Bivariate distributions

Table of Contents

Observations are often taken in pairs, leading to bivariate observations; or (X, Y). Often, we are interested in the relations between X and Y that have been measured in the same space.

Joint probability/density functions

Note that if two events A and B are dependent:

$$P(A \cap B) \neq P(A) \cdot P(B)$$

In the context of two discrete random variables X and Y:

$$P(X = x, Y = y) \neq P(X = x) \cdot P(Y = y)$$

Thus, we are introduced to:

> The **joint probability function** of X and Y is:
>
> $$f_{X,Y}(x, y) = P(X = x, Y = y)$$
>
> Further, note that the univariate density functions can be derived by:

> $$f_X(x) = \sum_{k \in Y} P(X = x, Y = k)$$
>
> $$f_Y(y) = \sum_{k \in X} P(X = k, Y = y)$$
>
> These are called the **marginal probability functions** in bivariate functions. If X and Y are continuous, integrals replace the summations.

The join density function becomes a bit more involved:

> The **joint density function** of continuous random variables X and Y is given by:
>
> $$\int \int_A f_{X,Y}(x, y) dx dy = P((X, Y) \in A)$$
>
> Thus, the joint density function becomes a plane/surface over some 3D space held by X, Y and $f_{X,Y}(x, y)$.

Example: Suppose that (X, Y) has the density function:

$$\frac{12}{7}(x^2 + xy)$$

for $x, y \in (0, 1)$. Find $P(X < 1/2, Y < 2/3)$

Solution:

$$\int_0^{\frac{1}{2}} \int_0^{\frac{2}{3}} \frac{12}{7}(x^2 + xy) dy dx$$

$$\frac{12}{7} \int_0^{\frac{1}{2}} \int_0^{2/3} (x^2 + xy) dy dx$$

$$\frac{12}{7} \int_0^{\frac{1}{2}} \left[x^3 y + xy^2/2\right]_0^{\frac{2}{3}}$$

$$\frac{12}{7} \int_0^{\frac{1}{2}} x^2 + x/3 dx$$

$$\dots$$

Double integration

Double integration is done by integrating and evaluating the inner integral, and then integrating once more. The inner bound represents the inner integral:

$$\int \int f(x) dy dx$$

Here, derive for y first:

$$\int F_y(x)dx$$

Then, integrate for x:

$$F_{x,y}(x, y)$$

If there are bounds, you must resolve them as you would any integral before doing the second integration.

Other results for the density function and bivariate probability

Generally, we just go from single summations/integrals to double summation/integrals.

Sum of probabilities

> Let $f_{X,Y}(x, y) \ge 0$ be the join probability/density function.
>
> If X, Y are discrete random variables then:
>
> $$\sum_{\text{all x}} \sum_{\text{all y}} f_{X,Y}(x, y) = 1$$
>
> If X, Y are continuous random variables then
>
> $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} = f_{X,Y}(x, y)$$

Cumulative distribution functions

> $$F_{X, Y}(x, y) = \begin{cases}

```
\sum_{u \le x} \sum_{v \le y} P(X = u, Y = v) & \text{discrete} \\
\int_{-\infty}^{y}\int_{-\infty}^{x} f_{X, Y}(u, v)\text{dudv} &
\text{continuous}
```

\end{cases} $$

Expected value

> If g is any function of X and Y
> $$ E{ g(X, Y)} = \begin{cases}

```
\sum_\text{all x}\sum_{\text{all y}}g(x, y)P(X = x, Y = y) & \text{discrete}
\\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f_{X, Y}(x,
y)\text{dxdy} & \text{continuous}
```

\end{cases}$$

Conditional probability

> If X and Y are discrete or discrete, then the conditional probability function of X given $Y = y$ is
>
> $$f_{X|Y}(x|Y = y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$
>
> or
>
> $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$
>
> If X and Y are discrete for any set A we have:
>
> $$P(Y \in A|X = x) = \sum_{y \in A} f_{Y|X}(y|X = x)$$
>
> $$P(a \le Y \le b|X = x) = \int_a^b f_{Y|X}(y|x)dy$$

Conditional variance

> The conditional variance of X given $Y = y$ is
>
> $$\mathrm{Var}(X|Y = y) = E(X^2|Y = y) - \{E(X|Y = y)\}^2$$
>
> where
> $$ E(X^2 | Y = y) = \begin{cases}

```
\sum_\text{all x}x^2P(X = x | Y = y) \\
\int_{-\infty}^{\infty} x^2 f_{X|Y}(x | y)dx
```

\end{cases}$$

Law of total expectation and variance

> **Law of total expectation:**
>
> $$\mathbb{E}(X) = \sum_{\text{all } x} \mathbb{E}(X \mid Y = y)\mathbb{P}(Y = y) \qquad \text{(discrete)},$$
>
> $$\mathbb{E}(X) = \int_a^b \mathbb{E}(X \mid Y = y)\, f_Y(y)\, \mathrm{d}x = \mathbb{E}\{\mathbb{E}(X \mid Y)\} \qquad \text{(continuous)}.$$

> **Law of total variance:**
>
> $$\text{Var}(X) = \mathbb{E}\{\text{Var}(X \mid Y)\} + \text{Var}\{\mathbb{E}(X \mid Y)\}$$

Independence

> Random variables X and Y are independent if and only if for all x, y:
>
> $$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$
>
> Similarly, for the CDF's:
>
> $$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$
>
> Similarly, for expected value:
>
> $$E(g(X)g(Y)) = E(g(X))E(g(Y))$$

Covariance

> The covariance of X and Y is
>
> $$\text{Cov}(X,Y) = E(X - E(X))(Y - E(Y)) = E((X - \mu_X)(Y - \mu_Y))$$
>
> where μ_X = E(X)$ and $\mu_Y = E(Y)$.

The covariance not just considers how X and Y vary about their means; but also how they very together *linearly*. The product of the differences from X and Y's means represents how linked X and Y are in deviation - if X and Y are both far apart from their mean; then the covariance will be larger.

If $\text{Cov}(X,Y) > 0$, then X and Y are positively associated; if X is likely to be large when Y is likely to be large, and vice versa.

> The following results can be found:
>
> 1. $\text{Cov}(X,X) = Var(X)$
> 2. $\text{Cov}(X,Y) = E(XY) - \mu_X\mu_Y$

3. If X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$
4. For arbitrary constants a, b

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X, Y)$$

Correlation

The **correlation** between X and Y is

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}} = \frac{\mathrm{Cov}(X, Y)}{\mathrm{sd}(X) \cdot \mathrm{sd}(Y)}$$

Thus, we standardise correlation in terms of covariance to be $-1 \le 0 \le 1$; if the correlation $= 0\$, then \$X$ and Y are uncorrelated.

1. $|\mathrm{Corr}(X, Y)| \le 1$
2. $\mathrm{Corr}(X, Y) = -1$ iff $P(Y = a + bX) = 1$, for some cosntants a, b such that $b < 0$. Flip the inequalities for $\mathrm{Corr}(X, Y) = 1$.

The above should be proved using $0 \le \mathrm{Var}(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y})$.

Bivariate Normal Distribution (Non-examinable)