MATH 2801 2016

1.

a) i) $P(X=0, Y=0) = P(X=0) \cdot P(Y=0)$

$$= \frac{9}{16} \times \frac{8}{16}$$

$$\frac{4}{16} \neq \frac{9}{32}$$

$\therefore$ Dependent.

ii)

A) Pairs for this to be true:
(1,1), (1,2) for (x,y).

$$\therefore P(XY) > 0 = P(X=1, Y=1) + P(X=1, Y=2)$$

$$= \frac{3}{16}$$

B) $E(XY) = 1 \times 1 \times \frac{3}{16} = \frac{3}{16}$

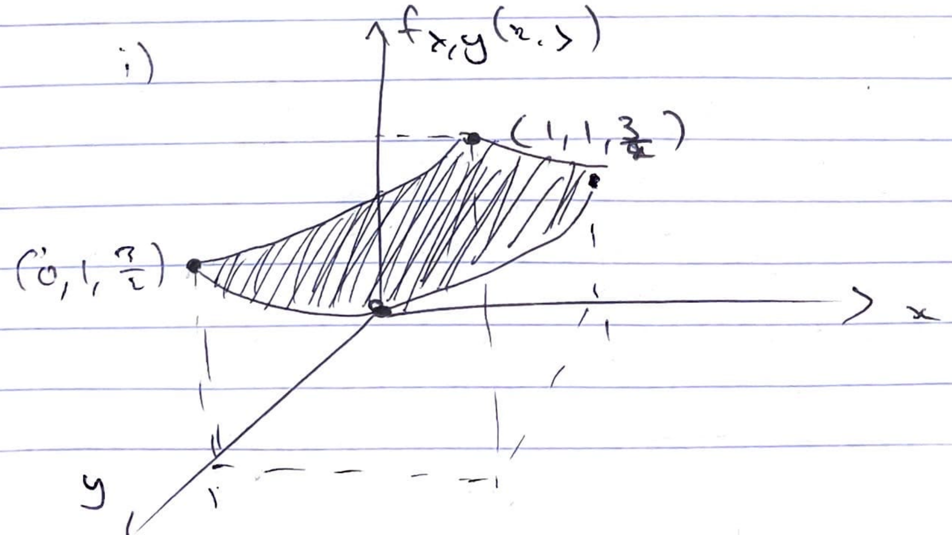
$$Var(XY) = E(X^2Y^2) - E(XY)^2$$

$$= \frac{3}{16} - \left(\frac{3}{16}\right)^2$$

$$= \frac{39}{256}$$

b) $f_{X,Y}(x,y) = \frac{3}{2}(x^2+y^2)$ for $0<x<1, 0<y<1$

i)



ii) CHECK BOUNDS.

$0<x<1$, $0<y<1$.

$$f_{X,Y}(x,y) = \frac{3}{2}(x^2+y^2) \text{ for } 0<x<1,\ 0<y<1.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_0^1 \frac{3}{2}x^2 + \frac{3}{2}y^2\,dx$$

$$= \left[\frac{x^3}{2} + \frac{3xy^2}{2}\right]_0^1$$

$$= \underline{\frac{1}{2} + \frac{3y^2}{2}}$$

iii) A) $$f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\frac{3}{2}(x^2+y^2)}{\frac{1}{2}(1+3y^2)}$$

$$= \frac{3x^2+3y^2}{1+3y^2}.$$

B) $$E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

$$= \int_0^1 \frac{3x^3+3xy^2}{1+3y^2}\,dx$$

$$= \left[\frac{\frac{3x^4}{4} + \frac{3x^2y^2}{2}}{1+3y^2}\right]_0^1 = \frac{\frac{1}{4}+\frac{3y^2}{2}}{1+3y^2}.$$

c) When $y=0.6$

$$\frac{\frac{1}{4}+\frac{3}{2}\times 0.6^2}{1+3\times 0.6^2} = \underline{0.380}.$$

iv) If independent then $f_{X|Y}(x|y) = f_X(x)$.

$$f_X(x) = \int_0^1 \frac{3}{2}x^2 + \frac{3}{2}y^2\,dy$$

$$= \frac{1}{2} + \frac{3x^2}{2}.$$

$\therefore$ Not independent.

2.

a)

i) $P(M,F) = \frac{1}{2}$

$P(C|M) = p_1 \qquad P(C|F) = p_2$

$P(C) = P(C|M)P(M) + P(C|F)P(F)$

$= \frac{1}{2}(p_1 + p_2)$

ii) Independent events... so...

$$\left(\frac{1}{2}(p_1+p_2)\right)^2$$

$$= \frac{1}{4}(p_1+p_2)^2$$

iii) $$P(W|2C) = \frac{P(W \cap 2C)}{P(2C)} = \frac{\frac{1}{4}p_2^2}{\frac{1}{4}(p_1+p_2)^2} = \frac{p_2^2}{(p_1+p_2)^2}.$$

b)

i) $f_{V,W}(v,w) = f_V(v)\cdot f_W(w)$ as independent.

ii) $P(V<W) = \int_0^{60}\int_0^{w} \frac{1}{3600}\,dv\,dw$

$= \int_0^{60} \frac{w}{3600}\,dw$

$= \left[\frac{w^2}{7200}\right]_0^{60} = \frac{1}{2}$

iii)

A) $F_T(t) = P(T \le t)$

$= P(W - V \le t)$

$= P(W \le t + V)$.

We need to be careful of ~~when~~ $W - t \ge 60$.

$W \ge 60 + t$.

$\cancel{\therefore P(W \le t+V) P(W \le 60+t) + P(}$

$\therefore P(W \le t+V) = P\cancel{(W \le t+V \mid W \le 60+t) P(W \le 60}+t)$

$+ P(W \le \cancel{t+V \mid W \ge 60+t) P(W \ge 60}+t)$.

$= \int_{60+t}^{60+t+V}\int_{0} \cancel{\frac{1}{3600}}\,dW\,dV$

iii) $F_T(t) = P(T \le t)$

$$= P(W - V \le t)$$

$$= P(W \le t + V)$$

$$= P(W \le t + V \mid V \le 60 - t)\, P(V \le 60 - t)$$
$$+ P(W \le t + V \mid V > 60 - t)\, P(V > 60 - t)$$

$$= \cancel{\int_0^{60}\int_0^{60-t} \frac{1}{3600}\, dv\, dw \times \int_0^{60-t} \frac{1}{60}\, dv}$$

$$+ \cancel{\left\{ 1 \times \int_{60-t}^{60} \frac{1}{60}\, dv \right.}$$

$$\cancel{w \le [\ge 60] = 1}$$

$$W \le 60 = 1$$

$$= \cancel{\int_0^{60} \frac{60-t}{3600}\, dw}$$

$$= \int_0^{60-t}\int_0^{t+v} \frac{1}{3600}\, dw\, dv \times \int_0^{60-t} \frac{1}{60}\, dv$$

$$+ \int_{60-t}^{60}\int \overbrace{\ldots}^{=1} \times \int_{60-t}^{60} \frac{1}{60}\, dv$$

$$= \int_0^{60-t} \frac{t+v}{3600}\, dv \times \frac{60-t}{60} + \frac{t}{60}$$

$$= \left[\frac{tv + \frac{v^2}{2}}{3600} \right]_0^{60-t} + \frac{t}{60}.$$

CBF.

B) JUST DIFFERENTIATE.

Explanation of A: $W \leq t + U$ causes some problems as $V + t \geq 60$ or $V + t \leq 60$. We can deal with this using law of total probability.

3.

a)

i) $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \ldots, n$.

$$m_X(u) = \sum_{x=0}^{\infty} \binom{n}{x} e^{ux} \times p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{\infty} \binom{n}{x} (pe^u)^x (1-p)^{n-x}.$$

By Binomial Expansion, which states

$$(a+b)^n = \sum \binom{n}{x} p^x (1-p)^{n-x}$$

We have:

$$= (1 - p + pe^u)^n.$$

ii) A) $m_Y(u) = E(e^{u2x}) = \sum \binom{n}{x} e^{u2x} p^x (1-p)^{n-x}$

$$= \sum \binom{n}{x} (pe^{2u})^x (1-p)^{n-x}$$

$$= (1 - p + pe^{2u})^n$$

~~B) By Discrete Transformation:~~

~~$f_Y(y) = \binom{n}{2x} p^{2x} (1-p)^{n-2x}$ for $x =$~~

B) $X = \frac{Y}{2}$.

By discrete transformation:

$$f_Y(y) = \binom{n}{y/2} p^{y/2} (1-p)^{n-y/2} \quad \text{for } y = 0, 2, \dots, 2N.$$

C) Nope. (I think).

b)

i) $H_0 = \mu = 7.5 \qquad H_1: \mu \neq 7.5$

ii) We assumple the samples are normally distributed. Check QQ-plot.

We assume the sample is random (this is given).

iii)
$$\frac{\bar{x} - \mu_0}{s / \sqrt{n}} \sim t_{n-1}$$

$$= \frac{\bar{x} - 7.5}{s / \sqrt{n}} \sim t_{n-1}$$

iv)
$$\frac{7 - 7.5}{2 / \sqrt{100}} \sim t_{99}$$

$$= -2.5$$

v) pt$(-2.5, 99) = 0.007$

vi) Yeah null hypothesis is cooked.

4.

a)

$$E(X) = \bar{X}$$

$$\hat{p} = \bar{X} \quad \square.$$

b) $L(x;p) = p^{x_1}(1-p)^{1-x_1} \times p^{x_2} \times (1-p)^{1-x_2} \times \dots \times p^{x_n}(1-p)^{1-x_n}$

$$= p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$$

$$l(x;p) = \ln(p^{\Sigma x_i}) + \ln((1-p)^{n-\Sigma x_i})$$

$$= \Sigma x_i \ln(p) + (n-\Sigma x_i)\ln(1-p)$$

$$l'(x;p) = \frac{\Sigma x_i}{p} - \frac{n-\Sigma x_i}{1-p}$$

$$l''(x;p) = -\frac{\Sigma x_i}{p^2} - \frac{n-\Sigma x_i}{(1-p)^2}$$

$$= -\left(\frac{\Sigma x_i}{p^2} + \frac{n-\Sigma x_i}{(1-p)^2}\right) < 0 \qquad 0 < p < 1.$$

$\therefore$ To find the MLE:

$$\frac{\Sigma x_i}{\hat{p}} = \frac{n-\Sigma x_i}{1-\hat{p}}$$

$$= \Sigma x_i - \cancel{\hat{p}\Sigma x_i} = n\hat{p} - \cancel{\hat{p}\Sigma x_i}$$

$$= \hat{p} = \frac{\Sigma x_i}{n}. \quad \square.$$

c) $I_n(p) = -E(l''(p))$

$$= -E\left(-\left(\frac{\sum x_i}{p^2} + \frac{n-\sum x_i}{(1-p)^2}\right)\right)$$

$$= E\left(\frac{\sum x_i}{p^2} + \frac{n-\sum x_i}{(1-p)^2}\right)$$

$$= \frac{\sum E(x_i)}{p^2} + \frac{n-\sum E(x_i)}{(1-p)^2}$$

$$= \frac{np}{p^2} + \frac{n-np}{(1-p)^2}$$

$$= \frac{n}{p} + \frac{n}{1-p}$$

$$= \frac{n}{p(1-p)}$$

d) $\cancel{\hat{p} \xrightarrow{P}}$

$\cancel{Var(p) = \frac{n}{p(1-p)}}$ $\quad Var(p) \approx \frac{1}{I_n(p)}$

$$\approx \frac{\hat{p}(1-\hat{p})}{n}$$

$$\hat{p} \xrightarrow{D} N\left(p, \frac{p(1-p)}{n}\right)$$

ii) EPIC DELTA METHOD

$g(\hat{p}) = \dfrac{\hat{p}}{1-\hat{p}}$

$g'(p)$ $\quad u = \hat{p} \quad v = 1-\hat{p}$

$\quad u' = 1 \quad v' = -1$

$\boxed{\dfrac{1}{(1-p)^2}}$

$\therefore$ ~~$g(\hat{p}) \xrightarrow{D} ($~~

$g(\hat{p}) \xrightarrow{D} \left(g(p), g'(p)\, se(\hat{p})\right)$

$g(\hat{p}) \xrightarrow{D} \left(\dfrac{p}{1-p}, \dfrac{1}{(1-p)^2} \times \dfrac{p(1-p)}{n}\right)$

$= g(\hat{p}) \xrightarrow{D} \left(\dfrac{p}{1-p}, \dfrac{p}{n(1-p)}\right)$

e) $\hat{p} = 0.75$.

$\left(0.75 \pm 1.96 \times \sqrt{\dfrac{0.75 \times 0.25}{100}}\right)$

$= (0.665, 0.835)$.