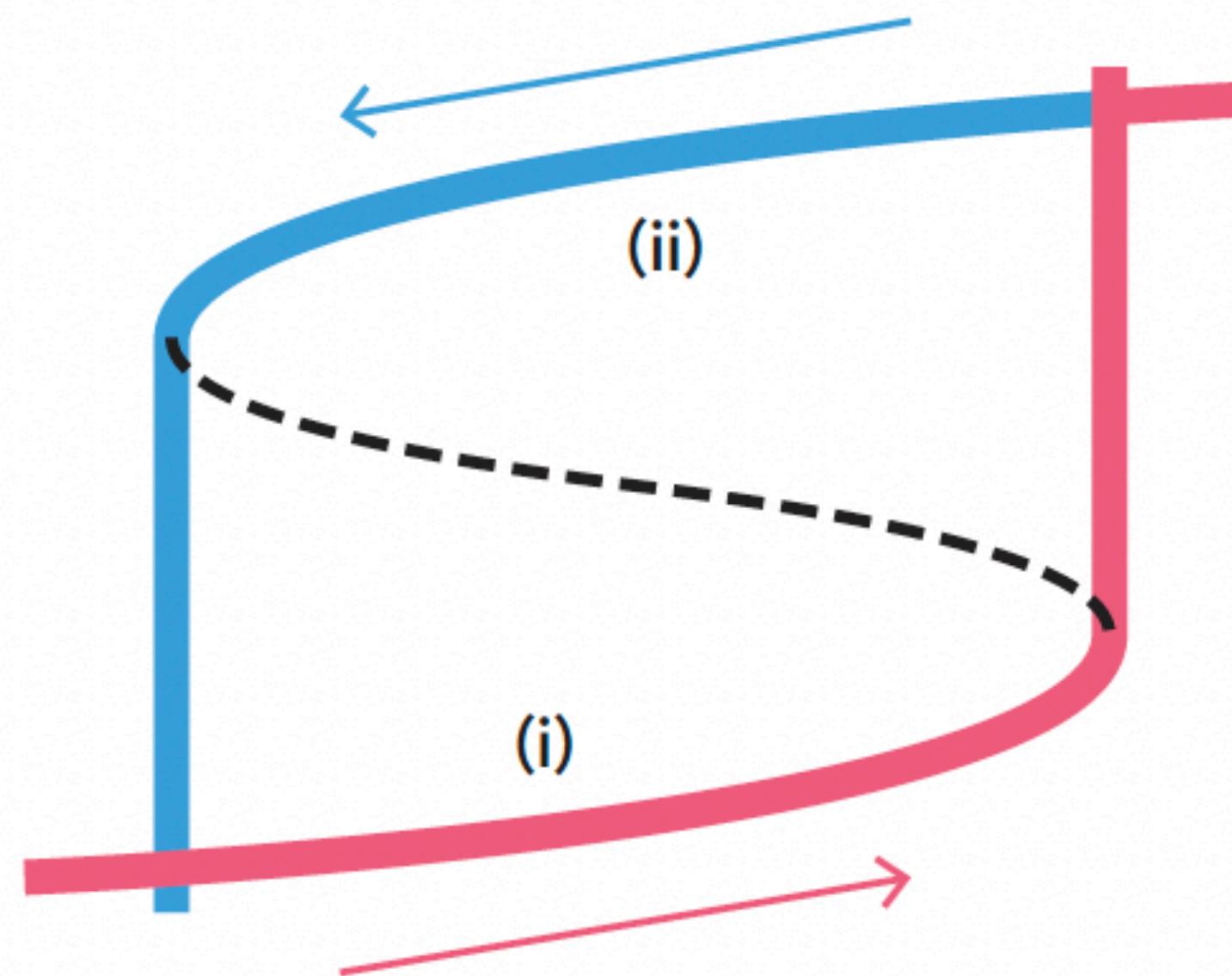
$$dX_t = f(X_t)dt + \sigma dW_t$$

$$X_0 = X(0)$$

Locally near steady state:

$$f(X_t) \sim -\gamma X_t$$

At saddle node: $\gamma = 0$



Stochastic Integrals

Kiyoshi *Itô* (1915-2008)



$$\int_0^T h(t) \, dW_t = \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

‘left end point’

Ruslan Stratonovich (1930-1997) ‘mid point’



$$\int_0^T h(t) \circ dW_t = \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} h\left(\frac{t_j + t_{j+1}}{2}\right)(W(t_{j+1}) - W(t_j))$$

both in the mean-square sense

$$\lim_{N \rightarrow \infty} E[(I - I_N)^2] \rightarrow 0$$

Evaluation of integrals

Problem: Evaluate $\int_0^T W_t dW_t$

Itô



$$f(W_T) - f(W_0) = \int_0^T f'(W_t) dW_t + \int_0^T \frac{1}{2} f''(W_t) dt$$

Take: $f(t) = t^2$

Answer: $\int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T)$

Solution of SDEs

$$f(t + dt, X_t + dX_t) - f(t, X_t) = f_1 dt + f_2 dX_t + \frac{1}{2}(f_{11}(dt)^2 + 2f_{12}dtdX_t + f_{22}(dX_t)^2) + \dots$$

Itô



$$dX_t = A_t dt + B_t dW_t$$

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2) ds + \int_0^t f_2 B_s dW_s$$

Local stochastic process near steady state

$$dX_t = -\gamma X_t dt + \sigma dW_t$$



$$A_t = -\gamma X_t ; \quad B_t = \sigma$$

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_1 + f_2 A_s + \frac{1}{2} f_{22} B_s^2) ds + \int_0^t f_2 B_s dW_s$$

Take: $f(t, x) = xe^{\gamma t}$

Result: $X_t = e^{-\gamma t} (X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$

Ornstein-Uhlenbeck process

Statistics OU process

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

Solution: $X_t = e^{-\gamma t} \left(X_0 + \sigma \int_0^t e^{\gamma s} dW_s \right)$



Autocorrelation: $E[X_t X_{t+s}], s > 0$

Result: $E[X_t X_{t+s}] = e^{-\gamma(2t+s)} \left(X_0^2 + \sigma^2 \frac{e^{2\gamma t} - 1}{2\gamma} \right)$

$$E[X_t X_{t+s}] \rightarrow \frac{\sigma^2}{2\gamma} e^{-\gamma s}, \quad t \rightarrow \infty$$

Spectrum: $S(\omega) = \frac{\sigma^2}{2\gamma} \mathcal{F}(e^{-\gamma s}) = \frac{\sigma^2}{\omega^2 + \gamma^2}$

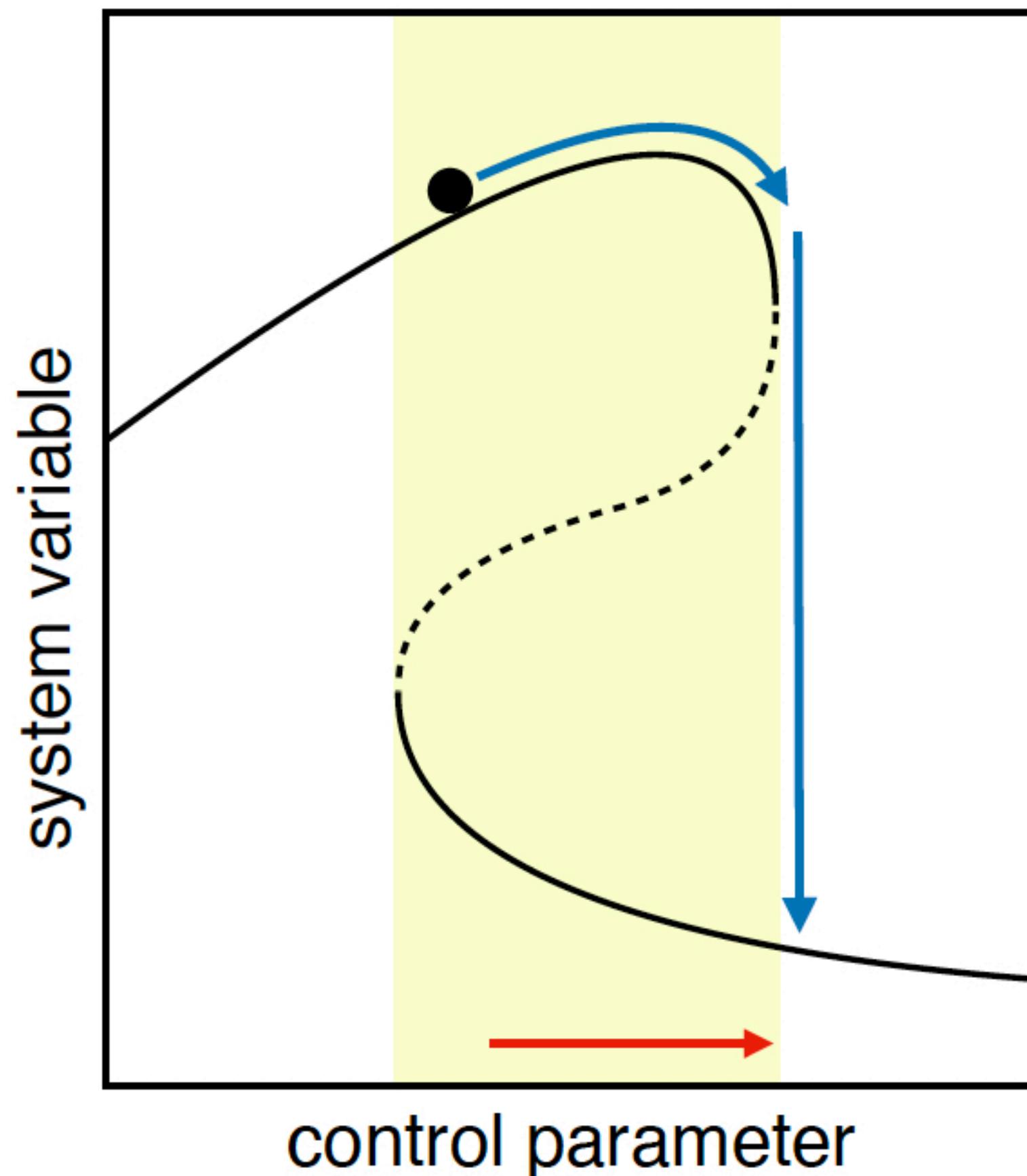


Early Warning Signals

$$dX_t = -\gamma X_t dt + \sigma dW_t$$



$$X_t = e^{-\gamma t} (X_0 + \sigma \int_0^t e^{\gamma s} dW_s)$$



AR(1):



$$X_{n+1} = \alpha X_n + Z_{n+1}$$

$$\alpha = e^{-\gamma \Delta t}$$

$$Var[X] = \frac{\sigma^2}{1 - \alpha^2}$$

$$\rho_n = \alpha^n$$

At saddle node: $\gamma \rightarrow 0$

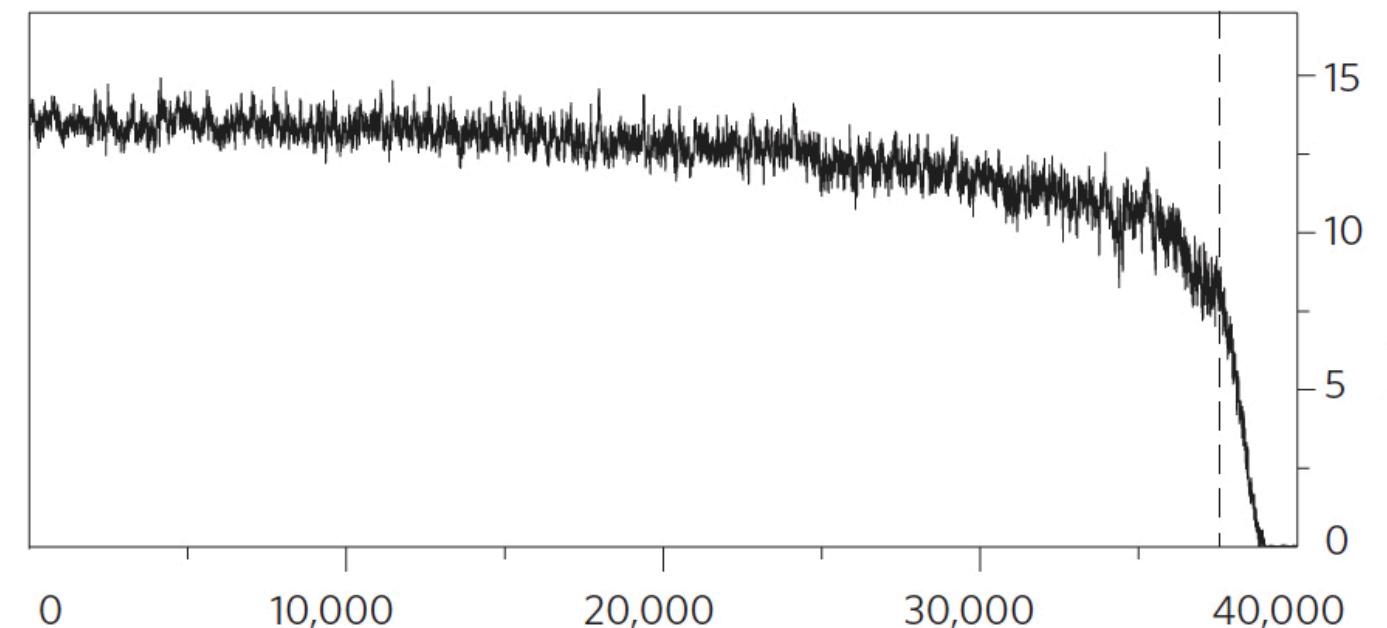


$Var[X] \rightarrow \infty ; \rho_n \rightarrow 1$

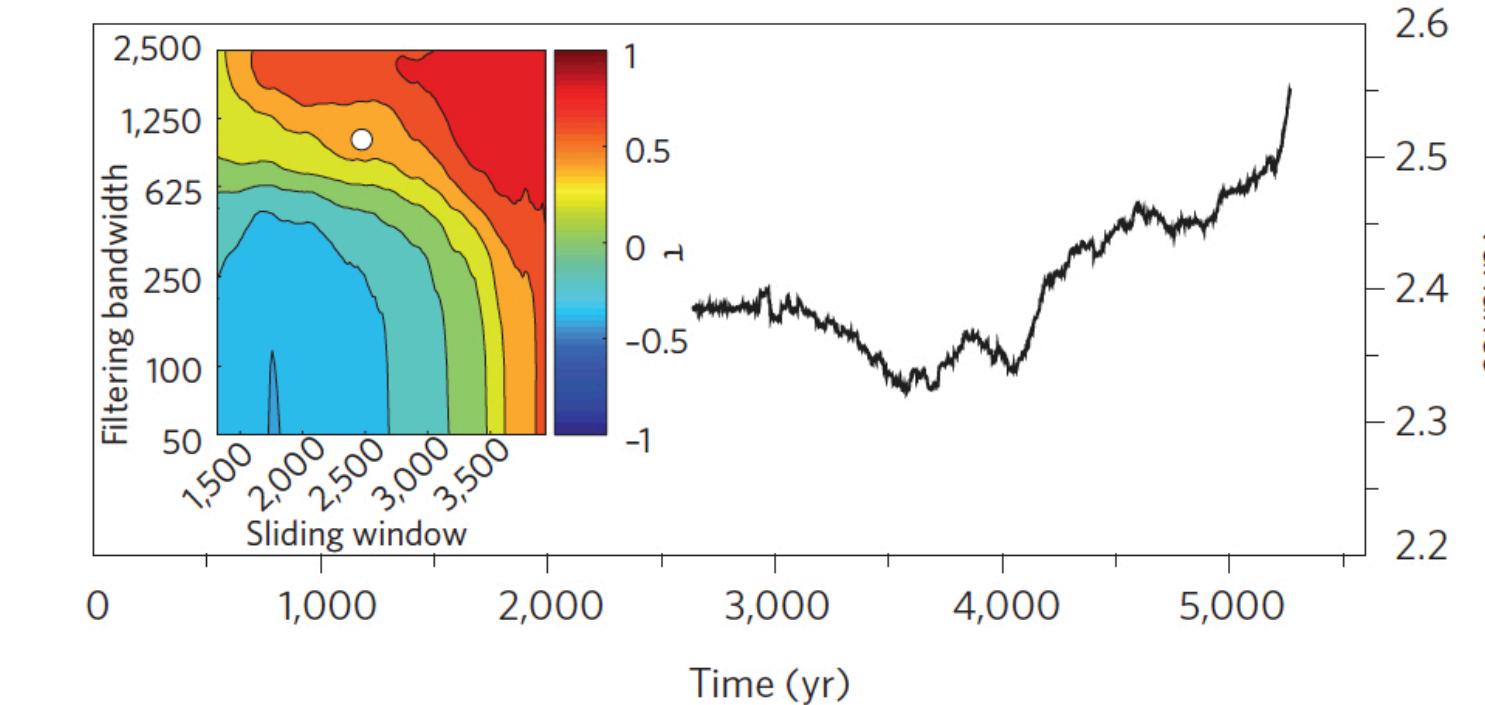
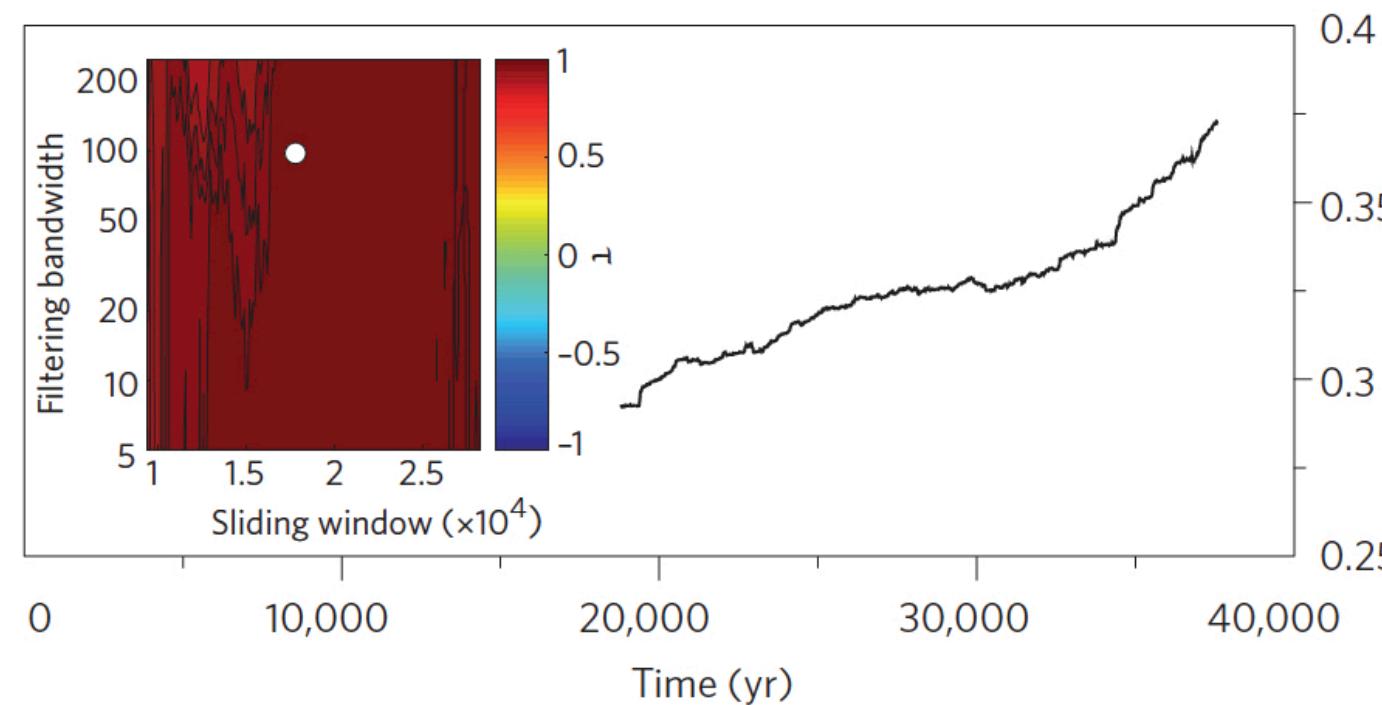
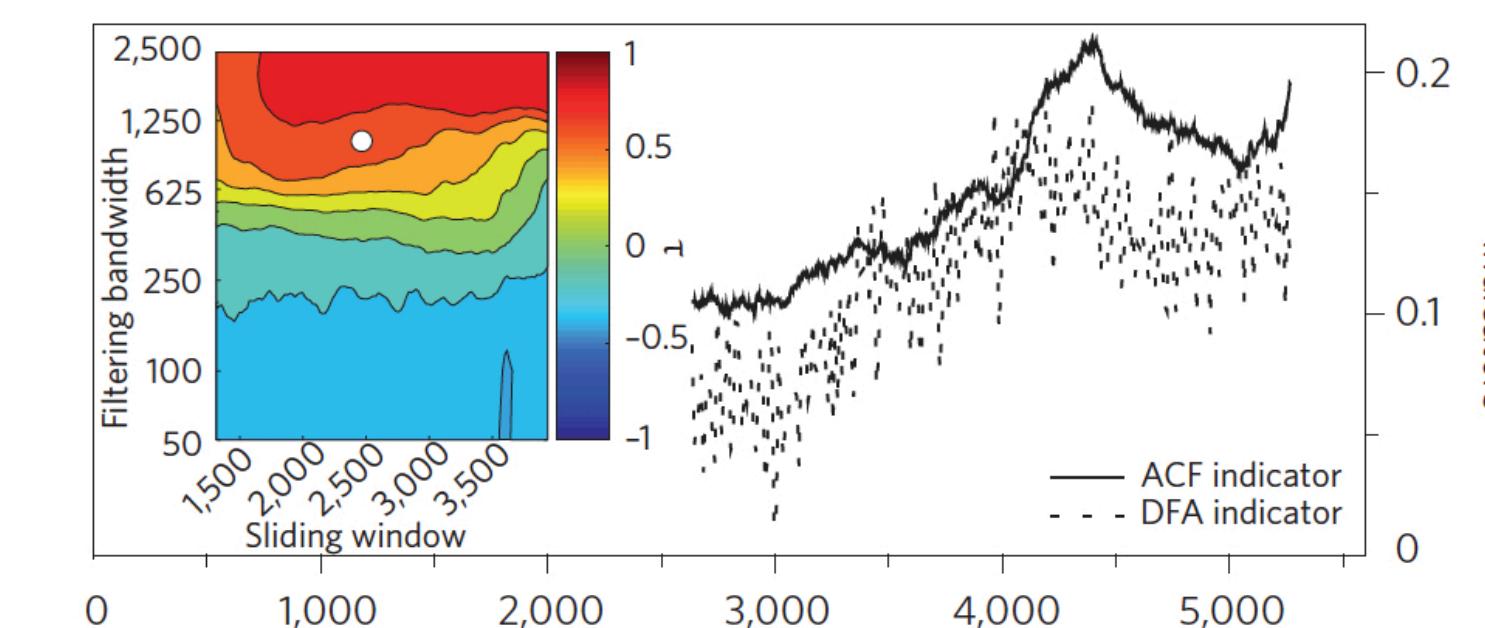
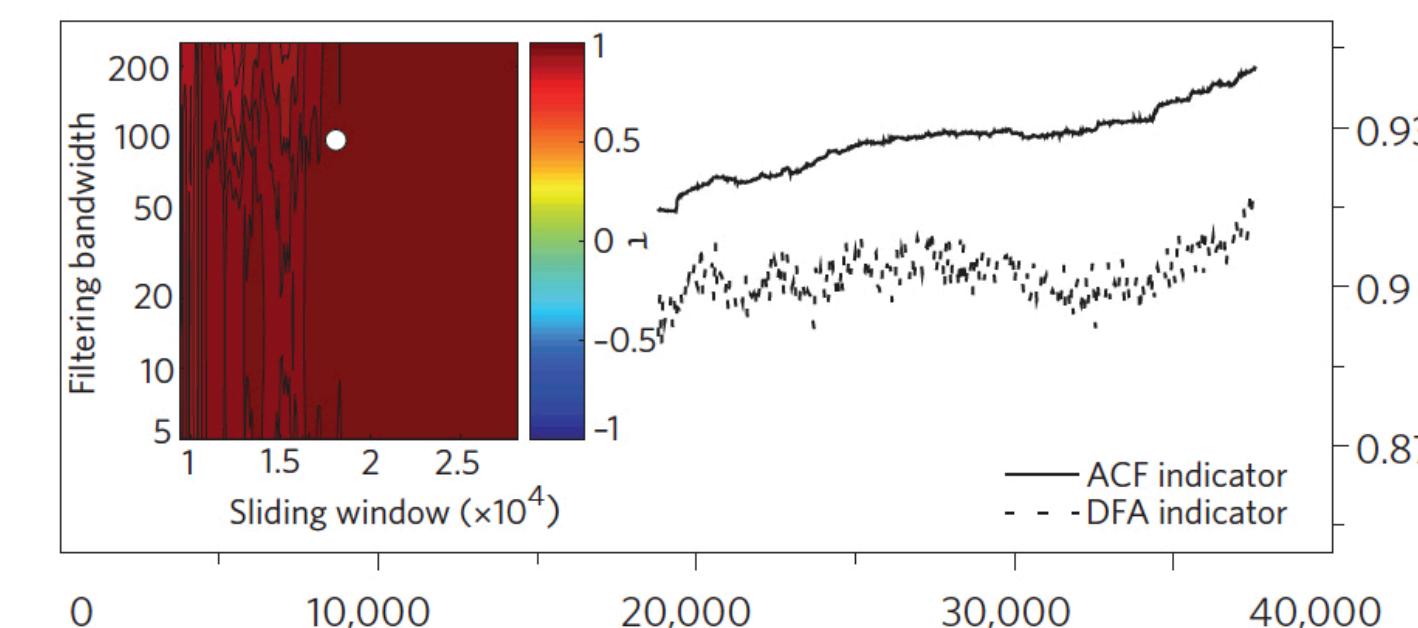
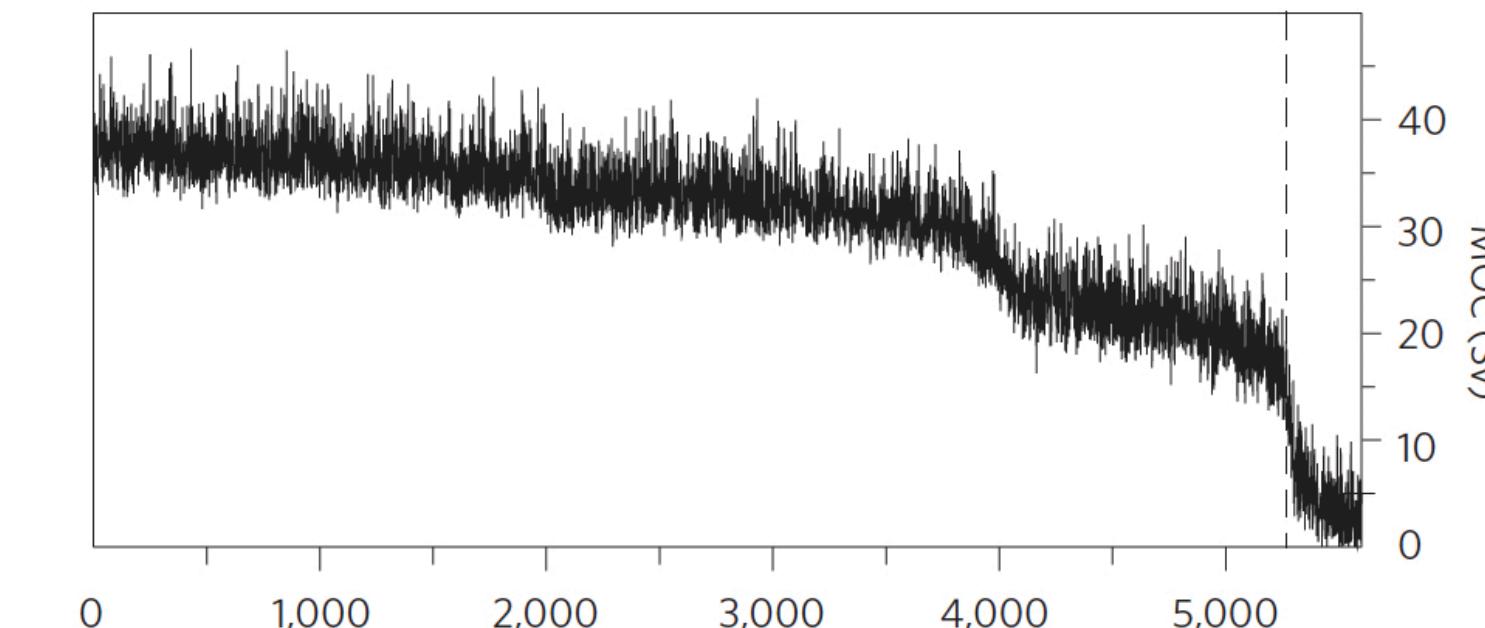
Example: AMOC

Tipping element: AMOC

a



b





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Exercises!

