2017 MATH2801

1. 3 coins in a bag, $\frac{1}{3}$ chance to choose one
a) of them.

i) Probability of winning a point is:

$$P(W) = P(W|C)P(C) + P(W|G)P(G) + P(W|S)P(S)$$

$$= 0.3 \times \frac{1}{3} + 0.5 \times \frac{1}{3} + 0.7 \times \frac{1}{3}$$

$$= 0.5$$

$$\binom{10}{0} 0.5^{10} = \underline{0.0010}$$

ii) $P(N=n) = \binom{10}{n} \times 0.5^{n} \times 0.5^{10-n}$

iii) Yes. 10 independent trials with a Bernoulli condition - either win or lose.

b)

i) $f_{X,Y}(x,y) = cxy$

$$\int_0^2 \int_0^x cxy \, dy \, dx$$

$$= \int_0^2 \left[\frac{cxy^2}{2}\right]_0^x = \int_0^2 \frac{cx^3}{2} dx = \left[\frac{cx^4}{8}\right]_0^2$$

$$= 2c = 1 \quad c = \frac{1}{2}$$

ii) $f_{X,Y}(x,y) = \frac{xy}{2}$

$$f_X(x) = \int_0^x \frac{xy}{2}\,dy$$

$$= \left[\frac{xy^2}{4}\right]_0^x$$

$$= \frac{x^3}{4}$$

iii) $$f_Y(y) = \int_0^2 \frac{xy}{2}\,dx$$

$$= \left[\frac{x^2y}{4}\right]_0^2 = y$$

$$f_Y(y) \cdot f_X(x) = \frac{x^3y}{4} \neq \underline{f_{X,Y}(x,y)}$$

<u>NO.</u>

~~iv)~~ CHECK BOUNDS.

$P(X \leq \frac{3}{2}, Y \leq \frac{1}{2})$

Y depends on X. $\therefore$ integrate first.

$$\int_{\cancel{1}\,0}^{\frac{3}{2}} \int_0^{\frac{1}{2}} \frac{xy}{2}\,dy\,dx = \int_0^{\frac{3}{2}} \left[\frac{xy^2}{4}\right]_0^{\frac{1}{2}}$$

$$= \int$$

iv) $P(X \le \frac{3}{2},\ Y \le \frac{1}{2})$

$$\cancel{\int_0^{\frac{3}{2}}\int_0^{\frac{1}{2}} \frac{xy}{2}}$$

$$\int_0^{\frac{1}{2}}\int_y^{\frac{3}{2}} \frac{xy}{2}\,dx\,dy$$

$$\cancel{= \int_0^{\frac{1}{2}} \frac{x^2y}{4}\,dy}$$

$$= \int_0^{\frac{1}{2}} \left[\frac{x^2y}{4}\right]_y^{\frac{3}{2}} dy$$

$$= \int_0^{\frac{1}{2}} \frac{9y}{16} - \frac{y^3}{4}\,dy$$

$$= \left[\frac{9y^2}{32} - \frac{y^4}{16}\right]_0^{\frac{1}{2}}$$

$$= \frac{17}{256}$$

2.

a) $X \sim \text{Exponential}(2)$

$$f_X(x) = \frac{1}{2} e^{-\frac{x}{2}}$$

$$F_X(x) = \int_0^x \frac{1}{2} e^{-\frac{x}{2}} \, dx$$

$$= \left[-e^{-\frac{x}{2}} \right]_0^x$$

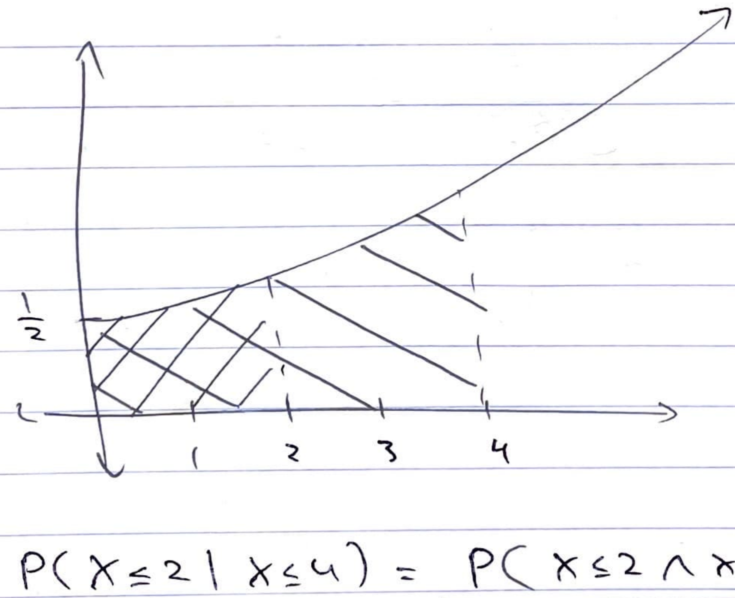
$$= 1 - e^{-\frac{x}{2}}$$

b)

i) $1 - e^{-1} = 0.632$

ii) $1 - e^{-2} = 0.865$

iii)



iv) $$P(X \le 2 \mid X \le 4) = \frac{P(X \le 2 \wedge X \le 4)}{P(X \le 4)}$$

$$= \frac{P(X \le 2)}{P(X \le 4)} = \frac{0.632}{0.865}$$

$$= \boxed{0.731}$$

c)

$$E(X) = \int_0^\infty x f_X(x)\,dx$$

$E(X) = 2$

$Var(X) = 4$

$Var(X) = E(X^2) - [E(X)]^2$

$4 = E(X^2) - 4$

$E(X^2) = 8$

d) $Y = X^2 + 4$

$X^2 = Y - 4$

$X = \pm\sqrt{Y-4}$ but $X \geq 0$.

$X = \sqrt{Y-4}$ $\qquad \frac{dX}{dY} = \frac{1}{2\sqrt{Y-4}}$

$= \quad F_Y(y) = P(Y \leq y)$

$= P(X^2 + 4 \leq y)$

$= P(X^2 \leq y - 4)$

$= P(X \leq \sqrt{y-4})$

$= \int_0^{\sqrt{y-4}} \frac{1}{2} e^{-\frac{x}{2}}\,dx$

$= \left[-e^{-\frac{x}{2}}\right]_0^{\sqrt{y-4}}$

$= 1 - e^{-\frac{\sqrt{y-4}}{2}}$

d) $Y = X^2 + 4$

$X^2 = Y - 4$

$\frac{d}{dY} X = \sqrt{Y-4}$ ← monotonic.

$$f_Y(y) = f_X(g^{-1}(Y)) \left| \frac{dX}{dY} \right|$$

$$= \frac{1}{2} e^{-\frac{\sqrt{Y-4}}{2}} \times \sqrt{Y-4}$$

$$= \frac{\sqrt{Y-4}\, e^{-\frac{\sqrt{Y-4}}{2}}}{2}.$$

3. $X \sim Exp(2)$ $\quad Y \sim Exp(2)$

$f_X(x) = \frac{1}{2} e^{-\frac{x}{2}}$ $\quad f_Y(y) = \frac{1}{2} e^{-\frac{y}{2}}$

i) A) $f_{X,Y}(x,y) = \frac{1}{4} e^{-\left(\frac{x}{2} + \frac{y}{2}\right)}$

B) $P(X+Y<4)$

$P(X < 4 - Y)$

$X > 0$

$4 - Y > 0$

$-Y > -4$

$Y < 4$.

∴ Consider the two events:

$P(X<4-Y \mid Y<4)\, P(Y<4) + P(X<4-Y \mid Y \geq 4)\, P(Y \geq 4)$.

$$= \int_0^4 \int_0^{4-Y} f_{X,Y}(x,y) \int_0^4 f_Y(y) + 0.$$

ii) A) $M_X(u) = \frac{1}{1-2u}$ $\quad M_Y(u) = \frac{1}{1-2u}$

B) $M_{X+Y}(u) = M_X(u) \times M_Y(u)$

$$= \frac{1}{(1-2u)^2}$$

c) $\text{Gamma}(2,2) = \frac{1}{(1-2u)^2}$

$\therefore\ X+Y \sim \text{Gamma}(2,2)$

b)

i) $$\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim N(0,1)$$

ii) $E(\bar{X}) = 2.$

$$Var(\bar{X}) = \frac{\sigma^2}{n} = \frac{4}{40} = \frac{1}{10}$$

$$\frac{2.3-2}{\sqrt{1/10}} = 0.949$$

$\Phi(0.949)$

4. a)

i) $E(\bar{X}) = \bar{X}$

$= \hat{\lambda} = \bar{X}$

ii)

$$L(x;\lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \times \frac{e^{-\lambda}\lambda^{x_2}}{x_2!} \times \dots \times \frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$

$$L(x;\lambda) = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!}$$

$$l(x;\lambda) = \ln(e^{-n\lambda}) + \ln(\lambda^{\sum x_i}) - \ln(\prod x_i!)$$

$$= -n\lambda + \sum x_i \ln(\lambda) - \ln(\prod x_i!)$$

$$l'(x;\lambda) = -n + \frac{\sum x_i}{\lambda}.$$

$$l''(x;\lambda) = -\frac{\sum x_i}{\lambda^2} < 0 \quad \lambda > 0.$$

Now, finding the MLE:

$$-n + \frac{\sum x_i}{\hat{\lambda}} = 0$$

$$\hat{\lambda} = \frac{\sum x_i}{n}.$$

iii) $I_n(\lambda) = -E(l''(\lambda))$

$$= -E\left(-\frac{\sum x_i}{\lambda^2}\right)$$

$$= \frac{\sum E(x_i)}{\lambda^2} = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

b)

i) $H_0: \mu = 0.05$ $H_1: \mu > 0.05$

ii) Assume the samples are normal.
Independently sampled.

iii) $\frac{\bar{x} - \mu}{s / \sqrt{n}} \sim t_{n-1}$

iv) $\frac{0.07 - 0.05}{0.0085 / \sqrt{4}} \sim t_3$

$= 4.705882353$

v) $1 - P(T < 4.705882353)$

vi) ~~F~~Null hypotesis is not epic.