Chapter 1

Introduction and Overview

In a linear world, the effects being always proportional to their causes, everything would be "simple" since the tools that would allow us to represent it as a *superposition* of elementary states are rather well mastered. Unfortunately the world is "complex." As a matter of fact, we often observe effects which saturate in spite of an increase of their causes, or which go in different and somehow unexpected ways. All this is ascribed to nonlinearities. Of course, if we succeed in determining the state of a nonlinear system, we immediately try to go back to a problem that we know how to handle by linearizing the dynamics around it and treating small departures from it by a perturbative approach, and next to reproduce this scheme as far as possible in order to reach other fully nonlinear states, eventually "far from" the initial one.

Most recently, our science of intrinsically nonlinear phenomena has made much progress (even though it rests in part on the use of linear tools) and investing this knowledge further in applications is of primary interest. The aim of these lecture notes is therefore to introduce the reader to this breakthrough by taking the problem of macroscopic instabilities as a pretext. In thermodynamic systems close to equilibrium the response to excitations is essentially linear, *i.e.* proportional to the (sufficiently small) amplitude of the applied stress with a proportionality factor called a *susceptibility*. As a consequence of linearity, the regime that develops is *unique*. When the applied stress increases, the system is driven farther from equilibrium and nonlinearities can no longer be neglected. This opens the possibility of *bifurcations* towards different regimes that can coexist and compete. As a result of such "catastrophes," losing reference to the initial state the dynamics becomes increasingly complicated, typically from "laminar" to "turbulent flow."

1.1 Dynamical Systems as a Context

In order to study the causes of the complexity induced by nonlinearities, we will concentrate our attention on problems defined in terms of the evolution of a set of *state variables* functions of a single independent variable called *time* (the general concept of dynamical system). Mechanical systems in the ordinary sense obviously belong to this class. Their archetype is the oscillator, and in its simplest linear expression, the *harmonic oscillator*[1] (Figure 1.1, left)

$$m\ddot{X} = F = -kX = 0\,, \tag{1.1}$$

where X measures the departure from the equilibrium length of the spring, m is the mass attached to its end, and F is the restoring force, here taken proportional to X *via* the stiffness coefficient k. As a result the oscillation period is independent of the amplitude: $T = T_0 = 2\pi/\omega_0$, with $\omega_0^2 = k/m$.

At this stage, nonlinearities can introduce themselves in two ways. First it is easy to imagine that the stiffness coefficient may not be a constant but rather a function of X itself. Assuming $k = k_0(1 + \alpha X^2)$ leads to what is known as the *Duffing oscillator*. Another possibility comes from the existence of mechanical constraints. The ideal rigid *pendulum*, a mass m at the end of a weightless rod of length l and revolving around a horizontal axis in the gravity field, is a good example (Figure 1.1, right). Parameterizing its position by the angle θ it forms with the vertical axis, one obtains:

$$J\ddot{\theta} + mgl\sin\theta = 0\,, \tag{1.2}$$

where $J = ml^2$ is the moment of inertia and $-mgl\sin\theta$ the torque exerted by its weight. The nonlinearity built in the sine function is a consequence of the topological constraint keeping the mass rigidly at a constant distance from the rotation axis and plays a role in case of large deviations. As an outcome, several equilibrium positions exist. The 'down' position is a stable equilibrium point, with small amplitude oscillations around it governed by the same equation as the harmonic oscillator. The 'up' position is unstable and the pendulum departs from it upon perturbations of any amplitude, even infinitesimal. Furthermore low energy states oscillate around the 'down' position with a period depending on the energy, while at larger energies, the system rotates always in the same direction, accelerating and slowing down periodically as it passes through the 'down' and 'up'

[1] As indicated at the end of the preface, when the independent variable is time, differentiation is denoted by dots on top of dependent variables.

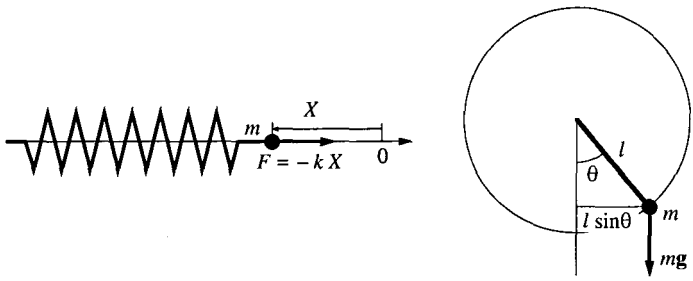

Fig. 1.1 Left: Linear spring. Right: Rigid pendulum.

positions, respectively. The behavior of nonlinear oscillators will be further considered in Chapter 2.

Such mechanical examples can serve us to introduce a geometrical description of the dynamics in a space called the *phase space*, here with coordinates the positions X or θ and the momenta $m\dot{X}$ or $J\dot{\theta}$, and to account for the dynamics in terms of trajectories in that space. In practice, any system involving a finite set of dependent variables serving to characterize its state, its *state variables*, governed by differential equations, belongs to the class we consider here. Examples are intensities and potential differences in an electrical circuit (Chapter 2, §2.3.1.2), concentrations of reactants in chemistry (Exercise 1.5.1), population densities in ecology (Exercise 1.5.2),... In analytical mechanics, a pair 'position + conjugate momentum' is called a *degree of freedom.* In a more general context, what is called a degree of freedom is simply a state variable.

In general, it is advantageous to write down the dynamical equations as a system of first order differential equations

$$\dot{X}_j = \mathcal{F}_j\left(\{X_i(t); i = 1, \ldots, d\}, t\right), \qquad j = 1, \ldots, d,$$

where the integer d is, in this context, called the *dimension* of the system.[2] Using the notations $\mathbf{X}$ and $\boldsymbol{\mathcal{F}}$ for the sets $\{X_j, j = 1 \ldots d\}$, and $\{\mathcal{F}_j, j = 1 \ldots d\}$, we have thus:

$$\dot{\mathbf{X}} = \boldsymbol{\mathcal{F}}(\mathbf{X}, t). \tag{1.3}$$

[2] Accordingly, the dimension of the phase space is twice the number of degrees of freedom in analytical mechanics whereas otherwise it is just the number of state variables. In the following we will try to avoid ambiguities arising from this terminology. See later, Chapter 2, §2.1.2.

For a system such as (1.3) time is a *continuous* variable. By contrast, a *discrete-time* system is defined as an iteration that we can write in the form

$$X_{j,k+1} = \Phi_j \left(\{X_{i,k}; i = 1, \ldots, d\}, k \right) , \qquad j = 1, \ldots, d .$$

or, formally:

$$\mathbf{X}_{k+1} = \mathbf{\Phi} \left(\mathbf{X}_k \right) . \tag{1.4}$$

For such a system, time is the discrete index k that serves to monitor the evolution. While appearing as a topic in itself in mathematics, the study of discrete-time dynamical systems turns out to be essential in physics and engineering owing to their occurrence as a result of the stroboscopic analysis of periodically forced continuous-time dynamical systems, or of a Poincaré section in self-oscillating systems. They can also introduce themselves as the outcome of modeling effort of specific phenomena, *e.g.* seasonal counts in population dynamics. All this will be considered in detail in Chapter 4.

Systems considered here all carry out the intuitive concept of *determinism* which imply the prediction of a state at time t or k beyond some time t_0 or k_0 at which some initial condition is specified. This *initial value problem* presents itself formally or explicitly as a computational problem, directly if it is defined as an iteration (1.4) or indirectly through some numerical approximation for a continuous-time system (1.3), *e.g.* Euler extrapolation,

$$\mathbf{X}(t_{k+1}) = \mathbf{X}(t_k) + \Delta t \, \boldsymbol{\mathcal{F}} \left(\mathbf{X}(t_k), t_k \right) , \tag{1.5}$$

since analytical integration is rarely possible. See Appendix B, §B.1, for an introduction.

Resuming the geometrical perspective introduced above, we are now interested in the properties of *trajectories* followed by the system in its phase space. This study rapidly points to the key role played by the concept of *stability* taken in the broad sense of *resistance to perturbations*. In practice, this vague definition refers to two different viewpoints:

The first one, rather quantitative, applies to specific trajectories: the solutions found, which depend on initial conditions, have to withstand small perturbations imposed either at the start or during the subsequent evolution, owing to unavoidable disturbances, either intrinsic (thermodynamic fluctuations) or extrinsic (noise). From this first viewpoint, only *stable* solutions (that resist) are physically observable. Further, a sufficiently large set of experimental conditions should make them attainable.

The second viewpoint is mode qualitative. It refers to the very definition of the system itself: when compared to the real world, any abstract implementation is intrinsically blurred with numerous approximations and its *control parameters* are not determined with infinite precision. Accordingly, in order to be of help, the analytical model must be *robust*, *i.e.* its predictions must not be too sensitive to these different sources of inaccuracy. This property indeed fails at *bifurcation points* where the system experiences qualitative changes of behavior. At such points one says that it is *structurally unstable*: the nature of the state depends sensitively on the perturbation.

It can be easily understood that these two facets of the term 'stability' are equally important for the applications. They will be at the heart of the most abstract part of the course, first in Chapter 2 and next in Chapter 4 where we will give a more precise meaning to the word 'prediction' when the considered system evolves chaotically, that is to say in a way *unpredictable* in the long term in spite of short term *determinism*, due to an instability of trajectories that is the essence of *chaos*.

1.2 Continuous Media as a Subject

Dynamical systems considered up to now were endowed with a supposedly small number of dependent variables, thus living in spaces of low enough dimensions. Once the microscopic structure of matter has been recognized, in principle one should turn to the study of systems made of a large number of components at the molecular scale, each with its own degrees of freedom in the mechanical sense. However a refined description of the microscopic configurations is generally useless (due to chaos at this scale) and our ignorance of dynamical details can be circumvented by adopting a statistical point of view from which the less probable (specific initial condition) evolves into the most probable (equilibrium compatible with conservation laws), founding the thermodynamic approach. In this framework, microscopic variables are replaced by average quantities such as energy, entropy, temperature or pressure.

In practice, *global* thermodynamic equilibrium is not of much interest since it accounts for a world eaten by the worm of the equiprobability of microscopic states, and in some sense completely dead. On the contrary, an inhomogeneous world traversed by various fluxes making it alive (which by the way allows us to study it!) is much richer and more interesting. We

are indeed confronted to a large class of transport processes in media that are out of equilibrium on a macroscopic scale and there is a wide range of time and space scales for which the concept of *continuous medium* is appropriate, that is to say the description of a system according to which its state variables are functions of the position in physical (three-dimensional) space, $\mathbf{X}(\mathbf{x}, t)$, governed by partial differential equations:

$$\partial_t \mathbf{X} = \mathcal{G}(\mathbf{X}, \boldsymbol{\nabla}\mathbf{X}, t)\,. \tag{1.6}$$

From a mathematical perspective, such systems are infinite-dimensional since we need to specify the value of these variables at every point in space. On the other hand the validity of the description relies on the definition of a mesoscopic scale inbetween the microscopic (molecular) level and the (macroscopic) size of the system taken as a whole. The concept of *material point* and the assumption of *local equilibrium* make sense precisely on this intermediate scale, sufficiently small to be considered as infinitesimal but large enough to contain as many molecules as necessary for the laws of thermodynamic to be relevant.

Equations such as (1.6) are in general derived from the macroscopic balance of extensive thermodynamic variables, say Z, that, in differential form, read

$$\partial_t \rho_z + \boldsymbol{\nabla} \cdot \mathbf{J}_z = \Sigma_z\,, \tag{1.7}$$

where ρ_z is the density of Z, $\mathbf{J}_z$ its flux, and Σ_z a source term that cancels when Z is a conserved quantity (momentum, energy). The simplest example is the Fourier heat equation governing thermal diffusion:

$$C\partial_t T = \chi \nabla^2 T\,, \tag{1.8}$$

governing the temperature field in a solid submitted to moderate gradients. It derives from the Fourier law

$$\mathbf{J}_Q = -\chi \boldsymbol{\nabla} T\,. \tag{1.9}$$

that links the heat flux to the temperature gradient through the thermal conductivity χ. Quantity C in (1.8) is the specific heat defined through the thermodynamic relation $\delta U = C\delta T$. Inserting this into the balance equation for the internal energy $\partial_t \rho_U + \boldsymbol{\nabla} \cdot \mathbf{J}_Q = 0$, where ρ_U is the corresponding density obtained as the limit of $\delta U/\delta V$ as the volume element $\delta V \to 0$, immediately yields (1.8).

The quantity $\kappa = \chi/C$ is the thermal diffusivity. Dimensionally, this coefficient is homogeneous to $[L]^2/[T]$, like the diffusivity D appearing in the Fick law governing molecular diffusion. In this context, an important quantity is the time scale τ for diffusive relaxation over a typical space scale ℓ, directly obtained from the dimensional relation above as $\tau = \ell^2/\kappa$. The relaxation of T is examined in the most elementary case in Exercise 1.5.3 which further illustrates a rare situation where one can find an analytic transform allowing the exact linearization of a parent nonlinear problem.

Equation (1.8) is indeed linear provided that the specific heat C and the thermal conductivity χ are constant, which is reasonable not to far from equilibrium. The Fourier law (1.9) is the prototype of a phenomenological relation linking a flux (the effect) to a gradient (the cause) at a linear level. In practice, the validity range of such linear law is generally rather wide, because constraints that we are able to apply to continuous media are usually weak when compared to molecular interactions. As counter-examples one could cite electronic systems containing active elements with threshold effects (diodes), or chemically reacting media, and *a fortiori* biological systems. However, in macroscopic media strong nonlinearities can in fact arise from global considerations. This is particularly the case in fluid systems where macroscopic flow deeply affects the transport properties, rendering a locally linear medium effectively nonlinear.

So, in hydrodynamics the concept of the *material point* transforms itself into that of *fluid particle*, the position of which becomes a function of time:

$$\mathbf{M}(t) = (x_{\mathrm{M}}(t), y_{\mathrm{M}}(t), z_{\mathrm{M}}(t)) \ .$$

It is most often useful to pass from this *Lagrangian* description, to an *Eulerian* approach in terms of velocity field:[3]

$$\mathbf{v} \equiv (v_x, v_y, v_z) \equiv \left(\tfrac{\mathrm{d}}{\mathrm{d}t} x_{\mathrm{M}}, \tfrac{\mathrm{d}}{\mathrm{d}t} y_{\mathrm{M}}, \tfrac{\mathrm{d}}{\mathrm{d}t} z_{\mathrm{M}}\right) \ .$$

The two descriptions are linked by the definition of the *material derivative* measuring the evolution of any physical quantity Z attached to the fluid particle as it is followed during its motion: $Z = Z\,(x_{\mathrm{M}}, y_{\mathrm{M}}, z_{\mathrm{M}}, t)$. Mathematically, material differentiation is thus a total differentiation with respect to time, $\frac{\mathrm{d}}{\mathrm{d}t} Z$, that is:

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Z = \partial_t Z + \partial_x Z \, \tfrac{\mathrm{d}}{\mathrm{d}t} x_{\mathrm{M}} + \partial_y Z \, \tfrac{\mathrm{d}}{\mathrm{d}t} y_{\mathrm{M}} + \partial_z Z \, \tfrac{\mathrm{d}}{\mathrm{d}t} z_{\mathrm{M}} = \partial_t Z + \mathbf{v} \cdot \boldsymbol{\nabla} Z \ . \quad (1.10)$$

[3]Here we no longer denote time differentiation by dots but come back with explicit derivatives in order to avoid ambiguities.

This relation allows a simple expression of the balance equation in differential form provided that the flux of Z is properly split into an irreversible diffusive part and a reversible part linked to the macroscopic motion:

$$\mathbf{J}_z = \mathbf{J}_{z,\mathrm{diff}} + \rho_z \mathbf{v}\,.$$

When studying the mechanism of natural convection in fluids originally at rest we will demonstrate the role of a global nonlinearity played by the advection term in spite of its apparent linearity in ρ_z, but before considering this example, let us recapitulate the equations governing the motion of a simple Newtonian fluid, *i.e.* a single component fluid with a constant viscosity. The first one is the continuity equation accounting for the conservation of matter. In full generality it reads:

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \mathbf{J}_\rho = 0\,,$$

(no source term). Noticing that diffusion of matter within itself does not make sense, we get $\mathbf{J}_\rho = \rho \mathbf{v}$ which leads to

$$\partial_t \rho + \mathbf{v} \cdot \boldsymbol{\nabla}\rho + \rho \boldsymbol{\nabla} \cdot \mathbf{v} \equiv \tfrac{\mathrm{d}}{\mathrm{d}t}\rho + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0\,. \tag{1.11}$$

In the following we will consider *incompressible flows* characterized by $\frac{\mathrm{d}}{\mathrm{d}t}\rho = 0$, so that the continuity equation more simply reads

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0\,. \tag{1.12}$$

On the other hand, for an incompressible fluid the compression viscosity drops out and there only remains a shear viscosity which, for an isotropic Newtonian fluid, relates the viscous stress tensor to the rate of strain in a linear way:

$$\sigma_{ij} = \mu(\partial_i v_j + \partial_j v_i)\,,$$

where μ is the *dynamical viscosity*. Once inserted into the momentum conservation equation, this relation leads to the *Navier–Stokes* equation

$$\rho(\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{v}\,. \tag{1.13}$$

The *kinematic viscosity* $\nu = \mu/\rho$ is the transport coefficient accounting for the diffusive relaxation of velocity gradients (Stokes law). At the inviscid limit ($\nu \to 0$) the Navier–Stokes equation is called the *Euler* equation. The opposite limit where the viscous dissipation dominates is called the *Stokes approximation*. Finally if the heating generated by viscous friction

is negligible, energy conservation yields the heat equation (1.8) where $\partial_t T$ is just replaced by $\frac{\mathrm{d}}{\mathrm{d}t}T \equiv (\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})T$, hence:

$$\partial_t T + \mathbf{v}\cdot\boldsymbol{\nabla}T = \kappa\boldsymbol{\nabla}^2 T\,. \tag{1.14}$$

1.3 From Simple to Complex

In a narrow vicinity of thermodynamic equilibrium, the response of a continuous system is simply proportional to the strength of applied constraints and displays the same symmetries, stationary or periodic in time, uniform or periodic in space, for example. However, far from equilibrium, when nonlinearities can no longer be neglected, the system can bifurcate towards solutions that breaks some of these symmetries.

The nature of the regime that develops depends on the value of *control parameters* measuring the relative intensity of the different contributions to the dynamics. In fluid mechanics, it will most often be the *Reynolds number* defined as $R = U\ell/\nu$ where U and ℓ represent a typical velocity and a length scale both characteristic of the flow under consideration while ν, the kinematic viscosity introduced above, is a fluid's property. In line with what has been said about the thermal diffusion time, it is enlightening to analyze the Reynolds number as the ratio of the viscous relaxation time over the length ℓ, $\tau_{\mathrm{v}} = \ell^2/\nu$, to the advection time required to carry velocity fluctuations over the same distance, $\tau_{\mathrm{a}} = \ell/U$. When the Reynolds number $R = \tau_{\mathrm{v}}/\tau_{\mathrm{a}}$ is small, viscosity has time to wipe out inhomogeneities, whereas when it is large, thermodynamic *dissipation* is too slow and the mechanical contribution of *advection* dominates.

The study of the stability of a given regime called the *base state* rests on the definition of perturbations, *i.e.* departures from this state, and their subsequent evolution. Figure 1.2 illustrate the general situation. Below a value of the control parameter R denoted R_{g}, 'g' for global, the base state is unconditionally stable: whatever the shape and the amplitude of the perturbation, it decays and the system returns to its base state (sufficient condition of stability). Above a second value R_{c}, 'c' for critical, the system is sensitive to at least one unavoidable perturbation and unconditionally departs from the base state. The determination of R_{c} appeals to the so-called *linear stability theory* dealing with the evolution of fluctuations that are *mathematically* infinitesimal. It is indeed sufficient that there exist such a perturbation that is amplified for the base state to be unstable. It may happen that R_{g} coincide with R_{c}, in which case $R \geq R_{\mathrm{c}}$ is a necessary and

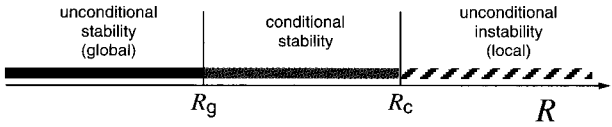

Fig. 1.2 Stability of the base state as a function of some control parameter R, *e.g.* the Reynolds number.

sufficient condition of instability but in general the interval $[R_g, R_c]$ has a finite width, which defines a range of *conditional stability*: the stability of the base state depends on the shape and amplitude of the (finite amplitude) perturbations to which it is submitted. Most of the time this fundamentally nonlinear problem remains unsolved.

In some favorable cases, one can succeed in computing the state that sets in above the threshold of the *primary* instability by a perturbation method. This so-obtained state is then promoted as a new base state, the stability of which is of interest. This state can in turn become unstable with respect to a *secondary* instability that complicates the dynamics, and so on up to a chaotic state which, for fluid systems, is usually called *turbulent*. In this perspective, turbulence is considered as resulting from various modes at the end of a *cascading process*. Instead of staying at this formal viewpoint let us rather examine a particularly simple and intuitive concrete mechanism responsible for the instability of a liquid layer heated from below and initially at rest.

1.3.1 *Thermal convection: the instability mechanism*

Let us consider a layer of fluid at rest presenting a density stratification due to heating from below (Figure 1.3). A temperature difference $\Delta T = T_b - T_t > 0$ is thus applied between the bottom 'b' and the top 't' of the layer. Since, with few exceptions, the density decreases with increasing temperature and this stratification with heavier fluid on top of lighter fluid is potentially unstable in a vertical gravity field (like the 'up' position of the pendulum).

When ΔT is small, the fluid stays at rest since the gravitational potential energy that would be gained by moving the heavier fluid to the bottom is not sufficient to counterbalance the energy loss by dissipation in that motion. More precisely, as long as the fluid is at rest, the heat transfer is entirely carried out by *conduction*. The temperature profile in the base

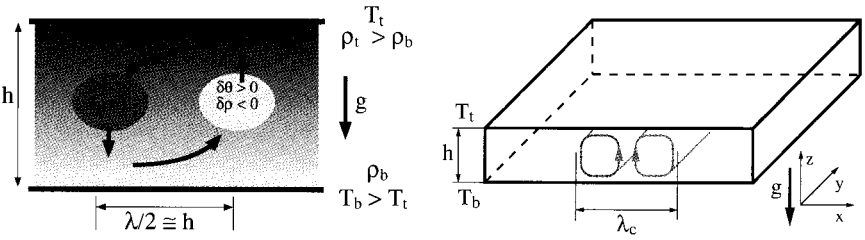

Fig. 1.3 Left: Instability mechanism of thermal convection. Right: Convection cells with wavelength $\lambda_c = 2\pi/k_c$.

state $T_0(z)$ is obtained from the Fourier equation (1.14) that simply reduces itself to $\mathrm{d}^2T_0/\mathrm{d}z^2 = 0$ since $\mathbf{v} \equiv 0$, hence $T_0(z) = T_b - \beta z$, where $\beta = \Delta T/h$ is the applied temperature gradient, h being the height of the layer.

Let us suppose a fluctuation localized inside a tiny droplet marked with a slightly higher temperature (perturbation $\theta > 0$). Since this bubble of hotter fluid is surrounded with colder denser fluid, it experiences an upward differential buoyancy force which makes it moving up, encountering ever colder fluid, which reinforce the motion: this is the destabilizing part of the mechanism. However, two stabilizing processes tend to oppose this process. First, the so-induced velocity tends naturally to decay owing to viscous friction. Second, thermal diffusion aims at ironing out the horizontal temperature gradient accompanying fluctuation θ. The fluid layer stays at rest as long as dissipation dominates but convection develops when the destabilization is sufficient, *i.e.* when ΔT is larger than some *critical* value ΔT_c called the *instability threshold*.

On the other hand, the stabilizing dissipative processes are diffusive in essence, so that their efficiency depends on the horizontal space dependence of the velocity and temperature modulations. Damping is fast on small scales and slow on large scales. It thus depends on the length-scale of the perturbation and it turns out that the onset of the instability corresponds to some optimum scale. For dimensional reasons this scale is of the order of the height h, as implied by the sketch in the left part of Figure 1.3. Accordingly the bifurcated convection state often sets in as a system of periodic rolls, with spatial period $\lambda_c \approx 2h$, as illustrated in its right part. The quantity $k_c = 2\pi/\lambda_c$ which appears when representing the plan-form in Fourier series is called the *critical wavevector*. The pattern formed has been called a *dissipative structure* by Prigogine [Glansdorff and Prigogine (1971)].

1.3.2 *Nonlinear convection and dynamical systems*

Here we account for the dynamics of convection close to the threshold by heuristic arguments that could be entirely supported by a rather complex detailed theoretical approach yielding the concrete value of the coefficients that we will introduce phenomenologically.

The space-time coherence introduced in the system by the instability mechanism allows us to describe the evolution of the convecting layer using a simple variable $A(t)$ measuring the amplitude of convection. We thus assume that, as far as their horizontal dependence is concerned, the velocity and temperature fields can be taken in the form

$$(v_z, \theta) \propto A(t) \sin(k_c x)\,, \tag{1.15}$$

where A plays the role of an *effective degree of freedom* and k_c is the critical wavevector introduced earlier. We are now interested in the phenomenological derivation of an evolution equation for this amplitude.

At the infinitesimal stage, the perturbation is governed by a differential equation that should read

$$\frac{\mathrm{d}}{\mathrm{d}t} A = \sigma A\,, \tag{1.16}$$

where coefficient σ accounts for the growth rate, negative below the threshold (damping) and positive beyond (amplification). Let us define a reduced control parameter $r = (\Delta T - \Delta T_c)/\Delta T_c$ and assume that the behavior of σ as a function of r is not singular, so that its expression can be restricted to the first term of its Taylor expansion in powers of r:

$$\sigma = r/\tau_0\,, \tag{1.17}$$

where τ_0, homogeneous to a time, characterizes of the natural evolution of relevant fluctuations.

Equation (1.16) where σ is given by (1.17) accounts for the evolution of coupled velocity-temperature fluctuations as long as the amplitude A is sufficiently small. This does not raise difficulties when $r < 0$ since A decays but when $r > 0$, in the unstable domain, it is exponentially amplified and does not stay small for a long time. In order to model the nonlinear effects, we will assume that (1.16) remains valid but that σ has to be corrected and becomes a function of A itself. Noticing that the change '$A \mapsto -A$' corresponds to a change of the spinning direction of the rolls (or to a translation of the structure by $\lambda_c/2$) and that the physics of the problem should not be sensitive to such a change, we are led to assume that the effective

growth rate σ_{eff} is an even function of A. At lowest order of an expansion in powers of A, it can be taken as:

$$\sigma_{\text{eff}} = (r - gA^2)/\tau_0 \,, \tag{1.18}$$

where g is called the *Landau constant*. In the simple case considered here, g is a positive quantity, *i.e.* $g = 1/\bar{A}^2$, $\bar{A}$ measuring the typical amplitude at which nonlinearities become effective. The experiments show (and the theory demonstrates) that the convection mechanism is indeed self-limiting, so that the effective growth rate decreases when the amplitude of convection increases. The nonlinear model accounting for convection in the neighborhood of the threshold then reads:

$$\tau_0 \frac{\mathrm{d}}{\mathrm{d}t} A = \mathcal{F}_r(A) = rA - gA^3 \,. \tag{1.19}$$

Its study is elementary and will be resumed later in a more general context. Here it is our first example of *effective dynamical system*. It governs a simple scalar variable A representing the macroscopic evolution of our system. Upon rescaling t by τ and A by $\bar{A}$, equation (1.19) can be recast in a universal form:

$$\frac{\mathrm{d}}{\mathrm{d}t} A = rA - A^3 \,. \tag{1.20}$$

The evolution of A from an initial value $A_0 = 10^{-3}$ for $r = 0.1$ is illustrated in Figure 1.4. As long as $rA \gg A^3$ the growth remains exponential, which appears graphically linear on the lin–log plot in the left part of the figure. When A increases, the slope of the curve σ_{eff} decreases. Ultimately, A saturates to a value $A_* = \sqrt{r}$. Taking a negative initial value would have led to $-A_*$, owing to the symmetry of the equation. The graph giving the nontrivial state as a function of the control parameter, depicted in the right part of Figure 1.4, is called a *bifurcation diagram*.

In the terminology of dynamical systems, variable A is the *degree of freedom*.[4] Accordingly, the real axis is here the *phase space*, while $\mathcal{F}_r(A)$ is the *vector field* defined on this space that governs the dynamics. Stationary solutions A_* are called *fixed points*, a stable fixed point being a simple example of *attractor*. The transition 'conduction → convection' is continuous with a stable nontrivial A_* tending to zero as the threshold is approached from above. One then speaks of a *supercritical* bifurcation. Owing to the shape of the graph in Figure 1.4 (right), it is further called a *fork bifurcation*. Equation (1.20) is thus the *normal form* for a supercritical fork bifurcation.

[4]Remember that we have left the field of analytical mechanics, see Note 2, p. 3.

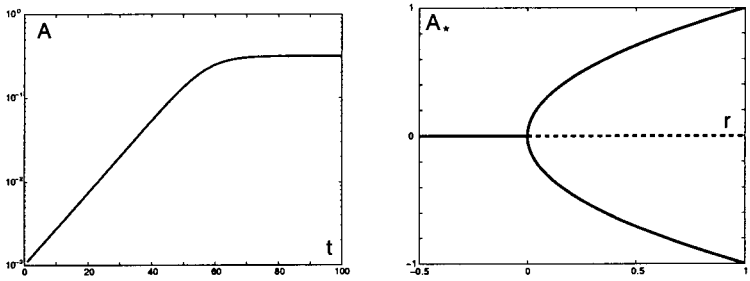

Fig. 1.4 Left: Evolution of the amplitude A as a function of time for model (1.20) with $r = 0.1$ (lin–log plot). Right: Bifurcation diagram giving the amplitude A of the steady state as a function of r. Conventionally, the unstable state $A = 0$ for $r > 0$ is indicated by dashes while a continuous thick line is used for the stable states, $A = 0$ for $r < 0$ and $A = A_* = \pm\sqrt{r}$ for $r > 0$.

In the opposite case, *i.e.* $g < 0$ in (1.19), the nontrivial solution appears below the threshold and is unstable, the bifurcation is then *subcritical*, and equation (1.20) has to be further completed to yield a meaningful model. This abstract approach will be resumed in Chapter 2 and 4.

Convection rolls that develop beyond the threshold of the first instability can themselves become unstable with respect to other mechanisms. One thus has to study the flow that results from the superposition of the saturated *primary* mode to the initial base state, using a similar but technically much more complicated approach. *Secondary* modes can then be detected yielding, upon saturation, new base states ready for subsequent destabilization as the control parameter is further increased.

The convecting layer is a continuous medium described by field variables. The reduction of the dynamics to a single scalar variable A, or a small set of similar variables when other instabilities have taken place, calls for justifications that will be examined in Chapter 4 as a precondition to the use of the concept of *chaos* for interpreting the increasing complexity of the dynamics all along the cascade of instabilities leading to turbulence. This approach will be all the more relevant when there are few convection cells, well confined by lateral walls at distances of the order of the height h itself. Its limitations will also have to be studied when *confinement effects* by far-apart lateral boundaries are too weak (Chapter 5).

1.3.3 *Stability and instability of open flows*

Natural convection has been the subject of many academic studies since it offers an ideal testing ground for ideas developed in nonlinear dynamics. However, systems of interest in engineering studies often display a supplementary feature: they involve *open flows* characterized by the existence of a mean current from upstream to downstream. Whereas the linear stability theory of open flows has a long history, dating back to the end of the Nineteenth Century, difficulties linked to this specific feature have delayed the nonlinear approach. Rather than a thorough account of results obtained in this field, Chapter 6 should thus be considered as a preliminary presentation aiming at better situating the problem in a nonlinear perspective and making its study easier.

Purely kinetic effects play an important role in open flows and can by themselves already be at the origin of instabilities. Accordingly, a first distinction can be made between flows that are *mechanically unstable* and those that are *mechanically stable.* The former display an inflection point in the base velocity profile which makes them unstable in the absence of viscous dissipation according to *Rayleigh's criterion.* Their prototype is the *mixing layer* illustrated in Figure 1.5 (left). By contrast, mechanically stable flows may become unstable only due to subtle feedbacks involving a transfer of momentum of viscous origin. A good example is given by the Blasius *boundary layer* flow depicted in Figure 1.5 (right). This classification will affect the whole process of transition to turbulence in an important way (Chapter 6, §6.2).

A second distinction has to be made from the very existence of a global flow of matter entering the region of interest and leaving it at the outlet,

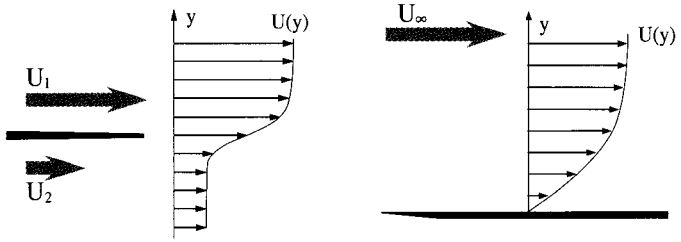

Fig. 1.5 Typical open flows. Left: Mixing layer created at the merging of two fluid veins with different velocities and maintained separated upstream by a splitting plate. Right: Boundary layer developing along a plate at some distance of its leading edge.

Fig. 1.6 Kármán vortex street emitted by a cylinder (courtesy P. Le Gal, IRPHE, Marseilles).

which makes implicit reference to a frame rigidly attached to the laboratory (obstacle, rigid wall). An important difference indeed appears between instabilities said to be *convective*, those that develop along the flow but are evacuated downstream and do not succeed in going upstream, and instabilities called *absolute* that are sufficiently intense to develop in spite of the global downstream transport. In the first case, the systems behaves as a *noise amplifier* and the result downstream essentially depends on the level of background fluctuations (residual turbulence). In the second case, one has rather to deal with a genuine *self-sustained oscillator* analogous to those studied in Chapter 2.

However, the situation is even more complicated by the fact that the intensity of instability mechanisms may vary in space. This is due to the fact that the velocity profile usually evolves downstream as a consequence of viscous dissipation that tends to smooth out the velocity gradients present at the entrance of the flow. Depending on the *local* intensity of the mechanism (linked to the magnitude of the shear) one can observe a change of the character of the mode usually from absolute to convective, hence the possibility of bifurcated states localized in a given region of space and usually called *global modes*. The wake of a blunt body inserted in an otherwise uniform stream is such an example (Figure 1.6). A recirculation takes place close to the obstacle, rendering the overall downstream transport locally sufficiently weak that regularly shed vortices can develop and stay attached to it (Kármán vortex street).

The understanding of this combined problem 'convective/absolute + local/global + linear/nonlinear' has made great progress recently but it requires sophisticated mathematical tools, especially in complex analysis, that would bring us far beyond the limited purpose of the present notes. The problem will just be evoked at a physically intuitive level in Chapter 6.

In the above presentation the instabilities were tacitly assumed to be linear and supercritical. It may however happen that the bifurcation is subcritical and that several different flow regimes coexist in some range of control parameter, the bifurcated state saturating only "far from" the base state as for the plane channel flow. The base state may also happen to remain stable against infinitesimal fluctuations so that the nontrivial (turbulent) regime cannot be reached by perturbation. Examples are the pipe Poiseuille flow and the plane Couette flow. Though these flows are known to be linearly stable for all Reynolds numbers, they are expected to become turbulent as the shear rate increases beyond all limit. The transition indeed happens and take a rather explosive turn: turbulent bursts developing intermittently from localized finite amplitude perturbations show up during the process, coexist with still laminar flow and then merge to fill the systems with fully developed turbulence. Figure. 1.7 displays such a turbulent spot in plane Couette flow. It shows that the pocket of turbulence is not structureless but, on the contrary, present a streamwise streaky pattern. This type of structure turns out to be omnipresent in turbulent wall flows. Their production and sustainment mechanisms are still the subject of current research (Chapter 6, §6.3.4).

Fig. 1.7 Mature turbulent spot induced by a localized perturbation of finite amplitude in plane Couette flow (courtesy S. Bottin and O. Dauchot, GIT, CE Saclay).

1.3.4 *Beyond the transition: fully developed turbulence*

Problems considered up to now all relate to the steps of the cascade leading from a simple and regular base state to a complex and irregular flow still partly ordered but chaotic, that may not yet be called turbulent. It seems indeed important to understand the steps of the laminar-turbulent transition just sketched in a deterministic framework. The underlying aim is of course to control it as best as we can, delay it if it is harmful, or advance it

if we need better mixing. However it turns out that beyond the transition we have to change our mind and take a fully statistical viewpoint. As a matter of fact, when we pull the considered system (flow) ever farther from equilibrium, more and more degrees of freedom become excited and it no longer makes sense to focus on their individual dynamics, like when passing from few-body systems well described by analytical mechanics to gases for which the thermodynamic approach is more appropriate.

This remark could lead to understand developed turbulence as a new macroscopic state of matter at a scale where the fluid would have transport properties very different from those of ordinary fluids in their laminar state where molecular chaos still control diffusion. Things are unfortunately less simple. The usual distinction between microscopic and macroscopic scales works well for thermodynamics because there is a wide gap between them, the precise reason for which the concept of material point (or fluid particle) makes sense. By contrast, in turbulence, relevant scales belong to a continuous range from the size of the flow domain to small scales. While the smallest eddies seem to evolve randomly like molecules in a gas, they are still coupled and, in fact, driven by the scales above them so that no decoupling is truly legitimate.

In Chapter 7, we will approach the theory of turbulence in a very preliminary way only, without trying to compete with numerous excellent books dealing with it, from conceptual problems to applications. Our limited aim will be to illustrate the idea of an *inertial cascade* (cf. Fig. 1.8) transferring energy from large scales where instability mechanisms generate the eddies, down to the smallest scales where viscosity successfully irons out the fluctuations. It is not difficult to catch an idea about this transfer by the advection term $\mathbf{v} \cdot \nabla \mathbf{v}$ of the Navier–Stokes equations. Considering an eddy motion locally described by a simple trigonometric line $v_x \sim \sin(kx)$, we get $\sin(kx)\partial_x \sin(kx) = k \sin(kx)\cos(kx) = \frac{1}{2}k \sin(2kx)$ that presents

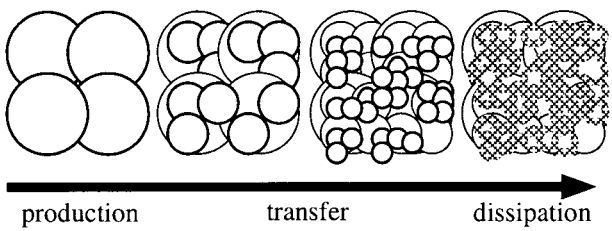

Fig. 1.8 Illustration of the Kolmogorov cascade from large scales to small scales where fluctuating eddies are blurred by viscous dissipation.

itself as a source term in the evolution equation for a mode with a spatial scale half of the initial one. The result of this transfer is a repartition of energy according to the celebrated $k^{-5/3}$ Kolmogorov law that will be derived heuristically.

In a second part, we will turn to the concrete problem of predicting the lowest order statistics of specific turbulent flows, obtaining the so-called *Reynolds averaged equations* governing the mean flow, which immediately opens the problem of the closure of the statistical description. As a matter of fact, the averaging of primitive equations introduces new higher order statistical quantities called the *Reynolds stresses* that remains to be evaluated in one way or another. We will then exploit the already mentioned analogy between the kinetic theory of gases and the statistical theory of turbulence to introduce a disputable but heuristically valuable approach to turbulent flows resting on the concept of *eddy diffusivity* patterned on that of *molecular diffusivity* (see Figure 1.9). This will serve us to make a first evaluation of the average properties of a turbulent flow, taking the turbulent boundary layer as an example, and obtaining the classical Kármán logarithmic law. The main shortcoming of this approach derives from the absence of scale decoupling mentioned above but we will not do much more than mentioning it, suggesting further that progress can be obtained in concrete situations by numerical simulations, and observing that the latter always imply the crucial step of sub-grid-scale modeling, *i.e.* the modeling of smallest scales that cannot be explicitly accounted for in the numerics.

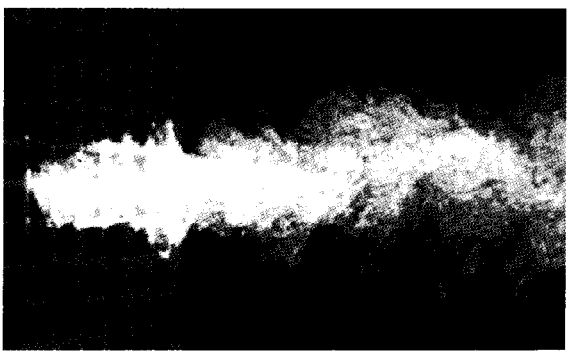

Fig. 1.9 Visualization of a turbulent flow downstream a grid. The fluid passing through a given mesh is seeded with a tracer that demonstrates turbulent diffusion in the flow. After a color picture by J.L. Balint, M. Ayrault and J.P. Schon (École Centrale de Lyon), courtesy M. Lesieur, in "La turbulence développée," La Recherche, n° 139 (1982), with permission.

1.4 Conclusion

Numerous books develop the topics to be touched upon. A partial list is given in the bibliography. At the risk of repetition, let us emphasize that we have tried to present a first approach to current problems, using a wide range of techniques, linear, nonlinear, deterministic, statistical, each with its own qualities and limitations, in order to lucidly face complicated situations encountered in our familiar environment, industrial or natural. Accordingly, in the concluding chapter we will evoke the problem of the Earth's climate as a concentration of the kind of topics examined at one or another moment, with special reference to the predictability problem and modeling issues involved in this nonlinear dynamical system with heterogenous space-time scales. Appendix A recalls some elementary and not so elementary results of linear algebra, while Appendix B works in roughly the same direction by giving rudiments of numerical simulation techniques expected to be of help in the understanding of complex processes (provided that we are able to build simplified models with valuable metaphoric value).

1.5 Exercises

1.5.1 *Chemical reactions*

Chemical kinetics is a field that has contributed much to the development of applied nonlinear dynamics in out-of-equilibrium systems. Here we consider a simplistic kinetic model introduced by Prigogine and Lefever in 1968, called the Bruxellator and accounting for a hypothetical reaction between two components A and B with two end products C and D, four steps and two intermediate compounds X and Y. Concentrations A and B of reactants A and B are the control parameters whereas those X and Y of the intermediate species X and Y are the variables. The global reaction:

$$\mathsf{A} + \mathsf{B} \rightarrow \mathsf{C} + \mathsf{D}$$

can be decomposed into successive steps:

$$\mathsf{A} \rightarrow \mathsf{X}, \qquad \mathsf{B} + \mathsf{X} \rightarrow \mathsf{Y} + \mathsf{C},$$
$$2\,\mathsf{X} + \mathsf{Y} \rightarrow 3\,\mathsf{X}, \qquad \mathsf{X} \rightarrow \mathsf{D}.$$

Assuming that the kinetic constants of each reaction is equal to 1, write down the equations for each step and show that the reaction is governed by

$$\dot{X} = A - (B+1)X + X^2Y\,, \qquad \dot{Y} = BX - X^2Y$$

This system will be further studied in Exercise 4.6.7.

Elements of chemical kinetics. Consider:

$$\sum n_i \mathsf{A}_i \quad \rightarrow \quad \sum n'_i \mathsf{A}_i$$

an elementary step between reactants A_i with concentration A_i [if specie A_i is absent from the left (right) hand side then $n_i = 0$ ($n'_i = 0$)]. The reaction rate corresponding to this step measures the number of reacting collisions per unit time. A collision is the meeting at a single physical point of all the components on the l.h.s. producing all those of the r.h.s. in the proportions given by the stoichiometric coefficients. The probability for reactant A_i to be there is proportional to the concentration and, if one neglects the correlations between species, one expects the probability of the collision as a product of individual probabilities and thus a reaction rate $k \prod_i A_i^{n_i}$, involving only participants on the l.h.s. and where k is the corresponding rate constant. When such a reaction takes place, the number of molecules of component A_i varies by $n'_i - n_i$, so that one gets:

$$\dot{A}_i = (n'_i - n_i) k \prod_i A_i^{n_i} .$$

The total variation of the concentration of a given specie A_i for a compound reaction is the sum of the variations at each step.

An oscillatory reaction, discovered by Belousov in the fifties and further studied by Zhabotinsky in the sixties (BZ reaction) has been the subject of intense laboratory work. It corresponds to the oxydization of a organic reducer (malonic acid) by BrO_3^- ions with a redox couple (*e.g.* $\mathsf{Ce}^{3+}/\mathsf{Ce}^{4+}$) as a catalyst. It involves about 15 chemical species coupled by an equivalent number of intermediate reaction steps. Simplified models of this reaction have been devised with only 5 steps and 3 intermediate variables (Field et Noyes, 1974).

1.5.2 *Prey-predator systems*

Prey and predators interact in much the same way as the molecules considered in the previous exercise. The dynamics of the population X of some specie X is usually described in terms of an effective growth rate:

$$\dot{X} = \alpha_{\text{eff}} X \tag{1.21}$$

where the expression of α_{eff} accounts for the balance between birth and death, which depends on control parameters and the population itself.

1) Assume that the effective growth rate is simply proportional to the amount of available food and interpret the different terms appearing in $\alpha_{\text{eff}} = \alpha - \beta X$, where α and β are two positive constants. The corresponding evolution equation is usually called the (continuous-time) *logistic equation*. What is the meaning of quantity $X_* = \alpha/\beta$. Derive the solution $X(t)$ of this 'limited growth' model starting with initial condition $X = X_0 > 0$ at $t = 0$ by explicit time integration for $X_0 < X_*$ and $X_0 > X_*$. Extract the long-term behavior ($t \to \infty$). Observe also that the evolution of X here is the same as that of A^2 where A is governed by (1.19) for $r > 0$, apart from scale changes on X and t to be determined [see Figure 1.4 (left) that would correspond to the case $X_0 < X_*$].

2) In the model considered above, X is the population of a predator X and the available food is determined by the population Y of its prey Y, the food is thus not indefinitely available and the term $-\beta X$ is no longer necessary. A simple assumption is that the growth of the population is associated with a food consumption that is proportional to the presence of prey (the equivalent of a 'reactive collision' in chemistry) and that its decay is due to some natural death rate. Argue in favor of expression

$$\alpha_{\text{eff}} = \alpha' Y - \alpha_0$$

for α_{eff} in (1.21). What should be the signs of the constants α' and α_0.

3) In order to close the system we need to write down an equation for the effective growth rate of Y. By arguments similar to previous ones, justify the expression

$$\dot{Y} = \gamma Y - \delta X Y \,.$$

This coupled system will be studied in Exercise 2.5.12.

Try to generalize the approach to describe more complicated prey-predator models, called Lotka–Volterra systems, by changing the assumptions, *e.g.* that the growth of the prey is also limited by competition for food.

1.5.3 *Diffusion equation*

We are interested in the decay of a solution to the diffusion equation [*e.g.* the heat equation (1.8)] in space dimension one:

$$\partial_t \vartheta = \kappa \partial_{xx} \vartheta, \tag{1.22}$$

starting from some initial condition $\vartheta_0(x)$.

1) Determine by substitution the relation between the growth rate s and the wavevector k of a periodic fluctuation taken in the form $\vartheta_k(x,t) = \bar{\vartheta}\exp(st)\exp(ikx)$ and derive its evolution as time goes on.

2) Adding boundary conditions $\vartheta(0,t) = \vartheta(\ell,t) = 0$, solve the evolution problem formally for a solution starting with an initial condition that can be expanded as a sine series: $\vartheta(x,0) = \sum_{n=1}^{\infty} \vartheta_n \sin(\pi n x/\ell)$. Extract its asymptotic solution in the limit $t \to \infty$.

3) Going back to the infinite medium, consider functions ϑ that are taken in the form

$$\vartheta(x,t) = \exp(\alpha w(x,t))\,. \tag{1.23}$$

Derive the partial differential equation governing $w(x,t)$, and next that governing $v = \partial_x w$. Find the value of α bringing this equation in the form

$$\partial_t v + v\partial_x v = \kappa \partial_{xx} v \tag{1.24}$$

called the Burgers equation.[5] The inverse of function change (1.23), called the Hopf–Cole transformation, offers an exact analytical way to linearize (1.24) and to put it back in the form (1.22).

[5] J.M. Burgers, *The nonlinear diffusion equation* (Reidel, 1974), see U. Frisch & J. Bec, "Burgulence," in [Lesieur *et al.* (2001)] for a recent review.