MATH2601 PRACTICE EXAM

1.

a) A census captures the population. A sample is a subset of the population.

b) $E(X) = \bar{x}$

The expected value of X is p.

$\therefore$

$$\hat{p} = \bar{x}$$

c) $f_X(x) = p^x(1-p)^{1-x}$

$$L(x;p) = p^{x_1}(1-p)^{1-x_1} \times p^{x_2}(1-p)^{1-x_2} \times \dots \times p^{x_n}(1-p)^{1-x_n}$$

$$= p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$$

$$l(x;\lambda) = \ln(p^{\Sigma x_i}) + (n-\Sigma x_i)\ln(1-p)$$

$$= \Sigma x_i \ln(p) + (n-\Sigma x_i)\ln(1-p)$$

$$l'(x;\lambda) = \frac{\Sigma x_i}{p} - \frac{n-\Sigma x_i}{1-p}$$

$$l''(x;\lambda) = -\frac{\Sigma x_i}{p^2} - \frac{n-\Sigma x_i}{(1-p)^2}$$

$$= -\left(\frac{\Sigma x_i}{p^2} + \frac{n-\Sigma x_i}{(1-p)^2}\right) < 0 \quad \text{for}$$

$\therefore$ Finding the MLE:

$$\frac{\Sigma x_i}{\hat{p}} = \frac{n-\Sigma x_i}{1-\hat{p}}$$

$$\Sigma x_i \cancel{- p\Sigma x_i} = n\hat{p} \cancel{- p\Sigma x_i}$$

$$\hat{p} = \frac{\Sigma x_i}{n}.$$

d) $I_n(p) = -E(l''(p))$

$$= -E\left(-\left(\frac{\sum x_i}{p^2} + \frac{n-\sum x_i}{(1-p)^2}\right)\right)$$

$$= E\left(\frac{\sum x_i}{p^2} + \frac{n-\sum x_i}{(1-p)^2}\right)$$

$$= \frac{\sum E(X_i)}{p^2} + \frac{n-\sum E(X_i)}{(1-p)^2}$$

$$= \frac{np}{p^2} + \frac{n-np}{(1-p)^2}$$

$$= \frac{n}{p} + \frac{n}{1-p}$$

$$= \frac{n-np+np}{p(1-p)}$$

$$= \frac{n}{p(1-p)}$$

e) $Var(\hat{p}) = \frac{p(1-p)}{n}$

$sd(p) = \sqrt{\frac{p(1-p)}{n}}$

~~as $\hat{p}_n \to \infty$~~,

as $n \to \infty$, $\hat{p} \xrightarrow{D} N\left(p, \frac{p(1-p)}{n}\right)$,

or more accurately:

$$\sqrt{n}\,\hat{p} \xrightarrow{D} N(p, p(1-p))$$

due to the limiting behavior of n.

f) $n = 678$

i) $\hat{p} = \frac{353}{678}$

$= 0.521$

ii) $\left(0.521 \pm 1.96 \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

$= \left(0.521 \pm 1.96 \times \sqrt{\frac{0.521 \times 0.479}{678}}\right)$

$= (0.483, 0.559)$

2.

a) Binomial Distribution. $n = 10$ $p = 0.2$.

i) $\binom{10}{2} \times (0.2)^2 \times (0.8)^8$

$= 0.302$

ii) $P(X \geq 2 \mid X \geq 1) = \frac{P(X \geq 2 \cap X \geq 1)}{P(X \geq 1)}$

By memoryless property:

$= \frac{P(X \geq 2)}{P(X \geq 1)}$.

$= \frac{1 - [P(X=0) + P(X=1)]}{1 - [P(X=0)]}$

$= \underline{0.699}$

b)

i) $X_1 \sim N(\mu, \sigma^2)$, $X_2 \sim N(\mu, \sigma^2)$.

ii) $m_{X_1+X_2}(u) = m_{X_1}(u) \times m_{X_2}(u)$

$$= e^{\mu u + \frac{1}{2}\sigma^2 u^2} \times e^{\mu u + \frac{1}{2}\sigma^2 u^2}$$

$$= e^{2\mu u + \sigma^2 u^2} := m_Y(u)$$

ii) $m_{N(2\mu, 2\sigma^2)} = e^{2\mu u + \sigma^2 u^2}$

By equality of mgf's, as $m_Y(u) = m_{N(2\mu, 2\sigma^2)}(u)$,
$Y \sim N(2\mu, 2\sigma^2)$. $\square$.

iii) If $P(-a < Y < a) = 0.9$

~~Then P~~

Then $[P(Y < -a) + P(Y > a)] = 0.1$

And by symmetry:

$P(Y < -a) = 0.05$. ← Find quantile of 0.05: qnorm (0.05)

$= -1.645$.

$\therefore$

$$\frac{-a}{2\sigma^2} = -1.645$$

$$\underline{a = 3.290\,\sigma^2}$$

iv) $\hat{\mu}_1 = \dfrac{X_1 + 2X_2}{3}$

$\hat{\mu}_2 = \dfrac{X_1 + X_2}{2}$

$$\text{bias}(\hat{\mu}_1) = E\left(\frac{X_1 + 2X_2}{3}\right) - \mu$$

$$= \mu - \mu$$

$$= 0$$

$$\text{bias}(\hat{\mu}_2) = E\left(\frac{X_1 + X_2}{2}\right) - \mu$$

$$= \mu - \mu$$

$$= 0.$$

$$\text{Var}(\hat{\mu}_1) = \text{Var}\left(\frac{X_1 + 2X_2}{3}\right)$$

$$= \frac{1}{9}\text{Var}(X_1) + \frac{4}{9}\text{Var}(X_2)$$

$$= \frac{5\sigma^2}{9}.$$

$$\text{Var}(\hat{\mu}_2) = \text{Var}\left(\frac{X_1 + X_2}{2}\right)$$

$$= \frac{1}{4}\text{Var}(X_1) + \frac{1}{4}\text{Var}(X_2)$$

$$= \frac{\sigma^2}{2}$$

$\text{MSE}(\hat{\mu}_1) = \dfrac{5\sigma^2}{9}$ $\quad\quad$ $\text{MSE}(\hat{\mu}_2) = \dfrac{\sigma^2}{2}.$

$\text{MSE}(\hat{\mu}_2) < \text{MSE}(\hat{\mu}_1)$, $\therefore$ uniformly better.

c) $f_W(w) = \frac{1}{w} \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(w)-\mu)^2}{2\sigma^2}}$, $w > 0$.

$$E(W) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln(w)-\mu)^2}{2\sigma^2}} dw$$

Let $t = \ln(w)$ $\quad$ $w = e^t$

$$\frac{dw}{dt} = e^t$$

$$dw = e^t dt$$

$$= \int_0^\infty \frac{e^t}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt.$$

This is the same as ~~the~~ mgf of $N(\mu, \sigma^2)$, where $a=1$.

$$\therefore E(W) = \text{mgf}_{N(\mu,\sigma^2)}(1)$$

$$= e^{\mu + \frac{1}{2}\sigma^2}.$$

3.

a) NOTE THE BOUNDS.

$0 < x < 2y$, $0 < y < 1$.

a) $$\int_0^1 \int_0^{2y} 2xy \, dx \, dy$$

$$= \int_0^1 \left[x^2 y\right]_0^{2y}$$

$$= \int_0^1 4y^3 = \left[y^4\right]_0^1 = 1. \quad \therefore \text{Valid.}$$

b) $f_{X,Y}(x,y) = 2xy$

$$\int_{\frac{x}{2}}^{1} 2xy \, dy$$

$$= \left[xy^2 \right]_{\frac{x}{2}}^{1}$$

$$= x - \frac{x^3}{4}$$

$$= x\left(1 - \frac{x^2}{4}\right)$$

c) $f_{Y|X}(y|X=x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

$$= \frac{2xy}{x\left(1-\frac{x^2}{4}\right)}$$

$$= \frac{2y}{1-\frac{x^2}{4}}$$

d) $\cancel{f_Y(y) = \int_{\frac{x}{2}}^{1} 2xy\,dx = \left[x^2 y\right]_{\frac{x}{2}}^{1} = y}$

$$f_Y(y) = \int_{0}^{2y} 2xy \, dx = \left[x^2 y \right]_{0}^{2y}$$

$$= 4y^3$$

If ind., then $f_{Y|X}(y|x) = f_Y(y)$. Contradiction found; $\therefore$ dependent.

e) $$E(Y \mid X=1) = \int_{\frac{1}{2}}^{1} \frac{2y^2}{1-\frac{1}{4}}\,dy$$

NOTE: We lower bound y by $\frac{1}{2}$ here as $y > \frac{x}{2}$. If x did not exist as a variable, we would not take this bound.

$$= \int_{\frac{1}{2}}^{1} \frac{2y^2}{\frac{3}{4}}$$

$$= \int_{\frac{1}{2}}^{1} \frac{8y^2}{3}\,dy$$

$$= \left[\frac{8y^3}{9}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{8}{9} - \frac{1}{9} = \frac{7}{9}.$$

f) $Z = X/2$

$X = 2Z$. Monotonic.

$\frac{dx}{dz} = 2.$

$$f_Z(z) = f_X(h^{-1}(z)) \left|\frac{dx}{dz}\right|$$

$$= 4z(1-z^2), \quad 0 < z < 1.$$

4. a) Matched pairs experiment.

b)

~~i) $H_0: \mu = 0$ $H_1: \mu$~~

Note the diff is neg.

i) $H_0: \mu_7 - \mu_{12} = 0$ $H_1: \mu_7 - \mu_{12} < 0$.

ii) $$\frac{\bar{x} - \mu}{s / \sqrt{n}} \sim t_{n-1}$$

iii) $$\frac{-5.583 - 0}{6.43 / \sqrt{12}} \sim t_{11}$$

$$= -3.0077\ldots$$

pt(-3, 11) = 0.00603.

v) The null hypothesis can be rejected, as the alternative hypothesis is extremely likely.

c) $$\left(\bar{x} \pm t_{11, 0.975} \times \frac{s}{\sqrt{n}}\right)$$

$$= \left(-5.583 \pm 2.201 \times \frac{6.430}{\sqrt{12}}\right)$$

$$= \left(-9.668, -1.498\right)$$

d) Random normal samples, independently selected. Normal sample $\Rightarrow$ QQ plot. Independence $\Rightarrow$ Random selection.