Chapter 2

First Steps in Nonlinear Dynamics

In this chapter we consider the dynamics of systems with a very small number of variables, as a prerequisite to the study of chaos, postponed to Chapter 4 after an examination of convection in Chapter 3, which will give us a physical motivation.

2.1 From Oscillators to Dynamical Systems

2.1.1 *First definitions*

Newtonian mechanics is the archetype of deterministic dynamical theories. Governed by an equation of the form

$$m\ddot{X} = F\,, \tag{2.1}$$

where X represents the position of a particle with mass m submitted to a force F, it accounts for processes that are invariants under a change of the arrow of time.

A traditional example of linear system is the *harmonic oscillator* (Fig. 1.1, left, p. 3) that describes the motion of a mass attached to an ideal spring with a restoring force proportional to the extension $F = -kX$ (Hooke's law). This elastic force, of internal origin, derives from a potential:

$$F = -\frac{\partial \mathcal{V}}{\partial X} \quad \text{with} \quad \mathcal{V} = \tfrac{1}{2}kX^2$$

depicted in Figure 2.1 (left). Here we have

$$m\ddot{X} + kX = 0\,. \tag{2.2}$$

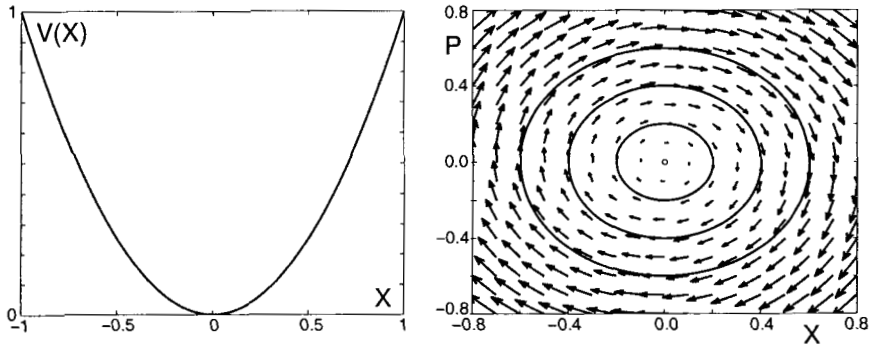

Fig. 2.1 Left: Potential for the harmonic oscillator. Right: Vector field and trajectories in the phase plane, the plane of variables $X_1 \equiv X$ and $X_2 \equiv P$.

The evolution is uniquely determined when initial conditions are specified:

$$X = X^{(0)} \qquad \text{and} \qquad \dot{X} = V^{(0)} \qquad \text{at} \quad t = 0\,.$$

(In the following, we will systematically use the superscript '(0)' to denote initial conditions.) The solution then reads:

$$X(t) = X^{(0)} \cos(\omega_0 t) + \left(V^{(0)}/\omega_0\right) \sin(\omega_0 t) \qquad \text{with} \quad \omega_0^2 = k/m\,.$$

Right now, it turns out advantageous to substitute a geometrical description (Fig. 2.1, right) to this analytical description. So, let us consider *trajectories* in the *phase space* where coordinates are the *position* X and the *momentum*

$$P = mV = m\dot{X}\,. \tag{2.3}$$

Equation (2.2) now reads

$$\dot{P} = -kX\,. \tag{2.4}$$

One can trace back the origin of this representation from the need to turn a high order differential equation into a first order differential system. On general grounds, this is done by introducing the successive derivatives as intermediate variables. For example, equation:

$$\mathrm{d}^n X/\mathrm{d}t^n = \mathcal{F}\left(\mathrm{d}^{n-1}X/\mathrm{d}t^{n-1}X, \ldots, \ddot{X}, \dot{X}, X; t\right) \tag{2.5}$$

is reduced to a first order system by setting $X_1 \equiv X$, $X_2 \equiv \dot{X}, \ldots$, $X_n \equiv \mathrm{d}^{n-1}X/\mathrm{d}t^{n-1}$, which yields:

$$\dot{X}_1 = X_2\,, \quad \dot{X}_2 = X_3\,, \ldots\,, \quad \dot{X}_n = \mathcal{F}\left(X_n, \ldots, X_3, X_2, X_1; t\right).$$

This operation is preliminary to any numerical implementation in view of simulation (Appendix B).

There is a slight disadvantage to start from mechanics[1] for an introduction of basic concepts of the theory of dynamical systems since Newton's equations endow the *phase space* with a specific structure and give it an even number of dimensions, $d = 2n$. As seen above, this space is indeed constructed as the product of the *configuration space* (variables X_i, $i = 1, \ldots, n$) and of the *momentum space* (variables P_i, $i = 1, \ldots, n$), each pair of *conjugate variables* (X_i, P_i) forming a *degree of freedom*. In the general case when the state of a system is specified using a supposedly sufficient number of variables, no longer grouped by pairs, we will call any of these state variable a 'degree of freedom'. Accordingly, the 'phase space' will then be the space where these variables "live" and the dimension of this space will just be the number of these variables. In agreement with the intuitive concept of *determinism*, the dimension d is also exactly the number of conditions necessary to specify any evolution uniquely. A way to avoid ambiguities would be to keep the terms 'phase space' and 'degrees of freedom' to mechanics in a strict sense and to use *state variables* and *state space* in all other cases, which we will not do since the risk of confusion is limited.

Anyway, let us consider the general case of a phase space $\mathbb{X}$ with dimension d spanned by d variables $\{X_1, \ldots, X_d\} \equiv \mathbf{X}$. The evolution of the so-defined variables is governed by a system that symbolically reads

$$\dot{\mathbf{X}} = \boldsymbol{\mathcal{F}}(\mathbf{X}; t)\,, \tag{2.6}$$

where $\boldsymbol{\mathcal{F}}$ is a set of d functions representing the components of a *vector field* defined on $\mathbb{X}$ that specifies the "velocity" of the point representing the system in its phase space.

When the properties of $\boldsymbol{\mathcal{F}}$ guarantee the existence and uniqueness of the solution to the initial value problem, in practice when the vector field $\boldsymbol{\mathcal{F}}$ is $\mathcal{C}^1$ (differentiable with continuous first derivative), one says that it defines a *flow* on phase space.

When t is explicitly absent from the definition of $\boldsymbol{\mathcal{F}}$, the system is said to be *autonomous*, otherwise it is *forced*. In practice, among forced systems, only periodically forced systems will be of interest to us, *i.e.* systems such that $\boldsymbol{\mathcal{F}}(\mathbf{X}; t + T) \equiv \boldsymbol{\mathcal{F}}(\mathbf{X}; t)$ for some minimal time interval T called the *period*. Within this class one often distinguishes *parametric forcing* for

[1] See later, §2.1.2, for a brief presentation of the formalism of analytical mechanics.

which the expression of $\mathcal{F}$ changes in time, *e.g.* the parametric linear oscillator (Mathieu equation):

$$\ddot{X} + (1 + a\sin(\omega t))X = 0\,, \qquad T = 2\pi/\omega\,,$$

from *external forcing* where an otherwise autonomous system is submitted to a periodic force independent of its state, *i.e.*

$$\ddot{X} + X = f\sin(\omega t)\,.$$

Noisy systems can be understood as particular forced systems with a *random* forcing. In case of additive noise, this defines the so-called *Langevin equation*

$$\dot{\mathbf{X}} = \mathcal{F}(\mathbf{X}) + \Xi(t)\,,$$

where $\Xi(t)$ is a random vector function. Quite different tools of statistical essence are then required which will not be introduced here since we want to stick to the deterministic point of view.

During its evolution, the system follows a phase space *trajectory* starting at $\mathbf{X}^{(0)}$ when $t = t^{(0)}$ and obtained by integration of (2.6):

$$\mathbf{X}(t) = \mathbf{X}^{(0)} + \int_{t^{(0)}}^{t} \mathcal{F}(\mathbf{X}(t'); t')\,\mathrm{d}t'\,. \tag{2.7}$$

The *orbit* is the set of points in $\mathbb{X}$ visited by the system in the course of a given trajectory. The description of a system's dynamics in terms of sets of orbits, called its *phase portrait*, as a function of its *control parameters*, is the field of *qualitative dynamics*.

Relation (2.7) allows us to define a map of $\mathbb{X}$ onto itself. Upon specifying an integration time τ, we get a *time-τ map*

$$\mathbf{X}(t+\tau) \equiv \mathbf{\Phi}_\tau\big(\mathbf{X}(t)\big) = \mathbf{X}(t) + \int_t^{t+\tau} \mathrm{d}t'\,\mathcal{F}\big(\mathbf{X}(t'); t'\big),$$

and, starting with an initial condition $\mathbf{X}^{(0)}$, we obtain a discrete sampling of the trajectory, $\mathbf{X}_0 \equiv \mathbf{X}^{(0)}$, $\mathbf{X}_1 = \mathbf{\Phi}_\tau(\mathbf{X}_0)$, $\mathbf{X}_2 = \mathbf{\Phi}_\tau(\mathbf{X}_1)$,..., $\mathbf{X}_{k+1} = \mathbf{\Phi}_\tau(\mathbf{X}_k)$...

The time-τ map is really interesting only when τ corresponds to some characteristics of the system. The most important case corresponds to periodic forcing with period $T = \tau$, in which case $\mathbf{\Phi}_T$ performs a *stroboscopic analysis* of the dynamics, *i.e.* takes pictures of the system at the period of a strobe signal in phase with the forcing. Later we will see another related way to arrive at such *discrete-time* systems as already mentioned in

Chapter 1, p. 4. The concepts of trajectories and phase portraits transpose immediately to systems written as first-order iterations $\mathbf{X}_{k+1} = \mathbf{\Phi}(\mathbf{X}_k; k)$.

From a practical viewpoint the determination of the trajectory issued from some initial condition $\mathbf{X}^{(0)}$ by numerical integration can be viewed as resulting from the iteration of a map $\mathbf{\Phi}_\tau$ integrating the field $\mathcal{F}$ over a time interval $\tau = \Delta t$ chosen from accuracy considerations specific to the numerical scheme (cf. Appendix B).

For the moment, let us consider autonomous systems and especially the simple case of the harmonic oscillator. Its phase space is the plane (X, P) isomorphic to $\mathbb{R}^2$. The right part of Figure 2.1 displays the corresponding vector field.[2] In this representation, trajectories follow elliptic orbits.

On general grounds it turns out useful to scale the variables as much as possible in order to cast the system into its most universal form, hiding its specificities inside the details of the variable change. Here such a transformation simply yields

$$\dot{X}_1 = X_2\,, \qquad \dot{X}_2 = -X_1\,. \tag{2.8}$$

Reversibility is one of the fundamental characteristics of ideal mechanical systems (*i.e.* without friction). The change '$t \mapsto -t$' indeed leaves the Newton equations invariant. This property is associated to energy conservation. Defining $E = \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2$ and computing $\mathrm{d}E/\mathrm{d}t$ using (2.8) one can check that E is a constant of motion. Things are different for a damped system that *dissipates* its energy. At a linear stage, the introduction of a *viscous friction* proportional to the rate of change of the variable leads to

$$\ddot{X} + 2\eta\dot{X} + X = 0\,, \tag{2.9}$$

where $\eta > 0$ measures the strength of the damping. Here the change $t \mapsto t/\omega_0$ has been performed in order to normalize the period of the ideal oscillator to 2π.

Trajectories are obtained in parametric form as

$$X(t) = \bar{X}\exp(-\eta t)\cos(\omega t - \varphi),$$

where $\bar{X}$ and φ can be computed from the initial conditions $X^{(0)}$ and $\dot{X}^{(0)}$ at $t = 0$ by identification, and where $\omega = (1 - \eta^2)^{1/2}$ corresponds to the angular frequency of a damped oscillation only when the friction is

[2] Here it has been obtained using the MATLAB macro `quiver`, explicitly:

```
X=[-0.8:0.1:0.8]; U=ones(1,size(X,2)); P=X'; FX=P*U; FP=-U'*X;
quiver(X,P,FX,FP).
```

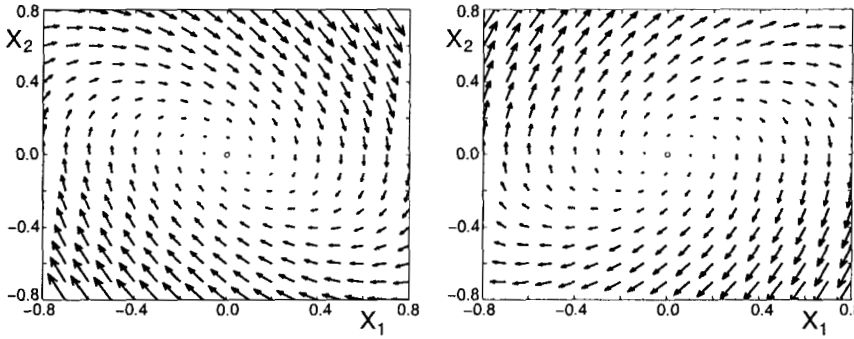

Fig. 2.2 Vector field of the non-ideal linear oscillator in its phase plane (X_1, X_2) for $\eta = 0.5$. Left: Damped oscillator ($\eta > 0$). Right: Driven oscillator ($\eta < 0$).

sufficiently weak, *i.e.* when $\eta < 1$ (the over-damped case $\eta > 1$ will be considered as an exercise). In the phase plane, equation (2.9) reads

$$\dot{X}_1 = X_2\,, \qquad \dot{X}_2 = -2\eta X_2 - X_1\,, \tag{2.10}$$

and the orbits now take the aspect of spirals converging towards the origin, see Figure 2.2 (left) that displays the corresponding vector field. The instantaneous dissipation rate of the total energy, still defined as $E = \frac{1}{2}(X_1^2 + X_2^2)$, is now given by $\mathrm{d}E/\mathrm{d}t = -2\eta X_2^2 < 0$.

For an excited system that would receive energy from the exterior world by viscous *driving*, thus governed by (2.9) but with $\eta < 0$, the integration of the vector field would give diverging spiral trajectories easily imagined from the vector field depicted in Figure 2.2 (right).

In fact, when defining a state physically, one never considers a mathematical point in phase space but rather a small, physically infinitesimal domain around this point. This remark suggests we focus on the future of sets of systems that at a given time belong to volume elements in phase space (surface elements in the specific two-dimensional case of interest here). For a general dynamical system (2.6) defined on a d-dimensional phase space $\mathbb{X}$, one then shows that the local evolution rate of the volume Δ of an infinitesimal domain around some point $\mathbf{X}$ is given by the divergence of the vector field $\mathcal{F}$ computed at this point (cf. Exercise 2.5.1):

$$\frac{1}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \mathrm{div}\,\mathcal{F} = \sum\nolimits_{j=1}^{d} \frac{\partial \mathcal{F}_j}{\partial X_j}\,. \tag{2.11}$$

For an ideal oscillator (2.8), one immediately obtains that the divergence is

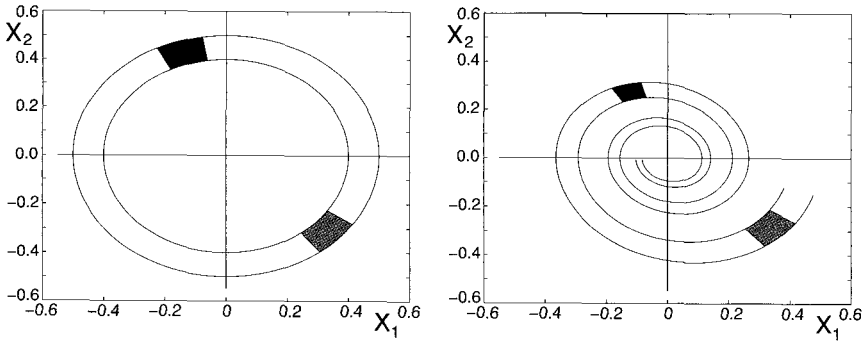

Fig. 2.3 Evolution of volumes in phase space. Left: Conservative system. Right: Dissipative system.

zero: the surface of a small surface element is thus conserved (Fig. 2.3, left). This property can be generalized to all frictionless mechanical systems, that are called *conservative* for that reason. The same calculation for the damped oscillator leads to $\operatorname{div} \mathcal{F} = -2\eta < 0$, thus to an indefinite erosion of areas (Fig. 2.3, right). This reduction is characteristic of *dissipative* systems whose permanent regimes, asymptotic states at the limit $t \to \infty$ after the damping out of transients, are described by *attractors*.

As suggested above, the phase space volumes give a measure of the number of accessible states. Asymptotically the damped oscillator always ends at its rest position $X_1 = X_2 \equiv 0$, a single state whatever the initial energy. The attractor is here a *fixed point* in phase space, 'fixed point' because the orbit of the trajectory starting there is just reduced to that point. By anticipation, we can say that it is a *stable* fixed point since trajectories starting in its vicinity converge to it as t tends to infinity. By contrast, the fixed point at the origin of the phase space associated to an driven oscillator in not an attractor but a *repellor*, it is unstable and trajectories move apart from it.

General systems are not conservative. They can be fueled in energy in some regions of their phase space while dissipating it in other regions. This feature, that cannot be achieved in the framework of linear systems with constant coefficients (sign of $\operatorname{div} \mathcal{F}$ fixed once for all), will turn out to be essential to the existence of self-sustained oscillations and more complicated time behavior.

2.1.2 *Formalism of analytical mechanics*

In order to ease the solution to some exercises, we give here an introduction to the analytical formalism of classical mechanics that allows one to pass from the Newton equations (second order in time) to the Hamilton equations (first order in time) giving to the intermediate variables so-introduced their status of conjugate momenta to the generalized coordinates.

For a system of n_{N} (subscript 'N' for Newton) material points with masses m_i and positions $\mathbf{X}_i \equiv (x_i, y_i, z_i)$ submitted to forces $\mathbf{f}_i$ deriving from a potential $\mathcal{V}(\{\mathbf{X}_i\}; t)$, the Newton equations read:

$$m_i \ddot{\mathbf{X}}_i = \mathbf{f}_i = -\frac{\partial \mathcal{V}}{\partial \mathbf{X}_i}, \quad (i = 1, \ldots, n_{\mathrm{N}}). \tag{2.12}$$

The kinetic energy is defined by:

$$\mathcal{T} = \frac{1}{2} \sum_{i=1}^{n_{\mathrm{N}}} m_i (\dot{\mathbf{X}}_i)^2,$$

and one easily checks that the total energy

$$E = \mathcal{T} + \mathcal{V}$$

is conserved. But this so-called *Newtonian formulation* makes the treatment of systems with constraints somehow awkward. The *Lagrangian formalism* answers this problem by introducing n_{L} ('L' for Lagrange) *generalized coordinates*.[3] Let

$$\mathbf{q} \equiv \{q_j; j = 1, \ldots, n_{\mathrm{L}}\}, \quad q_j = q_j(\{\mathbf{X}_i; i = 1, \ldots, n_{\mathrm{N}}\}; t), \tag{2.13}$$

be the change of variables from the $\mathbf{X}_i$s to the q_js expressing the constraints. The Lagrangian is then defined by

$$\mathcal{L} = \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{T} - \mathcal{V},$$

so that (2.12) can be rewritten as a set of Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial (\dot{q}_j)} - \frac{\partial \mathcal{L}}{\partial q_j} = 0, \qquad j = 1, \ldots, n_{\mathrm{L}}. \tag{2.14}$$

[3] $n_{\mathrm{L}} < n_{\mathrm{N}}$ and sometimes $\ll n_{\mathrm{N}}$, think of a solid body with many (n_{N}) rigidly linked particles and characterized just by the position of its center of mass (three coordinates) and its orientation (three Euler angles).

At this stage, the system is still governed by second order differential equations. One then defines the *momenta* $\mathbf{p} \equiv \{p_j\}$ conjugated to the coordinates $\mathbf{q} \equiv \{q_j\}$ by:

$$p_j \equiv \frac{\partial \mathcal{L}}{\partial(\dot{q}_j)}\,, \tag{2.15}$$

and the Hamiltonian by:

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = \sum_j p_j \dot{q}_j - \mathcal{L}\,. \tag{2.16}$$

The dynamical equations (2.14) then turn into the *Hamilton equations*:

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j}\,, \qquad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j}\,, \qquad j = 1,\ldots,n_{\mathrm{L}}\,. \tag{2.17}$$

The phase space $\mathbb{X}$ is the Cartesian product of the configuration space with coordinates q_j and the momentum space with coordinates p_j, thus with dimension $d = 2n_{\mathrm{L}}$. The vector field governing the dynamics then reads

$$\mathcal{F}_{q_j} \equiv \frac{\partial \mathcal{H}}{\partial p_j} \qquad \text{and} \qquad \mathcal{F}_{p_j} \equiv -\frac{\partial \mathcal{H}}{\partial q_j}\,, \tag{2.18}$$

expressions that provide it with a so-called 'symplectic' structure insuring the conservation of phase space volumes automatically ($\mathrm{div}\,\boldsymbol{\mathcal{F}} \equiv 0$, the Liouville theorem).

2.1.3 *Gradient systems*

Above, we spoke of 'forces deriving from a potential' in a strictly mechanical framework. Unfortunately, there is a risk of confusion when using the word 'derive' and 'potential' without care, because in some branches of dynamical systems theory, they have a somewhat different meaning that now warrants specification.

To do so, let us first consider an autonomous systems with a single real variable X:

$$\dot{X} = \mathcal{F}(X)\,. \tag{2.19}$$

Defining $\mathcal{G}$ from $\mathcal{F}$ by

$$\mathcal{G}(X) = -\int \mathcal{F}(X)\,\mathrm{d}X\,,$$

we can rewrite (2.19) as:

$$\dot{X} = -\mathrm{d}\mathcal{G}/\mathrm{d}X\,, \tag{2.20}$$

where $\mathcal{G}$ presents itself as a 'potential' from which the vector field $\mathcal{F}$ can be 'derived'. In higher dimensions, the natural extension of (2.20) for a function of several variables $\mathcal{G}(X_1,\ldots,X_d)$ through

$$\dot{X}_j = \mathcal{F}_j(X_1,\ldots,X_d) = -\partial\mathcal{G}/\partial X_j\,, \quad j=1,\ldots,d\,, \tag{2.21}$$

defines a large class of systems called *gradient flows*. It is also said that such systems are 'relaxational'. The origin of the latter term is to be found in the remark that (2.20) immediately leads to $\mathcal{G}$ being a monotonicly decreasing function of time:

$$\dot{\mathcal{G}} = \left(\mathrm{d}\mathcal{G}/\mathrm{d}X\right)\dot{X} = -\left(\mathrm{d}\mathcal{G}/\mathrm{d}X\right)^2 \leq 0\,. \tag{2.22}$$

Accordingly, the system evolves from almost all initial conditions so as to 'relax' toward one of the local minima of $\mathcal{G}$ where its stops asymptotically. In the d-dimensional case, the vector field $\mathcal{F}$ is seen from (2.21) to be everywhere perpendicular to the level curves of $\mathcal{G}$ so that:

$$\dot{\mathcal{G}} = \textstyle\sum_j \left(\partial\mathcal{G}/\partial X_j\right)\dot{X}_j = -\sum_j \left(\partial\mathcal{G}/\partial X_j\right)^2 \leq 0\,, \tag{2.23}$$

which again expresses the 'relaxation' toward one of its local minima.

Figure 2.4 displays the level lines of a two dimensional potential[4] $\mathcal{G} = -aX_1 - \frac{1}{2}(bX_1^2 + cX_2^2) + \frac{1}{4}(X_1^2 + X_2^2)^2$, with $a = 3/2$, $b = 13/4$, $c = 5/4$. The absolute minimum $\mathcal{G} = -5.5$ is at $\mathbf{M_1} = (2,0)$. There is a relative minimum $\mathcal{G} \simeq -0.14$ at $\mathbf{M_2} = (-1.5,0)$, a relative maximum $\mathcal{G} \simeq 0.36$ at $\mathbf{M_3} = (0.5,0)$, while points $\mathbf{S_1}$ and $\mathbf{S_2}$ at $(-3/4, \pm 0.83)$, later called *saddle points*, belong to the level line $\mathcal{G} = 0.171875\ldots$.

In mathematics, gradient flows appear in the theory of elementary *catastrophes* (see, *e.g.* [Poston and Stewart (1978)]). In thermodynamics, they offer a good framework for the Landau theory of *phase transitions*, see later the remark on p. 127 and also [Stanley (1988)].

Now, if $\dot{\mathbf{X}} = \mathcal{F}(\mathbf{X})$ derives from a potential $\mathcal{G}$ in the sense of (2.21), the components of $\mathcal{F}$ fulfill the relations:

$$\partial\mathcal{F}_j/\partial X_{j'} = \partial\mathcal{F}_{j'}/\partial X_j\,, \qquad \forall\, j, j'\,, \tag{2.24}$$

[4]For more detail, see P. M. & L. Tuckerman, "Phenomenological modeling of the first bifurcations of the spherical Couette flow," J. Physique **48** (1987) 1461–1469.

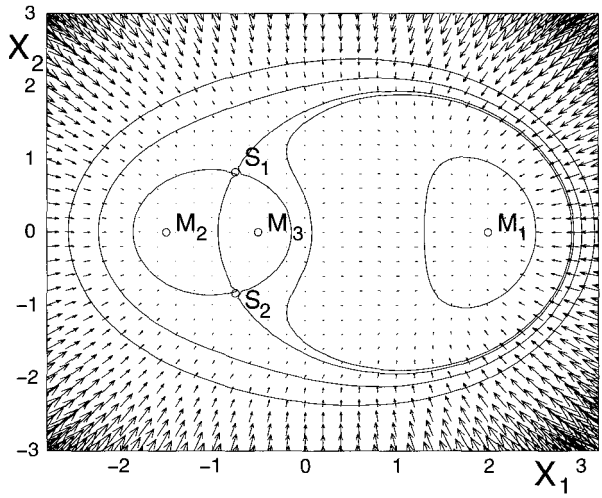

Fig. 2.4 For a gradient flow, the vector field $\mathcal{F}$ is everywhere perpendicular to the level curves of the potential $\mathcal{G}$ from which it derives.

as results from the Schwartz identity:

$$\partial^2 \mathcal{G}/\partial X_j \partial X_{j'} = \partial^2 \mathcal{G}/\partial X_{j'} \partial X_j \, .$$

Conditions (2.24) are consequently necessary to the existence of a potential. In the general case, the components of the vector field $\mathcal{F}$ do not fulfill such conditions and one can expect an evolution that is richer than a "simple" relaxation towards a local minimum of some putative $\mathcal{G}$. This is of course the case of ideal mechanical systems for which the total energy is a constant of the motion so that the vector field given by the Hamilton equations is everywhere parallel to the surfaces of constant total energy in phase space (Figure 2.1, right). For profound reasons to be discussed later, we will have to wait for the study of three-dimensional systems before any dynamics more complicated than a relaxation towards fixed points or periodic oscillations can be observed.

2.2 Stability and Linear Dynamics

The first piece of information of interest about the regime attained by a given system under specific conditions relates to its stability, *i.e.* the way it reaches the state and responds to perturbations. In practice one can only

exceptionally determine the response to perturbations of arbitrary amplitude. Except for Lyapunov functions generalizing the potentials introduced in Section 2.1.3 (see also Exercise 2.5.2), we have no tools to attack the problem in full generality and we must restrict ourselves to a study of the evolution of infinitesimal perturbations, for which all the resources of *linear* analysis are available.

2.2.1 *Formulation of the linear stability problem*

In what follows, we are mainly interested in specific time-independent states of autonomous systems:

$$\dot{\mathbf{X}} = \boldsymbol{\mathcal{F}}(\mathbf{X})\,. \tag{2.25}$$

These states are thus solutions to:

$$\boldsymbol{\mathcal{F}}(\mathbf{X}_\mathrm{f}) = 0 \tag{2.26}$$

and are represented by *fixed points* in phase space, hence the subscript 'f' serving to denote them. Equation (2.26) is the formal writing of a system of d nonlinear equations with d unknowns that has a discrete and finite set of solutions in general.

In out-of-equilibrium macroscopic systems, especially continuous media, it is natural to first consider the solution that belongs to the *thermodynamic branch*, defined as the branch of solutions that can be followed from thermodynamic equilibrium by continuity, but what will be said also holds for any other time-independent solution, even if it is more difficult to obtain.

Let $\mathbf{X}_\mathrm{f}$ be the fixed point of interest. Inserting $\mathbf{X} = \mathbf{X}_\mathrm{f} + \mathbf{X}'$ in (2.25), one expands that equation in powers of $\mathbf{X}'$. Noting that order 0 is identically fulfilled by the fixed point condition and keeping only first order terms, one gets:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}' = \boldsymbol{\mathcal{L}}\mathbf{X}'\,. \tag{2.27}$$

The (Jacobian) operator $\boldsymbol{\mathcal{L}}$ resulting from this linearization is represented by a matrix with elements

$$l_{ij} = \partial \mathcal{F}_i / \partial X_j \big|_{\mathbf{X}_\mathrm{f}}\,,$$

all partial derivatives being evaluated at the fixed point $\mathbf{X}_\mathrm{f}$ as indicated by the notation. In this context, the linear stability problem amounts to an integration of equation (2.27) and thus to the evaluation of the action of operator $\exp(t\boldsymbol{\mathcal{L}})$ on some initial condition $\mathbf{X}'^{(0)}$. Infinitesimal perturbations

are said to live in the *tangent space* at the fixed point. Their evolution is governed by the *tangent dynamics* (2.27).

A simple way to justify the introduction of the exponential of operator $\mathcal{L}$ consists in considering the scalar equation $\dot{X} = aX$ with $X(0) = X^{(0)}$, in writing recursively

$$X(t) = X^{(0)} + \int_0^t aX(t')\mathrm{d}t'$$
$$= X^{(0)} + \int_0^t \left[X^{(0)} + \int_0^{t'} aX(t'')\mathrm{d}t''\right]\mathrm{d}t' = \ldots,$$

and in integrating explicitly all what can be integrated. This yields:

$$X(t) = X^{(0)} + atX^{(0)} + \frac{1}{2}a^2t^2X^{(0)} + \cdots + \frac{1}{n!}a^nt^nX^{(0)} + \cdots = \exp(at)X^{(0)}.$$

The extension of this approach to solving (2.27) involves what is precisely defined as the exponential $\exp(t\mathcal{L})$, *i.e.* the limit of a power series. In general, the solution of (2.27) rests on turning $\mathcal{L}$ to its diagonal form, or more precisely to its Jordan normal form. See Appendix A, §A.2, for a reminder.

2.2.2 *Two-dimensional linear systems*

We first turn to the case of two variables since it contains the essentials of nontrivial aspects of the problem and allows us to introduce the core of the terminology. The extension to dimension d will be alluded to in the next subsection. Thus consider the linear dynamical system:

$$\dot{X}_1 = a_{11}X_1 + a_{12}X_2\,, \tag{2.28}$$
$$\dot{X}_2 = a_{21}X_1 + a_{22}X_2\,, \tag{2.29}$$

with initial condition, $X_j = X_j^{(0)}$, $j = 1, 2$, at $t = 0$ as a generalization of the harmonic oscillator studied previously. Since this system can be studied for itself and not as a tangent problem, we have dropped the primes indicating that its variables originally measured the departures from the fixed point of some primitive nonlinear system and forgotten that the coefficients derive from the evaluation of some Jacobian operator $\mathcal{L}$ by changing the notations to anonymous as. Solutions to (2.28, 2.29) are searched in the form

$$X_i = \bar{X}_i \exp(st).$$

By mere substitution, this leads to an *eigenvalue problem*:

$$s\bar{X}_1 = a_{11}\bar{X}_1 + a_{12}\bar{X}_2\,,$$
$$s\bar{X}_2 = a_{21}\bar{X}_1 + a_{22}\bar{X}_2\,,$$

that has nontrivial solutions $\bar{\mathbf{X}} \not\equiv 0$ only if the the growth rate s satisfies the compatibility condition:

$$s^2 - (a_{11} + a_{22})\, s + a_{11}a_{22} - a_{12}a_{21} = 0\,, \tag{2.30}$$

also called the *characteristic equation*. In full generality, this quadratic equation has two roots, distinct or not, real or complex conjugate (see Appendix A, Exercise A.4, p. 348).

Eigen-solutions corresponding to eigenvalues either positive or complex with positive real part depart exponentially fast from the origin. In the negative case they converge to it.

The general solution is a superposition of eigen-solutions. Hence when both roots are negative or complex with negative real parts, the origin is said to be *linearly stable* whereas it is sufficient that one of the roots be real and positive or complex with positive real part to render the origin instable, which immediately yields a universal classification.

2.2.2.1 *Two real distinct roots*

Without changing notations, in the eigen-basis we have:

$$\dot{X}_j = s_j X_j \qquad \Rightarrow \quad X_j(t) = X_j^{(0)} \exp(s_j t)\,, \qquad j = 1, 2\,,$$

and we can distinguish two sub-cases:

i) The two roots have the same sign ($s_1 s_2 > 0$): the origin is called a *node*, stable when s_1 and s_2 are negative, unstable in the opposite case. Orbits around the origin have a parabolic shape obtained by elimination of t between the different components of the solution, here between $X_1(t) = X_1^{(0)} \exp(s_1 t)$ and $X_2(t) = X_2^{(0)} \exp(s_2 t)$, which yields $X_2/X_2^{(0)} = (X_1/X_1^{(0)})^{s_2/s_1}$ with a positive exponent s_2/s_1. They open in the direction of X_1 or X_2 according to the relative magnitude of s_1 and s_2, see Figure 2.5(a).

ii) The two roots have opposite signs ($s_1 s_2 < 0$): the origin is a *saddle*, stable along the direction of the negative root and unstable along the other one and thus always unstable. Orbits have a hyperbolic shape as depicted in Figure 2.5(b).

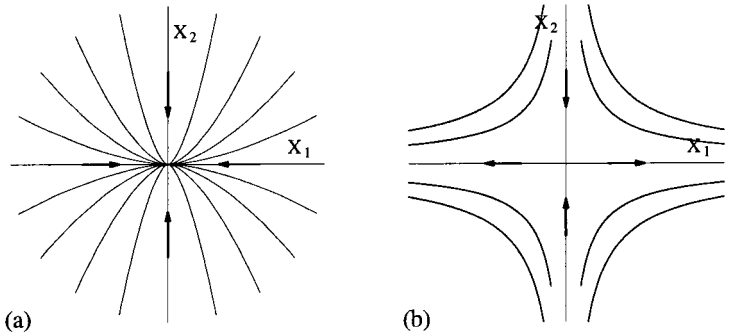

Fig. 2.5 Real roots s_1 and s_2. (a) Stable Node, $s_2 < s_1 < 0$. (b) Saddle, $s_2 < 0 < s_1$.

2.2.2.2 *A pair of complex conjugate roots*

Let us write the roots as $s = \sigma \pm i\omega$. In such a case there are no real eigenvectors but by a linear variable change one can cast the system into the form

$$\dot{X}_1 = \sigma X_1 + \omega X_2 \,, \tag{2.31}$$

$$\dot{X}_2 = -\omega X_1 + \sigma X_2 \,, \tag{2.32}$$

which, by integration, leads to

$$X_1(t) = \exp(\sigma t)\Big[X_1^{(0)} \cos(\omega t) + X_2^{(0)} \sin(\omega t)\Big] \,, \tag{2.33}$$

$$X_2(t) = \exp(\sigma t)\Big[- X_1^{(0)} \sin(\omega t) + X_2^{(0)} \cos(\omega t)\Big] \,. \tag{2.34}$$

Trajectories thus spiral around the origin that is called a *spiral point*, or a *focus*, stable or unstable according to the sign of the real part σ, see Figure 2.6(a).

In the *marginal* case $\sigma = 0$, the fixed point is called a *center* or an *elliptic point*, Figure 2.6(b). This is of course the case of the harmonic oscillator considered previously. One immediately observes that this property is not *robust*. The introduction of a damping or a driving, as weak as they could be, converts the center into a focus, stable or unstable [cf. (2.9): eigenvalues $s = -\eta \pm i(1-\eta^2)^{1/2}$, pure imaginary only for $\eta = 0$]. In such a point of the space of control parameters, the system is said to be *structurally unstable*.

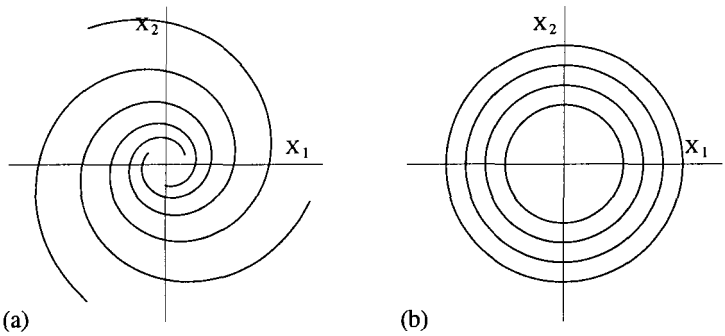

Fig. 2.6 Complex roots $s = \sigma \pm i\omega$. (a) Focus. (b) Center ($\sigma = 0$).

2.2.2.3 *Double roots*

The last case is when the characteristic equation has (real) double roots, which happens when $(a_{11} - a_{22})^2 + 4a_{12}a_{21} = 0$. Usually, the system has only one eigen-direction so that is cannot be cast into diagonal form by a linear variable change but only into what is called its *Jordan normal form*, here:

$$\dot{X}_1 = sX_1 + X_2\,, \tag{2.35}$$

$$\dot{X}_2 = sX_2\,. \tag{2.36}$$

Integrating (2.36) one gets $X_2(t) = X_2^{(0)} \exp(st)$, which is further inserted in (2.35) that now reads:

$$\dot{X}_1 - sX_1 = X_2^{(0)} \exp(st)\,, \tag{2.37}$$

which point out the *resonant* character of the right hand side since it evolves at the same rate as that defined by the left hand side. Equation (2.37) is easily integrated by the Lagrange method of *variation of the constant*: solving the homogeneous problem, one gets $X_1 = \tilde{X} \exp(st)$, where $\tilde{X}$ is an integration constant. Assuming that this "constant" is now a function of time, $\tilde{X}(t)$ and introducing this expression into (2.37) one gets

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{X} = X_2^{(0)},$$

which leads to

$$\tilde{X} = X_2^{(0)}\, t + X_1^{(0)}\,,$$

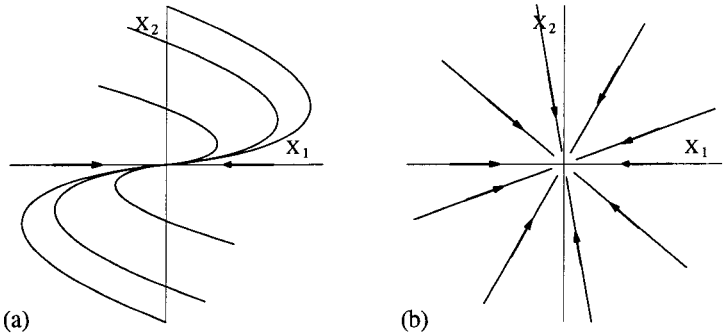

Fig. 2.7 Double real root. (a) Improper node. (b) Star (diagonalizable case).

and thus to

$$X_1(t) = \left(X_1^{(0)} + X_2^{(0)}\, t\right) \exp(st) \tag{2.38}$$

where the second, sub-dominant, term inside the parentheses is called a *secular term.*[5]

This fixed point is called an *improper node.* The corresponding phase portrait is displayed in Figure 2.7(a) which shows that it is indeed intermediate between a node and a focus.

When the eigenvalue is double but the system still diagonalizable, which occurs when the operator $\mathcal{L}$ is symmetric, all directions in the plane (X_1, X_2) are eigen-directions, which is another limit case of a node called a *star*, see Figure 2.7(b).

2.2.3 *Stability of a time-independent regime*

Let us come back to a d-dimensional system linearized around one of its fixed points $\mathbf{X}_f$. At the linear stage, the general solution of the perturbation problem can be expressed as a *superposition* of solutions corresponding to each eigenvalue s_j, with dominant exponential behavior $\exp(s_j t)$. These eigenvalues can be ordered by decreasing value of their real part, and while in the short term one may observe a complicated evolution of generic perturbations made of arbitrary superpositions of eigenmodes, due to linear

[5]The word originates from the Nineteenth Century studies of the period of planets by perturbation methods, when corrections to the relative position of the planets were found to show up on time scales of the order of hundreds of years.

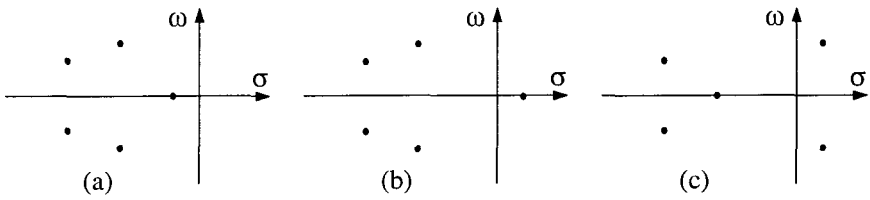

Fig. 2.8 Stability of the fixed point $\mathbf{X_f}$ of a hypothetical 5-dimensional system $\dot{\mathbf{X}} = \mathcal{F}(\mathbf{X})$: The operator $\mathcal{L}$ obtained by linearization of $\mathcal{F}$ around $\mathbf{X_f}$ has 5 real or complex eigenvalues and $\mathbf{X_f}$ can be linearly stable (a), unstable against a stationary mode (b), unstable against an oscillatory mode (c). Instability against several modes is also possible but one is usually interested in the first instability of a state, *i.e.* the transition from (a) to (b) or (c), which can be achieved by varying just one control parameter in a sufficiently limited range.

interferences between them (cf. §A.3, p. 343), in the long term only the contribution corresponding to the eigenvalue with largest real part survives.

The considered base state associated to the fixed point is thus linearly stable if all the eigenvalues have negative real parts, Figure 2.8(a), and linearly unstable if at least one of the eigenvalues has positive real part, Figure 2.8(b,c). When the unstable root is real, the corresponding unstable mode is said to be *stationary*, Figure 2.8(b), and when it is complex, one speaks of *oscillatory* mode, Figure 2.8(c).

Here we have made use of the concept of *asymptotic stability*, the amplitude of the perturbation tending to zero as t increases. When the eigenvalue is zero or has zero real part, the corresponding mode is *marginal* or *neutral*. Linear theory then does not allow us to draw any conclusion about the stability of the base state and nonlinearities have to be taken into account. Methods with a more *global* flavor have to be used, in the spirit of the *energy method* introduced in Exercise 2.5.2. In the long time limit, perturbations evolve more slowly than exponentially, generally as some power of time. They may relax, in which case the base state is still asymptotically stable, or grow, in which case it is unstable. This type of situation can be dealt with using the concept of *orbital stability* which is weaker than that of asymptotic stability since it only requires that the perturbed state can permanently depart, but in a controlled way, from the base state, thus remaining in its vicinity. A typical example is that of an elliptic point in a mechanical system, locally equivalent to a harmonic oscillator: trajectories circle around the fixed point without approaching it as would be the case is the eigenvalue had negative real part and the fixed point be asymptotically

stable. So, an elliptic point with purely imaginary eigenvalue, marginal at a linear level, can be orbitally stable or unstable at the nonlinear stage.

The persistent occurrence of purely imaginary eigenvalues often results from symmetry conditions and especially the invariance of the dynamics upon time reversal, which is characteristic of mechanics. One can indeed observe that reversibility implies an exchange '$s \leftrightarrow -s$' upon the change '$t \mapsto -t$', and thus either to complex conjugation in case of purely imaginary roots or an exchange 'stable $\leftrightarrow$ unstable' within a pair of real eigenvalues.[6]

Otherwise the presence of neutral modes must be considered as accidental: since the system usually depends on control parameters, this circumstance only occurs at specific locations in the parameter space. At such points the system is said to be *structurally unstable* since a slight modification of its definition can turn the considered fixed point from stable to unstable. It is ready for a *bifurcation* associated with a qualitative change of its phase portrait.

The breadth of possibilities increases with the dimension of the system but the terminology introduced for two-dimensional systems can be extended straightforwardly. For example, one still speaks of a node when all the eigenvalues are real and have the same sign. In the same way, a fixed point can be called a saddle-focus if it has a pair of complex conjugate eigenvalues with real parts of one sign and its other eigenvalues real with the opposite sign. The case of a three-dimensional system is considered in Exercise 2.5.3.

2.3 Two-dimensional Nonlinear Systems

In this section we study the dynamics of autonomous nonlinear systems that evolve in a two-dimensional phase space. In a first instance, we take an essentially qualitative view point and use what precedes to draw phase portraits. Then we attack the problem of the explicit quantitative determination of the period of oscillators, presenting several methods in a computational perspective that is however indispensable to the classical nonlinear culture.

[6]The existence of quadruples $\pm\sigma \pm i\omega$ is not ruled out in the general case, but the symmetry does not change the fact that the system is unstable since, within a quadruple, there is always a pair of roots with positive real parts, see Exercise 2.5.4.

2.3.1 *Two examples of oscillators*

2.3.1.1 *The rigid pendulum*

The first classical example of nonlinear oscillator that we consider is the simple *rigid pendulum* already introduced in Section 1.1, see Figure 1.1 (right) on p. 3. The potential energy from which the external gravity force can be derived, $\mathcal{V}(\theta) = mg(1 - cos\theta)$, is illustrated in Figure 2.9 (left). After a convenient choice of the time unit (the angular frequency of small oscillations being given by $\omega_0^2 = g/l$, one performs the change $t \mapsto t/\omega_0$), the evolution equation can be written as a two-dimensional first order system:

$$\dot{\theta} = \varphi\,, \qquad \dot{\varphi} = -\sin\theta\,. \tag{2.39}$$

The corresponding phase space $\{\theta;\varphi\}$ is a cylinder $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ is the unit circle (the one-dimensional sphere, hence the letter '$\mathbb{S}$') parameterized by the cyclic variable θ, *i.e.* $\theta + 2\pi \equiv \theta$ (the interval $[-\pi, +\pi[$ is made periodic by identifying $-\pi$ with $+\pi$). On the other hand $\varphi \in \mathbb{R}$ can take its values from $-\infty$ to $+\infty$. This cylinder being open along the generatrix $\theta = \pi$, one obtains a reduced representation of the phase space as an band of width 2π in the θ direction and infinite length in the φ direction, with identification of the sides at $\theta = \pm\pi$.

As already noticed, it is here the geometrical constraint fixing the distance of the mass to the rotation axis that generates the nonlinearity. The global character of this constraint is reflected in the topological structure

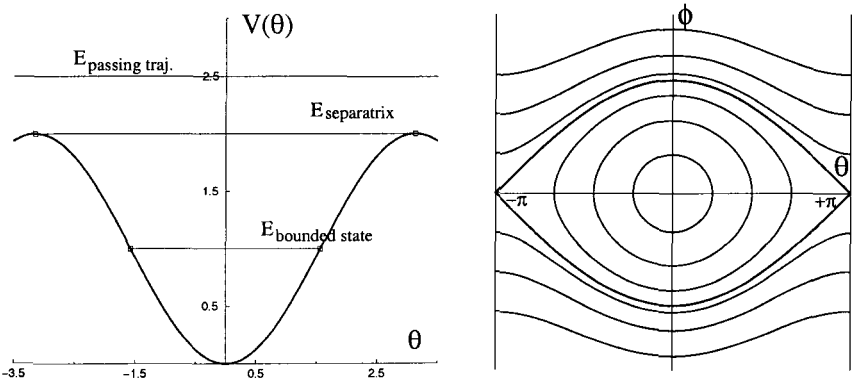

Fig. 2.9 Left: Potential energy of the rigid pendulum. Right: Phase portrait in reduced representation with indication of bounded states, the separatrix, and passing orbits.

of the phase space. Nonlinearities reveal themselves when large amplitude motions are considered, whereas linearization remains legitimate close to the origin. Classically, the distinction is made between *passing trajectories* at sufficiently high energy and *bounded trajectories* at low energy, see Figure 2.9 (right).

Fixed points of the system are given by $\varphi = 0$ and $\theta = k\pi$, $k \in \mathbb{N}$. The study of the dynamics close to these points is a straightforward application of the linear approach developed in previous sections. So, it appears that point $(\theta = 0, \varphi = 0)$ is a center with eigenvalues $\pm i$ and corresponds to the small-oscillation regime. By contrast, the point $(\theta = \pm\pi, \varphi = 0)$ is a saddle with eigenvalues ± 1. Specific trajectories called *separatrices* link one of these points to the other, thus separating the domain of bounded orbits from that of passing orbits.

Breaking the Hamiltonian character of the dynamics, we now consider the effect of viscous friction. The second equation of (2.39) is therefore completed by a term proportional to the angular velocity $\varphi = \dot{\theta}$:

$$\dot{\varphi} = -\eta\varphi - \sin\theta\,.$$

This case is further illustrated in Figure 2.10 using an extended representation that no longer takes advantage of the limitation to the periodized interval $[-\pi, \pi[$. As a matter of fact, this representation makes it easier to understand how the pendulum returns to its rest position after a certain number of complete turns around the axis, a number that depends on the initial energy.

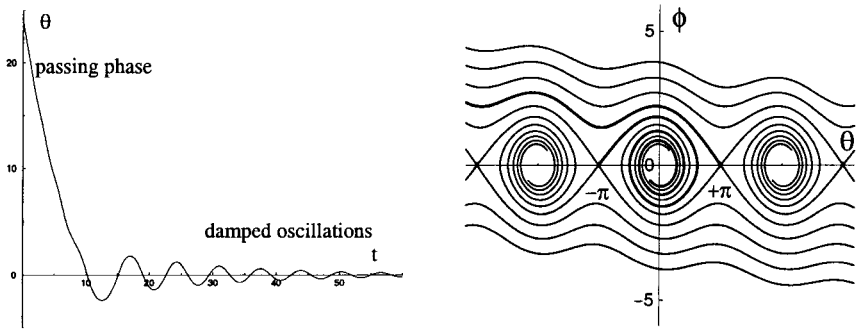

Fig. 2.10 Damped pendulum. Left: Decay of a high energy trajectory. Right: Phase portrait in extended representation.

It can be observed that the centers persist but are converted into stable spiral points. On the other hand unstable points at $\theta = \pm(2k+1)\pi$ do not change their nature but trajectories that emerge from them along their unstable direction miss the next point and spiral towards the foci, whereas trajectories that arrive to them along their stable direction need a slightly larger energy than in the conservative case, as expected. Other similar systems are proposed to the study in Section 2.5.

2.3.1.2 *Van der Pol oscillator*

Examples of oscillatory processes are numerous in fields other than mechanics, from electronics (the case we consider now) to ecology (prey-predator systems, Exercise 2.5.12) or economics (expansion-recession cycles, see [Anderson *et al.* (1988)]).

In the RLC circuit described in Figure 2.11, the Joule effect in resistor R, accounting for dissipation, is described by a standard ohmic voltage-intensity relation, $U_A - U_B = RI$. The other elements, the coil with inductance L and the capacitor with capacity C introduce the equivalent of an inertia at the origin of oscillations by setting the intensity and the voltage out of phase: the charge of the capacitor is given by $Q = C(U_C - U_G)$, with $Q = \int I dt$ while, when submitted to a varying intensity, the coil responds by building a voltage difference $U_B - U_C = L\dot{I}$.

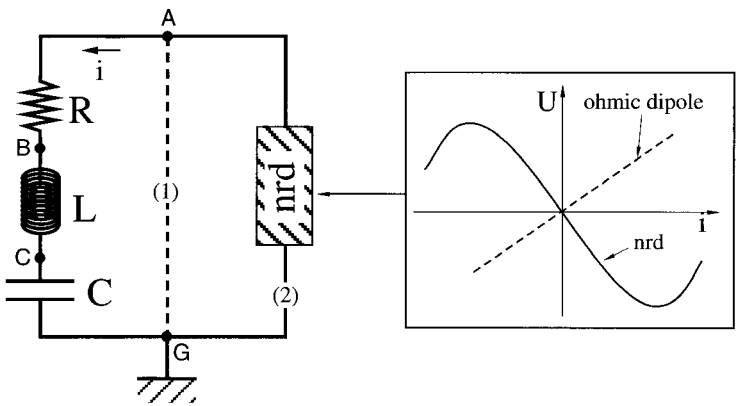

Fig. 2.11 Sketch of the RLC circuit modeling the van der Pol oscillator. Connection (1) is a simple short circuit. Connection (2) is through an active dipole with a voltage-intensity characteristic displaying a range of negative resistance (negative-resistance dipole = nrd, the gray line corresponds to an ordinary ohmic resistor).

For a series circuit we get:

$$U = U_A - U_G = \frac{Q}{C} + R\dot{Q} + L\ddot{Q}\,. \tag{2.40}$$

The capacitor discharge after closing the loop along (1), *i.e.* $U_A = U_G$, is then given by:

$$\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{LC}Q = 0\,.$$

This equation governs a damped linear oscillator $\propto \exp(-i\omega t)$ with a complex angular frequency ω solution to:

$$\omega^2 + i\gamma\omega - \omega_0^2 = 0\,,$$

where the resonance angular frequency ω_0 is given by $\omega_0^2 = 1/LC$ and the damping factor γ by $\gamma = R/L$.

The energy initially stored in the capacitor is dissipated in the resistor, at the origin of the damping. If one succeeds in injecting energy in the system so as to compensate the losses, one can obtain self-sustained oscillations. To achieve this aim, the circuit is closed on a negative-resistance dipole [loop (2)]. This "active element" is concretely implemented with an operational amplifier that draws its energy from an external electric supply maintaining the whole system in a permanent out-of-equilibrium state.

The voltage-intensity relation accounting for the active element is supposed to be ohmic but with a negative resistance coefficient. In practice, saturation effects come and limit the validity of the "anomalous" Ohm law so that we may take (cf. Fig. 2.11, right):

$$U_{G'} - U_{A'} = -R_0 I + bI^3 \qquad \text{with} \quad R_0 > 0 \quad \text{and} \quad b > 0\,.$$

Using $I = \dot{Q}$ as a variable rather than Q itself, the equation governing the circuit then reads:

$$\left[L\dot{I} + RI + (1/C)\int I\mathrm{d}t\right] + \left[-R_0 I + bI^3\right] = 0\,,$$

or, upon differentiation with respect to time:

$$L\ddot{I} + \left[(R - R_0) + 3bI^2\right]\dot{I} + I/C = 0\,. \tag{2.41}$$

As long as $R > R_0$ the coefficient of $\dot{I}$ is positive and dissipation plays a normal role: oscillations are damped. This is no longer the case when $R < R_0$: a small perturbation (*i.e.* such that $3bI^2$ is negligible when compared to $|R - R_0|$) is amplified and the oscillation develops. As soon as

the amplitude of the oscillation is large enough, the nonlinear dissipation term plays a normal role and stops the divergence. Decreasing R, we can therefore control the bifurcation from a time-independent steady state (oscillatory perturbations are damped) towards the regime of self-sustained oscillations.

Performing the changes $I \mapsto X$ and $t \mapsto t/\omega_0$ in equation (2.41) we get one of the forms of the van der Pol model

$$\ddot{X} - (r - gX^2)\dot{X} + X = 0\,, \tag{2.42}$$

where $r \propto R_0 - R$ is the control parameter and where $g > 0$ is a measure of the strength of nonlinearities that could have been suppressed by a rescaling of X, ending with $g = 1$. Figure 2.12 presents the results of the numerical integration of (2.42) for two different initial conditions in the quasi-harmonic regime (top line) or strongly anharmonic regime (middle line). In both cases one can observe that orbits spirals toward a closed curve called a *limit cycle*, either from the inside or from the outside depending on the initial condition. This special orbit is nearly elliptical in the first case and rather quadrangular in the second. To them correspond nearly sinusoidal or on the contrary highly anharmonic oscillations, which is also illustrated in the Fourier spectra that have a higher level of harmonics in the second case than in the first. The bottom graphs in Figure 2.12 illustrate the deformation of the limit cycles as the parameter r is varied from 0.1 to 2.0.

The van der Pol limit cycle is an example of attractor that is not trivially reduced to a single point (as was the case for the damped pendulum). It should be noted that, by contrast with ideal mechanical oscillators that do not have attractors and for which the amplitude of the motion is fixed by the total energy (kinetic+potential) in the initial condition, here it is the competition between the destabilization by energy injection and the dissipation that fully determines the characteristics (amplitude and period) of the regime achieved beyond the instability threshold.

In Chapter 4 we will come back to the description of this bifurcation when r changes from negative to positive values. For the moment let us determine in an approximate way the amplitude of the cycle in the quasi-harmonic regime, close to the threshold.

Inserting $X \simeq X_{\rm m}\cos(t)$ in equation (2.42) we get

$$-X_{\rm m}\cos(t) + \left(r - gX_{\rm m}^2\cos^2(t)\right) X_{\rm m}\sin(t) + X_{\rm m}\cos(t) = 0\,.$$

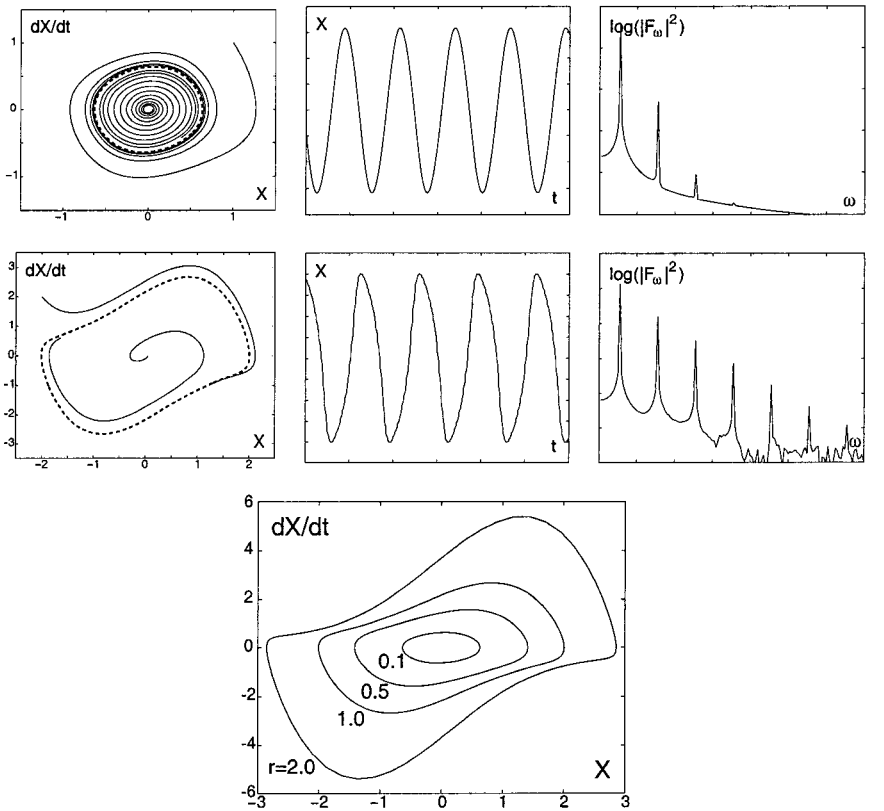

Fig. 2.12 Van der Pol oscillator (2.42) for $r = 0.1$ (quasi-harmonic, first line) and $r = 1.0$ (strongly anharmonic, second line), both with $g = 1$. Left: Phase portraits showing the convergence toward a limit cycle indicated by a dashed line. Middle: Corresponding time series of the intensity signal. Right: Corresponding Fourier spectra (lin–log plot of the modulus of the Fourier amplitudes F_ω squared). Bottom: Limit cycles for increasing values of control parameter r.

Let us restrict to a *first harmonic approximation*. It consists in demanding that the equation be identically fulfilled for terms in $\sin(t)$ and $\cos(t)$, without worrying about higher harmonics generated by the nonlinearities. It is easily observed that the compensation is automatic for the cosine terms. For the sine terms, using the classical formulas $\cos^2(t) = \frac{1}{2}(1+\cos(2t))$ and $\cos(2t)\sin(t) = \frac{1}{2}(\sin(3t) - \sin(t))$, neglecting the $\sin(3t)$ term, we get:

$$\left(r - \tfrac{1}{4}gX_{\mathrm{m}}^2\right)\sin(t) = 0\,,$$

which leads to:

$$r - \tfrac{1}{4} g X_{\mathrm{m}}^2 = 0 \qquad \text{that is} \qquad X_{\mathrm{m}} = 2\sqrt{r/g}\,. \tag{2.43}$$

This relation fixes the amplitude of the cycle and it can further be checked that the amplitude of term in $\sin(3t)$ generated by the nonlinearities is of order X_{m}^3, hence $\sim r^{3/2} \ll r^{1/2}$ for r small, thus justifying the expression 'quasi-harmonic', see Figure 2.12 (top-right).

Plotting X_{m} as a function of r, we get exactly the same bifurcation diagram as for A_* in Figure 1.4 (right), p. 14, but restricted to its upper branch, which makes sense since the change $t \mapsto t + \pi$ (legitimate since the system is autonomous) brings the branch $X_{\mathrm{m}} < 0$ on top of the other one.

Getting the result this way is rather crude. In order to improve the solution, one should try to fulfill the equation harmonic by harmonic (a special case of the so-called *Galerkin method*). This would lead to an infinite nonlinear algebraic system, the lowest order consistent truncation of which is precisely (2.43). Here, an additional implicit assumption has been that the nonlinearities did not change the angular frequency. This property turns out to be correct at lowest order in r for the van der Pol model but not necessarily in other cases, which opens the problem of the general determination of the period of nonlinear oscillators to be examined now.

2.3.2 *Amplitude and phase of nonlinear oscillators*

2.3.2.1 *The Duffing oscillator: period from a direct computation*

Let us come back to the special case of the harmonic oscillator and note that the linear relation between the restoring force and the elongation may not stay indefinitely valid. In general the microscopic characteristics of the elastic forces induce nonlinearities. If the spring is "hard" the force necessary to obtain a specific elongation grows faster than just being proportional to it and we can assume $F = -kX(1 + cX^2)$ with $c > 0$. (The opposite case of a "soft" spring with $c < 0$ is studied as part of Exercise 2.5.7.) In the absence of friction, choosing the time scale so that the angular frequency of the unperturbed oscillator is equal to 1, we get:

$$\ddot{X} + X + cX^3 = 0\,, \tag{2.44}$$

which called the *Duffing oscillator*.

As long as the amplitude is small, the nonlinear term is negligible and the harmonic approximation is satisfactory. When the amplitude increases,

the oscillator "feels" the effects of the nonlinearity and the motion becomes anharmonic. The period and the shape of the orbits change progressively. However, as long as the amplitude remains small enough, corrections to the linear solution may be obtained using *perturbation methods.*

Here the case is particularly simple. Let us multiply (2.44) by $\dot{\mathbf{X}}$ and rewrite the result as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}(\dot{X})^2 + \tfrac{1}{2}X^2 + \tfrac{1}{4}cX^4\right] = 0\,. \tag{2.45}$$

The quantity between the brackets is clearly the total energy, the sum of the kinetic energy and the potential energy $\mathcal{V}(X) = \frac{1}{2}X^2 + \frac{1}{4}cX^4$ from which the elastic restoring force derives. Integrating (2.45) we obtain

$$\tfrac{1}{2}\dot{X}^2 + \mathcal{V}(X) = E\,,$$

called a *first integral* of (2.44). The differential order of the problem has indeed decreased by one since we can rewrite this equation as

$$\dot{X} = \pm\sqrt{2(E - \mathcal{V}(X))}\,, \tag{2.46}$$

provided that the quantity under the root sign be non-negative. The condition $\mathcal{V}(X_\mathrm{t}) = E$ defines the *turning points* of the problem, which correspond to points with maximal elongation and zero velocity, whereas the region '$|X| > X_\mathrm{t}$' is forbidden[7] since it corresponds to a negative kinetic energy.

In the case of the nonlinear spring considered here, we get

$$\tfrac{1}{2}X_\mathrm{t}^2 + \tfrac{1}{4}cX_\mathrm{t}^4 = E\,,$$

which, for c small enough, yields:

$$X_\mathrm{t} \approx \sqrt{2E}\,\left(1 - \tfrac{1}{2}cE\right)\,,$$

so that the maximum elongation for a hard spring is reduced when compared to that of a harmonic oscillator with the same energy.

The system oscillates between its two turning points. The period can thus be computed by integration of (2.46) between them:

$$\frac{1}{2}T = \int_{-X_\mathrm{t}}^{X_\mathrm{t}} \frac{\mathrm{d}X}{\sqrt{2(E - \mathcal{V}(X))}}\,. \tag{2.47}$$

The value of this integral, analytically defined as a 'complete elliptic integral of the first kind' can be found in tables or numerically computed. Here it is

[7](in classical mechanics, not in quantum mechanics)

more interesting to find its expression at low energy when c small, and thus for $X_t \ll 1$. Performing the variable change $X = X_t \sin(\varphi)$ and using the parity of the quantity to be integrated, through an expansion truncated at first order we get:

$$T \simeq 4 \int_0^{\pi/2} \left[1 - \tfrac{1}{4} c X_t (1 + \sin^2 \varphi)\right] d\varphi \,,$$

hence

$$T \simeq 2\pi \left(1 - \tfrac{3}{4} cE\right) \quad \text{or} \quad \omega = \frac{2\pi}{T} \simeq 1 + \tfrac{3}{4} cE \,. \tag{2.48}$$

For a hard spring the period therefore decreases when the energy increases, which is easily understood by writing $X + cX^3 = X\left(1 + cX^2\right)$ and observing that $(1 + cX^2)$ plays the role of an effective elastic constant, the average value of which is always larger than that of the reference linear oscillator.

After this example of explicit calculation from (2.47), made possible by the Hamiltonian context (see also Exercise 2.5.10) let us examine three methods for obtaining the value of the period in cases where nonlinear and/or dissipative effects can be considered as perturbations to a harmonic oscillator with intensities scaled by some small parameter.

2.3.2.2 *Averaging method*

In this first method, it is assumed that the amplitude of the oscillations is modulated on a long time scale that allows one to determine an effective equation for the modulation by *averaging*. Its intuitive simplicity makes it a reasonable first choice but it cannot easily be improved beyond lowest order. As a result we re-obtain the approximate solution to the van der Pol problem previously derived but here with a bonus.

To explain the method let us come back to the weakly damped linear oscillator (2.9) rewritten here with $\varepsilon = 2\eta$:

$$\ddot{X} + X = -\varepsilon \dot{X} \,, \tag{2.49}$$

with initial conditions

$$X^{(0)} = 1 \qquad \text{and} \qquad \dot{X}^{(0)} = 0 \,.$$

The solution reads

$$X(t) = A^{(0)} \exp\left(-\tfrac{1}{2}\varepsilon t\right) \cos\left(\omega t - \varphi^{(0)}\right) \tag{2.50}$$

with $\omega^2 = 1 - \frac{1}{4}\varepsilon^2$, $A^{(0)} = 1/\omega$, $\varphi^{(0)} = \arctan(\varepsilon/2\omega)$. This expression can be written as $X(t) = A(t)\cos(t + \varphi(t))$ where $A(t) \propto \exp(-\varepsilon t/2)$ appears as the average of the amplitude over a pseudo-period, which varies slowly provided that $\varepsilon \ll 1$. On the other hand, the argument of the cosine, $\omega t - \varphi^{(0)}$, can be written as $t + \varphi(t)$ with $\varphi(t) = (\omega - 1)t - \varphi^{(0)}$, a quantity that is also slowly varying since $\omega - 1 \sim \frac{1}{8}\varepsilon^2$.

The problem is now to generalize these notions of average amplitude $A(t)$ and phase $\varphi(t)$, supposed to be sufficiently *slowly varying*, when the system presents itself as a second order differential equation close to that governing the ideal linear oscillator, *i.e.* when it can be written as:

$$\ddot{X} + X = -\varepsilon f(X, \dot{X}), \qquad \varepsilon \ll 1. \tag{2.51}$$

The solution is searched in the form:

$$X(t) = A(t)\cos(t + \varphi(t)), \tag{2.52}$$

where $A(t)$ and $\varphi(t)$ are two unknown functions of time. Differentiating this expression we get:

$$\dot{X} = -A\sin(t + \varphi) + \left[\dot{A}\cos(t + \varphi) - A\dot{\varphi}\sin(t + \varphi)\right]. \tag{2.53}$$

In the absence of modulation ($\dot{A} \equiv 0$, $\dot{\varphi} \equiv 0$), we would have:

$$\dot{X} = -A\sin(t + \varphi),$$

so that it appears natural to reduce the freedom introduced in replacing the original unknown X by the two unknowns A and φ by forcing the quantity between the brackets to cancel identically. The first equation linking A to φ is therefore:

$$\dot{A}\cos(t + \varphi) - A\dot{\varphi}\sin(t + \varphi) = 0. \tag{2.54}$$

Differentiating (2.52) once more and taking (2.54) into account we get

$$\ddot{X} = -A\cos(t + \varphi) - \left[\dot{A}\sin(t + \varphi) + A\dot{\varphi}\cos(t + \varphi)\right]$$

that we insert in (2.51) to obtain

$$\dot{A}\sin(t + \varphi) + A\dot{\varphi}\cos(t + \varphi) = \varepsilon g(A, \varphi), \tag{2.55}$$

where $g(A, \varphi) \equiv f(X, \dot{X}) = f(A\cos(t + \varphi), -A\sin(t + \varphi))$. In order to isolate $\dot{A}$ and $\dot{\varphi}$ we can combine (2.54) and (2.55) using the usual trigonometric relations: computing (2.54)$\times\cos(t + \varphi)$ + (2.55)$\times\sin(t + \varphi)$ and

$(2.55)\times\cos(t+\varphi) - (2.54)\times\sin(t+\varphi)$, we obtain:

$$\dot{A} = \varepsilon f\big(A\cos(t+\varphi), -A\sin(t+\varphi)\big)\,\sin(t+\varphi)\,, \tag{2.56}$$

$$A\dot{\varphi} = \varepsilon f\big(A\cos(t+\varphi), -A\sin(t+\varphi)\big)\,\cos(t+\varphi)\,. \tag{2.57}$$

Up to now, everything is exact and one can notice that (2.56, 2.57) would result from a change to cylindrical coordinates $X = A\cos(\theta)$, $Y = A\sin(\theta)$, $\theta = -t-\varphi$, in a phase plane where the unperturbed second-order equation $\ddot{X} + X = 0$ would be replaced by a system of two first order equations[8] $\dot{X} = Y$, $\dot{Y} = -X$.

The approximation comes in as soon as one assume that $A(t)$ and $\varphi(t)$ are slowly variable on the short time scale $T \simeq 2\pi$. If this is the case, we can integrate the above equations over an approximate period while considering A and φ as constants, which leads to

$$\dot{A} = \frac{\varepsilon}{2\pi}\int_0^{2\pi} f\big(A\cos(t), -A\sin(t)\big)\,\sin(t)\,\mathrm{d}t\,, \tag{2.58}$$

$$A\dot{\varphi} = \frac{\varepsilon}{2\pi}\int_0^{2\pi} f\big(A\cos(t), -A\sin(t)\big)\,\cos(t)\,\mathrm{d}t\,. \tag{2.59}$$

Let us apply these formulas to the van der Pol oscillator, a non-conservative system here taken in the form

$$\ddot{X} - \varepsilon(1-X^2)\dot{X} + X = 0\,. \tag{2.60}$$

This equation is slightly different from (2.42) where the scaling of variable X was more adapted to the problem of the bifurcation when r goes through zero. The scaling chosen here is such that the nonlinearity contributes to the solution when $X \sim \mathcal{O}(1)$ and is valid only for $\varepsilon > 0$. With $f = -(1-X^2)\dot{X}$, we get:

$$\dot{A} = \frac{\varepsilon}{2\pi}\int_0^{2\pi} \big(1 - A^2\cos^2(t)\big)\,A\sin^2(t)\,\mathrm{d}t = \tfrac{1}{2}\varepsilon A\left(1 - \tfrac{1}{4}A^2\right)\,,$$

$$A\dot{\varphi} = \frac{\varepsilon}{2\pi}\int_0^{2\pi} \big(1 - A^2\cos^2(t)\big)\,A\sin(t)\,\cos(t)\,\mathrm{d}t = 0\,.$$

The second equation shows that there is no correction to the angular frequency (at least at this order) which implicitly justifies the choice made when developing the first harmonic approximation on p. 49. By contrast,

[8] The minus signs in the definition of θ arise from the fact that the so-defined two-dimensional vector field generates trajectories that rotate clockwise, opposite to the trigonometric convention.

the first equation governing the amplitude (the bonus alluded at the beginning of the section) is non-trivial and may serve us to study the convergence towards the limit cycle corresponding to $A_* = 2$ which nothing but (2.43) in the scales turning (2.42) in the form (2.60).

2.3.2.3 *The Poincaré–Lindstedt method*

The second method considered here is called the *Poincaré–Lindstedt* expansion. The solution is now assumed to be periodic with some unknown period close to that of the reference oscillator, and both the period and the solution are searched through an expansion in powers of ε. For simplicity, it is explained in the linear case but can easily be applied when nonlinearities are present (Exercise 2.5.11).

Let us consider two harmonic oscillators with nearly equal angular frequencies, the first one is governed by $\ddot{Y} + Y = 0$, with angular frequency $\omega_Y = 1$, and the second one by $\ddot{X} + (1-\varepsilon)X = 0$, with angular frequency $\omega_X = (1-\varepsilon)^{1/2}$. Starting with identical initial conditions are $Y = X = 1$, $\dot{Y} = \dot{X} = 0$, they will progressively drift out of phase. Choosing the first oscillator as a reference, one can interpret this phase shift as the result of *secular terms* that appear already in the first order expansion of the solution for the second oscillator $X(t) = \cos(\omega_X t)$. From $\omega_X \approx 1 - \frac{1}{2}\varepsilon$, one gets $\cos(\omega_X t) = \cos(t)\cos\left(\frac{1}{2}\varepsilon t\right) - \sin(t)\sin\left(\frac{1}{2}\varepsilon t\right)$, so that for sufficiently short times such that $\frac{1}{2}\varepsilon t \ll 1$, with $\cos\left(\frac{1}{2}\varepsilon t\right) \sim 1$ and $\sin\left(\frac{1}{2}\varepsilon t\right) \sim \frac{1}{2}\varepsilon t$, one finds $\cos(\omega t) = \cos(t) + \frac{1}{2}\varepsilon t \sin(t)$, which points out the secular term correction explicitly (cf. p. 41). It is not difficult to obtain this solution directly from the equation by a perturbation expansion to which we now turn.

For the second oscillator, the problem reads:

$$\ddot{X} + X = \varepsilon X$$

and the solution is searched for as a power expansion in ε:

$$X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots \tag{2.61}$$

We are led to a series of simple linear problems:

$$(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} + 1)X_0 = 0\,, \tag{2.62}$$

$$(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} + 1)X_1 = X_0\,, \tag{2.63}$$

$$(\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} + 1)X_2 = X_1\,, \tag{2.64}$$

$$\dots = \dots$$

The solution to (2.62) with initial conditions $X = 1$, $\dot{X} = 0$ is nothing but $X_0 = \cos(t)$ so that the inhomogeneity in the problem for X_1 just contains a term that is resonant with the left hand side. A special solution to (2.63) is obtained by identification as $X_1 = \frac{1}{2} t \sin t$, which is indeed responsible for the secular growth of the phase shift at the considered order.

Any expansion obtained in this way has thus a limited validity in time. One says that it is *non-uniform*. The aim of the rest of this section and the next one is to obtain a *uniformly valid* expansion. Here the origin of the discrepancy is obvious: we must correct the clock and pass from the original time t to a new one τ for which secular terms would be absent. This is precisely the essence of the *Poincaré–Lindstedt method*.

Let us set $\tau = \omega t$ and look for the relation between τ and t in the form of an expansion of ω in powers of the small perturbation parameter ε:

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots \tag{2.65}$$

Denoting differentiation with respect to variable τ as $\frac{\mathrm{d}}{\mathrm{d}\tau}$, we get $\frac{\mathrm{d}}{\mathrm{d}t}X \equiv \omega\frac{\mathrm{d}}{\mathrm{d}\tau}X$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}X \boxminus \omega^2\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}X$, and thus

$$\left[(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots)^2\tfrac{\mathrm{d}^2}{\mathrm{d}\tau^2} + 1\right](X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots) = \varepsilon(X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots)$$

leading to a new series of linear problems

$$(\tfrac{d^2}{d\tau^2} + 1)X_0 = 0\,, \tag{2.66}$$

$$(\tfrac{d^2}{d\tau^2} + 1)X_1 = X_0 - 2\omega_1\tfrac{d^2}{d\tau^2}X_0\,, \tag{2.67}$$

$$(\tfrac{d^2}{d\tau^2} + 1)X_2 = X_1 - \left(\omega_1^2 + 2\omega_2\right)\tfrac{d^2}{d\tau^2}X_0 - 2\omega_1\tfrac{d^2}{d\tau^2}X_1\,, \tag{2.68}$$

$$\dots = \dots$$

The structure of (2.66, 2.67, ...) is similar to that of (2.62, 2.63, ...), except that inhomogeneities on the right hand side now contain free parameters that can be fixed so as to "kill" all the resonant terms that generate secular terms at the origin of the time non-uniformity in the initial expansion. This operation is an application of the *Fredholm alternative* stipulating that, when the kernel of a linear operator $\mathcal{L}$ is non-trivial, the problem $\mathcal{L}\mathbf{X} = \mathbf{F}$ has solutions only if the right hand side $\mathbf{F}$ is orthogonal to the kernel of the adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$ (see appendix A, §A.3.2).

Here the unperturbed problem (2.66) is self-adjoint. Its kernel is generated by the trigonometric lines $\cos(\tau)$ and $\sin(\tau)$. Inserting $X_0 = \cos(\tau)$ in

(2.67) we get on the right hand side $F_1 = (1 + 2\omega_1)\cos(\tau)$. The Fredholm alternative imposes us to cancel the coefficient of $\cos(\tau)$, *i.e.* $\omega_1 = -\frac{1}{2}$. At second order we gets $F_2 = (\omega_1^2 + 2\omega_2)\cos(\tau)$ that yields $\omega_2 = -\frac{1}{2}\omega_1^2 = -\frac{1}{8}$, and so on. In this way, by reconstructing ω from its expansion we obtain $\omega = 1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2$, which is the beginning of the Taylor expansion of $\omega = (1-\varepsilon)^{1/2}$, as expected from the direct calculation. In Exercise 2.5.11 the method is applied to the Duffing oscillator as a typical example of a nonlinear system.

2.3.2.4 *The method of multiple scales*

The Poincaré–Lindstedt method does not allow for amplitude and phase modulations that were essential in the averaging method. We are thus led to the last and most general approach called the *method of multiple scales* that lifts the restrictions of both previous methods by introducing a hierarchy of time scales.

The natural relevance of the Poincaré–Lindstedt method is to Hamiltonian systems with a single degree of freedom in the mechanical sense, *i.e.* two-dimensional dynamical systems for which energy conservation implies the periodicity of bounded states, a key feature of the direct calculation above, p. 51. Difficulties appear for non-conservative systems since the occurrence of strict periodicity is then a much less trivial matter. In order to understand how to escape this problem, let us consider again the damped oscillator.

Solution (2.50) $\propto \mathcal{R}e\left\{\exp[(-\frac{1}{2}\varepsilon + i\omega)]\right\}$ can of course be written as $\propto \mathcal{R}e\left\{\exp[i(\omega + i\frac{1}{2}\varepsilon)t]\right\}$, where $\omega + i\frac{1}{2}\varepsilon$, though a complex quantity, can be understood as the angular frequency of some oscillation. In much the same way as turning to amplitude and phase in the averaging method comes to a change for cylindrical coordinates, the natural extension of the Poincaré–Lindstedt method thus suggests to change from a Cartesian parameterization of the phase space to complex one. As a matter of fact, using (2.65) and applying the previous procedure to (2.49) without assuming that ω is a real quantity, one gets at first order

$$\left(\tfrac{d^2}{d\tau^2} + 1\right)X_1 = -2\omega_1 \tfrac{d^2}{d\tau^2}X_0 - \tfrac{\mathrm{d}}{\mathrm{d}\tau}X_0\,,$$

instead of (2.67). Injecting the complex solution $X_0 = A\exp(it)$ in this equation, one finds the compatibility condition $2\omega_1 - i = 0$, *i.e.* $\omega = 1 + \omega_1\varepsilon = 1 + \frac{1}{2}i\varepsilon$, which correctly accounts for the expected damping at this order.

However, already in this simple problem, we may note the existence of two different time scales: the damping time of order $1/\varepsilon$, and another time scale, the inverse of the frequency shift, of order $1/\varepsilon^2 \gg 1/\varepsilon$ when $\varepsilon \ll 1$. This suggests to lay the most general method on a hierarchy of time scales, suitable to correct the lack of synchronization between a reference linear oscillator and the nonlinear system at hand. The aim of this strategy is to make the approximation uniformly valid in time, much like the introduction of leap years helps us to adjust the calendar according to a complicated algorithm that is more and more complicated as longer and longer periods of time are considered.

Let us look for the solution of the problem in the form

$$X(t) = X(t_0, t_1, t_2, \dots) \quad \text{with} \quad t_0 = t\,, \quad t_1 = \varepsilon t\,, \quad t_2 = \varepsilon^2 t\,, \dots$$

For ε small enough, the time scale measured by t_1 is indeed slow with respect to that by t_0 since when t_0 varies by a quantity $\mathcal{O}(1)$, the arguments of X in t_1, t_2,... vary by $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon^2)$,...

The differentiation with respect to time is then given by

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} &= \tfrac{\mathrm{d}}{\mathrm{d}t} t_0 \partial_{t_0} + \tfrac{\mathrm{d}}{\mathrm{d}t} t_1 \partial_{t_1} + \tfrac{\mathrm{d}}{\mathrm{d}t} t_2 \partial_{t_2} + \dots \\ &= \partial_{t_0} + \varepsilon \partial_{t_1} + \varepsilon^2 \partial_{t_2} + \dots\,, \end{aligned}$$

since $\mathrm{d}t_0/\mathrm{d}t \equiv 1$, $\mathrm{d}t_1/\mathrm{d}t \equiv \varepsilon$,.... In the same way the second derivative reads:

$$\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} = \partial_{t_0^2} + 2\varepsilon \partial_{t_0} \partial_{t_1} + \varepsilon^2 \left(\partial_{t_1^2} + 2 \partial_{t_0} \partial_{t_2} \right) + \dots\,.$$

The solution is taken in the form

$$X(t) = X_0(t_0, t_1, t_2, \dots) + \varepsilon X_1(t_0, t_1, t_2, \dots) + \dots\,,$$

and we have to insert these expansions in the motion equation and isolate the different orders in ε. Here we consider the specific cases of the Duffing and van der Pol oscillators as two complementary illustrations of the method.

- The Duffing Oscillator:

We have:

$$\ddot{X} + X = -\varepsilon X^3\,,$$

so that at order ε^0 we find

$$(\partial_{t_0^2} + 1) X_0 = 0\,,$$

the solution of which reads:

$$X_0(t_0, t_1, t_2, \dots) = A_0(t_1, t_2, \dots) \cos\big(t_0 + \varphi_0(t_1, t_2, \dots)\big) .$$

At order ε^1 we obtain:

$$(\partial_{t_0^2} + 1) X_1 = -X_0^3 - 2\partial_{t_0}\partial_{t_1} X_0 .$$

The strategy is to find conditions on A_0 and φ_0 such that the r.h.s., which expands as:

$$\begin{aligned} &-X_0^3 - 2\partial_{t_0}\partial_{t_1} X_0 \\ &\quad = -A_0^3 \cos^3(t_0 + \varphi_0) + 2(\partial_{t_1} A_0) \sin(t_0 + \varphi_0) \\ &\qquad\quad +2A_0(\partial_{t_1}\varphi_0) \cos(t_0 + \varphi_0) \\ &\quad = \big(-\tfrac{3}{4} A_0^3 + 2A_0(\partial_{t_1}\varphi_0)\big) \cos(t_0 + \varphi_0) + 2(\partial_{t_1} A_0) \sin(t_0 + \varphi_0) \\ &\qquad\quad -\tfrac{1}{4} A_0^3 \cos(3(t_0 + \varphi_0)) , \end{aligned}$$

contains no term in resonance with the l.h.s. Clearly only the $\cos(3(t_0+\varphi_0))$ term on the last line is naturally non-resonant, while the terms on the previous line are. We thus get two conditions, one from cosine, the other from the sine (the kernel is two-dimensional):

$$\begin{aligned} -\tfrac{3}{4} A_0^3 + 2A_0\partial_{t_1}\varphi_0 &= 0 , \\ \partial_{t_1} A_0 &= 0 , \end{aligned}$$

that can both be integrated. The second condition means that A_0 does not depend on t_1 and thus, at best, on t_2, t_3,..., *i.e.* $A_0 = A_0(t_2, \dots)$. The first equation in turn gives the lowest order correction to the phase:

$$\varphi_0 = \tfrac{3}{8} A_0^2 t_1 + \bar{\varphi}_0(t_2, \dots) .$$

The non-resonant part can then be computed, if one wants to stop at this order, the full solution can be reconstructed by setting $t_1 = \varepsilon t$ in the result (compare with the output of the Poincaré–Lindstedt method used in Exercise 2.5.11).

- The van der Pol oscillator

  The first steps of the computation do not change. At order ε^1 we get:

$$(\partial_{t_0^2} + 1) X_1 = \big(1 - X_0^2\big)\partial_{t_0} X_0 - 2\partial_{t_0}\partial_{t_1} X_0 ,$$

the right hand side of which expands as

$$\begin{aligned}&\big(1 - A_0^2 \cos^2(t_0 + \varphi_0)\big)\big(-A_0 \sin(t_0 + \varphi_0)\big) + 2(\partial_{t_1} A_0) \sin(t_0 + \varphi_0) \\ &\qquad + 2A_0(\partial_{t_1}\varphi_0)\cos(t_0 + \varphi_0) \\ &= \big(-A_0\big(1 - \tfrac{1}{4}A_0^2\big) + 2\partial_{t_1} A_0\big)\sin(t_0 + \varphi_0) + 2A_0(\partial_{t_1}\varphi_0)\cos(t_0 + \varphi_0) \\ &\qquad + \tfrac{1}{4}A_0^3 \sin\big(3(t_0 + \varphi_0)\big)\,.\end{aligned}$$

The conditions that annihilate the resonant terms now read:

$$\begin{aligned}-A_0\big(1 - A_0^2/4\big) + 2\partial_{t_1} A_0 &= 0\,, \\ 2A_0\partial_{t_1}\varphi_0 &= 0\,.\end{aligned}$$

The second condition, $\partial_{t_1}\varphi_0 = 0$, hence $\varphi_0 = \bar{\varphi}_0(t_2, \dots)$, shows as before that there is no correction to the angular frequency at this order. On the other hand, the first equation is seen to govern the evolution of A_0, like in the averaging method, except for the presence of the factor ε that is incorporated in the definition of t_1 and would reappear if we were to come back to the independent variable t.

The method of multiple scales can be pursued at higher order in a completely systematic way. The price to be paid is increasingly heavy computations that are greatly eased by the use of formal algebra softwares such as MAPLE or MATHEMATICA. It can be adapted to treat the problem of periodically forced nonlinear oscillators, consult *e.g.* [Nayfeh and Mook (1979)] for further reference.

2.4 What Next?

The notion of determinism implemented in time-continuous dynamical systems considered up to now implies that a unique trajectory goes through any regular point in phase space (trajectories do not intersect since a given state cannot have several pasts and futures). The consequence is of utmost importance for two-dimensional systems which experience a strong topological constraint, so that their behavior remain "simple" in a sense to be reexamined in Chapter 4. In higher dimensions, by contrast, trajectories have enough space to wind one around each other, which is ultimately at the origin of "complicated" behavior. Before considering this problem, we examine in the next chapter how high dimensional systems, especially continuous media, may behave as effective low dimensional systems so as to render the tools that we have begun to introduce of great practical use.

2.5 Exercises

2.5.1 *Evolution of volumes in phase space*

Consider a continuously differentiable ($\mathcal{C}^1$) two-dimensional map

$$Y_1 = \mathcal{G}_1(X_1, X_2) ,$$
$$Y_2 = \mathcal{G}_2(X_1, X_2) .$$

1) Determine the transform of a phase space element $[X_1, X_1 + \delta X_1] \times [X_2, X_2 + \delta X_2]$ under map $\mathcal{G}$ and compute its surface at lowest significant order [Hint: Figure 2.13].

2) Assuming that $\mathcal{G}$ is the time-τ map of some continuous-time system

$$\dot{X}_1 = \mathcal{F}_1(X_1, X_2) ,$$
$$\dot{X}_2 = \mathcal{F}_2(X_1, X_2) ,$$

over an infinitesimal time interval $\tau = \delta t$, compute $\mathcal{G}$ from $\mathcal{F}$ and infer that the variation of phase space volumes is locally given by the divergence of the vector field $\mathcal{F}$, formula (2.11).

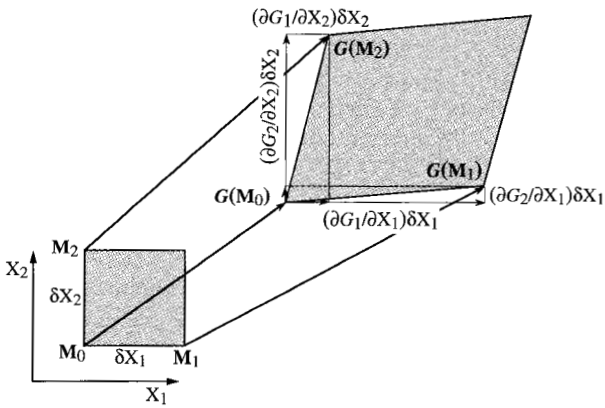

Fig. 2.13 Transformed domain $[X_1, X_1 + \delta X_1] \times [X_2, X_2 + \delta X_2]$ under map $\mathcal{G}$. Exercise 2.5.1, first part.

2.5.2 *The energy method and its application*

The energy method is an example of *global* method for studying stability problems without reference to specific perturbations, *e.g.* infinitesimal ones.

1) As a preliminary, consider the linear system $\dot{V} = -\mu V$ and check that the "kinetic energy" $E = \frac{1}{2}V^2$ is governed by:

$$\dot{E} = -\mu V^2 \leq 0\,,$$

which implies a monotonic return to equilibrium, illustrating the concept of *asymptotic* stability.

2) Consider next the oscillator defined as

$$\dot{X} = Y + X\sqrt{X^2+Y^2}\,, \qquad \dot{Y} = -X + Y\sqrt{X^2+Y^2}\,,$$

where the nonlinear terms come and modify a marginally stable dynamics in the neighborhood of the origin (a center) and for which the weaker notion of *orbital* stability has been introduced.

2a) Define $E = \frac{1}{2}(X^2+Y^2)$, compute $\frac{\mathrm{d}}{\mathrm{d}t}E$ and conclude that the trivial solution is orbitally unstable. [Answer: $\frac{\mathrm{d}}{\mathrm{d}t}E = (X^2+Y^2)^{3/2} > 0$ for any $(X,Y) \neq (0,0)$).]

2b) Determine the evolution of $Z = \sqrt{2E}$ (distance to the origin) starting from some initial condition $Z_0 > 0$ at $t = 0$ and show that that Z diverges at $t_* = 1/Z_0$, *i.e.* the corresponding trajectory spirals away to infinity in a finite time.

3) We now search to extend the concept of energy used up to now, in such a way that the study of its variations allows one to decide about stability or instability of a given state, conveniently taken as the origin of coordinates:

• *Definition*: Let $\mathcal{G}(\mathbf{X})$ be a function of point $\mathbf{X}$ in phase space $\mathbb{X}$, taking its values in $\mathbb{R}^+$, *definite positive*, *i.e.* such that $\mathcal{G}(\mathbf{0}) = 0$ and $\mathcal{G}(\mathbf{X} \neq \mathbf{0}) > 0$ (in practice a definite positive quadratic form). A sufficient condition of asymptotic stability, is that the amplitude of the perturbation, as measured by $\mathcal{G}$ decreases in the course of time and tends to zero as t goes to infinity. This will be the case if $\dot{\mathcal{G}} = \sum_j \partial_{X_j}\mathcal{G}\,\dot{X}_j = \sum_j \partial_{X_j}\mathcal{G}\,\mathcal{F}_j$ is *negative definite* as a function of $\mathbf{X} \in \mathbb{X}$ taking its values in $\mathbb{R}$, such that $\dot{\mathcal{G}}(\mathbf{0}) = 0$ and $\dot{\mathcal{G}}(\mathbf{X} \neq \mathbf{0}) < 0$. Such a $\mathcal{G}$ is called a *Lyapunov function*.[9]

[9]This definition can be extended to the case of continuous media described in terms of fields (velocity, temperature, concentration,...), in which case the Lyapunov function becomes a functional of the fields.

• *Application:* Consider the system defined as:

$$\dot{X} = XY - X^3\,,$$
$$\dot{Y} = -Y - 2X^2\,.$$

Show that the origin is the unique fixed point and further that it is marginally stable. Since no conclusion can be drawn at the linear stage, by generalizing the energy E to a function in the form:

$$\mathcal{G}(X,Y) = \tfrac{1}{2}\left(\alpha X^2 + \beta Y^2\right),$$

determine the conditions under which $\mathcal{G}$ is a Lyapunov function, thus proving the asymptotic stability of this state.

2.5.3 *Linear stability and bifurcations in dimension three*

As a useful preliminary to Exercise 3.3.3 in Chapter 3, consider a real, three-dimensional, linear dynamical system in the form

$$\dot{X}_i = \sum_{j=1}^{3} a_{ij} X_j\,, \quad i = 1,2,3\,.$$

1) Recall the relations between the coefficients and the roots of its characteristic equation written as $s^3 + as^2 + bs + c = 0$. Further identify all the possible non-degenerate or degenerate cases (real or complex nature of the roots, their sign or the sign of their real part, simple and multiple roots). Write down the Jordan normal form of the operator corresponding to each situation (see Appendix A, §A.2 for a reminder) [Answer: see Fig. A.1(b)].

2) When the control parameters vary, the eigenvalues of the system move in the complex plane. Determine the remarkable relations fulfilled by the coefficients of the characteristic equation in the marginal cases and sketch the spectrum of the operator in the complex plane before, at, and after a stationary bifurcation ($s = 0$) or an oscillatory bifurcation ($\mathcal{R}e(s) = 0$). [Answer: see Fig. 2.14.]

2.5.4 *Coupled linear oscillators*

Consider the following system of two coupled oscillators

$$\ddot{X} + X = Y\,,$$
$$\ddot{Y} + \Omega^2 Y = -cX\,.$$

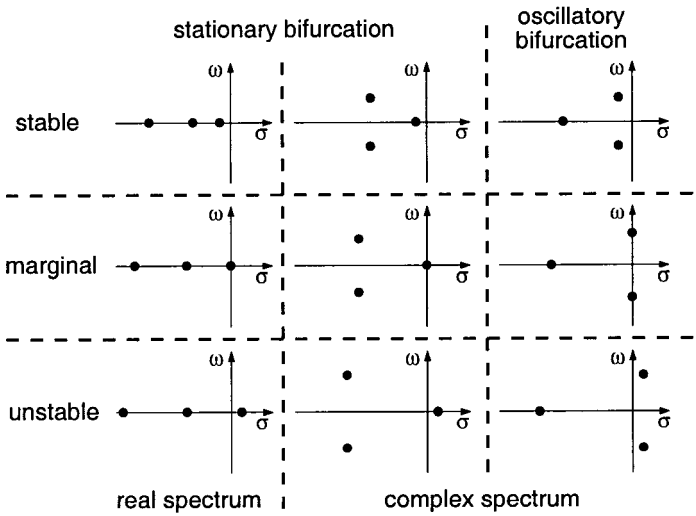

Fig. 2.14 Typical bifurcations in the three-dimensional case, Exercise 2.5.3.

Find the equation governing the angular frequencies of the eigenmodes taken as $(X, Y) = (X_0, Y_0)\exp(i\omega t)$, observe that the solutions appear as pairs of opposed eigenvalues. Determine the domain in the plane of parameters Ω^2 and c where the eigenmodes are stable. [Hint: remember that for a mechanical system, stability means orbital stability, with purely imaginary eigenvalues.]

2.5.5 *Logistic equation*

Reexamine the limited growth model (logistic equation) introduced in Chapter 1, Exercise 1.5.2 about population dynamics. Study the stability of its two fixed points and draw its phase portrait.

2.5.6 *Dynamical systems and solitons*

1) Consider a dynamical system with a single degree of freedom in the sense of analytical dynamics (a pair of conjugate variables) with a force deriving from the potential

$$V(X) = -\frac{X^2}{2} + \frac{X^3}{3}. \tag{2.69}$$

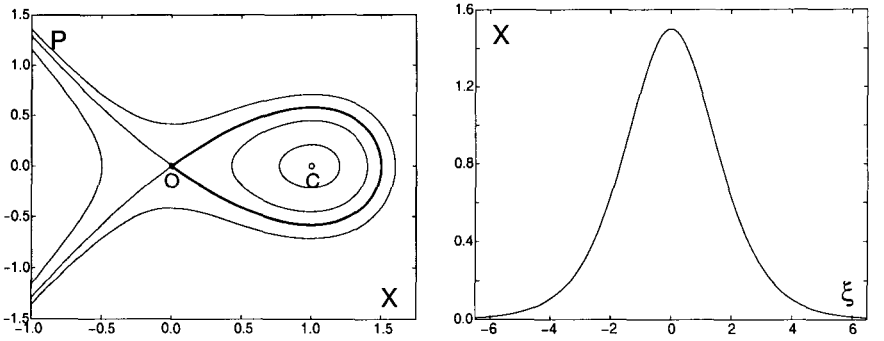

Fig. 2.15 Left: Phase portrait of the system with a force deriving from potential (2.69) Right: Profile of the soliton corresponding to the separatrix on the left part.

1a) Write down the second order differential equation governing X (mass $m = 1$) and then the first order system for X and $P = \dot{X}$. Find the fixed points, compute their eigenvalues and eigenvectors (if any). Sketch the phase portrait in the phase plane (X, P) [Answer: Figure 2.15, left].

1b) Draw the graph of the potential $V(X)$ and determine the energy corresponding to the separatrix, the special trajectory that limits the domain of orbits bounded around the minimum of the potential. Find the turning point and the equation accounting for this trajectory. Determine its solution by identification with

$$X(t) = X_0/\cosh^2(t/\tau)\,.$$

Why is this form "natural"?

1c) Sketch the phase portrait of the system when perturbed by the introduction of a weak viscous friction, taking the weakly damped pendulum as an example.

2) The Korteweg–de Vries equation reads:

$$\partial_t h + h\partial_x h + \beta\partial_{xxx} h = 0\,.$$

This is a non-dissipative equation with a nonlinearity of hydrodynamic type, identical to that of the Burgers equation considered in Chapter 1, Exercise 1.5.3. It is integrable and its solution can be expressed in terms of a superposition of interacting solitons. Here we are interested in a solution with a single soliton such that $h \to 0$ when $x \to \pm\infty$ moving without deformation at speed c, and thus only function of the combination $\xi = x - ct$.

2a) Determine the equation in ξ governing such a moving solution (make the substitutions $\partial_x \mapsto \mathrm{d}/\mathrm{d}\xi$ and $\partial_t \mapsto -c\,\mathrm{d}/\mathrm{d}\xi$). Integrate this equation and find the value of the integration constant that corresponds to the soliton solution.

2b) By appropriate variable changes put this equation in the form considered in the first part of the exercise and deduce the analytical expression of the soliton. Discuss the relation between the speed, the height, and the width of the soliton [Hint: Figure 2.15, right].

2.5.7 *Variants of the Duffing oscillator*

1) Consider the system

$$\ddot{X} + X(a + X^2) = F\,,$$

where a is a coefficient with unspecified sign for the moment, and F a quantity playing the role of an external force.

1a) Find the potential from which the dynamics can be derived in the sense of mechanics, *i.e.* $\mathcal{V}$ such that $m\,\ddot{X} = -\partial\mathcal{V}/\partial X$, and the time independent response to a force $0 < F \ll 1$ (observe that the roots of an equation in the form $x^3 - px + q = 0$ with $q \ll 1$ and $p \sim 1$ are approximately given by $px \sim q$ for x "small" and $x^3 \sim px$ for x "large."

1b) Discuss the dynamics in the phase plane $(X, Y \boxminus \dot{X})$ as a function of the sign of parameter a. Locate the fixed points and determine their stability properties (when solving the linearized problems, neglect the nonlinearities for the "small" solution and the constant term for the "large" one). Draw a few typical orbits. In which sense can one speak of the system as of an oscillator.

2) Consider now

$$\ddot{X} + X + cX^3 = 0 \qquad \text{with} \quad c < 0 \quad \text{(soft spring)}.$$

2a) Find the elastic potential from which the dynamics can be derived and discuss the behavior of the system in its phase plane as above.

2b) For which set of initial conditions is the system a physically well posed one (bounded orbits)? How should one correct the model in order to avoid the divergence of some trajectories? Interpret this limitation by considering the theory leading to the approximate expression (2.48) for the angular frequency of small oscillations when $c > 0$ and assuming that it is still valid for large amplitude orbits when $c < 0$.

2.5.8 *Carriage with a spring*

Consider the mechanical system built with an ideal spring of equilibrium length ℓ_0 fixed by one of its ends at point P as described in Figure 2.16(a). Its other end is attached at M to a small carriage with mass m sliding along a horizontal line forming the x axis. Point P is at a distance ℓ_1 from this axis. The intensity of the restoring force is proportional to the elongation $|\mathbf{F}| = k(\ell - \ell_0)$.

1) Determine the components of the forces exerted on the carriage at abscissa X and the corresponding equation of motion. Taking ℓ_0 as length unit, $\sqrt{m/k}$ as time unit, and setting $\ell_1 = \lambda\ell_0$, rewrite this equation in dimensionless form as a system of two equations for X and $Y = \dot{X}$.

2) Compute the potential $\mathcal{V}(X)$ from which the force $F(X)$ derives, *i.e.* $F(X) = -\partial\mathcal{V}/\partial X$. Find the fixed points as a function of λ, study their stability by linearizing the system.

3) Sketch the phase portraits in the two cases $\lambda < 1$ and $\lambda > 1$. What can be said from the trajectory issued from the vicinity of the origin along the unstable direction in the case $\lambda < 1$. Using the fact that the system is frictionless, determine the corresponding turning point quantitatively.

2.5.9 *Carriage sliding on a rotating hoop*

Consider now a carriage sliding along a rail in the form of a hoop with radius a, itself rotating around a vertical diameter (cf. Fig. 2.16b). The rotation period if $T = 2\pi/\omega$. The motion is governed by the following

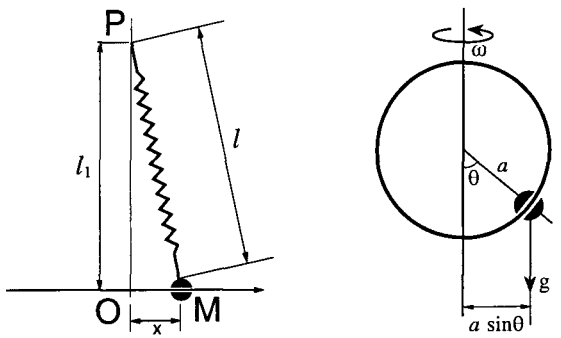

Fig. 2.16 Left: Spring and carriage sliding along a horizontal rail. Right: Carriage sliding along a rail in the form of a hoop further rotating around a vertical diameter.

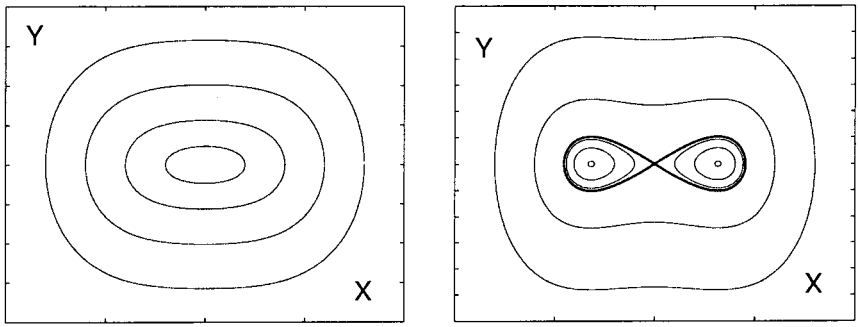

Fig. 2.17 Phase portrait of the hoop system for $\omega^2 = 0.8$ (left) and $\omega^2 = 1.25$ (right).

equation:

$$a\frac{d^2\theta}{dt^2} = -g\sin(\theta) + a\omega^2\cos(\theta)\sin(\theta)$$

where g is the gravitational acceleration.

1) Justify this equation by taking advantage of the elements of analytical mechanics recalled in §2.1.2. To this aim, compute first the gravitational potential energy and then the kinetic energy resulting from the superposition of the two independent rotation motions, around the axis and within the plane of the hoop. Set the equation in the form of a two-dimensional differential system for θ and $\varphi = \dot{\theta}$.

2) Find the fixed points and study their stability as a function of the angular speed ω. Sketch the phase portraits in the different cases [Answer: Figure 2.17].

3) Examine the effects of a slight viscous friction proportional to φ.

2.5.10 *Period of an oscillator in a quartic potential*

Consider a strongly anharmonic oscillator with a restoring force deriving from the potential

$$V(X) = V_0 + \tfrac{1}{4}X^4 .$$

From equation (2.47) giving the period of an oscillator, show without explicit calculation that one gets $T \propto E^{\alpha}$ with an exponent α to be determined (perform the change of variables that applies the interval between the turning points onto $[-1, 1]$).

2.5.11 *Averaging and Poincaré–Lindstedt methods*

The Duffing oscillator is taken in the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 1\right)X = -\varepsilon X^3 .$$

1) Extract the relation between the amplitude and the angular frequency of the oscillator by making use of equation (2.59) for the phase. What is the meaning of equation (2.58) for the amplitude.

2) Following the Poincaré–Lindstedt method (§2.3.2.3), derive the complete solution at first order in ε starting from the initial condition $X^{(0)} = A$, $\dot{X}^{(0)} = 0$ at $t = 0$. Using expansions (2.61–2.65), show first that at the relevant order the problem simply reads

$$\left(\frac{d^2}{d\tau^2} + 1\right)X_1 = 2\omega_1 X_0 - X_0^3 ,$$

and next that the elimination of resonant terms leads to the same result as the first order Taylor expansion of the solution obtained by a direct calculation (2.48). [Observe that, at lowest order, the averaging method and the Poincaré–Lindstedt method involve exactly the same computations.]

3) Find the complete solution at first order, once the Fredholm alternative is fulfilled.

2.5.12 *Prey–predator system*

Consider the system derived in Chapter 1, Exercise 1.5.2:

$$\dot{X} = \mathcal{F}_X(X,Y) = (\alpha' Y - \alpha_0)X , \tag{2.70}$$

$$\dot{Y} = \mathcal{F}_Y(X,Y) = (\beta - \gamma X)Y , \tag{2.71}$$

in the physical quadrant $X \geq 0$, $Y \geq 0$ (X and Y are population counts).

1) Draw a qualitative phase portrait by studying the vector field along lines defined by $\mathcal{F}_X(X,Y) = 0$ and $\mathcal{F}_Y(X,Y) = 0$ (vertical and horizontal isoclines respectively). Search and study the characteristics of the fixed points (roots of $\mathcal{F}_X(X,Y) = 0 = \mathcal{F}_Y(X,Y)$).

2) Show that the quantity

$$H(X,Y) = \alpha' Y - \alpha_0 \log(Y) + \gamma X - \beta \log(X)$$

is conserved along a trajectory and that H displays a minimum at the nontrivial fixed point.

3) Consider the successive intersections of a trajectory with line $Y = \alpha_0/\alpha'$

and conclude from the behavior of H that all trajectories are periodic orbits. Notice that they are not limit cycles and explain why the elliptic character of the non-trivial fixed point is a deficiency of the model. Try to correct it by considering the most general model of the form $\mathcal{F}_X = X\tilde{\mathcal{F}}_X(X,Y)$, $\mathcal{F}_Y = Y\tilde{\mathcal{F}}_Y(X,Y)$, at most quadratic in X and Y. In particular find the conditions to be fulfilled in order to reproduce the qualitative features characteristic of a prey–predator system.