FINS2624: Portfolio Management

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Contents

I. Markowitz Portfolio Theory

A. Return and risk

Portfolio management is all about *assets* and their *returns*. By nature, asset returns are **stochastic** - they are random. We can have predictions about the future using data from the past, but they are not guaranteed.

We generally care about two characteristics of returns:
- Expected return: what do we expect the return to be on average?
- Risk: The expected average dispersion of an asset around the expected return - generally given by standard deviation of returns (volatility).

Holding period return (HPR) and annual percentage rate (APR)

Investor returns from holding an asset come from two basis sources:
- Income received periodically such as interest (debt security) or dividends (equity security)
- Capital gains/losses from the price of the asset increasing/decreasing.

Holding period return (HPR) is the return on an asset during the period it is held.

$$\text{HPR}_{0\to T} = \frac{P_T - P_0 + I_T}{P_0}$$

where:
- P_T is the price at time T
- P_0 is the initial price
- I_T is the total income received in the holding period (e.g dividends)

Given T is in years, the **Annual Percentage Rate (APR)** gives the average annual return of the investment over the holding period

$$\text{APR}_{0\to T} = \frac{\text{HPR}_{0\to T}}{T}$$

Expected return

The *expected* return from a list of scenarios s, and the corresponding probability of each scenario $p(s)$, and the return of the scenario $r(s)$, is given by

$$\mathbb{E}(r) = \sum_s p(s) \times r(s)$$

Variance and standard deviation

The *variance* from a list of scenarios s, and the corresponding probability of each scenario $p(s)$ and the return of the scenario $r(s)$, is given by

$$\text{Var}(r) = \sigma_r^2 = \sum_s p(s) \times [r(s) - \mathbb{E}(r)]^2$$

The standard deviation is the square root of variance

$$\text{Std}(r) = \sigma_r = \sqrt{\sigma_r^2}$$

Ex-ante (after the fact) expected return and standard deviation
Consider the following scenarios of a stock's return:

- 25% return with 20% probability
- 15% return with 40% probability
- 5% return with 30% probability
- -5% return with 10% probability

The expected return can be calculated as such:

$$\begin{aligned}\mathbb{E}(r) &= 0.2 \cdot 0.25 + 0.4 \cdot 0.15 \\ &\quad + 0.3 \cdot 0.05 + 0.1 \cdot -0.05 \\ &= 0.12\end{aligned}$$

Therefore we have an expected return of 12%. Then, we can find the variance of returns by

$$\begin{aligned}\text{Var}(r) &= 0.2 \cdot (0.25 - 0.12)^2 + 0.4 \cdot (0.15 - 0.12)^2 \\ &\quad + 0.3 \cdot (0.05 - 0.12)^2 + 0.1 \cdot (-0.05 - 0.12)^2 \\ &= 0.0081\end{aligned}$$

Thus the standard deviation of returns is $\sigma_r = \sqrt{0.0081} = 0.09$ or 9%.

We can estimate *ex-post* (before-the-fact) expected returns and standard deviation using historical realised average returns and standard deviations.

We then have the concept of *historical average return* and *historical variance*, which can be used as predictors of expected returns and volatility.

Historical average return
For a given period $t = \{1, \ldots, N\}$, and there associated returns $r_i = \{r_1, r_2, \ldots, r_N\}$, the historical average return is given by

$$\hat{r} = \frac{1}{N} \sum_{i=1}^{N} r(i)$$

Historical variance and std. dev.
For a given period $t = \{1, \ldots, N\}$, and there associated returns $r_i = \{r_1, r_2, \ldots, r_N\}$, with expected return $\hat{r}$, the historical variance is given by

$$\hat{\sigma}_r^2 = \frac{1}{N-1} \sum_{i=1}^{N} (r(i) - \hat{r})^2$$

Of course, the standard deviation is given by $\hat{\sigma}_r = \sqrt{\hat{\sigma}_r^2}$.

Thus, under the assumption that the past predicts the future, we have that

$$\mathbb{E}(r) = \hat{r} \text{ and } \sigma_r = \hat{\sigma}_r$$

B. Portfolios

What is a portfolio?
A portfolio is a collection of assets defined by:

- the assets that are in the portfolio
- the amount invested in each asset

It is possible to have negative amounts invested in an asset through borrowing cash and shorting stocks.

Portfolio return
Since portfolios can have many assets, the return of an portfolio is found by dividing the change in portfolio value by the initial portfolio value.

Given a two-asset example, A and B, with initial investment A_0 and B_0 and returns R_A and R_B, the portfolio return is

$$R_P = \frac{A_0}{A_0 + B_0} R_A + \frac{B_0}{A_0 + B_0} R_B$$

Given weights N assets with weights $w = \{w_1, w_2, \ldots, w_N\}$ and returns $r = \{r_1, r_2, \ldots, r_N\}$, we have the portfolio return

$$r_P = \sum_{i=1}^{N} w_i r_i$$

Thus, the expected portfolio return is similarly given (through the linearity of expectations) by

$$E(r_P) = \sum_{i=1}^{N} w_i E(r_i)$$

Variance on the other hand, becomes more tricky. Since assets often have correlations to eachother, we must consider the relationships of the returns *between* the assets.

Covariance
Covariance measures how assets move with one another. It is defined as the product of each asset's expected deviation from it's mean.

$$\begin{aligned}\text{Cov}(X, Y) &= \mathbb{E}\left[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]\end{aligned}$$

Two assets that tend to trend higher than their mean at the same time have *positive* covariance.

On the contrary, two assets that tend to trend in opposite directions from their mean at the same time have *negative* covariance.

You may notice that covariance is not normalised, so the covariance of two different pairs are generally not comparable. This motivates the usage of *correlation*.

Correlation
Correlation is a normalised value that captures covariance-like trends

$$\rho_{xy} = \frac{\text{Cov}(r_X, r_Y)}{\sigma_X \sigma_Y}$$

Thus, $-1 \leq \rho_{x,y} \leq 1$.

- Two assets that tend to be higher than their means simultaneously have positive correlation
- Two assets that move in different directions with respect to their means simultaneously have negative correlation

There are a few properties of covariance that are useful mathematically.

Properties of covariance

1) An assets variance is the covariance of an asset with itself

$$\text{Var}(X) = \text{Cov}(X, X)$$

2) The covariance of asset A with asset B is the same as that of asset B with asset A (commutative)

$$\text{Cov}(X, Y) = \text{Cov}(X, Y)$$

3) You can multiplicatively distribute the two terms inside a covariance, forming a summation of covariances (distributive). Constants can be taken out of covariance.

$$\begin{aligned}\text{Cov}(\alpha X + \beta Y, \gamma Z) &= \text{Cov}(\alpha X, \gamma Z) + \text{Cov}(\beta Y, \gamma Z) \\ &= \alpha\gamma\text{Cov}(X, Z) + \beta\gamma\text{Cov}(Y, Z)\end{aligned}$$

We then return to the analysis of portfolios and their returns, with the knowledge of variances and covariances. How do we find the variance of a *two asset portfolio?*

Return of a portfolio
Let the fraction invested in asset A be w_A and the remainder, invested in B be $w_B = 1 - w_A$. The return of a portfolio is given by

$$r_P = w_A r_A + w_B r_B$$

and thus the expected return is given by

$$\mathbb{E}(r_P) = w_A \mathbb{E}(r_A) + w_B \mathbb{E}(r_B)$$

Variance of a two asset portfolio
Let the fraction invested in asset A be w_A and the remainder, invested in B be $w_B = 1 - w_A$. We can find the variance of portfolio return as follows

$$\begin{aligned}\text{Var}(r_P) &= \text{Var}(w_A r_A + w_B r_B) \\ &= \text{Cov}(w_A r_A + w_B r_B, w_A r_A + w_B r_B) \\ &= \text{Cov}(w_A r_A, w_A r_A) + \text{Cov}(w_B r_B, w_B r_B) \\ &+ 2\text{Cov}(w_A r_A, w_B R_B) \\ &= w_A^2 \text{Var}(r_A) + w_B^2 \text{Var}(r_B) \\ &+ 2 w_A w_B \rho_{A,B} \sigma_A \sigma_B\end{aligned}$$

The final element can be reasoned by eh fact that

$$\rho_{A,B} \sigma_A \sigma_B = \text{Cov}(A, B)$$

If there are two assets, then there is one covariance required. Note that the number of covariances required to be computed grows by

$$\binom{N}{2}$$

Thus *covariance matrices* are utilised to simplify the utilisation of covariances for portfolio mathematics. Covariance matrices have each asset as rows and columns, and has the covariance of each row-pair as the element.

$$\Sigma = \begin{bmatrix} \text{Var}(A) & \text{Cov}(A, B) & \text{Cov}(A, C) \\ \text{Cov}(B, A) & \text{Var}(B) & \text{Cov}(B, C) \\ \text{Cov}(C, A) & \text{Cov}(C, B) & \text{Var}(C) \end{bmatrix}$$

Remember that covariances are commutative. For example, for the above three asset example for some portfolio P, we can find the variance of the portfolio by essentially finding each pair of weights and (co)-variances.

$$\begin{aligned}\sigma_P^2 &= w_A^2 \sigma_A^2 + w_A w_B \text{Cov}(A, B) + w_A w_C \text{Cov}(A, C) \\ &+ w_B w_A \text{Cov}(A, B) + w_B^2 \sigma_B^2 + w_B w_C \text{Cov}(B, C) \\ &+ w_C w_A \text{Cov}(A, C) + w_C w_B \text{Cov}(B, C) + w_C^2 \sigma_C^2\end{aligned}$$

Note we have essentially just followed the structure of the covariance matrix, and applied the necessary weight pairs. Noting that covariances are commutative, and substituting the correlation identity, we have

$$\begin{aligned}\sigma_P^2 &= w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + w_C^2 \sigma_C^2 \\ &+ 2 w_A w_B \rho_{A,B} \sigma_A \sigma_B + 2 w_B w_C \rho_{B,C} \sigma_B \sigma_C \\ &+ 2 w_A w_C \rho_{A,C} \sigma_A \sigma_C\end{aligned}$$

With the above identity, we notice that since $-1 \leq \rho \leq 1$, that the correlations between assets can be a variance *reducing* factor.

Diversication and diversified portfolios
Consider the two-asset portfolio example. If the two portfolios are correlated with $\rho = 1$, then

$$\sigma_P^2 = (w_A \sigma_A + w_B \sigma_B)^2$$

If they are uncorrelated ($\rho = 0$), then

$$\sigma_P^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2$$

If they are negatively correlated ($\rho = -1$), then

$$\sigma_P^2 = (w_A\sigma_A - w_B\sigma_B)^2$$

This is why *diversification* is so important, having low/negative correlated assets reduces the risk in the portfolio.

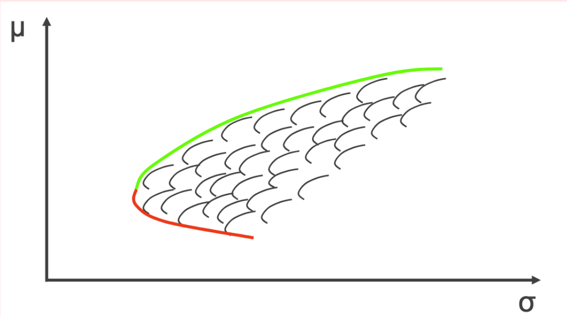
Combining many assets with low correlations can maximise the diversification benefit.

The diversification benefit
The diversification benefit is quantitatively given by the standard deviation reduced (or increased) by the inter-correlation between assets in a risky portfolio.

Given N assets and $\sigma = (\sigma_1, \ldots, \sigma_N)$ the vector of asset volatilities, $w = (w_1, \ldots, w_N)$, the asset weights and portfolio volatility σ_P, the diversification benefit is given by

$$\text{Diversification benefit} = \sum_{i=1}^{N} w_i\sigma_i - \sigma_P$$

The efficient frontier and portfolio choices
The *efficient frontier* represents portfolios with the highest expected return for a given level of risk (σ).



In the above diagram, the green and red line represent the efficient frontier - but the red portfolios for a *risk-averse* investor are dominated.

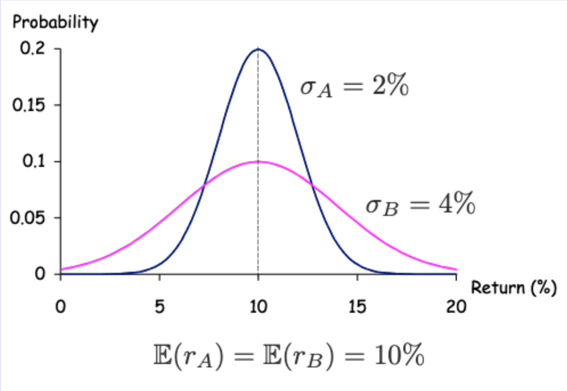
The green line, represents the best returns given by a combination of different portfolio weights for a given level of risk σ. Each black "half-egg" is a sub-optimal portfolio.

C. Preference and Utility

In previous sections, we have focused on risk and return. Maximising wealth often comes with risk - which in finance means that the realised outcomes could be better or worse than what is expected.

Risk cannot be wholly avoided - but we wish to take appropiate and measured risk.

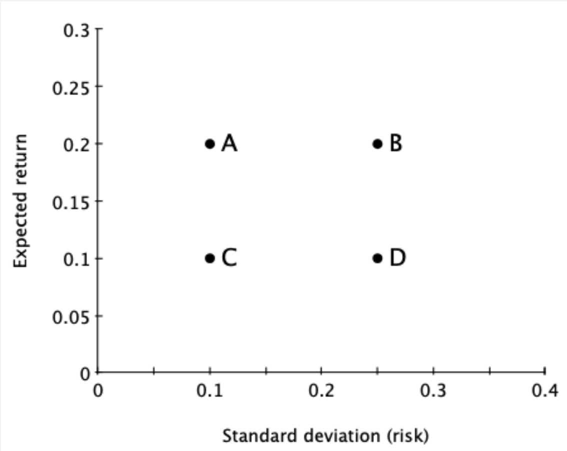
Asset dominance
If we are only concerned with expected returns and standard deviations, some investment decisions become trivial.



Which asset is better, A or B? For an investor who is solely focused on risk and return, it is clear A is a clearly better investment.

We say that the asset A **dominates** B, if every *reasonable* investor would always choose to invest in A (and not in B).

Asset domination and investor's preference
Consider the below risk and return profiles of assets A, B, C and D.



A reasonable investor would **always** choose:

- A over B, C and D; as it comes with a better return/risk
- B over D, as it comes with better return for the same risk
- C over D, as it comes with the same return for less risk

But what about between B and C? B returns more but is also riskiers.

Mean-variance criterion
'Mean-variance' analysis concerns itself with expected returns and risk. The **Mean-Variance (M-V) Criterion** is the selection of portfolios based on the means and variances of their returns.

- Choose the highest expected return portfolio for a given level of variance
- Choose the lowest variance portfolio for a given expected return

There are different tolerances to *risk* amongst investors.

- *Risk averse* investors weigh both return and risk
- *Risk neutral* investors judge assets solely by their expected return
- *Risk seeking* investors prefer higher levels of risk

Economic utility and utility functions
Utility is a measure of *satisfaction* of an investor. When an investor prefers asset A over asset B, we say that asset A provides the investor with greater *utility*.

We use utility functions to model preferences mathmetically.

- A utility function assigns a value to each outcome so that preferred outcomes get higher values
- We generally model utility as a function of only wealth.

Conditions for investor preference utility functions

For *investor preference* utility functions, **two** conditions must be met:

1) Per unit of wealth, utility must increase (monoticity)
2) The rate of increase for U must decrease per unit of wealth

In mathmetical terms, the two conditions are:

$$\frac{dU}{dW} > 0$$
$$\frac{d^2U}{dW^2} < 0$$

where U is a function of wealth W.

An example of a utility function used to model investor utility is *quadratic* utility. Given an investors level of risk-aversion A, and utility U:

$$U = \mathbb{E}(r) - \frac{1}{2}A\sigma^2$$

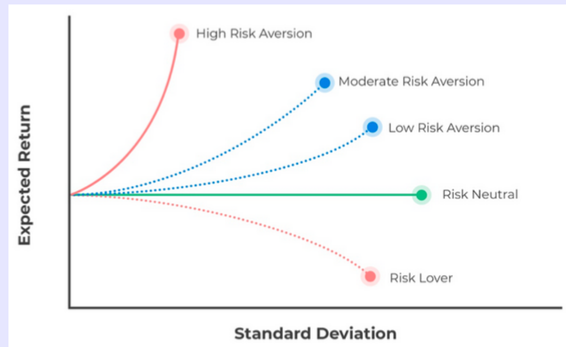
The utility is viewed as a certainty equivalent return.

- For a riskless investment, the expected return is known and the risk is zero.
- For a risky investment, quadratic utility gives the riskless return with which investors would be *equally* happy.

Indifference curves
Indifference curves show us portfolios with which investors are equally satisfied.

- All portfolios in mean-variance space with a given utility are connected by a curve
- For example, the indifference curve for $U = 0.1$ contains all combinations of $\mathbb{E}(r)$ and σ^2 that yield a $U = 0.1$.

Therefore indifference curves show us portfolios with which investors are *equally satisfied*.



Indifference curves is a graphical representation of different levels of risk and return that offer the same *utility*. With **higher risk aversion**, the curve becomes *steeper*, as more return is required to reason the risk.

D. Minimum variance/efficient frontier and optimal portfolios with no risk-free asset

By combining risky assets in a portfolio in different proportions, we can constructu portfolios with the minimum variance given a desired expected return.

Minimum Variance Frontier (MVF)
The *minimum variance portfolio* for some desired return C can be found by the following constrained optimisation problem

$$\min_{w_i} \mathrm{Var}(r_p) = \mathrm{Var}\left(\sum_{i=1}^{N} w_i r_i\right)$$

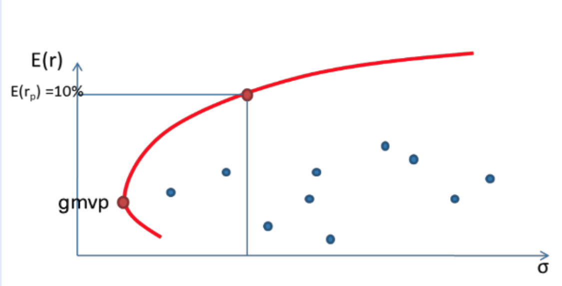
Subject to the constraint

$$\mathbb{E}(r_p) = \mathbb{E}\left(\sum_{i=1}^{N} w_i r_i\right) = C$$

Where

- w_i is the weight of the i-th asset.
- r_i is the return of the i-th asset.
- r_p is the return of the portfolio.

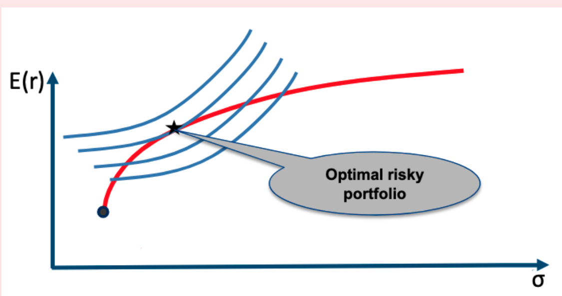
The collection of all minimum variance portfolios for some range of returns forms the *Minimum Variance Frontier (MVF)*.



We can see in the above minimum variance frontier that there are some portfolios that are dominated - namely the ones that are below the 'turning point'. The turning point is called the **Global Minimum Variance Point (GVMP)** - the portfolio that has the smallest variance.

The efficient frontier: what portfolio to choose?
Since all of the portfolios below the global minimum variance point are dominated, we can discard them. The frontier remaining is called the *efficient frontier*.

Risk averse investors should only choose portfolios on the efficient frontier.



The investor/manager should pick the asset portfolio weighting combination which provides the *highest utility*.

- This is equal to finding the indifference curve tangential to the efficient frontier

E. Complete portfolios including a risk-free asset

An important caveat to realistic portfolio management is that many investors will blend *risky* portfolios with risk-free assets; like treasury bills.

The risk-free asset
Short-term government bills (T-bills) are often considered **the risk-free asset**, as they have (almost) no default risk and limited interest rate risk.

- No default risk due to government backing (so more secure)
- Short term, so interest rate Δ risk is reduced

The return on the risk free asset is called the **risk-free rate**; denoted r_f. The risk free asset has the following characteristics

1) $E(r_f) = r_f$
2) $\text{Var}(r_f) = 0$
3) $\text{Cov}(r_f, r_i) = 0$, for any risk asset i

Blending the risk free asset with risky portfolios means that we have a "y-intercept" to our risk-return profile; a guaranteed return.

Risk premium of an investment
The risk premium (premia) of an investment is the excess-return above the risk-free rate.

$$\text{Risk premium} = E(r_a) - r_f$$

where r_a is the return of the risk asset.

In a realistic portfolio, we then blend the risky and risk-free assets. Assume we have a complete risky portfolio P, and it's return r_P; we weight the portfolio with weight y% and thus the risk-free asset with $(1-y)$%. We then get the complete portfolio C with expected return characteristics

$$\begin{aligned}\mathbb{E}(r_C) &= (1-y)r_f + y\mathbb{E}(r_P)\\ &= r_f - yr_f + y\mathbb{E}(r_P)\\ &= r_f + y(\mathbb{E}(r_P) - r_f)\end{aligned}$$

Or the expected return of the complete portfolio C is the risk free rate + the risk premium of the risky portfolio. The risk is given by

$$\begin{aligned}\sigma_C^2 &= y^2\sigma_P^2 + (1-y)^2\sigma_{r_F}^2\\ &\quad + 2y(1-y)\text{Cov}(r_P, r_f)\\ &= y^2\sigma_P^2\\ \sigma_C &= y\sigma_P\end{aligned}$$

Therefore we have that

$$y = \frac{\sigma_C}{\sigma_P}$$

Thus, the expected return becomes

$$\mathbb{E}(r_C) = r_f + \frac{\sigma_C}{\sigma_P}[\mathbb{E}(r_P) - r_f]$$

Consider the right hand side element; this is the *Sharpe* ratio of the risky portfolio multiplied by a unit of risk in the complete portfolio!

Sharpe ratio
The Shape ratio is a measure of risk-adjusted return. It considers the amount of return the investment returns per-unit of risk it has.

$$\text{Sharpe ratio} = \frac{\mathbb{E}(r_A) - r_f}{\sigma_A}$$

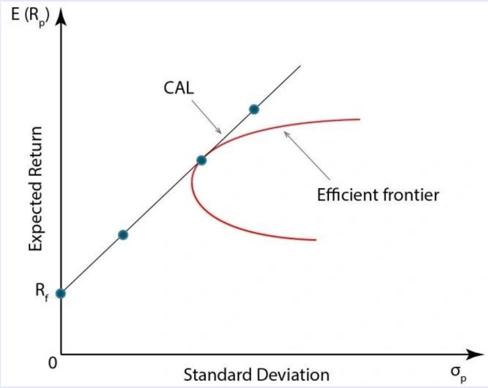
where

- r_A is the return of the risky asset A
- r_f is the risk-free return
- σ_A is the risk of the asset A

Capital Allocation Line (CAL)
Earlier we found the expected return of a complete portfolio with risk-free assets

$$\mathbb{E}(r_C) = r_f + \frac{\sigma_C}{\sigma_P}[\mathbb{E}(r_P) - r_f]$$

When the line connects with some risky portfolio P, it is known as the **Capital Allocation Line (CAL)**.



Consider some scenarios for the weighting of the complete portfolio

- When $y = 1$, we put the entire complete portfolio into the risky portfolio, and thus is the tangent to the efficient frontier
- When $y > 1$, we take leverage - the CAL actually gets 'flatter' due to borrow costs
- When $y < 1$, the complete portfolio is a blend of the risk portfolio and the risk-free asset.

F. Separation and the optimal risky portfolio

The capital allocation line defines how we can allocate portfolio capital between the risk-free and risky asset.

The optimal risky portfolio P^*
Given some risk free rate, we are able to form infinite amounts of capital allocation lines which intersect with the 'cloud of portfolios'. But what is the *optimal* risky portfolio?

We obviously do not consider anything that is not on the efficient frontier, as these portfolios are dominated.

The optimal risky portfolio P^* is the portfolio tangential with the capital allocation line - as it offers the highest Sharpe ratio (the slope).

Optimal allocation along the CAL
Now we have the choice of the optimal risky portfolio P^*, what is the optimal blend of risky and risk-free assets?

The blend is chosen by the *risk-averseness* of the investor, by their given utility for some risky weighting y. We can derive the optimal weighting using quadratic utility

$$\begin{aligned}\text{Max}(U) &= \mathbb{E}(r_C) - \frac{1}{2}A\sigma_C^2 \\ &= r_f + y[\mathbb{E}(r_{P^*}) - r_f] - \frac{1}{2}Ay^2\sigma_{P^*}^2\end{aligned}$$

We take the first derivative w.r.t y

$$\begin{aligned}\frac{\partial \text{Max}(U)}{\partial y} &= \mathbb{E}(r_{P^*}) - r_f - Ay^*\sigma_{P^*}^2 \\ 0 &= \mathbb{E}(r_{P^*}) - r_f - Ay^*\sigma_{P^*}^2 \\ y^* &= \frac{\mathbb{E}(r_{P^*}) - r_f}{A\sigma_{P^*}^2}\end{aligned}$$

Therefore y^* defines the optimal complete portfolio given a investors risk-averseness A.

Complete portfolio with higher borrow rate
In reality, borrowing will generally cost more money than lending will yield. Consider the following characteristics of the risky portfolio.

- $r_f = 0.04$
- $E(r_P) = 16\%$
- $\sigma_P = 20\%$
- $A = 2.5$

Furthermore consider that there exists a **higher borrowing rate** of $r_B = 0.05$. We then have

$$\text{CAL}_P = 0.04 + w \cdot 0.12$$

For weight w into the risky portfolio. Under quadratic utility, we can find the optimal weight.

$$\begin{aligned}U &= 0.04 + (0.16 - 0.04)w - 0.05w^2 \\ \frac{dU}{dw} &= (0.16 - 0.04) - 0.1w \\ w^* &= \frac{0.16 - 0.04}{0.1} \\ &= 1.2\end{aligned}$$

Note the optimal weight requires us to leverage, and borrow wealth. Therefore, we must adjust the risk free rate to reflect the borrowing rate.

$$\begin{aligned}w^* &= \frac{0.16 - 0.05}{0.1} \\ &= 1.1\end{aligned}$$

Now the expected return is given by

$$E(r_P) = 0.05 + 1.1 \cdot 0.11 = 0.171$$

Thus the capital allocation line (CAL) given some risk free rate r_f and a group of risk assets is **universal to all investors**. The allocation shifts by the virtue of the investor's risk-averseness.

Furthermore, when the borrowing rate is higher, we first consider whether the optimal portfolio takes leverage, and then adjust the optimal portfolio to use the borrowing rate.

Separation theorem
The Separation Theorem states portfolio optimisation may be separated into two independent steps:

1) Determine the CAL and optimal risky portfolio P^* (**common to all investors**)
2) Determine the share of wealth which will be invested in P^* based on individua risk aversion (**unique to investors**)

The theory emphasises that all investors invest in the same risky portfolio P^* and the risk-free asset, but differ in their wealth allocation to them.

We then discuss the idea of the market portfolio, which arises from the separation theorem.

The market portfolio and overperformance
Assuming market equilibrium - that is, supply = demand

1) All investors hold the same risky portfolio P^*
2) The investors must collectively hold all risky assets in the market
3) Therefore the aggregate portfolio is the market portfolio M
4) Therefore $P^* = M$, as everyone holds the same portfolio
5) Thus the rational way to increase return and risk is to vary the exposure to M

Therefore, the attractiveness of some constituent of the market S is considered by it's return proportional to it's market weight.

We can see from the above why a diversified market portfolio such as taking an index ETF is considered attractive under the assumption of market equilibrium.

G. Capital Market Line (CML)

Under separation theorem, every rational investor invests along the CAL, regardless of risk aversion. This means that all investors invest in the same optimal risky portfolio P^*. If *everyone holds* P^*, this means that it must be the market portfolio $\mathbf{M}$, which is weighted by the asset's total value divided by the market's total value.

Under the assumption of the market portfolio $\mathbf{M}$, the capital allocation line is referred to as the **Capital Market Line (CML)**.

- A more risk-taking investor invests more into $\mathbf{M}$
- A more risk-averse investor invests more into treasury bills

A practical implementation of CML is the investment into US short-term treasury bills + S&P 500 ETF. The market portfolio $\mathbf{M}$ is described by the weight of the i-th asset

$$w_i = \frac{\text{Market cap}_i}{\sum_{i=1}^n \text{Market cap}_i}$$

Market risk premia and variance decomposition
To decompose the market's risk premium and variance into indivdiual assets, we can apply the following derivations.

$$\begin{aligned}
R_{\mathrm{M}} &= \sum_{i=1}^n w_i R_i \\
\mathbb{E}(R_{\mathrm{M}}) &= \sum_{i=1}^n w_i \mathbb{E}(R_i) \\
\sigma_{\mathrm{M}}^2 &= \mathrm{Cov}(R_{\mathrm{M}}, R_{\mathrm{M}}) \\
&= \mathrm{Cov}\left(\sum_{i=1}^n w_i R_i, R_{\mathrm{M}}\right) \\
&= \sum_{i=1}^n w_i \mathrm{Cov}(R_i, R_{\mathrm{M}})
\end{aligned}$$

and therefore we can see the individual asset contribution to $\mathbf{M}$'s risk premia and variance.

Reward-to-risk ratio of risk premia
We commonly wish to compare investments with eachother - it is understood that *returns* must be considered *relative* to volatility.

The **reward-to-risk** ratio for an asset A is given by

$$\text{Reward-to-risk} = \frac{E(R_A) - r_f}{\sigma_A^2}$$

H. Capital Asset Pricing Model (CAPM)

CAPM is a model for deriving expected returns on risky assets under equilibrium conditions.

CAPM Assumptions
CAPM has a few assumptions about investor behaviour:

- Investors are rational, mean-variance optimisers
- Investors are price takers - no investor is large enough to influence equilibrium prices
- Investors common planning horizon is a single period
- Investors have homogenous expectations on the statistical propoerties of all assets

And some further assumptions about market structure:

- Investors can borrow and lend at a common risk-free rate with no borrowing constraints
- All assets are publicly held and traded on public exchanges

- Perfect capital markets (no financial frictions e.g short selling constraints)

CAPM Derivation
In market *equilibrium*, **M** would have the highest Sharpe ratio and would therefore also have the highest reward-to-risk ratio, given by

$$\frac{\mathbb{E}(R_{\text{M}})}{\text{Cov}(R_{\text{M}}, R_{\text{M}})} = \frac{\mathbb{E}(R_{\text{M}})}{\sigma_{\text{M}}^2}$$

Furthermore, any individual assets reward-to-risk ratio should equal to M's reward-to-risk; otherwise the price of the asset would adjust until the ratio is in parity with other assets.

$$\frac{\mathbb{E}(R_i)}{\text{Cov}(R_i, R_{\text{M}})} = \frac{\mathbb{E}(R_{\text{M}})}{\sigma_{\text{M}}^2}$$

We then re-arrange to derive

$$\begin{aligned}
\mathbb{E}(R_i) &= \frac{\text{Cov}(R_i, R_{\text{M}})}{\sigma_{\text{M}}^2} E(R_{\text{M}}) \\
\beta_i &= \frac{\text{Cov}(R_i, R_{\text{M}})}{\sigma_{\text{M}}^2} \\
\mathbb{E}(R_i) &= \beta_i \mathbb{E}(R_{\text{M}}) \\
\mathbb{E}(R_i) - r_f &= \beta_i(\mathbb{E}(R_{\text{M}}) - r_f) \\
\mathbb{E}(R_i) &= r_f + \beta_i(\mathbb{E}(R_{\text{M}}) - r_f)
\end{aligned}$$

CAPM Interpretation
We derived the CAPM for a individual asset i to be

$$\mathbb{E}(R_i) = r_f + \beta_i(\mathbb{E}(R_{\text{M}}) - r_f)$$

This means that the expected return of a stock is driven b

- r_f: the risk free rate
- β_i: the individual assets sensitivity to the market portfolio **M**'s return
- $\mathbb{E}(R_{\text{M}}) - r_f$: the risk-free return of **M**

This suggests that for all assets in the market, that there is **one price of risk** - and the only change in expected return is given by the assets sensitivity to the market's return.

CAPM Critiques and Application
CAPM has been empirically shown to be questionable - and in particular the 'idiosyncratic' return from the CAPM has shown to be *explainable* (which suggests that the CAPM is missing some factors).

There are some strong applications of CAPM:

- **Portfolio construction**: avoid idiosyncratic risk by holding a well-diversified portfolio
- **Investment performance evaluation**: evaluate the performance of a portfolio relative to the market
- **Capital budgeting**: CAPM required returns can be used as a cost of capital to value estimated future cash flows on assets/projects

I. Systematic and unsystematic (idiosyncratic) risk

We can notice that the CAPM suggests that there exists a linear relationship between individual asset returns and the market portfolio **M**'s returns.

$$\mathbb{E}(r_i) = r_f + \beta_i(\mathbb{E}(R_{\text{M}}) - r_f)$$

At some time t, if we regress some stock i's returns to **M**'s risk-free return, we have the model

$$r_{i,t} = r_f + \beta_i(r_{\text{M},t} - r_f) + \epsilon_{i,t}$$

If we now consider the *risk* (variance) of the stock's return, we find

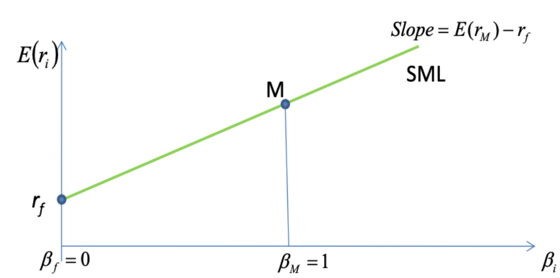
$$\begin{aligned}
\sigma_i^2 &= \text{Var}(\beta_i(r_{\text{M},t} - r_f) + \epsilon_{i,t}) \\
&= \text{Var}(\beta_i(r_{\text{M},t} - r_f)) + \text{Var}(\epsilon_{i,t}) \\
&+ 2 \cdot \text{Cov}(\beta_i(r_{\text{M},t} - r_f), \epsilon_{i,t}) \\
&= \beta_i^2 \sigma_{\text{M}}^2 + \sigma_{\epsilon_i}^2 + 0 \\
&= \beta_i^2 \sigma_{\text{M}}^2 + \sigma_{\epsilon_i}^2
\end{aligned}$$

Therefore for both **return** and **risk**, we can decompose into systematic (explained) and idiosynratic (unexplained) returns/risk.

$$\begin{aligned}
r_{i,t} &= \text{Risk-free} + \text{Systematic} + \text{Idiosyncratic} \\
\sigma_i^2 &= \text{Systematic} + \text{Idiosyncratic}
\end{aligned}$$

J. Security Market Line (SML)

CAPM prices systematic risk - that is, it explains an individual assets tendency to move relative to the market. The *Security Market Line (SML)* explains the return on an asset relative to it's leveraging of β.



Therefore the SML shows us that we can only find more return by taking on more risk relative to the market portfolio **M**, and with the slope of the risk premium of **M**.

II. Market Models & Market Efficiency

A. Single index model (SIM): a market portfolio proxy

Since CAPM is an *equilibrium* model, that is largely theoretical, SIM is an *empirical* model. It replaces the market portfolio **M** with a proxy - most commonly a market index like the ASX 200 for Australia, or S&P 500 for the United States.

Single Index Model (SIM)
The **CAPM** was stated as

$$r_{it} - r_{ft} = \beta_i(r_{Mt} - r_{ft}) + \epsilon_{it}$$

Without strong assumptions of equilibrium, we can no longer guarantee that $\sum \epsilon_{it} = 0$. In **SIM**, we introduce the alpha term α

$$r_{it} - r_{ft} = \alpha_i + \beta_i(r_{Mt} - r_{ft}) + \epsilon_{it}$$

α_i can be seen as the *average* of the residuals over time; a persistent over/underperformance of the market.

We can then say that CAPM is a subset of SIM with the strict assumptions that:

- $\alpha_i = 0$
- $\mathbb{E}(\epsilon_i) = 0$
- $\text{Cov}(\epsilon_i, r_M) = 0$
- $\text{Cov}(\epsilon_i, \epsilon_j) = 0$

Jensen's α
So what is α?

- It is the *average* return which is not explained by β (unsystematic)
- If $\alpha \neq 0$, we say that the asset is mispriced *relative* to CAPM

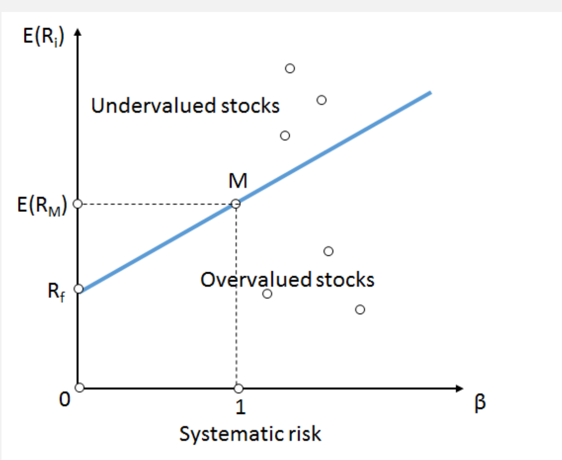
We can say that when:

1) $\alpha > 0$, the asset is under-priced, and thus overperforms
2) $\alpha < 0$, the asset is over-priced, and thus under-performs

The α is the y-intercept when regressing market returns versus asset returns.

$$\alpha_i = R_i - \beta_i R_{\text{M}}$$

Identifiying relative valuation with SML



We can then say that for a unit of risk relative to the market, measured by β:

- Assets above the SML have *more return* per unit of market risk
- Assets below the SML have *less return* per unit of market risk

Now we consider risk measures using SIM. We wish to split risk, like returns, into systematic and unsystematic components.

Risk decomposition of SIM; R^2, ratio of risk
For SIM, the total variance is similar and is composed of systematic and unsystematic risk

$$\sigma_i^2 = \beta_i^2 \sigma_{\text{M}}^2 + \sigma_{\epsilon_i}^2$$

The *ratio* of systematic and total risk is the R^2 given by

$$R^2 = \frac{\text{Systematic risk}}{\text{Total risk}} = \frac{\beta_i^2 \sigma_{\text{M}}^2}{\sigma_i^2} = (\rho_{i,M})^2$$

R^2 is the proportion of the movement in asset i that is explainable by the market movements. We can estimate the covariance between two assets under SIM by

$$\text{Cov}(r_i, r_j) = \beta_i \beta_j \sigma_{\text{M}}^2$$

The correlation between two assets is the product of their correlation with the market

$$(\rho_{i,j}) = \frac{\beta_i \beta_i \sigma_{\sigma_M}^2}{\sigma_i \sigma_j} = (\rho_{i,\text{M}}) \cdot (\rho_{j,\text{M}})$$

B. *Active investing*

CAPM suggests that *passive investing*, the idea of following the market portfolio **M** is the optimal choice for all investors - but in reality, there exists mispriced assets.

Optimal risky portfolios with mispriced assets
Given that assets can be mispriced, the market portfolio is no longer the optimal risky portfolio.

- We should aim to give more weight to under-priced assets
- and less weight to over-priced assets

We wish to maximise the Sharpe ratio of our complete portfolio given by

$$\max_{w_A} S_P = \frac{\mathbb{E}(r_P) - r_F}{\sigma_P}$$
$$\text{where } r_P = w_A r_A + (1 - w_A) r_{\text{M}}$$

where r_A is the active (risky) return, and r_{M} is the passive (market) return.

We can then say that the active return component r_A is an overlay onto the passive return, as the assets are chosen from the same universe.

Reward-to-risk of α
Let an asset A be a single under-priced asset with

$\alpha_A > 0$

$$r_{At} - r_{ft} = \alpha_A + \beta_A(r_{Mt} - r_{ft}) + \epsilon_{At}$$

There is a trade off in trading assets for α:

1) An *upside* for additional return over the market
2) A *downside* for unexampled, unsystematic risk

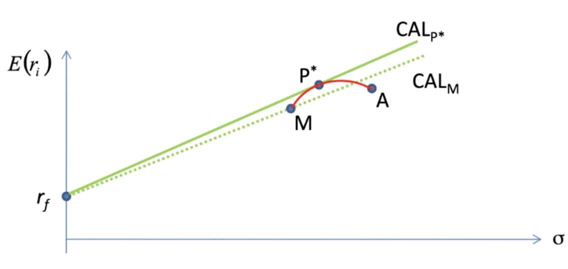
Therefore, the reward-to-risk ratio for a mispriced asset is given by

$$\text{Reward-to-risk} = \frac{\alpha_A}{\sigma^2_{\epsilon_A}}$$

Finding the weight of a mispriced asset To build the optimal risky portfolio P^* with a mispriced asset A, we must require the following:

1) Calculate the asset's reward-to-risk ratio
$$\frac{\alpha_A}{\sigma^2_{\epsilon_A}}$$
2) Calculate the *unadjusted* weighting in A, based on it's reward-to-risk ratio relative to the market
$$w_A^0 = \frac{\alpha_A/\sigma^2_{\epsilon_A}}{\mathbb{E}(R_{\mathrm{M}})/\sigma^2_{\mathrm{M}}}$$
3) Find the *optimal weight* in the mispriced asset by adjusting for the diversification benefit arising from combining the asset A with the market
$$w_A^* = \frac{w_A^0}{1 + (1 - \beta_A)w_A^0}$$

Now there exists two Capital Allocation Lines (CAL); one of the original market portfolio's expected return, and an active + market portfolio expected return.



An example of an active + market portfolio
Suppose that:

- There exists a risky asset A and market portfolio **M**
- $\mathbb{E}(r_{\mathrm{M}}) = 0.1, \sigma_{\mathrm{M}} = 0.25$
- $\mathbb{E}(r_A) = 0.13, \sigma_A = 0.4, \beta_A = 1.2$
- What is the optimal risky portfolio P^*?

We first find the α of the investment.

$$\begin{aligned}\mathbb{E}(r_A) - r_f &= \alpha_A + \beta(\mathbb{E}(r_M) - r_f) \\ &= 0.13 - 0.02 - 1.2(0.10 - 0.02) \\ &= 1.4\% \\ &= \alpha_A\end{aligned}$$

We then consider the *unsystematic* risk

$$\begin{aligned}\sigma^2_\epsilon &= \sigma^2_A - \beta^2\sigma^2_{\mathrm{M}} \\ &= 0.4^2 - 1.2^2 \cdot 0.25^2 = 0.07\end{aligned}$$

We can now compute the reward-to-risk ratio

$$\frac{\alpha_A}{\sigma^2_{\epsilon_A}} = \frac{0.014}{0.07} = 0.2$$

Then find the *naive* weight w_A^0

$$\begin{aligned}w_A^0 &= \frac{0.2}{\frac{0.1-0.02}{0.25^2}} \\ &= 0.15625\end{aligned}$$

Then find the adjusted optimal weight w_A^*

$$\begin{aligned}w_A^* &= \frac{0.15625}{1 + 0.15625(1 - 1.2)} \\ &= 0.16129\end{aligned}$$

Therefore we invest 16.13% of our risky asset portfolio into the mispriced asset and the remaining 83.87% into the market portfolio.

Exploiting multiple mispriced assets
There may be multiple mispriced assets, which we can combine into an optimal active portfolio.

Each mispriced asset in the active portfolio is given a weight proportional to it's contribution to the total reward-to-risk ratio

$$w_i = \frac{\alpha_i/\sigma^2_{\epsilon_i}}{\sum(\alpha_j/\sigma^2_{\epsilon_j})}$$

Then the active portfolio **AP** has α and β given by the weighted averages of the individual asset α and β

$$\alpha_{\mathrm{AP}} = \sum_i w_i \cdot \alpha_i$$

$$\beta_{\mathrm{AP}} = \sum_i w_i \cdot \beta_i$$

Furthermore, the total unsystematic risk of the portfolio is given by

$$\sigma^2_\epsilon = \sum_{i=1}^n w_i^2 \sigma^2_{\epsilon_i}$$

where covariances are not considered as unsystematic risk is considered to be independent.

The information ratio
Not all alphas are equal, even if the added returns are the same. We adjust an α's quality through the introduced unsystematic risk, with the **Information Ratio** (IR)

$$IR_i = \frac{\alpha_i}{\sigma_{\epsilon_i}}$$

Optimally combining a mispriced asset i with the market portfolio **M** to form P^* yields a squared sharpe ratio S such that:

$$S^2_{P^*} = S^2_{\mathrm{M}} + \mathrm{IR}^2_i$$

Creative an active portfolio with multiple mispriced assets
When dealing with the creation of an active portfolio, you must undertake two steps:

1) Find the weight(s) in the active portfolio, using the risk-adjusted α
2) Find the weight of the active portfolio and market portfolio, by adjusting for diversification benefit

After finding individual α and unsystematic risk σ^2_ϵ, you can combine these into a single active α and risk by:

$$\alpha_{\text{active}} = \sum_{i=1}^{n} w_i \alpha_i$$

$$\sigma^2_{\epsilon_{\text{active}}} = \sum_{i=1}^{n} w_i^2 \sigma^2_{\epsilon_i}$$

We can then apply the previously mentioned weighting of the active portfolio relative to the diversification benefit - that is, the active portfolio will be treated as a 'single asset' for weighting purposes.

C. Multi-factor models

CAPM is a single-factor model; it has a single predictor, the risk-premia of the market return. Models generally perform better when there are more explainers; particularly when these explainers are *orthogonal*.

Factor models
A general factor model expresses the **excess** return R_i on an asset as follows

$$R_i = \alpha_i + \beta_{1,t} F_{1,t} + \beta_{2,i} F_{2,t} + \cdots + \beta_{j,i} F_{j,t} + \epsilon_{i,t}$$

where:

- α_i is the average return not explained by the factors
- F_j is the j-th systematic factor
- $B_{j,i}$ is the loading of asset i on the j-th factor
- ϵ_i is the idiosyncratic return.

Define a **risk premium** as the expected return on a factor, $\mathbb{E}(F_j) = \lambda_j$. This gives a multi-factor expected excess return

$$\mathbb{E}(R_i) = \alpha_i + \beta_{1,i}\lambda_1 + \beta_{2,i}\lambda_2 + \cdots + \beta_{j,i}\lambda_j$$

Self-financing portfolios and market neutrality
When a portfolio is constructed such that the net position of the total portfolio is 0%. An example is holding a 100% long position and a 100% short position.

Commonly, these portfolios are used for *market neutral* positions, such that they have $\beta \approx 0$. These are called `equity market neutral` strategies.

Fama-French-Carhart 4 Factor Model
The Fama-French-Carhart 4 Factor model uses self-financing portfolios as factors

$$\mathbb{E}[R_i] = \alpha_i + \beta_{\mathrm{M},i} \cdot \lambda_{\mathrm{M}} + \beta_{SMB,i} \cdot \lambda_{SMB} + \beta_{HML,i} \cdot \lambda_{HML} + \beta_{MOM,i} \cdot \lambda_{MOM}$$

- Market portfolio **M**, is the difference between the return on a value-weighted stock market index and the risk-free-rate.
- Small-Minus-Big portfolio SMB, consists of a long position in the market's smallest 50% firms by market cap and a short position in the largest 50% by market cap (big).
- High-Minus-Low HML, consists of a long position in the top 30% of firms by Book-to-Market ratio (value stocks), and a short position in the bottom 30% of firms (growth stocks)
- Momentum portfolio MOM, consists of taking a long position in the top 30% of firms with the highest returns in the preceding year (winners), and a short position in the bottom 30% (losers).

The Fama-French-Carhart 4 factor model has shown to be empirically and statistically significant, with a high degree of explanatory power.

D. The efficient market hypothesis (EMH)

In trading and portfolio management, `information` is an extremely precious commodity.

- Knowing something that the rest of the market *doesn't* know and/or hasn't priced in yet, gives investors an advantage
- Profits from informational advantages can be significant
- The number of trades who *look for informational advantage* is significant and growing
- Due to the sheer number of trades looking to make a profit by gaining informational advantage, **market prices are likely to reflect all available information**.

The efficiency of asset prices
When an asset price factors in all *available* information, then we call the price of the asset *efficient*.

The price at time T, should reflect the information set I available to investors. This information set I contains:

$$I = \text{current value} + \text{future expected value}$$

If prices are efficient, then only *new* information can move the price. Of course, the new information set I_* cannot be inferred from the current information set I. Thus it is **random**.

Efficient Market Hypothesis (EMH)
The efficient market hypothesis (EMH) states that market prices are *efficient*, reflecting all currently available information, and changes in market prices follow a *random walk*.

There are three versions of EMH.

Weak EMH Current price reflects all *publicly available* historical trading information about a firm such as price patterns/trends and trading volume.

Semi-strong EMH Current price reflects all *public* information about a firm.

Strong EMH Current price reflects all available information about a firm, including public information and non-public information.

Note that:

$$\text{Weak EMH} \subset \text{Semi-strong EMH} \subset \text{Strong EMH}$$

Thus, stronger forms of EMH must satisfy the requirements of the weaker forms.

To consider the *hurdles* of each EMH form, we consider three different types of trading/analysis.

Technical analysis
Technical analysis is a form of analysis where historical **price patterns** are used to generate hypotheses about future market prices.

Fundamental analysis
Fundamental analysis is a form of analysis where fundamental and valuation data is synthesised to create a 'more accurate' valuation for a company.

Insider trading
Using non-public/inside information to gain an information advantage over the market.

The following observations follow

- If technical performance $>$ market performance, than weak EMH does not hold.
- If fundamental performance $>$ market performance, than semi-strong EMH does not hold.
- If insider trading $>$ market performance, than strong EMH does not hold.

E. *Event studies and how to apply them*

Before we consider how to apply *event studies* to test market hypotheses, we will first strictly define events.

(Market) events
Events refer to major ("market moving") announcements which are likely to have a material impact on the stock price, including *earnings* and *dividend* announcements, buybacks, merges and takeovers.

How do we study the 'abnormalness' of an event?

Cumulative Abnormal Return (CAR)
To study events, we use residual returns ϵ to some empirical model H. Consider the hypothesis H that

$$r_t = \alpha + \beta r_{Mt} + \epsilon_t$$

The *abnormal return* is given by

$$\epsilon_t = r_t - (\alpha + \beta r_{Mt})$$

We then add the abnormal returns for some given post-event period to find the **Cumulative Abnormal Return (CAR)**.

There exists some limitations to testing market efficiency:

- Noise: it is hard to obtain sufficient statistical power due to stock volatility
- Magnitude: minor mispricing may be worthwhile to explot for manages with large capital
- Selection bias: only unsuccessful investment schemes are made public; successful schemes remain private
- Sampling error/data mining: sample period may not reflect future periods.

Furthermore, the empirical hypothesis H that we choose *must be true* for the validity of **abnormal returns**.

F. *Market anomalies and effects*

There are persistent market *anomalies* or *effects* which are utilised by investors to gain excess returns over the market. There are **two schools of thought** to explain market anomalies:

1) They are market inefficiencies
2) They are additional βs which should be included in the empirical model, and is not captured by CAPM/SIM.

Momentum and reversal effect
Momentum and reversal are a *weak-form* anomaly.

- Momentum: good or bad recent price performance tends to trend
- Reversal: Episodes of under-or-over shooting followed by correction

Size effect
The size effect is a *semi-strong form* anomaly.
The size effect is simply the obersvation that *small cap* outperforms *large cap*.

- An explainer for the size effect could be more risk-premium is required for illiquid stocks due to limited investor attention

Value effect
The value effect is a *semi-strong form* anomaly.
'Value' stocks on high Book/Market ratios *outperform* 'growth' stocks on low Book/Market ratios.

- The P/E ratio is also similarly used.

Post-earnings drift (PEAD)
The post-earnings drift effect is a *semi-strong* anomaly.
PEAD hypothesises that stocks are slow to fully capture news, and thus there exists a 'drift' in post-earnings returns that trend in the same direction as post-earnings.

G. Behavioural biases

We have previous modelled 'estimates' as some `mean` informed estimate, and a random `error` from this estimate

$$x_i = \mu + \epsilon_i$$

The average of these estimates is given by

$$\begin{aligned}\bar{x} &= \frac{1}{N}\sum_{i=1}^{N}(\mu + \epsilon_i) \\ &= \mu + \frac{1}{N}\sum_{i=1}^{N}\epsilon_i\end{aligned}$$

Whilst market efficiencies require that random errors cancel out, traders may undertake systematic errors/biases, such that the residuals do not sum to 0.

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\epsilon_i &= \frac{1}{N}\sum_{i=1}^{N}(\text{Bias} + \tilde{\epsilon}_i) \\ &= \text{Bias} + \frac{1}{N}\sum_{i=1}^{N}\tilde{\epsilon}_i\end{aligned}$$

Therefore, the average estimate becomes

$$\begin{aligned}\bar{x} &= \mu + \text{Bias} + \frac{1}{N}\sum_{i=1}^{N}\tilde{\epsilon}_i \\ &= \mu + \text{Bias}\end{aligned}$$

H. Limits to arbitrage

Since the long-term average of an investor with bias is given by $\bar{x} = \mu + \text{Bias}$, we could trade on this investors bias and take the contrarian position against it.

Given that the market always prices the asset fairly for $\bar{x} = \mu$, we then have a **arbitrage** trade.

Arbitrage
Arbitrage is a risk-free profit opportunity.

When we speak of arbitrage in markets, we generally speak of `statistical arbitrage`, which is a statistically estimated profit opportunity instead of a truly risk-free one.

In real trading markets, arbitrageurs face limitations to arbitrage.

Fundamental risk of arbitrage

- Markets can stay irrational for very long periods of time; longer than the investor can hold
- Uninformed traders or systematic biases may keep market value from converging to intrinsic value
- An arbitrage has to be right, and right *quickly*.

Implementation and execution risk of arbitrage

- Transacton and carry costs can limit arbitrage activity
- Restrictions on short selling in some countries make it difficult to arbitrage
- If all legs of the trade are not executed in unison, then the arbitraguer has Δ risk

Model risk of arbitrage

- (Statistical) arbitrage hinges on the fact that the model is *correct*
- Arbitrage often attempts to exploit minor mispricings, requiring large capital and leverage to be profitable.

III. Bond pricing and term structure of interest rates

What is a bond?
Bonds are debt obligations for a fixed sum between issuers (borrowers) and bondholders (lenders).

- Borrowers are typically corporates, governments, etc
- Lenders are typically fund managers

For the receiving of cash, the borrower pays *interest* and *principal* payments on designated dates.

Legally, **indenture** is the contract between the issuer and the bondholder describing the terms and conditions of the bond including the coupon rate, maturity date, per value and provisions.

What is default risk?
Default risk is the risk that the issuer will *not* repay their debt obligations.

- A default implies that the issuer is no longer able to receive expected cash flows

A. Pricing bonds

Bond pricing theory
When pricing bonds P, there are five key parameters:

1) Term (T): the period of time to maturity of the bond
2) Face value (FV) or par value: the principal or *loan amount* of the bond, typically repaid in full as one large cash flow at maturity
3) Coupon (C): series of smaller cash flows paid before maturity
4) Coupon frequency: the number of times per annum the coupon is paid.
5) Yield to maturity (YTM): the interest rate applied to discount the cash flows from the bond

The pricing model for an annual-coupon bond is

$$P_B = \frac{FV}{(1+y)^T} + \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

where

- P_B is the price of the bond
- C_t is the interest or coupon payments
- T is the number of periods to maturity
- FV is the face value of the bond
- y is the yield to maturity

Adjusting for n-coupons
For n-coupons per period (where 1 is annual, 2 is semi-annual, etc), we adjust the parameters as such

1) $C_t \to \frac{C_t}{n}$
2) $y \to \frac{y}{n}$
3) $T \to T \cdot n$

Annuity
An *annuity* is a financial product you can buy with a lump sum, to receive a series of regular guaranteed payments.

Pricing bonds as annuities
We now price bonds as an annuity to simplify the pricing equation. Consider the coupon component.

$$\sum C = C_t \left(\frac{1}{1+y} + \frac{1}{(1+y)^2} + \ldots + \frac{1}{(1+y)^T} \right)$$

This is a geometric series with $a = \frac{1}{1+y}$ and $r = \frac{1}{1+y}$.

The sum is then

$$\sum C = C_t \left(\frac{\frac{1}{1+y}\left(1 - \frac{1}{(1+y)^T}\right)}{1 - \frac{1}{1+y}} \right)$$

Simplifying within the brackets, this becomes

$$\sum C = C_t \cdot \frac{1}{y} \left(1 - \frac{1}{(1+y)^T}\right)$$

The par value component stays the same. Therefore, the price of the bond is

$$P_B = C_t \cdot \frac{1}{y} \left(1 - \frac{1}{(1+y)^T}\right) + \frac{FV}{(1+y)^T}$$

The **intuition** behind the formula comes from how the future value of cash is more than the present value of cash.

Therefore, the value of each yearly coupon is discounted at each payment period by the growth of the previous value of the coupon.

An example question of pricing a bond
Given that the 5-year yield is 4%, price a \$1000 5-year, 2.5% annual coupon treasury note.

First, the coupon value is $1000 \cdot 0.025 = 25$. We can then use the annuity formula derived previously:

$$\begin{aligned} P_{TN} &= 25 \cdot \frac{1}{0.04} \cdot \left(1 - \frac{1}{(1+0.04)^5}\right) + \frac{1000}{(1+0.04)^5} \\ &= \$933.22 \end{aligned}$$

We now consider the three types of bonds with reference to their provided yields.

Discount, par and premium bonds
Relative to the market yield y_M:

1) If $y_B < y_M$, then the bond is **discount**
   - $P_B < P_{\text{Par}}$
2) If $y_B = y_M$, then the bond is **par**
3) If $y_B > y_M$, then the bond is **premium**
   - $P_B > P_{\text{Par}}$

Given certain parameters, the yield-to-maturity rate can be derived.

Deriving the yield-to-maturity (y)
Given P_B (PV), FV, C_t and T we can derive the yield to maturity rate (as it is a single unknown variable).

You can do this in excel by applying the following formula

```
RATE(T, C_t, -PV, FV)
```

Note that yield-to-maturity does not reflect the actual realised return on a bond. This is because whilst the yield-to-maturity was chosen at the point of purchase, interest rates change! To consider the realised yield, we consider a separate process.

Holding Period Return (HPR)
A simple measure for the returns from a bond B, is the following

$$\text{HPR} = \frac{\text{Total bond proceeds}}{P_0} - 1$$

where P_0 is the price the bond was bought at.

Realised compound yield
To deal with changing interest rates, we want a more accurate measure of the yield we get on a bond B. We do this by:

1) Reinvesting all interm cash flows (coupons) to the end of the holding period
2) Calculating the aggregate cash flows (total bond proceeds) to the end of the holding period
3) Calculate HPR
4) Annualise the return $(1 + HPR)^{\frac{1}{T}}$

Example of calculating a realised compound yield
Consider purchasing a 5-year \$1000 annual coupon bond at 4.0% for \$1045.80. You predict the future rates in the next i-th years are

$$r_1 = 3.0\%, r_2 = r_3 = 3.5\%, r_4 = r_5 = 4.0\%$$

1) What is the YTM of the bond at $T = 0$?
2) Assuming the rate predictions are correct, what is the HPR and realised compound yield?

For question 1, we use

```
RATE(5, 0.04 * 1000, -1045.80, 1000)
```

and find the yield-to-maturity is 3%. Now consider the total bond proceeds

$$\begin{aligned} \text{TBP} = 40 \cdot (1.035^2 \cdot 1.04^2) + {} & \text{[Year 1]} \\ 40 \cdot (1.035 \cdot 1.04^2) + {} & \text{[Year 2]} \\ 40 \cdot (1.04^2) + {} & \text{[Year 3]} \\ 40 \cdot (1.04) + {} & \text{[Year 4]} \\ 1040 & \\ = 1215.99 & \end{aligned}$$

Therefore, the HPR is

$$\text{HPR} = \frac{1215.99}{1045.80} - 1 = 16.27\%$$

Now, annualising this return

$$\text{Realised yield} = (1 + 0.1627)^{\frac{1}{5}} - 1 = 3.06\%$$

Compared to the yield-to-maturity, this bond was a *premium* bond.

Pricing bonds with arbitrage
Consider that a bank provides a borrowing/lending rate of 8%.

1) What is the fair price (arbitrage-free) for a bond with $FV = 100$, $T = 2$ and $C = 20$?
2) How would an arbitrageur make a risk-free profit *if* the bond was priced at \$120?

First, find the fair price $\widehat{PV}$

$$\begin{aligned} \widehat{PV} &= \frac{100}{1.08^2} + \frac{20}{1.08^2} + \frac{20}{1.08} \\ &= \$121.40 \end{aligned}$$

Now, if the *actual price* $PV < \widehat{PV}$, then we should:

- **Buy** the bond
- **Sell (borrow)** cash, such that we can pay off the loan with the bond's cash flows

At $t = 1$, there is a cash flow of \$20 (coupon), and at $t = 2$ there is a cash flow of \$120 (coupon + redemption). Therefore

$$\frac{120}{1.08^2} = \$102.88$$
$$\frac{20}{1.08} = \$18.52$$

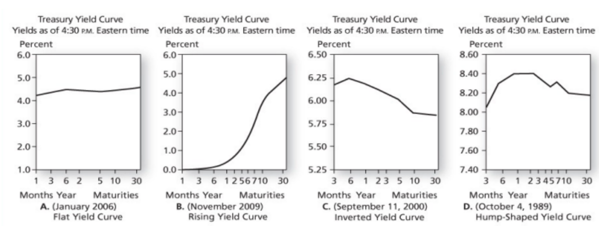
Thus, we make a risk-free profit of \$1.40.

B. Term structure of interest rates

What is term structure?
The term structure of interest rates refers to how interest rates vary of different interest horizons.

It is commonly referred to as the **yield curve**, which displays the relationship between *yield* and time to maturity.

Treasury yield curves are defined by time-to-maturity (x-axis) and yield (y-axis).



Upward sloping curves would then indicating the future short-term interest rates are expected to be higher than the present, and vice-versa.

Spot rates and pure yield curves
The spot rate y_t is the interest rate prevailing *today* at time 0, for a t-period investment.

The spot rate is then, the yield-to-maturity of zero-coupon bonds.

The **pure yield curve** is derived from spot rates on zero-coupon bonds of differing maturities.

On-the-run yield curve
The on-the-run yield curve, unlike the pure yield curve uses recently issued coupon bonds selling at or near par.

On-the-run yield curves are more widely used, as coupon bonds are much more common than zero-coupon bonds.

We now turn to the methodology of yield curves given the observation of recently sold bonds (coupon and zero-coupon).

Spot rates from zero-coupon bonds
At the sale of a zero coupon bond, we know:

1) P_0, the price of the bond
2) FV, the value paid out at the end of the bond
3) t, the time to maturity

We can then find the spot rate y_t, given that

$$P_0 = \frac{FV}{(1+y_t)^t} \implies y_t = \frac{FV}{P_0}^{1/t} - 1$$

Spot rates from coupon bonds: bootstrapping
Now, consider how we can find spot rates for multiple periods t, given information from a 1-year zero-coupon bond and a 2-year coupon bond.

For the first year $t = 1$, we can find the spot rate y_1. Then, our goal is to find y_2.

For the coupon bond, the pricing is

$$P_B = \frac{c_1}{1+y_1} + \frac{FV + c_2}{(1+y_2)^2}$$

We know c_1, c_2, FV and y_1, so we can now also infer y_2!

$$y_2 = \sqrt{\frac{FV + c_2}{P_B - \frac{c_1}{(1+y_1)}}} - 1$$

Note that we can now roll this forward for 3-year, 4-year, n-year coupon bonds. This method is called **bootstrapping**.

C. Future rates (rates under certainty)

We are eable to replicate long-term bond cash flows by re-investing short term bond cash flows.

Investing into multiple short-term bonds should be **equivalent** to investing into a single long-term bond under no-arbitrage.

Notation of interest rates and yields: $y_{s,t}$
The notation of an interest rate/yield $y_{s,t}$ denotes the yield/interest rate an investor would receive from period $s \to t$.

Thus, $y_{0,t}$ is the spot rate for an investment of period t.

An example of pricing future interest rates
Consider two investments A and B - both are ZCBs. Investment A is a 2-year bond, and investment B is a $2\times$ 1-year bond. Thus under no-arbitrage:

$$\begin{aligned} 1 + R_A &= 1 + R_B \\ (1+y_{0,2})^2 &= (1+y_{0,1})(1+y_{1,2}) \\ y_{1,2} &= \frac{(1+y_2)^2}{(1+y_1)} - 1 \end{aligned}$$

We can generalise for t periods.

Future rates
The future rate $y_{0,t}$ should be equal to *combining shorter-term*, 1 period investments for t periods.

$$y_{0_t} = [(1+y_{0,1})(1+y_{1,2})\dots(1+y_{t-1,t-2})]^{1/t} - 1$$

Under certainty, $y_{t-1,t}$ is often called the **short rate** r_t. We can get the short rate at some time t with the following

$$r_t = \frac{(1+y_t)^t}{(1+y_{t-1})^{t-1}} - 1$$

An important side not is that **interest rates in future** are impossible to guess correctly with high precision:

- We call these implied rates future rates when the interest rate changes are known for the future
- We call these implied rates *forward rates* when the interest rate changes are not known

D. Forward rates (rates under uncertainty)

No-one (not even the most accurate of models) can predict $y_{s,t}$ until we are at time s. Thus, the best guess we have is to infert the information on future spot rates based on the yield curve - which is the information set I at time t_0.

Implied forward rates
The implied forward rate $f_{s,t}$ is the interest rate implied by two *distinct* spot rates.

$$(1+y_{0,s})^s(1+f_{s,t})^{t-s} = (1+y_{0,t})^t$$

$$f_{s,t} = \left[\frac{(1+y_t)^t}{(1+y_s)^s}\right]^{\frac{1}{t-s}} - 1$$

Therefore the forward rate is a prediction of an interest rate between times s and t.

To distinguish between spot rates, future rates and foward rates:

1) Spot rates are the yield on zero-coupon bonds contracted **now** and invested **now**
2) Future rates are yield on zero-coupon bonds contracted **in the future** and invetsted **in the future** (look-ahead into the future)
3) Forward rates are yields on securities that are contracted **now** and invested **in the future**

E. Expectations Hypothesis

Previously we stated that the yield on a two-year ZCB should equal to combining two one-year ZCBs:

$$(1 + y_{0,2})^2 = (1 + y_{0,1})(1 + y_{1,2})$$

In a realisitc market scenario, $y_{1,2}$ must be forecasted and thus an *expectation*

$$(1 + y_{0,2})^2 = (1 + y_{0,1})(1 + \mathbb{E}(y_{1,2}))$$

Expectations Hypothesis (EH)
The *Expectations Hypothesis (EH)* states that only the *market expectations* on the future interest rates shape the yield curve.

Since forward rate are inferred from the term structure, a common way to denote EH is

$$f_{s,t} = \mathbb{E}(y_{s,t})$$

The implication is that the yield curve is **purely** function of expected forward rates - without investor bias, risk premiums, etc.

Thus the expectations hypothesis states that the yield curve is just an expectation of forward rates.

Price/liquidity risk
Consider investing in a 1-year ZCB versus a 2-year ZCB and selling at $t = 1$.

The 1-year ZCB has a fixed risk-free $y_{0,1}$, but what about the 2-year ZCB?

$$P_0 \cdot (1 + HPR) = P_1$$

$$HPR \implies \frac{P_1}{P_0} - 1$$

$$P_1 = \frac{FV}{1 + y_{1,2}}$$

Therefore there exists some *uncertainty* in our 2-year ZCB option, as it bakes in the expected cash flows for $t = 1 \to 2$. This is called price or liquidity risk.

Re-investment risk
Consider now investing in a 2-year ZCB versus $2\times$ 1-year ZCB, re-investing at $t = 1$.

The 2-year ZCB guarantees $y_{0,2}$, but what about the $2\times$ 1-year ZCB?

$$(1 + HPR)^2 = (1 + y_1)(1 + y_{1,2})$$

But $y_{1,2}$ is not known, and thus has risk. This is called reinvestment risk.

F. Liquidity preference hypothesis

Typically we assume that investors can pick a bond that matches their desired investment horizon.

Liquidity preference hypothesis
Investors prefer short-term investments, and thus require a *premium* for holding long-term investments

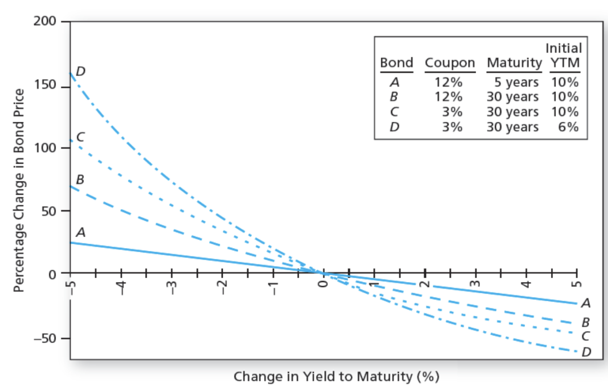
$$f_{s,t} = \mathbb{E}(y_{s,t}) + LRP$$

where LRP is the liquidity risk premium.

Thus, an *upward yield curve* does not necessarily represent an expectation of rates to rise - but could also represent an increasing liquidity risk premium for investors.

G. Bond pricing sensitivity to underlying rate Δ

Bond prices are sensitive to interest rate movements, as the value of a bond relies on it's yield versus the current market yield.



In the above figure, we can see the bond price changes of 4 different bonds relative to the underlying interest rate changes. We can take a few observations:

- Bond prices and yields are *inversely related*
- Bond prices are more sensitive to interest rate falls than increases (convexity)
- Long-term bond prices are more sensitive to rate Δ than short-term $T \uparrow \Delta P \uparrow$
- As maturity increases, price sensitivity to interest rate to interest rate changes increases at a decreasing rate
- High coupon bond prices are *less* interest-rate sensitive $C \uparrow \Delta P \downarrow$
- Higher YTM bond prices are less interest rate sensitive than lower YTM bond prices $YTM \uparrow \Delta P \downarrow$

H. Effective measures of bond maturity: duration

Duration
Duration is the `effective` maturity of a bond.

- Cash flows throughout the bond's maturity is not equal (earlier cash flows are less discounted)
- Duration is a *weighted-average* until until cash flows are received

Duration is a key concept for three main reasons:

1) A measure of the effective maturity of a bond
2) A measure of the interest rate sensitivity of a portfolio
3) An essential tool in immunising portoflios from interest rate risk

Macaulay Duration
The Macaulay Duration is the weighted average time that cash flows on a bond are received (in years)

$$\mathbf{D}_{\text{Macaulay}} = \sum_{t=1}^{T} \frac{t}{P} \cdot \frac{CF_t}{(1+y)^t}$$

where

- t is the time period when a cash flow is received
- CF_t is the cash flow received at time t
- y is the yield-to-maturity
- P is the market price of the bond

Note that the component

$$w_t = \frac{CF_t}{P \cdot (1+y)^t}$$

Determines the **weight** that the future cash flow at time t has on the present value of the bond. Thus, we have the new form

$$\mathbf{D}_{\text{Macaulay}} = \sum_{t=1}^{T} t \cdot w_t$$

Duration as measured by the Macaulay Duration, is actually just a measure of interest rate risk.

$$\mathbf{D} = -\frac{\frac{\partial P}{P}}{\frac{\partial y}{1+y}}$$

To prove this, consider each component of the division first.

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial \sum_{t=1}^{T} CF_t (1+y)^{-t}}{\partial y} \\ &= \sum_{t=1}^{T} CF_t \cdot (-t) \cdot (1+y)^{-t-1} \\ &= -\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+y)^{t+1}}\end{aligned}$$

Therefore

$$\begin{aligned}-\frac{\frac{\partial P}{P}}{\frac{\partial y}{1+y}} &= -\frac{\partial P}{\partial y}\frac{1+y}{P} \\ &= \sum_{t=1}^{} \frac{\frac{CF_t}{(1+y)^t}}{P} \cdot t = D\end{aligned}$$

Thus duration is a measure of the interest rate risk of a bond.

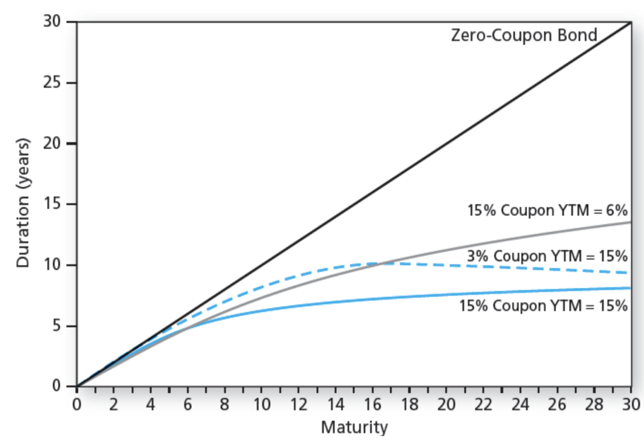
Calculating the duration of a bond
Consider a five-year, 10% coupon bond with FV of \$100 and 10% YTM. What is the duration of this 5-year coupon bond?

Note YTM = coupon rate, and thus P = FV = 100. Thus, we can calculate the weights for each t.

$$\begin{aligned}w_1 &= \frac{10/1.1}{100} = 0.0909 \\ w_2 &= \frac{10/1.1^2}{100} = 0.0826 \\ w_3 &= \frac{10/1.1^3}{100} = 0.0751 \\ w_4 &= \frac{10/1.1^4}{100} = 0.0683 \\ w_5 &= \frac{110/1.1^5}{100} = 0.683\end{aligned}$$

Thus, duration is equal to

$$\begin{aligned}D &= 0.0909 \cdot 1 + 0.0826 \cdot 2 + 0.0751 \cdot 3 + 0.0683 \\ &\cdot 4 + 0.683 \cdot 5 \\ &= 4.16\end{aligned}$$



- Higher coupons means more of the cash flow is paid out faster, therefore duration is lower
- Higher yield-to-maturity means future cash flows are more heavily discounted, and thus duration is lower
- Zero-coupon means the entire cash flow is paid at the end of maturity, and thus duration is maturity.

Modified duration: duration as ameasure of interest rate risk

Earlier, we found that

$$\frac{\partial P}{P} = -D \cdot \frac{\partial y}{(1+y)}$$

We can instead have a new *modified* duration definition

$$D^* = \frac{D}{1+y}$$

and thus we have

$$\frac{\partial P}{P} = -D^* \cdot \partial y$$

Using modified duration to approximate bond price changes

Consider a ZCB with 4.2814 years left to maturity. The YTM is 10%.

1) Calculate the duration and modified duration
2) Estimate the relative change if the interest rate rose by 0.1%

$D = T = 4.2814$, as it is a ZCB. Thus

$$D^* = \frac{4.2814}{1+0.1} = 3.892$$

Therefore, the approximate bond price movement is

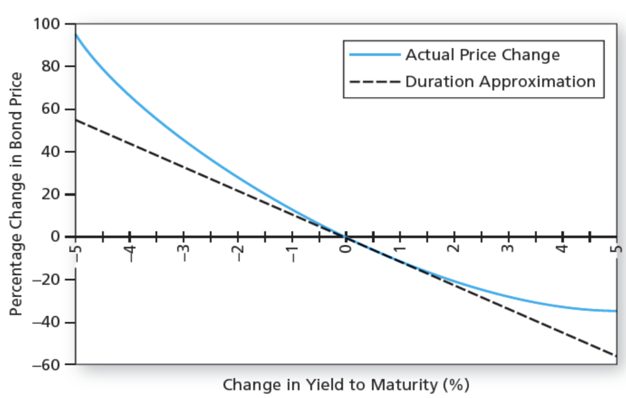
$$\frac{\Delta P}{P} \approx -D^* \cdot \Delta y = -3.892 \times 0.1\% = -0.3892\%$$

I. Convexity

Modified duration approximates a linear *negative* relationship of bond prices with interest rate changes

$$\frac{\Delta P}{P} = -D^* \cdot \Delta y$$

The approximated price change given by modified duration is always lower than the actual price changes, on both positive and negative price movements.



Convexity and a more precise change in bond price

Bond price movements relative to the approximations given by modified duration, overestimate negative price movements and underestimed positive price movements.

This effect is called *convexity*. Convexity is given by the second derivate of bond price w.r.t yield

$$\frac{d^2P}{dy^2} = \sum_{t=1}^{T} CF_t t(t+1)(1+y)^{-t-2}$$

Let $w_t = \frac{CF_t}{P \cdot (1+y)^t}$ again. We then have

$$\text{Convexity} = \frac{1}{(1+y)^2} \sum_{t=1}^{T} w_t \cdot (t^2 + t)$$

More precise bond price changes with Taylor series

We can then include the convexity into our bond price Δ formula, given modified duration using Taylor series.

$$\frac{\Delta P}{P} \approx -D^* \cdot \Delta y + \frac{1}{2} \cdot \text{Convexity} \cdot (\Delta y)^2$$

Desirability of higher convexity

More convex bonds have increased upside and decreased down side compared to less convex bonds. This makes convexity *desirable* to investors.

J. Holding a portfolio of bonds: duration and immunisation

Now we consider characteristics of a bond applied to a portfolo of bonds.

Portfolio Duration

We define portfolio duration as the price-weighted duration of all the bonds within the portfolio.

$$\mathbf{D}_{\text{Portfolio}} = \sum_{i=1}^{n} \mathbf{D}_i \cdot w_i$$

where

$$w_i = \frac{P_i}{\sum_j P_j}$$

To consider why duration analysis and more are important to portfolio managers, we must consider the rational investor's dislike for added risk.

Immunisation

Immunisation refers to techniques designed to shield financial status from interest rate risk

- Immunisation may be accomplished by matching the duration of liabilities and that of assets
- As interest rate changes, the portfolio may no longer be immunised, and must be rebalanced to match the durations

The **two key requirements** to immunise a liability are:

- The (weighted) present values of the immunisation portfolio and liability must be equal
- The (weighted) *duration* of the immunisation portfolio and liability must be equal

An example of immunising a portfolio
Consider holding a liability L of \$100 maturing in 3 years. Assume a flat interest rate of 8%. You have access to two bonds:

- A ZCB b_1 with a face value of \$100 and time-to-maturity of 2 years
- A ZCB b_2 with a face value of \$100 and a time-to-maturity of 6 years

We first match the present value of the liability using the present value of the two bonds. Let w_1 and w_2 be the dollar amounts invested into b_1 and b_2.

$$w_1 \cdot \frac{100}{1.08^2} + w_2 \cdot \frac{100}{1.08^6} = \frac{100}{1.08^3}$$

Then, we must match the duration of the two bonds and the liability. Note we normalise by the sum of the dollar amounts, as the units are different (time versus money)

$$\frac{w_1 \cdot 2 + w_2 \cdot 6}{w_1 + w_2} = 3$$

This leads to $w_1 = 0.248$ and $w_2 = 0.743$, and thus for each bond we must invest

$$\text{Invest}(b_1) = 0.248 \cdot 100 = \$24.8$$
$$\text{Invest}(b_2) = 0.743 \cdot 100 = \$74.3$$

IV. Options and Derivatives

What is a derivative?
Derivatives are a financial instrument whose value *depends* on an observable price.

- Options: a contract that grants the hlder the right, but not the obligation, to transct in an asset
- Futures: A contract to buy or sell a commodity at a point in the future.
- Swaps: A contract in which two parties exchange cash flows from different financial instruments

A. Introduction to options

What is an option?
Options give the holder the *right, but not the obligation* to trade the underlying asset at a non-market price. Options have several characteristics:

- The option type is generally either a `call` or `put`.
- A call allows the option holder to buy the underlying, whereas a put allows the holder to sell
- The `strike price` indicates the price at which the option holder can transact shares
- Using (buying/selling the underlying) the option is called `exercising` the option.
- The `expiration date` indicates the last date that an option can be exercised

Rights and obligations depend on whether you are the writer/issuer of an option - by writing/issuing an option, you are selling the option.

Rights and obligations to buyers and sellers
For a `call option`:

- Holder (buyer) has the right to buy the underlying security at a fixed price
- Writer (seller) has the obligation to sell the security, should the holder exercise

For a `put option`:

- Holder (buyer) has the right to sell the underlying security at a fixed price
- Writer (seller) has the obligation to buy the security, should the holder exercise

So options always come with a counter-party, who as a writer, must transact with you as a holder if you decide to exercise the option. What about the prices of options? What specifically happens when an option is exercised?

Option premiums and settlement
The purchase price of an option is called the `option premium`. When an option is exercised, two different *settlements* can occur:

1) Physical: The writer of the option must deliver the underlying asset, selling shares to the option holder.
2) Cash: The writer pays out the *intrinsic value* of he option at the time of exercise.

Whilst we have talked about options so far as if they only grant the right to one unit of an asset, it is often the case that options grant the right to purchase/sell *multiple* units of the asset. When can options be exercised?

Exercising an option: European and American
Options come in two main types:

- `European options`: can only be exercised **at** the expiration date. This makes valuation easier.
- `American options`: can be exercised **at any time** before and including the expiration date, making valuation difficult.

B. Concepts and terminology in options trading and pricing

We begin with the following notation:

- t is the current time
- T is when the option expires
- S_t is the price of the underlying stock at time t
- X is the exercise price of the option

We now consider a few key concepts about options and their pricing. First, how do we describe options that are profitable, even or unprofitable at the current time t?

Moneyness of options
Options are referred to as for the current time t_0:

- `In the money`: the option can be exercised at t_0 for a profit
- `At the money`: $S_{t_0} = X$
- `Out of the money`: the holders would realise a loss if they were to exercise the option at t_0

We refer to an option as *deep in/out-of-the-money* when the difference between the strike and underlying is large.

We have referred to the **intrinsic value** of an option so far - but what does this mean?

Intrinsic value: the value if exercised now
The intrinsic value of an option is the value it would have if exercised immediately, at time t_0. For a call option, the intrinsic value is

$$\max(0, S_t - X)$$

And for a put option, the intrinsic value is

$$\max(0, X - S_t)$$

You may have noticed that there exists a concept of the value of *time* when it comes to options - if the option is out of the money, and there exists more time $T - t$ until expiration, then the option should be more expensive (more probability it hits the strike price).

Time value of options (θ)
An option's time value derives from the possibility that the moneyness can change in the future.

The probability of the option being in the money at exercise/expiration increases with:

- Time to expiration: more time means more probability that the stock hits the strike price
- Volatility: more volatility means more possible movement to the strike price

C. Options pricing: binomial option pricing

Option pricing models are generally built around the fact that options are *redundant* - that is, the payoff of an option can be replicated (synthetic) by existing securities.

First, consider some terminology important to our derivation.

Hedge ratio
The hedge ratio H or delta of an option indicates the number of shares of the underlying security required to replication option values in the next period.

A portfolio with H shares and an appropiate position in the risk-free asset will replicate the payoff of the option.

Consider we wish to replicate a call option on some stock.

- Today's stock price is S_0
- In one step, it moves either

$$S_u = S_0 \cdot u \qquad S_d = S_0 \cdot d$$

- Risk free gross return over the step is $R = 1 + r$
- The option payoff in the up/down states is P_u and P_d

Thus we must find weight H in the stock and **cash position**B_0 in the risk free asset to replicate the option payoff

$$P_u = H \cdot S_u + B_0 \cdot R$$
$$P_d = H \cdot S_d + B_0 \cdot R$$

W

We can solve for H and B_0 to get

$$H = \frac{P_u - P_d}{S_u - S_d}$$
$$B_0 = \frac{P_u - HS_u}{R} = \frac{P_d - HS_d}{R}$$

Then, the price of the option today V_0, is defined by constructing the synthetic option with the above portfolio weights

$$V_0 = H \cdot S_0 + B_0$$

A simpler pricing formula comes from the elimination of H for the `risk-neutral` valuation. We instead consider the probability of the u scenario as p and the d scenario as $1-p$. The future expected price is then

$$RS_0 = pS_u + (1-p)S_d$$

Note that we are assuming in this case that there exists no *risk-premium* for the stock, and that the return of the stock *equals the return of the risk-free rate*. We then solve for p

$$\begin{aligned} RS_0 &= pS_u + S_d - pS_d \\ p(S_u - S_d) &= RS_0 - S_d \\ p &= \frac{RS_0 - S_d}{S_u - S_d} \\ &= \frac{RS_0 - dS_0}{uS_0 - dS_0} \\ &= \frac{R-d}{u-d} \end{aligned}$$

Similarly, in a risk-free world, the option valuation grows in parity to the risk-free asset. Letting $q = \frac{R-d}{u-d}$, we find

$$RV_0 = pP_u + (1-p)P_d$$
$$V_0 = \frac{1}{R}(qP_u + (1-q)P_d)$$

We can then recursively find an n-period binomial price for options, by winding down from the end state.

An easy way to understand binomial option pricing, is that we consider every (discrete) possibility of the underlying price, and then try to reconstruct the option price from this.

Risk-neutral probability
We have heard a lot about the "risk-neutral" probability. Option pricing models generally assume that every asset's expected return equals the risk-free rate. Therefore:

- Given some positive return factor U over years t
- and some negative return factor D over years t
- and a risk-free rate r

We have the risk-neutral probability

$$p = \frac{e^{r\Delta t} - D}{U - D}$$

Applying the binomial option pricing model
A stock is currently priced at \$20. In a *given 4-month period*, the price can either:

- Go up 18.91%
- Go down 15.9%

The (annual) volatility is $\sigma = 0.3$. The risk-free rate $r_f = 0.04$. Price a European call option of strike price \$12 with an 8 months expiry.

Consider C_S is the payoff of the call option at state S. The following should be clear:

$$\begin{aligned} C_{uu} &= \$20 \cdot 1.1891 \cdot 1.1891 - \$12 = \$16.28 \\ C_{ud} &= \$20 \cdot 1.1891 \cdot 0.841 - \$12 = \$8 \\ C_{du} &= \$20 \cdot 0.841 \cdot 1.891 - \$12 = \$8 \\ C_{dd} &= \$20 \cdot 0.841 \cdot 0.841 - \$12 = \$2.15 \end{aligned}$$

Now, before we work backwards, we must find the risk-neutral probability of the upward price movement for a 4-month period

$$p = \frac{1.013 - 0.841}{1.1891 - 0.841} \approx 0.495$$

Thus, we can price C_u and C_d with a 4-month risk-free rate $r \approx 0.013$

$$C_u = \frac{0.495 \cdot \$16.28 + 0.505 \cdot \$8}{1.0133} = \$11.94$$
$$C_d = \frac{0.495 \cdot \$8 + 0.505 \cdot \$2.15}{1.0133} = \$4.98$$

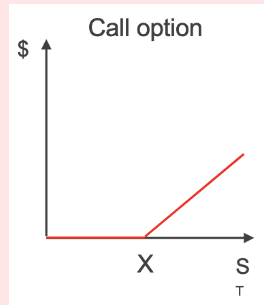
Now, we can find the call option's price

$$C_\emptyset = \frac{0.495 \cdot 11.94 + 0.505 \cdot 4.98}{1.0133} = \$8.38$$

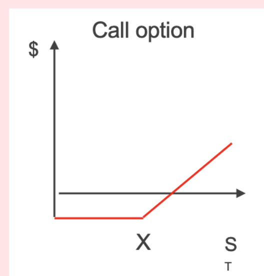
D. *Option strategies*

We now consider different investment strategies utilising options. We first consider some ways we can visualise options and their characteristics.

Payoff diagrams and profit diagrams
For options, it is useful to plot the payoff and profit against the strike price of the underlying. This is because the payoff and profit have a piecewise-linear relationship with the strike price.



The above is a *payoff* diagram. We have positive payoff for a call option once the underlying is above the strike price. However, options come with a premium tht a holder must pay.



The above is a *profit diagram*. The piecewise function for option profit is

$$\text{Profit} = \begin{cases} -C & \text{if } S_t \leq X \\ S_t - X - C & \text{otherwise} \end{cases}$$

Now that we have an understanding for how to price options, we turn to the question of *why options*, or more specifically how are options used.

Uses for options
There are two main uses of options in financial markets:

- Hedging; to reduce risk in existing portfolio positions without having to rebalance
- Speculation; to establish positions independent of existing portfolio positions

We now go through some common investment 'patterns' or 'combinations' of options and stocks which investors can use to achieve desired payoffs.

Protective put
A protective put is a position that consists of:

- One unit long of stock S
- One unit long of a put on stock S

At time t, the stock price is S_t. If we buy a put such that $X = S_t$ at a premium C, then our profit function is

$$\text{Profit} = \begin{cases} X - C & \text{if } S_t \leq X \\ S_t - C & \text{if } S_t \end{cases}$$

To explain the above cases:

- If the stock is less than the strike price, we can exercise the put and sell our stock at the strike price
- If the stock is more than the strike price, we do not exercise the put, and sell our stock for the market price

This gives investors a floor on risk.

Covered call
A **covered call** is a position that consists of buying a stock and writing a call option. This means we are

- One unit long of a stock S
- One unit short of a call on a stock S

If the stock S rises, our only payoff is the option premium C, as we will be exercised to sell our unit of the stock S. The profitfunction is, where S_t is the price of the stock at time t, X is the strike, and C is the option premium

$$\text{Profit} = \begin{cases} S_t + C & \text{if } S_t \leq X \\ X + C & \text{if } S_t > X \end{cases}$$

Straddle
A straddle (call) is a position which speculates on the *volatility* of a stock. The investor holds:

- One long of a call option on stock S at strike X
- One long of a put option on stock S at strike X

Note the premiums of the call and put as C_0 and P_0. Therefore, we have a V-shaped profit diagram, in the form of

$$\text{Profit} = \begin{cases} S_t - X - (C_0 + P_0) & \text{if } S_t > X \\ -(C_0 + P_0) & \text{if } S_t = X \\ X - S_t - (C_0 + P_0) & \text{if } S_t < X \end{cases}$$

We therefore want the stock price to be very volatilie and move significantly beyond the strike price - as the hurdle to make a profit is two premiums.

Collar
A **collar** has a "collared" or limited profit range, constructed by:

- One unit long in a stock
- One unit long in a put position with strike price X_P
- One unit short in a call position with strike price $X_C > X_P$

The goal is to cover the price of the put position with writing the call position, but writing the call position means that there is a limited upside to the underlying increasing (as we can be exercised).

Note the prices of the call and put premiums as C_0 and P_0.

$$\text{Profit} = \begin{cases} X_P - P + C & \text{if } S_t \leq X_P \\ S_t - P + C & \text{if } X_P < S_t \leq X_C \\ X_C - P + C & \text{if } X_C < S_T \end{cases}$$

To explain, we have this range $X_P \to X_C$.

- We are limited on downside risk with the long position in the put
- We are also limited on upside with the short position in the call
- But the put premium exposure is reduced by collecting the call premium

E. Put-call parity

Compare the payoffs of a protective put (+1 stock, +1 put) and a call option + risk-free asset that yields \$X at expiration.

- When $S_t \leq X$, the pay-off for both is X
- When $S - t > X$, the pay-off for both is S_t

Thus, note that under no-arbitrage, we have

$$S_t + P_t = C_t + \frac{X}{1+r}$$

This is called **put-call parity**, and only applies to European options (due to the risk-free asset having to yield \$X at expiration).

Pricing a put by put-call parity
The S&P 500 index price \$1000 and the effective 6-month interest rate is 2%. Suppose the price on a 6-month S&P 500 call is \$109.20 for a strike price of \$970. What is the price on a 6-month *put* with the same strike price?

$$S_0 = 1000$$
$$X = 970$$
$$r = 2\%$$
$$C_0 = 109.20$$

Therefore, we can re-arrange the put-call parity relationship to be

$$P_t = C_t + \frac{X}{1+r} - S_t$$
$$= 109.20 + \frac{970}{1.02} - 1000$$
$$= 60.18$$

From the put-call parity and no-arbitrage, there exists implicit bounds on the prices of calls and puts.

Bounds on a European call (put) option
The payoff of a call option at expiration t is

$$C_t = \max(0, S_t - X)$$

The prices of course must be non-negative

$$C_t \geq 0$$

The price of the call cannot exceed the stock price (other, we have the right to buy a stock for more, which makes no sense)

$$C_t \leq S_t$$

There is an upper bound of arbitrage - that the payoff of a call should not exceed the payoff of owning the stock and shorting a $X yielding risk-free asset

$$C_t \geq S_t - PV(X)$$

Then there exists the put-call parity relationship

$$C_t = P_t + S_t - PV(X)$$

The above can be re-arranged or inversed for a put.

F. The impact of early excercise on an option

In comparison to European options, American options give the right to excerise option contracts *early*, before the expiration date. We now consider in what cases it is optimal to execute early.

It is never optimal to early exercise on a call option
The lower bound on a stock's value with price S, strike X and dividend(s) D is

$$C \geq S - PV(X) - PV(D)$$

For a non-dividend paying stock, this means that

$$C \geq S - PV(X)$$

Since present values are (generally) less than future values, we know that

$$C \geq S - X$$

Thus, the call value is always greater than or equal to the value realised on excercise. This indicates that we should always hold.

Early exercise of an American call option may be optimal on dividend-paying stock
Consider a stock that may pay a large dividend whose ex-dividend date is before expiration.

- Call option values are reflective of the underlying stock price
- Therefore, it can be optimal to exercise before the ex-dividend date

If d is the time of ex-dividend, and $S_{d-1} = 40$, $S_d = 37.5$ yet $S_t = 39$, it was clearly more optimal to exercise the call before the dividend.

Early exercise of an American put option may be optimal for both dividend-paying and non-paying stock
Once a put option is in-the-money ($S_t \leq X$), early exercise may yield greater value than waiting and giving the stock a chance to appreciate in value

- Consider a stock with $S_{t-\Delta d} = 0$
- We could exercise at $t - \Delta d$ and capture the risk-free rate for Δd
- This is more optimal rather than holding the put

G. Black-Scholes model for pricing (European) options

Black-Scholes is a pricing model for options which treats options as a *redundant asset* that can be re-constructed by a underlying stock and risk-free asset.

- Stock prices follow a Geometric Brownian Motion (GBM), which dictates stocks move in a "random walk"

$$dS_t = \mu dt + \sigma S_t dW_t$$

- Black-Scholes is equivalent to the binomial pricing model for discrete time slices which become infitesimally small.
- Black-Scholes model only prices European options

Black-Scholes pricing model for a call option
Consider the following variables

- S_t: the current price of the underlying asset
- X: the strike price of the option
- $T-t$: the time in years remaining until the option expires
- r: the risk-free rate as a continously compounded rate
- σ: the return volatility of the underlying asset
- $\phi(x)$: the cumulative probabiltiy function for $X \sim N(0,1)$

The Black-Scholes pricing model for a call option is

$$C_t = S_t N(d_1) - X e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{S_t}{X} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

It is important to understand what the different components of the Black-Scholes model explain in the pricing of options

The components of the Black-Scholes pricing model

1) $\phi(d_1)$ measures the sensitivity to the stock price, or Δ (delta)
2) $\phi(d_2)$ is the probability of exercise under the risk-neutral measure

Thus

1) $S_t\phi(d_1)$ is the present value of the expected stock received if exercised
2) $PV(X)\phi(d_2)$ is the present value of expected payment for the strike

But what about put options? Put options under Black-Scholes are priced with put-call parity, such that

$$P_t = C_t - S_t + \frac{X}{1+r}$$

where $C_t = S_t\phi(d_1) - Xe^{-r(T-t)}\phi(d_2)$. It is also interesting to consider what variables are observable to the writer of an option at time t.

Implied volatility

Volatility (σ) is the only variable not "readily" observable in the B-S model. *Implied volatility (IV)* is the volatility required for the underlying asset, for it to match the observed option price.

- In the B-S model, σ is assumed to be constant, but
- Empirical evidence suggests that IV $\propto S_t$
- Generally, $IV_{\text{put}} > IV_{\text{call}}$

Thus there is some *empirical scruitiny* against the B-S model.

The effect of B-S parameters on option price

If the variable Increases…	the value of a call option
Stock Price, ***S***	Increases more in the money
Strike price, ***X***	Decreases the in the money region
Volatility, ***σ***	Increases
Time to expiration, ***T***	Increases
Interest rate, ***rf***	Increases as it reduces the present value of the strike ($C_t = P_t + S_t - PV(X)$)
Dividend payouts	Decreases as stock price goes down

Options markets can be highly illiquid or *unavailable* due to various factors:

- Not a big enough market for them, and thus no volume
- Regulatory restrictions on options trading in some countries

Dynamic hedging

Dynamic hedging is a strategy whereby the payoff of an option is replicated by trading a portfolio containing the underlying asset and a risk-free asset.