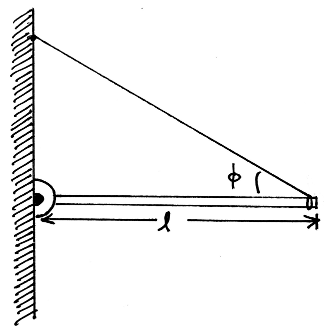
Problem Set 5
Due Friday, March 11

Challenge Problems

CP21.

A thin rod of mass M and length ℓ is connected to a wall by a hinge. The rod is horizontal, and it is held stationary by a massless cable running from its free end to the wall. The cable makes an angle ϕ with the horizontal.

(a) What is the tension in the cable?
(b) What are the components of the force of the rod on the hinge?



Solution.

(a) The tension in the cable can be determine by examining only the torques acting on the rod. If we pick the origin of our coordinates to be at the hinge, and if we pick the positive x-direction point to the write in the figure and let the positive y-direction point upward, then the net torque on the rod in the z-direction is

$$\sum \tau_z = T \sin\phi \ell - \frac{Mg\ell}{2} \tag{1}$$

On the other hand, the angular acceleration of the rod is zero, so we get

$$T \sin\phi \ell - \frac{Mg\ell}{2} = 0, \tag{2}$$

and therefore

$$\boxed{T = \frac{Mg}{2\sin\phi}} \tag{3}$$

(b) To determine the force of the rod on the hinge, we analyze the forces on the rod to first determine the force of the hinge on the rod, and then we note that the force of the rod on the hinge has the same components but with opposite sign because of Newton's Third Law. Drawing a free body diagram for the rod gives

$$\sum F_x = -T\cos\phi + N_x = 0 \tag{4}$$

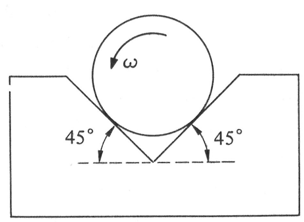
$$\sum F_y = T\sin\phi - Mg + N_y = 0 \tag{5}$$

where N_x, N_y are the components of the force of the hinge on the rod, both of which can be either positive or negative (we don't know their directions quite yet). This is two equations in two unknowns N_x, N_y since we already solved for the tension in part (a). Solving these equations and plugging in our previous result for the tension gives

$$\boxed{N_x = \frac{Mg}{2\tan\phi}, \qquad N_y = \frac{Mg}{2}}. \tag{6}$$

CP22.

A cylinder of mass M and radius R is rotated in a uniform V-shaped groove with constant angular velocity ω. The coefficient of kinetic friction between the cylinder and each surface is μ_k. What is the magnitude and direction of the torque that must be applied to the cylinder to keep it rotating at that angular velocity? You may assume that whatever agent is applying the torque is applying zero net force on the cylinder.



Solution.

We want the angular acceleration of the cylinder to be zero, and this means that $\tau_{\text{net},z} = 0$, where we compute our torques relative to an origin on the z-axis, and we take the z-axis as the axis of rotation.

Excluding the applied forces that keep the cylinder rotating (the ones whose corresponding torques we are trying to solve for) there are five forces that can potentially contribute to the net torque : the normal force on the left contact point $\mathbf{N}_1$, the kinetic friction force on the left contact point $\mathbf{f}_1$, the normal and friction forces on the right contact point $\mathbf{N}_2$, $\mathbf{f}_2$, the force due to gravity $M\mathbf{g}$.

The torque due to gravity vanishes because it can be treated as acting at the center of mass for the purposes of computing torque. The torque due to the normal forces vanish as well because they point along the same lines as their position vectors $\mathbf{r}$ relative to the origin. This leaves us with the torques due to the friction forces. The friction forces oppose the rotation and point tangent to the cylinder in the clockwise direction at a distance R from the center of mass, so assuming the x-y plane is perpendicular to the axis of rotation, their respective torques are

$$\boldsymbol{\tau}_1 = -f_1 R\mathbf{k}, \qquad \boldsymbol{\tau}_2 = -f_2 R\mathbf{k} \tag{7}$$

The sought-after applied torque must sum with these frictional torques to zero, so we get

$$\boldsymbol{\tau}_{\text{applied}} = (f_1 + f_2)R\mathbf{k}. \tag{8}$$

Now we just need to compute f_1 and f_2. We assume that that whatever agent is supplying the external torque supplies no net force, and supposing that the positive y-axis points up, Newton's Second Law in the x- and y-directions tells us that

$$\frac{N_1}{\sqrt{2}} - \frac{f_1}{\sqrt{2}} - \frac{f_2}{\sqrt{2}} - \frac{N_2}{\sqrt{2}} = 0 \tag{9}$$

$$\frac{f_1}{\sqrt{2}} + \frac{N_1}{\sqrt{2}} + \frac{N_2}{\sqrt{2}} - \frac{f_2}{\sqrt{2}} - Mg = 0 \tag{10}$$

and the relationship between kinetic friction and normal force tells us that

$$f_1 = \mu_k N_1, \qquad f_2 = \mu_k N_2 \tag{11}$$

This constitutes four equations in four unknowns f_1, f_2, N_1, N_2. The results for f_1 and f_2 are

$$f_1 = \frac{Mg}{\sqrt{2}} \frac{\mu_k(1+\mu_k)}{1+\mu_k^2} \tag{12}$$

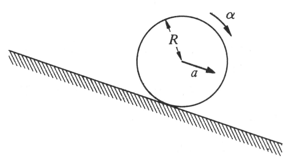
$$f_2 = \frac{Mg}{\sqrt{2}} \frac{\mu_k(1-\mu_k)}{1+\mu_k^2} \tag{13}$$

Adding these together and plugging into (8) gives

$$\boxed{\boldsymbol{\tau}_{\text{applied}} = \sqrt{2}Mg\left(\frac{\mu_k}{1+\mu_k^2}\right)R\,\mathbf{k}} \tag{14}$$

CP23.

A cylindrical drum of radius R rolls down a slope *without slipping*. Its center has velocity v and acceleration a parallel to the slope. What are the drum's angular velocity ω and angular acceleration α in terms of v and a? Justify your answer with an argument that uses a diagram and math. Try watching a cylindrical object roll without slipping on a surface to get some intuition for this.



Solution. In a time Δt, the angle $\Delta\theta$ for which the drum has rolled equals $\Delta s/R$ where Δs is the amount of its circumference has moved along the slope which gives

$$\Delta s = R\Delta\theta. \tag{15}$$

On the other hand, because the drum is rolling without slipping, drawing a picture convinces one that the amount Δs that the circumference has rolled along the ground in time Δt equals the displacement Δx of the center of the drum;

$$\Delta s = \Delta x. \tag{16}$$

If we combine these two facts into one equation and divide both sides by Δt, then we get

$$\frac{\Delta x}{\Delta t} = R\frac{\Delta\theta}{\Delta t}. \tag{17}$$

Taking the limit $\Delta t \to 0$ on both sides therefore gives

$$\boxed{v = R\omega}. \tag{18}$$

Taking the derivative of both sides with respect to time then gives

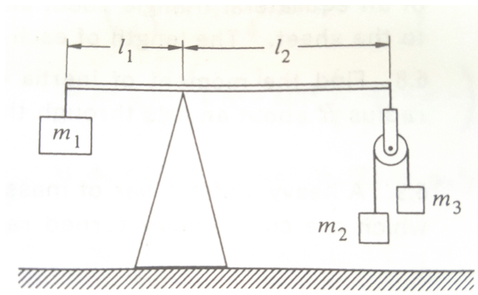
$$\boxed{a = R\alpha}. \tag{19}$$

CP24.

A pivoted beam has mass m_1 suspended from one end and an Atwood's machine suspended from the other with masses m_2 and m_3 suspended on either side. The frictionless pulley has negligible mass and size. Find the relation between.

(a) Find the relation between m_1, m_2, m_3, ℓ_1, and ℓ_2 which will ensure that the beam has no tendency to rotate just after the masses are released.

(b) What would you predict the relation would be in the case that all three masses are equal? Does your answer from part (a) agree with that prediction?



Solution.

(a) We take the location of the vertex of the fulcrum to be the origin and let the positive x-direction point to the right and the positive y-direction point vertically upward. The tension in the string holding mass m_1 is $m_1 g$, so that tension exerts a torque on the left hand side equal to

$$\boldsymbol{\tau}_L = m_1 g l_1 \mathbf{k} \tag{20}$$

On the right hand side, denoting the tension connecting the masses T, noting that the accelerations of the two masses are equal and opposite, and applying Newton's Second Law in the y-direction to each mass gives

$$T - m_2 g = m_2 a, \qquad T - m_3 g = -m_3 a \tag{21}$$

On the other hand, applying Newton's second Law to the pulley, and using the fact that it's massless, we find that the tension $'$ in the rope holding the pulley satisfies

$$T' - 2T = 0. \tag{22}$$

These equations can be solved for the tension T' to give

$$T' = 4g\frac{m_2 m_3}{m_2 + m_3} \tag{23}$$

The torque due to the string on the right hand side is therefore

$$\boldsymbol{\tau}_R = -T' l_2 \mathbf{k} = -4g\frac{m_2 m_3}{m_2 + m_3} l_2 \mathbf{k} \tag{24}$$

Since the angular momentum of this system doesn't change while it's in equilibrium, the net external torque is zero;

$$\boldsymbol{\tau}_L + \boldsymbol{\tau}_R = 0 \tag{25}$$

which gives the desired relationship:

$$\boxed{m_1 l_1 = 4\frac{m_2 m_3}{m_2 + m_3} l_2} \tag{26}$$

(b) When all three masses are equal (say to m), the tension in the left string will be mg while the tension on the right string will be $2mg$ (there will be no accelerations of the right-hand masses in this case). For the torques to balance, the tension that is twice as small needs to be acting twice as far away on the rod to compensate, so I would expect the relation to reduce to $l_1 = 2l_2$. In this special case, the relationship we just derived gives

$$m l_1 = 4\frac{m^2}{2m} l_2 \tag{27}$$

and this implies $l_1 = 2l_2$ as expected! Can you think of any more limiting cases to check?

CP25.

A uniform, solid sphere of mass M and radius R and a uniform, solid cylinder of the same mass and radius are released simultaneously at rest from the top of an inclined plane of length ℓ that makes an angle θ with the horizontal.

Which reaches the bottom first if they both roll without slipping? Justify mathematically.

Solution.

Using conservation of kinetic plus potential energy from the top to the bottom of the inclined plane (there is no net non-conservative work done by the force of static friction), we find that

$$Mg(\ell\sin\theta + R\cos\theta) = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 + MgR\cos\theta \tag{28}$$

where we have assigned the bottom of the inclined plane to be $y = 0$, V is the speed of the cylinder at the bottom of the incline, and ω is its angular speed at the bottom of the incline.

Notice also that we have included an $MgR\cos\theta$ term on either side of the conservation equation due to the fact that the center of mass of the cylinder is actually a distance $\ell\cos\theta$ above the plane at any given point, but this term cancels on both sides, so it could have been omitted from the beginning. We also note that since the cylinder is rolling without slipping, we have the constraint

$$V = R\omega \tag{29}$$

Plugging this into the conservation of energy equation and solving for V gives

$$v = R\sqrt{\frac{2Mgd\sin\theta}{MR^2 + I}} \tag{30}$$

to determine the velocity of the cylinder at the bottom, we simply plug in the moment of inertial of the cylinder which is

$$I = \frac{1}{2}MR^2, \tag{31}$$

and we get

$$\boxed{v_{\text{cyl}} = \sqrt{\frac{4}{3}g\ell\sin\theta}}. \tag{32}$$

Doing the same computation with the moment of inertia of the sphere gives

$$\boxed{v_{\text{sph}} = \sqrt{\frac{10}{7}g\ell\sin\theta}}. \tag{33}$$

The acceleration of the center of mass of each object is constant along the inclined plane, so the one that has the higher final speed is the one that gets to the bottom first. We determine which is larger by taking their ratio;

$$\frac{v_{\text{sph}}}{v_{\text{cyl}}} = \sqrt{\frac{10}{7}\frac{3}{4}} = \sqrt{\frac{15}{14}} > 1. \tag{34}$$

Hence the sphere gets to the bottom faster.