Looking Iteratively Like Archimedes

By J. H. DeBlois

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(Subtitle: "Iterative Algorithm without Algebra - Archimedes' Use of Greek Math and Numbers to Iteratively Determine the Ratio of the Circumference of a Circle to its Diameter (Known as Pi) in Five Fastly Converging Steps using Circumscribed and Inscribed Polygons Translated from the Greek")

Abstract.

This paper examines new translations of Archimedes text "Dimensions of a Circle," Proposition Gamma (Proposition 3), which presents Archimedes' calculation of the ratio of the circumference of a circle to its diameter (the ratio is known as "Pi") and seeks to determine what Archimedes actually wrote, as nearly as possible. The two versions of text were transcribed by ancient scholars from papyrus to parchment, but there is good agreement between them. The translations are compared to how Proposition 3 has been translated in the past. The vehicle is a large spreadsheet of data. From the data, we surmise what Archimedes would tell us the algorithm is, and we do this without straying too far from what is written. Knowledge of common Greek math habits and the ancient Greek manner of writing numbers using Greek characters was required to complete the translations. The result is a detailed algorithm for the iterative calculation of "Pi", which Archimedes carried out for five iterations. The algorithm uses geometric constructions, not algebra, and rests elegantly on the proofs of Euclid. Archimedes' way of describing how his results are obtained lines up well with the modern human-machine interface designs supported by functional programming languages like LISP. The algorithm can be carried out as many times as desired or to whatever degree of accuracy is desired. Once the basic picture is understood, the algorithm is simple to roll out. It would be fun to write a pictoral simulation. A more direct conversion to programming language might be possible.

I. Purpose.

Many mathematicians in many countries over many centuries have wanted to know the ratio of the circumference of a circle to its diameter – the quantity we call "Pi" and write using the 16th letter of the modern Greek alphabet "Π". In the west, Archimedes is credited with calculating it for the first time in 253 BCE. (Footnote 1) Archimedes was a Greek philosopher from Syracuse, Sicily who worked in Alexandria, Egypt. In the third proposition of his treatise called "Dimensions of the Circle", Archimedes calculated the ratio to have a value between two numbers, describing the ratio as "less than 3 1/7 but greater than 3 10/71"(Footnote 2). He showed how he could get the numbers to move closer and closer to each other by bisecting the central angle in a triangle he had constructed as an overlay to a circle. Studying infinite streams in LISP and reading Professor Offner's paper about "Pi" lead me to wonder what exactly Archimedes had written in his proposition. I obtained the work in Greek and set to translating it. My purpose was to imagine how Archimedes did this. We have become much more visual as a people due to the use of computers and computer icons. I wondered what pictures Archimedes drew, and how they would compare to the pictures in Offner's paper.(Footnote 3). I drew various sketches of the relationships in a circle, but notations were all in trigonometry. Later, I learned from Professor Offner that the Greeks did not know algebra. Thus, Archimedes used visual methods and geometry to prove the proposition.

II. The Translations.

There are (at least) two known documents in Greek containing the "Dimensions of the Circle" by Archimedes. Both documents have been transcribed onto parchment from papyrus, so we recognize we are at least one step away from knowing precisely what Archimedes wrote. The more complete document is the one called Heiberg, which Google books has made available on the web (Footnote 4). It contains both the Greek treatise of Archimedes and a page-by-page translation into Latin including footnotes about the Greek version. The other document is called Palimpsest, and has only been translated in the past ten years, after being sold at auction by Christie's (Footnote 5). The Palimpsest is actually a religious treatise with the text by Archimedes underneath – an amazing discovery! A progression of light ray operations were used to illuminate the undertext. The anonymous owner allowed the work to be on display at the Baltimore museum in 2011.

In making a sentence-by-sentence comparison of the two texts of Proposition 3, with the exception of a number of occurrences of a few characters being different, many of which form some part of one of the numbers, the two documents are in total agreement (Footnote 6). This suggests that the scribes were faithful and skilled.

The translation that I have worked on is presented in a spreadsheet. It includes the Greek from the Heiberg text, sentence by sentence; some unicode characters, a direct translation using letters similar to what Archimedes used and the Berggren translation. There are also many notes interspersed throughout explaining the numbers, and other features.

The proposition actually includes two algorithms. The first calculates the upper bound for "Pi" using a polygon that circumscribes the circle. The second calculates the lower bound for "Pi" using a polygon that is inscribed in the circle. The ratio is circumference to diameter, so it makes sense that the larger ratio would refer to the construction for the larger polygon.

Both Greek texts contain two figures, one for each algorithm, but I have not seen the figures from the Palimpsest. Those figures exist, but are still privately held. The two figures in the Heiberg document illustrate the third proposition text, showing part of a circle and the appropriate triangle and labels. At first glance, it appears that the first figure is not drawn to scale. But in actuality, the figure is pretty well protrayed. The letters on the relevant points in the figure are in Greek, so the translation of the letters is also necessary. Archimedes is not consistent on lettering the two figures, so perhaps he wrote them at different times. He is also not consistent in putting a value on the square root of 3, which has two different approximations.

The document begins with a statement of the full proposition to be proved (and we add a second fraction in '[...]' to help compare the limits): "The ratio of the circumference of any circle to its diameter is less than 3 1/7 [which is 3 10/70] but greater than 3 10/71."(Footnote 7) Then there are five sections with about 45 sentences total.

III. The Circumscribed Polygon.

For the first algorithm, imagine that Archimedes is collecting his thoughts to instruct us on the relationship of the circumference of the circle approximated by a polygon outside the circle to the diameter of the circle approximated by its radius. In proposition one, Archimedes proved that the area of a circle is equal to the area of a right triangle whose short vertical side is the length of the radius and whose long horizontal side is the length of the full circumference of the circle. (Footnote 8) From this we know that Archimedes will most likely be building from triangles because that is how he has already envisioned the circumference of the circle.

A. What to Draw First.

Puzzling about this, I imagine that Archimedes draws a circle, draws a horizontal radius from center to the left edge, and then plays around with unfolding the length of the circumference. He hits upon a key right triangle to investigate. He builds the triangle with the radius as its base (Footnote 9). The triangle setup has surrogates for both dimensions of interest: its base is the radius drawn as a horizontal line to the left of center (the radius is the surrogate for the diameter) and its left side is the vertical tangent line touching the circle at the end of the radius (this tangent line will be come a surrogate for the circumscribing polygon because it will be one side).

Now Archimedes labels the center E and the left end of the radius G. The vertical line that is tangent extends above and below G. Archimedes selects a point Z (the symbol 'Z' is Greek for the capital letter eta) above the radius and outside the circle on the line tangent to the circle at G such that the line from Z to the center E makes a thirty (30) degree angle at the center E.

If that triangle ZEG were duplicated below the horizontal radius using a point Z', the two segments of the tangent line ZE and EZ' will make one side of the hexagon that circumscribes the circle and has six sides with angles of sixty (60) degrees.

B. Plan for the Iterations.

At this point, Archimedes tells us his plan to iterate toward the actual value of "Pi". He decides to bisect the central thirty (30) degree angle four times in succession. Accordingly, the number of sides in the polygon is increased from 6 to 12, 12 to 24, 24 to 48, and from 48 to 96. The central smallest angle in the triangle keeps getting smaller as the number of sides increases: 30 to 15, 15 to 7 1/2, 7 1/2 to 3 3/4, 3 3/4 to 1 7/8 degrees. The polygon perimeter, calculated by the length of a side times the number of sides keeps on getting slightly smaller (more sides, but a length of side that is slightly less than half what it was before). The ratio of the polygon perimeter to the diameter is getting closer to the actual ratio "Pi" from above.

We do not yet know the length of the first polygon side. Archimedes will show us, and he relies on a proof from Euclid VI. 3, which is described below. We will agree we know the lengths of all the sides in the upper triangle ZEG in relative terms. Archimedes has the numbers for such a triangle. He has no reason to use a unit circle. He uses a circle for which he has measurements, and identifies the actual lengths of the sides by pairs of the letters assigned to the triangle. The numbers are expressed in a Greek number system (Footnote 10).

IV. The Greek Number System.

The Greek number system uses the 24 Greek letters of the Greek alphabet and four additional characters, dubbed as 'Phoenician' in one reference. (). (Footnote 11). A close inpsection of where the 'Phoenician' characters occur shows a curious pattern of missing ninth items (as well as missing the digit for six). The mystery of why/when the 'Phoenician' characters were added (and how the math got extended) will have to wait for a further research effort; it is likely that the advent of trade brought the need for the expansion of the role of digits to include 9, 90, 900, etc., but what the math was before and after is not clear.

The following table contains the 24 letters by letter name (as per the character map program of windows), symbols (capital letter and small letter) and meaning in the ancient Greek number system. It also includes the additional Greek symbols (included in the Microsoft character map unicode subset "Greek") needed to complete the number system which Archimedes used:

The Greek Number System In Use by Archimedes

Alphabet section:

Name:	Capital Letter:	Small Letter:	Role in Greek Number System:
alpha	Α	α	one;
beta	Β	β	two;
gamma	Γ	γ	three;
delta	Δ	δ	four;
epsilon	Ε	ε	five;
Note 1: skip six -- see additional characters section below;			
zeta	Ζ	ζ	
se v en			
eta	Η	η	eight;
theta	Θ	θ	nine;
iota	Ι	ι	ten
kappa	Κ	ϰ	twenty
lamda	Λ	λ	thirty;

mu		μ	forty (but only for lower case);
		Μ	ten-thousand, but only for upper case, see Note 6 ;
nu	Ν	ν	fifty;
xi	Ξ	ξ	sixty;
omicron	Ο	ο	seventy;
pi	Π	π	eighty;

Note 2: skip ninety -- see below;

rho	Ρ	ρ	one-hundred;
sigma	Σ	σ	two-hundred;
tau	Τ	τ	three-hundred;
upsilon	Υ	υ	four-hundred;
phi	Φ	φ	five-hundred;
chi	Χ	χ	six-hundred;
psi	Ψ	ψ	seven-hundred;
omega	Ω	ω	eight-hundred;

Note 3: skip nine hundred -- see below;
Note 4: one-thousand is written by placing the letter for one (alpha) in front of the 1-3 digits for 1-999; but I do not know how to write simply one thousand!
Note 5: two-thousand to nine-thousand is written by placeing the letter for 2-9 in front of the 1-3 digits for 1-999; same caveat for numbers ending in zero;
Note 6: ten-thousand is given above: M;
Note 7: units of M are augmented by placing a single letter for 1-9 above or preceding M;
Note 8: no numbers using upper case were found in the Archimedes document, so it is possible that the upper case letters are not used for numbers except for M, which may give a clue to forming even higher numbers using upper case letters!
Additional characters section:

Name:	Capital:	Small:	Role:
stigma		ϛ	s
s			
digamma		F	s
koppa		ϟ	ninety;
sampi		ϡ	nine hundred

Additional character not in unicode: (I have no reference for how fractions work yet!)

?	L	fraction? 1/2? (it is followed by double number sign)
?	L" δ''	1/4?

V. The Circumscribed Polygon Algorithm.

Archimedes selects the right triangle ZEG with point Z on the tangent line outside the circle. In this right triangle, the hypotenuse EZ makes a 30 degree angle at the center E. It is the Z end of the hypotenuse that will be moving down with each iteration and smaller central angle. Archimedes gives us the measurements of the triangle as ratios of the sides. He uses the center of the circle E as the top of the reference triangle. Then, the sides become: side EZ (the longer side, which is the hypotenuse of the right triangle) and side EG (the shorter side, which is the side adjacent to the 30 degree angle in the right triangle). The base of the reference triangle is the segment outside the circle (the side opposite to the 30 degree angle in the right triangle).

The algorithm has nine parts in total, including parts repeated for the five iterations. The first two parts give known measurements of two important quantities. There are also nine parts in the second algorithm, but there are some differences: the overlaid triangle is wholly within the circle, the overlaid triangle includes the diameter of the circle, not just the radius, and it is not the hypotenuse that is moving down; instead, it is the adjacent side.

A. Measuring the First Part: the Ratio of the Hypotenuse to Angle ZEG's Opposite Side (Ratio ^).

Archimedes gives his first measurement. In Proposition 3, the second sentence in section 2 is translated from the Greek as:

"The EZ to ZG surely is as 307 is to 153."

Note that Archimedes specifies the hypotenuse as "EZ", which starts at E (the center of circle) and goes to Z (the point on the tangent line outside the circle). He specifies the length of the line EZ in ratio to the vertical segment ZG. He never really says he's using a circle of a particular radius size. But we notice that 2 * 153 = 306, which is about 307. We know from Euclid that a 30-60-90 right triangle with angle ZEG 30 degrees has an hypotenuse/opposite side ratio of 2:1. Using 307/153 increases the ratio slightly above 2:1. But since we are computing an upper bound for "Pi", this error does not affect the algorithm adversely (Footnote 12).

The second point to review regarding the quoted measurement is that it is counter-intuitive in geometric terms. From the simple idea of a triangle ZEG (which could be called EGZ or GZE) and is easy to visualize, we are asked to visualize just two line segments. Assuming order matters, we are to envision EZ first, which means the hypotenuse drawn from center to outer point. That is a left-going, rising segment. Then we are asked to drop down from the point Z until we meet the radius at G. This is a downward going line stroke, very natural if it were the first segment, but somewhat disconcerting as the second. The image is a jarring, two-sided figure (an incomplete triangle).

The third point is that the quoted measurement is a ratio of two three-digit numbers. For us, the ratio in this type of triangle is usually given by the simpler ratio 2:1, but Archimedes regards it as 307:153. Okay, we say, Archimedes, you want us to actively know that you measured some circle's circumscribed (or overlaid) triangle's hypotenuse as 307 and the resulting vertical side came out to 153.

B. Measuring the Second Part: the Ratio of Angle ZEG's Adjacent Side to Opposite Side (Ratio L).

Archimedes gives us his second measurement. In Proposition 3, the third sentence in section 2 is translated as:

"But the other side EG to vertical side GZ surely is as which 265 to 153 which is less."

Now we are asked to visualize the radius EG (which is left-going from center E to point G) and the rising vertical segment GZ on the line tangent to the circle at G. This image makes an 'L' which we are asked to mentally draw going the reverse direction. The direction is necessary to get the ratio correct. Archimedes wants us to travel 265 units to the left along the radius segment and then 153 units upward along the vertical segment until we reach point Z.

Berggren suggests that Archimedes uses these particular numbers because he has already worked out the value of the ratio of the these lengths, which is not an easy ratio because it equates to the square root of three. (Footnote x). The three-digit numbers give a certain amount of precision.

Archimedes points out that this ratio (265:153) is "less" than the previous ratio (307:153). We check it: the first ratio divides out to 2 remainder 1/153 compared to second ratio, which divides out to 1 remainder 112/153 and we agree that the second ratio is less than the first. Put another way, this means that the radius side is shorter than the hypotenuse, which is of course true.

C. Setting Up

1. Assigning Symbols for Working with the Overlaid Right Triangle.

Let's put this in symbols so we can summarize easily. The ZEG triangle is:

◺

The first part's (counter-intuitive) draw symbol is:

↓◺↖

and we can use '^' for a shortened version indicating ratio EZ to ZG.

The second part's draw symbol is:

↑◺←

and we can use 'L' for a shortened version indicating ratio EG to GZ.
We know that the second part's ratio is less than the first part's ratio, so we write:

ratio L < ratio ^

where L and ^ are ratios of longer sides of the reference triangle to its base.

D. Measuring the Third Part: Adjacent Side (Radius) to New Opposite Side (Ratio L')

After exploring what is known about the overlaid triangle, Archimedes takes the step to advance the process of iterating. The next overlaid triangle will be created, and it too will have a reference triangle with top angle at the center. Using it, Archimedes is able to calculate the ratio for the adjacent side to the new opposite side. Three steps (and a section to resolve a discrepancy) are required.

1. Bisecting the Angle ZEG.

First, Archimedes will bisect angle ZEG and assign letter (H) to the new point on the tangent line. The direct translation is:

"Bisect therefore the angle ZEG and draw EH to divide the angle evenly."

The new right triangle HEG is no longer a 30-60-90 right triangle. It is a 15-75-90 right triangle with the smallest angle again at the center E. How much of the above argument still holds?

Here is Archimedes' next statement translated directly:

"Therefore it is, as far ZE to EG, as ZH to HG and also alternately and also connected."

With this cryptic statement, Archimedes works his way toward the new base segment size by applying Euclid. The first ratio ZE to EG for the first reference triangle (with top at E) is the ratio of the two sides (hypotenuse to adjacent side in overlay triangle ZEG). Note that we did not look at this ratio before. The second ratio ZH to HG, is the ratio of the two newly-defined segments within the base of the first reference triangle. We would like to know the measurement of HG, also called GH.

And what about the surrogates for circumference and diameter? Radius EG has not changed, but the new vertical segment GH in place of GZ will give a better surrogate for the circumference. Can Archimedes deliver the new ratio of the polygon circumference to circle diameter in terms of the prior known quantities?

2. Examine the Construction and Apply Euclid VI. 3 the First Time.

Comparing the vertical bases in the two reference triangles, we see that the second one (HG) is shorter than the first one (EG) because the point H has moved down the tangent segment. However, H is NOT halfway down the tangent segment. It is outside the circle, on the same tangent line, closer to Z than to G. So HG is the larger lower segment of base ZG.

Archimedes does not mention Euclid. The translation to Latin mentions some other mathematicians known to Archimedes. (Footnote) Berggren mentions Euclid VI. 3 which proves that the ratio of the sides of the reference triangle is equal to the ratio of the base segments. If the sides were equal, the base segments would be equal, but here they are not. How can Archimedes get a measurement?

Above Archimedes told us above that the ratio of these two base segments to each other is the same as the ratio of the two longer (unequal) sides. This is supported by a direct application of Euclid VI. 3, which states:

If an angle of a triangle is bisected by a straight line cutting the base, then the segments of the base have the same ratio as the remaining sides of the triangle; and, if segments of the base have the same ratio as the remaining sides of the triangle, then the straight line joining the vertex to the point of section bisects the angle of the triangle. (Footnote 13)

so we know that the base segments ZH and HG in the first triangle ZEG have the same ratio as the triangle sides ZE (hypotenuse) and EG (radius). Okay, that's clear. We know a value for the ratio

ZH to HG. But in addition, we know that HG is the larger segment; this is because the hypotenuse is longer than the radius, so the given ratio is less than one. We could compute the ratio ZE to EG from the first two parts discussed above, and thus have the value for the ratio ZH to HG, but we do not need this ratio. The goal is to get a value for the next smaller base (HG) of the new reference triangle.

3. Apply Euclid VI. 3 Again, Elegantly.

Next, Archimedes uses Euclid VI. 3 again and moves the argument in an original direction. In the previous section, he was looking at ratios of separate continguous segments (ZE to EG, ZH to HG). Now he looks at ratios of composite: separate ratios (ZE+EG: EG, ZH + HG, HG). We infer that Archimedes added the second side (EG or HG) to the first side in both ratios:

$$ZE + EG:EG = ZG:HG$$

which is easy to see on the left-hand side, but trickier to see on the right-hand side. There, ZH (upper part) has been replaced by ZG (the whole segment, ie ZG =ZH+HG). This is an elegant extension of Euclid's result.

Furthermore, we infer that Archimedes used the equivalence a/b=c/d <-> a/c=b/d to rearrange the above result so it expresses the ratio of sides in the new triangle HEG in terms of the known sides in the triangle ZEG. We note that these steps are all contained within Archimedes Sentence 6! Sentence 6 translated directly states:

"As far as therefore ZE connected to EG in ratio to ZG is the same as EG in ratio to GH." and knowing that the 'connection' is addition gives:

"Therefore, these are the same ratios: ZE plus EG in ratio to ZG and EG in ratio to GH." which could be put more graphically as:

"Slant side (hypotenuse) plus radius (adjacent side) is to base (side opposite starting angle) as radius (adjacent side) is to next base."

What a switch! Here's what it looks like now using lettered segments from above (compare to the prior equation):

$$ZE + EG: ZG = EG: HG$$

From here, Archimedes reverses the letters of the target segment HG to GH and puts the two addends of the numerator separately over the same denominator:

$$(ZE:ZG) + (EG:ZG) = EG:GH$$

which (with another letter reversal) is recognized as the sum of the first part EZ:ZG and the second part EG:ZG. Using the named ratios, this gives:

ratio ^ + ratio L = EG:GH

which (with another letter reversal) is recognized as the ratio of the adjacent side to the new base. This is the new better ratio of surrogates computed from prior parts. We name this new ratio as follows:

ratio L' = ratio^ + ratio L

and note that could be computed easily from the first two parts given above:

ratio L' = 307/153 + 265/153 = 572/153

which is just what is needed.

Next, compare this to what Archimedes writes in section 2, sentence 7 (Footnote 14):

"ωστε ή ΓΕ προς ΓΗ μείζονα λογον εχει, ήπερ φοά προς ρνγ."

which translates directly as:

"Thus, GE in ratio to GH greater is for sure, more than <number φοά> in ratio to <number ρνγ>" where the numbers are:

φοά which is letter/numbers: phi 500, omicron 70, alpha 1; and

ρνγ which is letter/numbers: rho 100, nu 50, gamma 3;

which gives a minimum ratio:

>571 in ratio to 153

and in which the phrase "greater is for sure" is referring to the prior ratio of surrogates. So Archimedes is saying that the new ratio L' is greater than the prior ratio L, ie 571:153 is greater than 265:153, which it is. (There is, however, the issue of 572 vs. 571, which merits discussion next.)

4. Resolve the Issue with Letter/Numbers for Part Three.

Here we have to pause for a moment on the math. Why is Archimedes writing 571, not 572 as expected? Clearly, the numbers in the Greek above spell out 571 because the one's digit is alpha which stands for 'one.' Looking back at the Part One and Part Two, we decide to check the Greek. We retrieve the letters for the ratios from the Greek text, and translate them using the table given above.

For Part One (in V. A. above), the second sentence in section two includes the two numbers for the ratio ^ (hypotenuse to opposite side). The ratio is given as:

τζ προς ρνγ΄

which gives us numbers translated using the above table as (bold, highlighting shows the number of interest):

τζ

which is letter/numbers: tau 300,**zeta 7**; and

ρνγ which is letter/numbers: rho 100,nu 50;gamma 3

which gives the ratio 307 to 153. But in section V. A. above, we did note that 153*2=306, not 307! We conclude that it is more likely that Archimedes is not using the special character for six. Dropping that character out, gives (highlighting and bold on the number of interest):

τζ which is letter/numbers: tau 300,**zeta 6**;

which makes the ratio 306 to 153. This combined with the given Part Two result adds correctly to 571. For completeness, we will check Part Two.

For Part Two, the third sentence in section two includes the two numbers for the ratio L (adjacent side to opposite side). The ratio is:

σξέ προς ρνγ΄.

which gives us numbers and translations:

σξέ which is letter/numbers:sigma 200, chi 60, epsilon 5; and

ρνγ which is letter/numbers: rho 100, nu 50, gamma 3;

which gives the ratio 265 to 153. All digits are low enough that the placement of the special characters (for six, for ninety and for nine-hundred) do not come into play.

Therefore, we disagree with the translation in Berggren for sentence two in section two. It is not 307, but 306. The reference from which the table for the Greek math was constructed is not precisely what Archimedes was using. We will exercise care in translating any number above five! Some future work could get to the bottom of this curiosity. It is likely that the number systems were somewhat defined on the fly, but we emphasize that the argument and the convergence of the algorithm are unaffected by this detail. This observation underlines the robustness of the iterative geometric/ratio method that Archimedes employs, at least for this first pass in Part Three.

E. Measuring the Fourth Part: New Hypotenuse to New Opposite Side (Ratio ^').

For the next iteration (assuming there is one), in addition to new ratio L' computed above, Archimedes will also need the new ratio ^', which can be computed using the Pythagorean theorem.

Archimedes does not mention Pythagorus, but accomplishes the goal in two simple sentences. In section 2, his sentence 8 is:

ή ΕΗ άρα προς ΗΓ δυνάμεί λογον εχει, όν Μ ͵ϑυν΄ προς Μ ͵γυϑ΄.

which is similar in form to some earlier sentences. We see that the subject is ratio EH to HG (new hypotenuse to new opposite side). The next word ('δυνάμεί') means strengthen, so it the new ratio is for sure strengthening or increasing compared to the prior ratio.

Then we see the numbers for the ratio. This time both numbers start with M, the indicator for ten-thousand, so the numbers are large. After studying the matter, it appears that in the text, the multipliers for the ten-thousands have been omitted. But to begin, we translate the letter/numbers that are written for each number in the ratio. In each, the numbers following the Ms give the less significant digits of the numbers. For the first one, those are: theta upsilon nu and for the second one: gamma upsilon theta. In the first one, the theta is less than the upsilon, so we have theta thousands, followed by upsilon nu, which gives: nine thousand + four hundred + fifty. In the second one, the gamma is less than the upsilon, so we have gamma thousands, followed by upsilon theta, which gives: three thousand + four hundred + nine. Without the lost superscripts, this gives the ratio 19,450 to 13,409.

We infer that Archimedes has applied the Pythagorean theorem to calculate new ratio ^' from the other two sides. This is confirmed by analysis of the next sentence. In section 2, sentence 9, Archimedes writes:

μήκει άρα, φϟ ά ή΄ προς ρνγ΄.

which contains (after the first two words) the ratio that reads: phi koppa alpha (phi 500, koppa 90, alpha 1) in ratio to rho nu gamma (the usual) or: 591/153. Working backwards, how did Archimedes get the 591, the numerator in the value for the ratio?

Archimedes seeks the ratio EH to HG, but does not know EH, an hypotenuse. He squares the ratio and replaces EH squared by (HG squared + EG squared). This gives:

(EH:HG) squared =EH squared: HG squared = (HG squared + EG squared): HG squared

so the sides to be squared are known quantities (from ratio L'):

>(571 squared + 153 squared): 153 squared

which gives:

>(326,041 + 23,409): 23,409 = 349,450: 23,409

and we notice that the digits after the 10,000s place in both the numerator and the denominator match what was translated above. (I have the impression that the palimpsest translation may have included superscripts, so I could go back and check that. We need '34' in the first and '2' in the second. That would be superscripts lamda delta and beta. For the moment, the agreement is assumed sufficient.)

We note that this ratio ^', given as 591/153 above, is indeed larger than the prior ratio ^, which was: 571/153.

E. Repeating the Sequence for Additional Iterations (Parts Five and Six, Seven and Eight, and Nine) and Stopping.

Next, Archimedes continues with the second bisection of the initial 30 degree central angle. As illustrated above, Part One and Part Two (ratio L = r/ZG = 265:153) were used to calculate Part Three:

ratio L' = ratio ^ + ratio L

ratio L' = r/HG = 571:153

and also Part Four was calculated from Part Three:

ratio ^' = f(ratio L')

and these methods could be repeated with a different set of inputs. Thus, it follows that Parts Five and Six calculate:

ratio L'' = r/TG = 1162:153

from ratio ^' and ratio L' in the same manner as Part Three, and ratio ^" from ratio L" in the same manner as Part Four.

Similarly, for the third bisection, Archimedes gives us Parts Seven and Eight that calculate:

ratio L'" = r/KG = 2334:153

from ratio ^" and ratio L", and ratio ^'" from ratio L'".

Finally, for the fourth bisection, Archimedes gives us Part Nine that calculates:

ratio L"" = r/LG = 4673:153

from ratio^'" and ratio L'". Because of stopping, ratio ^"" is not calculated.

In each case, the numbers for the ratios compound according to the algorithms. The iterative process could be continued farther, of course. Thus, Archimedes constructed the first part of the first iterative algorithm for the calculation of "Pi", stopping after four bisections.

F. Getting the Result: the Value of "Pi".

Then Archimedes reviews the angle bisections, stating that with a central angle starting at 1/3 of a right angle and four bisections, the central angle at the end is 1/48 of a right angle. Then he constructs a triangle identical to LEG below the radius using point M, and calls it triangle MEG. He combines the two triangles to make one section of a polygon. He tells us that the central angle for both triangles combined is 1/24 of a right angle. Also, this triangle has a tangent segment LM that is 1/96 of the perimeter.of the polygon.

At this point, Archimedes writes a sentence that is the longest of any by far, which sums up the insight needed to relate the overlaid triangles to the circle using the surrogates. The section 2, sentence 22 translates directly as:

"Consequently, the radius EG in ratio to the GL all the more so is worth 4673 to 153, then double radius EG giving diameter AG, then GL (the half side) doubled gives LM (the side), and the ratio of the diameter AG to the 96 sided perimeter of polygon is greater than 4,673 to 14,688."
which shows that Archimedes sees that the ratio of the radius to the half edge of the polygon would be the same as the ratio of the diameter to the full edge (4673:153). What is required after that is multiplying by 96, which gives ratio 4673:14688. This ratio is for diameter to perimeter of the polygon, but the definition of "Pi" requires the reverse: circumference of the circle to diameter. Thus far, the ratios have been greater than the true answer, but when the ratio is reversed, the ratio is less than the true answer. Then, in sentence 23, Archimedes reverses the ratio by stating, translated directly:

"But this is threefold, exceeding it by ratio of 667 to 4,673 less than the threefold plus seventh. "
in which Archimedes has divided 14,688 by 4,673 to get 3 remainder 667/4,673, and reduced the fraction to 1/7. The actual result for 4,673/667 is 7 remainder 4/667, which means that the 1/7 is 1/7.007 so the answer is greater than 3 by less than 1/7.

In the final sentence 25, translated directly, Archimedes concludes:

"Thus the polygon to around the circle of its diameter is threefold but less than to a seventh greater than."
and that concludes Archimedes' first algorithm, the circumscribed case.

VI. The Inscribed Polygon.

For the second algorithm, imagine that Archimedes is collecting his thoughts to instruct us on the relationship of the circumference of the circle approximated by a polygon inside the circle to the diameter of the circle itself.

A. What to Draw First.

Archimedes draws a circle with a horizontal diameter AG (actually GA, so the G is at the left end and the A is at the right end). This time, Archimedes selects an overlay triangle that includes the diameter as one of its sides (for the circumscribed algorithm, the overlay triangle included only the radius). Archimedes' triangle has the same 30 degree starting angle, but not at the center of the circle; instead, it is at A. The triangle is again a right triangle, but the right angle is now at the top, where the line making the 30 degree angle at A on the diameter going up and leftward touches the circle. We call that point B. The third side is the shortest side, and it will be the edge of the polygon. Thus, there are no surrogates this time.

Being a man of few words, Archimedes accomplishes all this with one sentence, section 3, sentence 1:

ἔστω κύκλος, καὶ διαμετρος ἡ ΑΓ, ἡ ὑπὸ ΒΑΓ τρίτον ὀρθῆς.

which translates directly as:

"We have this circle, with diameter the AG, the angle BAG being one-third of a right angle." which requires the additional information of how the points are labelled, which is conveyed in the figure.

B. Plan for the Iterations.

As before, one point of the triangle will drop lower as the 30 degree angle is bisected. It it the top point of the triangle and on the circle edge that drops lower. The top angle is always the right angle, so the hypotenuse of the triangle is the diameter and does not move.

VII. The Inscribed Polygon Algorithm.

As before, the algorithm has nine parts in total, including parts repeated for the five iterations. The first two parts give known measurements of two important quantities. As before, the third and fourth parts are the parts repeated for the iterations. However, as before, the key ratio the delivers the answer, is the ratio L, which is the ratio of the side adjacent to the 30 degree angle to the side opposite to it.

The figure in the text has the overlaid triangle drawn above the diameter. It is possible, however, to (flip it about the diameter and) draw it below the diameter. In which case, the ratios pictures are entirely analagous to the ones for the circumscribed case. In this paper, no figures are included, so we will continue to imagine the overlaid triangle placed above the diameter.

A. <u>Measuring the First and Second Parts (Ratio ^ and Ratio L) in Triangle BAG.</u>

This time Archimedes is brief about the basic measurements. He approximates the square root of 3 with a different number than before. He may have written (or designed) this algorithm at a different time, or just have been accustomed to using different sets of ratios on different days!

Sentence 2 reads (first phrase):

ἡ ΑΒ ἄρα πρὸς ΒΓ ἐλάσσονά λόγον ἔχει, ἡ ὃν ͵ατνα´ προς ψπ´

which translates as:

The ratio of AB to BG (adj side to opp side) is on the lesser side than, being 1,351 to 780

and continues (second phrase):

[ἡ δὲ ΑΓ ἄρα πρὸς ΓΒ, ὃν ͵αφξ´ προς ψπ´].

which translates as:

[the ratio of AG to GB (hyp to opp side) being 1560 to 780].

and which tells us:

1. the diameter is AG (actually GA, with point A at the right-hand end);
2. the top point of the triangle is B;
3. there is a right angle at B;
4. there is a 30 degree angle at BAG (on the right-hand end);
5. thus the first overlaid triangle can be referred to as triangle BAG;
6. ratio L (adjacent side AB to opposite side GB) is 1351: 780; and
7. ratio ^ (hypotenuse AG to opposite side GB) is 1560: 780, which is the larger ratio, as it should be for the hypotenuse.

B. <u>Measuring the Third Part: New Adjacent Side to New Opposite Side (Ratio L').</u>

After exploring the first overlaid triangle, Archimedes takes the step to advance the process of iterating towards "Pi". He creates the next overlaid triangle, and it too will a second view as a reference triangle with top angle the bisected angle, at the right-hand end of the diameter. For the inscribed case, the two overlaid triangles delineate three additional similar triangles that Archimedes can use.

Using these similar triangles and Euclid VI. 3, Archimedes is able to calculate the ratio for the new adjacent side to the new opposite side. Three steps are required.

1. Bisecting the Angle (First Time: Angle BAG).

Archimedes bisects angle BAG by drawing an appropriate line from A to the circle. Then he assigns a letter (first time: H, Greek eta) to the new point on the circle. This creates a second overlaid triangle, which can be called HAG.

As the line is drawn from point A on the circle at the right-hand end of the diameter up and leftwards toward H on the circle, it crosses the shortest side of triangle BAG. Archimedes calls that intersection point Z (zeta). The line AZ divides the triangle BAG into triangles BAZ and GAZ.

When the third side of this second overlaid triangle HAG is drawn (from the left-hand end of the diameter G upwards to the point H), another small triangle GHZ is created.

Thus, the first overlaid triangle BAG and second overlaid triangle HAG can be separated into three 15-75-90 triangles:

1. small triangle GHZ: has right angle at H on the circle;
2. second overlaid triangle HAG: has right angle at H and overlaps triangle BAG, and;
3. medium triangle BAZ: has right angle at B on the circle, and is the upper half of triangle BAG.

Archimedes needs to find a measurement for the side HG in the small triangle GHZ. To do so, he establishes that the three 15-75-90 triangles listed are all similar.

2. Examine the Construction and Apply Euclid VI. 3 the First Time.

In the triangles, Archimedes finds five ratios that are the same:

1. In Triangle HAG: consider ratio AH:HG (adjacent side to opposite side or ratio L');
2. In small Triangle GHZ: consider ratio HG:GZ (adjacent side to opposite side)
3. In medium Triangle BAZ: consider ratio AB:BZ (adjacent side to opposite side)
4. In Triangle BAG: consider ratio AB:BZ (adjacent side to upper part of opposite side)
5. In Triangle BAG: consider ratio AG:GZ (hypotenuse to lower part of opposite side)

Archimedes uses similar triangle arguments for the equality of the first three ratios. Then he uses identity for the third and fourth, but notices that the fourth is a side:part of base ratio, that can be used to apply Euclid VI. 3. That gives the ratio AG: GZ, which includes the diameter and the hypotenuse of the small Triangle GHZ, but not the needed adjacent side, segment GH.

3. Apply Euclid VI. 3 Again, Elegantly.

Archimedes then uses the same argument as in the case of the circumscribed polygon. He expands the ratio by adding a similar side to the numerators. But instead of needing to rearrange inner numerator and denominator, he has only to notice that the two base segments make the third side. From VII. A. above, we know ratio L= adj/opp= BA:GB =1351:780.and ratio ^ =hyp/opp=AG:GB =1560:780. From VII. B. 2 above, using 1 and 4.:

ratio L' = AB: BZ = AB + AG: BZ + GZ = AB + AG:BG = AH:HG (new adj.: new opp.)

and so:

ratio L' = AB:BG + AG:BG = 1351:780 + 1560: 780 = 2911:780

which means:

ratio L' = AH:GH < 2911:780

C. Measuring the Fourth Part: Hypotenuse to New Opposite Side (Ratio ^').

From ratio L' above and using the Pythagorean theorem as in the case of the circumscribed polygon, Archimedes can find AG:GH, which is ratio ^'. We have:

ratio ^' = AG:GH < 3013 1/4: 780

F. Repeating the Sequence for Additional Iterations (Parts Five and Six, Seven and Eight, and Nine) and Stopping.

Next, Archimedes continues with the second bisection of the initial 30 degree angle. Then, the results for B and C above can be used to calculate parts five and six, ratio L" and ratio ^".

First a review:

1. From A. above:

6. ratio L (adjacent side AB to opposite side GB) is 1351: 780; and

7. ratio ^ (hypotenuse AG to opposite side GB) is 1560: 780

2. From B. Measuring the Third Part: New Adjacent Side to New Opposite Side (Ratio L').

ratio L' = AH:GH < 2911:780

3. From C. Measuring the Fourth Part: Hypotenuse to New Opposite Side (Ratio ^')

ratio ^' = AG:GH < 3013 1/4: 780

Now using 2. and 3. we get (and the first one is not in Berggren, but the other two are):

ratio L"=AT:GT=<5924 1/4: 780=<1823:240

ratio ^"=AG:GT=<1838:240

This is repeated again for the third bisection, giving parts seven and eight, with ratio L"' and ratio ^"'. (Here I got numbers that are too small (17:66, 11 1/40:66, 19 1/6:66), so I am relying on Berggren):

ratio L"'=AK:GK=<1007:66

ratio ^"'=AG:GK=<1009 1/6:66

Then Archimedes does his fourth bisection of the angle, giving part nine, with ratio L"".
(Here my number agrees with Berggren):

ratio L""=AL:GL=<2016 1/6:66

Thus, Archimedes constructed the second part of the first iterative algorithm for the calculation of "Pi", stopping after four bisections.

G. Getting the Result: the Value of "Pi".

Then Archimedes reviews the angle bisections and gets the ninety-six sided polygon. He reverses the ratio and multiplies 66 * 96 = 6336, so the final ratio is: 6336:2017 1/4. Removing three, leaves 284 1/4 to 2017 1/6, which is greater than 10/71, setting the lower bound for "Pi".

Thus the circumscribed algorithm gave us greater than 3 by less than 1/7 or 10/70; and the inscribed algorithm gave us greater than 3 by more than 10/71. The fraction 10/71 is 10 out of more than 70, so it is smaller than 10/70. Thus Archimedes has bounded "Pi" in five iterations to an interval of size 10/71 to 10/70.

VIII. Conclusions and Further Work.

We thus conclude:

1) Archimedes had a gift for getting to the heart of the problem geometrically: For each algorithm he selected an appropriate right triangle to overlay the circle with which he was working.

2) This corresponds to selecting the variable with which to count iterations. Archimedes did not count them, however; he simply repeated the actions required to generate the new result. In the circumscribed algorithm, he used bisections of the central angle that corresponded to shorter and shorter triangles with left sides tangent to the circle at the end of the radius. In the inscribed

algorithm, he used bisections of the angle between diameter and a line to the circle's edge that corresponded to shorter and shorter triangles with left sides between diameter and circle's edge.
3) The top corner of each triangle dropped down, but never as much as half; thus the calculation of the upper bound was properly controlled to remain greater than the true valueof "Pi". For the inscribed algorithm, the lower bound was similarly well-controlled.
4) The ratio of surrogate quantities was computed directly from the previous set. Ratio L' was computed first, giving the new result. If the iteration was to continue, ratio ^' would be computed to allow the algorithm to have both quantities to pass along. The inscribed algorithm did not require surrogates because the radius and the side of the polygon were calculated directly.
5) Archimedes' procedure is remarkably like the use of an infinite stream with delay to calculate "Pi" because the functional programming algorithm is approximaged by the position and handling of the next overlay triangle construction, the next ratios manipulation, and the calculation of the necessary surrogate ratio and its expansion into next result. Only if another iteration is selected, would the second ratio be calculated.
6) In visual terms, the result is given without needing the actual numbers. The ratio L' simply is the quantity desired, as are ratio L'', ratio L''', ratio L''''. Staying in triangle geometry and ratios permits this simple algorithm to unroll, brilliantly.
7) Actual numbers can be generated, of course. They increase in number of places as the central angle is bisected and the ratios are calculated more precisely from above and below by the two algorithms. Further work could address the number of digits generated and how.

IX. Bibliography.
The sources include:

1. Abelson, Harold and Gerald Jay Sussman with Julie Sussman, Structure and Interpretation of Computer Programs, 2nd Edition (MIT Press, Cambridge MA 1996, 657 pp.) - see Ch. 3, "Modularity, Objects and State", 3.5.2 Infinite Streams pp 326-334 and 3.5.3 Exploiting the Stream Paradigm", pp 334- 346 including calculation of Pi slowly and more rapidly.

2. Berggren, Lennart, Jonathan Borwein and Peter Borwein, Pi: A Source Book, 2nd Edition, (Springer: New York, 2000, 735 pp.) - see Ch 3, "Archimedes. Measurement of a Circle", p7 and Ch 4, "Phillips. Archimedes the Numerical Analyst (1981)", p 15.

3. Gibbons, Jeremy, "Unbounded Spigot Algorithms for the Digits of Pi", Source: The American Monthly, Vol. 113, No. 4 (Apr., 2006), pp. 318-328, Published by: Mathematical Association of America, http://www.jstor.org/stable/27641917.

4. Offner, Carl D, "Computing the Digits in Pi", September 6, 2009, 59pp. - private paper.

5. Google books:
http://math.nyu.edu/~crorres/Archimedes/Books/Brown/Heiberg_I_Google_Leland.pdf which contains the text of Dimensions of the Circle in alternating pages of Latin and Greek, p 275 if accessed via the pdf and pages 259-270 if read from the page numbers. Google states the pdf has no US copyright, is in the public domain, but could be copyrighted in another country. The editor of the book is J. L. Heiberg and it was published in 1879.

6. Baltimore Museum: http://www.archimedes.palimpsest.net – the home site for the palimpsest document translation, which is privately owned, and for which credit goes to the Baltimore Museum. - the translation files are many, but the Dimensions of A Circle is in Folios 0000-171v (Arch68r), 0000-171r (Arch68v) and 1777r-172v(Arch69r).

7. Web: http://www.mlahanas.de/Greeks/counting.htm – this site provides a good explanation of Greek number system in operation.

8. wikipedia /wiki/Meas_of_a_Circle -- explains palimpsest.net -- all the data is made available by the owner, who is anonymous.

9. Google tech talk: Infinite possibilities: Archimedes on the web.

10. DeBlois: http://www.cs.umb.edu/~hdeblois/a/arch.xhtml and http://www.cs.umb.edu/~hdeblois/a/arch.pdf .

X. Footnotes.

1. Source 8, wikipedia, has excellent background information on Archimedes 287BC-212 BC.

2. Many sources. Berggren translates the proposition quite literally, as verified by the author and many others. See Berggren, p. 9 and DeBlois spreadsheet, section 1, sentence 1. Palimpsest and wikipedia article also agree.

3. Prof. Offner, source 4, gave an excellent lecture on the delayed computation using streams in LISP.

4. Source 5 Google Heiberg was provided by the librarian at UMASS Boston in response to my request to read Dimensions of a Circle by Archimedes in Greek. The Latin was a pleasant surprise, and I found that it had a quite different approach to the math, as well as many scholarly footnotes about the Greek.

5. Source 6, recommended by Prof. Offner, is a wealth of information, with the translation in xml including markup regarding the accuracy of each word translated. I was not able to get the .xsd files to work properly, so I read the xml directly. If this paper were to be published, I am to write to the museum in Baltimore and notify them that I am using and referencing the work.

6. Prior to the spreadsheet, I printed the Google Heiberg document, source 5, and marked it up for sentence numbering. When Prof. Offner brought the palimpsest document, source 6, to my attention, I endeavored to read it, but found it confusing. The tech talk source 9 was a good starting point. Later I came back to it and did figure out what was where. As I read through the undertext, I marked up the hardcopy with the corresponding passages. By the end of reading the three sections in the palimpsest, all sections of Proposition 3 had been marked up.

7. Source 5 is five pages: p.262, 264, 266, 268 and 270 and the first quote is from p. 262.

8. Source 5, proposition a, is on p. 260.

9. Source 5, proposition gamma, p. 264.

10. In source 5, proposition 3, the first occurrence of a number occurs on page 264. In the first weeks of this project, I could not recognize numbers in the Greek text because they are written with letters and I was assuming that my dictionary skills were not sufficient to find the words. After awhile, as documented on my spreadsheet, odd multiple occurrences of consonants began to catch my attention. There were also many numbers in translated versions, which I thought at first were all inferred! It was great fun to go back and decipher all the numbers, but I did not update my spreadsheet completely. In fact, for the last part of the work, there is only hardcopy notations,

nothing in the spreadsheet yet. The process of putting the Greek into the spreadsheet using the character map in Windows is time-consuming.

11. Source 7 was found by Prof. Offner, who came up with a google search term that found it. I had been trying without success. It was a glorious day when I got this little reference!!!

12. Archimedes does not spell out everything in detail, but he does deserve credit for writing in a concise, balanced manner, so you know what is in need of your attention. I have included many of my 'direct translations' from the spreadsheet in Source 10, and I am not going to footnote them individually. By direct, I mean as word for word as possible, so the sentence structure is often halting; this allows me to find the minimal statement for an algorithm. The spreadsheet also quotes Berggren, Source 2, to give a more polished translation. I feel the more polished translations eliminate some of the visual quality of Archimedes' work. Berggren did, however, identify the source of some implied math knowledge as Euclid. I googled Euclid to find Euclid VI.3, and enjoyed reading quite a bit of Euclid's work. Somewhere I saw a comment that Archimedes considered Euclid a more basic mathematician.

13. A google search brought up many copies of Euclid's work, including VI.3.

14. Source 5, p264.

15. This matter of the discrepancy in whether Archimedes meant 6 or 7 actually indicates that I need to do more research on the various Greek number systems, but I include it here for two reasons: 1) to illustrate how easy it is to be wrong, given that Archimedes lived long ago and isn't here to verify anything; and 2) to show that Archimedes' work is somewhat independent of the numbers due to his use of ratios. There are many ways to cross-check, and the net effect is that his thought and his algorithms are quite robust. No need to think that without algebra the Greeks were limited! I am continually struck by the thought that we continue to layer computer technology without using algebra except for the deepest levels, and of course there it is essential. So this gap between tech and non-tech actually has several chasms that we need to bridge.

XI. Gratitude. The author thanks Prof. Offner for this research opportunity and his invaluable and cheerful help in shaping this work. Errors are of course my personal responsibility. JHDeB. 11 May 2012 11:50 pm Gloucester MA USA