2023 T1

Q2)

Proof. Suppose $0 \leq x \leq \sqrt{\frac{36}{17}}$. Notice that the Maclaurin series of f is given by:

$$f(x) = \sum_{k=0}^{\infty} A_k x^{2k+1}$$

Solution:

First, consider that the Maclaurin series of e^x :

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{x^n}{n!} \\
e^{\frac{-17t^2}{12}+7} &= e^7 \sum_{k=0}^{\infty} \frac{\left(-\frac{17t^2}{12}\right)^k}{k!} \\
&= e^7 \sum \int_0^x \frac{\left(-\frac{17}{12}\right)^k t^{2k}}{k!} dt \\
&= e^7 \sum \frac{x^{2k+1}\left(-\frac{17}{12}\right)^k}{(2k+1)k!} dt
\end{aligned}$$

therefore, A_k is given by:

$$\frac{\left(-\frac{17}{12}\right)^k e^7}{(2k+1)k!}$$

An alternate solution (Thanks to amber on CSESoc Discord):
Maple has a function that allows us to efficiently find this using:
`convert(integral, FormalPowerSeries)` (This is super useful!)

Continuing on with the question:
By Taylor's theorem with the Lagrange formula for the remainder,
there exists some $0 \leq c \leq x$ such that:

$$\begin{aligned}
|f(x) - P_3(x)| = |R_4(x)| &= \frac{f^{(4)}(c)x^4}{4!} \\
&= Bx^4 e^{-17/12c^2+7}
\end{aligned}$$

Find B.

Solution:

We've been given the 4th derivative in the maple output at the beginning of the question:

`289/12*exp(7)*x*exp(-17/12*x^2)-4913/216*exp(7)*x^3*exp(-17/12*x^2)`

We can then remove all the exp(7) and exp(-17/12x^2), and the sub in x for c, to find B.

Continuing on with the question:

However, notice that for such c we must have

$$
\begin{aligned}
0 \le |3 - \frac{17}{6}c^2| \le 3 \\
0 \le \exp\left(-\frac{17}{12}c^2 + 7\right) \le \exp(7)
\end{aligned}
$$

Solution (Thanks to one eight seven on CSESOC discord):

For $3 - \frac{17}{6}c^2$, we see that it is constantly decreasing, and therefore, we simply check the endpoints.

It is seen that 0 produces the maximum value, which is 3.

For $\exp\left(-\frac{17}{12}c^2 + 7\right)$, it is similar in the fact that it is monotonically decreasing. Therefore, check the end points once more, and we find that exp(7) is the maximum

Note: $0 \le x \le \sqrt{\frac{36}{17}}$. Therefore $0 \le c \le x$.

Q3)

a) Consider the different trigonometric identities until they fit:

$$
\begin{aligned}
\cos^2 x + \sin^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
\cosh^2 x - \sinh^2 x &= 1 \\
1 - \tanh^2 x &= \text{sech}^2 x
\end{aligned}
$$

b) Same as above, but with a side note:

$$
1 - \tanh^2 x = \text{sech}^2 x
$$

Don't be fooled into thinking we can use the negative tanh as a substitution!

c) Same as above, we instead of $\tan x$, we end up using $\cot x$, as well as a cosech^x identity.

Q4)

a) T_1 is a linear transformation

```
Consider if the zero vector exists within the linear transformation.
<0.a, 0.b> = <0, 0>, therefore, true.

Consider if the linear transformation is closed under addition.

Two arbitrary vectors, x and y, in R3, are applied to the linear
transformation:

T(x) + T(y) = <x.a, x.b> + <y.a, y.b> = <x.a + y.a, x.b + y.b>

= <a.(x + y), b.(x + y)>

= T(x + y)

Consider the scalar multiplication of an arbitrary vector within the
linear transformation:

T(cx) = <(cx).a, (cx).b>

= c.<x.a, x.b>
```

b) T_2 is not a linear transformation

```
Consider the zero element:

sqrt(0 + 0) = 0
Consider closure under addition with two arbitrary vectors a and b in
R2.

T(a) + T(b) = sqrt(8*a1^2 + 6*a2^2) + sqrt(8*b1^2 + 6*b2^2)

T(a + b) = sqrt(8*(a1 + b1)^2 + 6*(a2 + b2)^2)

Therefore, not closed under adition
```

c) T_3 is not a linear transformation

```
T(p(x)) = p(x) + 5x^2 - 2x

Consider the existence of zero:

p(0) + 5(0)^2 - 2(0)

= p(0), not guaranteed to be zero.
```

Q4)

You must check both the standard eigenvectors of multiple 1, and also, the fact that there can be a linear combination of the vectors.

A matrix A has eigenvalues and eigenvectors

$$\lambda = 9\,, \quad \mathbf{v} = s\begin{pmatrix}1\\0\\3\end{pmatrix} + t\begin{pmatrix}-2\\1\\2\end{pmatrix} \qquad \text{and} \qquad \lambda = -1\,, \quad \mathbf{v} = u\begin{pmatrix}1\\3\\-1\end{pmatrix}.$$

Which of the following give a correct diagonalisation $A = MDM^{-1}$? **[Select all correct answers.]**

Multiple selection advice. In a multiple selection question, marks are deducted for incorrect selections (but you cannot get less than zero marks). You are advised to only select options that you are sure about.

- [x] $D = \begin{pmatrix}9&0&0\\0&9&0\\0&0&-1\end{pmatrix}, \quad M = \begin{pmatrix}1&2&3\\0&1&9\\3&14&-3\end{pmatrix}.$
- [x] $D = \begin{pmatrix}9&0&0\\0&-1&0\\0&0&9\end{pmatrix}, \quad M = \begin{pmatrix}-2&1&1\\1&3&0\\2&-1&3\end{pmatrix}.$
- [ ] $D = \begin{pmatrix}9&0&0\\0&9&0\\0&0&-1\end{pmatrix}, \quad M = \begin{pmatrix}1&2&1\\0&0&3\\3&6&-1\end{pmatrix}.$
- [ ] $D = \begin{pmatrix}-1&0&0\\0&9&0\\0&0&9\end{pmatrix}, \quad M = \begin{pmatrix}1&-2&1\\0&1&3\\3&2&-1\end{pmatrix}.$
- [x] $D = \begin{pmatrix}9&0&0\\0&9&0\\0&0&-1\end{pmatrix}, \quad M = \begin{pmatrix}1&-2&1\\0&1&3\\3&2&-1\end{pmatrix}.$
- [ ] none of the above, A is not diagonalisable.

Q6)

Three axioms are contained within vector spaces:

a) Addition: Given two elements x, y in X, one can form the sum x + y, which is also an element of X.

b) Inverse: Given an element x in X, one can form the inverse -x, which is also an element of X.

c) Scalar multiplication: Given an element x in X and a real number c, one can form the product cx, which is also an element of X.

1. For all vectors u and v and all scalars lambda, we have $\lambda(u+v) = \lambda u + \lambda v$
   This is axiomatic
2. There exists a vector **0** such that for all vectors v, we have $\mathbf{0v} = \mathbf{0}$
   This is not true, as it refers to the **vector 0**.
3. For all vectors **u** and **v** and **w**, we have (u + v) + w = v + (w + u)
   This is not axiomatic, but it can be proved through axioms
4. For any vector v, if 0 is the zero scalar and **0** is the zero vector, then $0\mathbf{v} = \mathbf{0}$.
   This is not axiomatic, but it can be proved through axioms
5. For all vectors u and v and all scalars $\lambda \cap \mu$, we have $(\lambda + \mu)(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$
   This is false, and breaks the distributive axiom
6. For all vectors **u** and **v** and all scalars λ, if $\lambda\mathbf{u} = \lambda\mathbf{v}$ then $\mathbf{u} = \mathbf{v}$.
   This is visibly not true.

Q7)

a) Consider that the matrix is in form:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

Consider that a, b, c $\in \{0, 1, 2, \ldots, 43\}$. Therefore, there are $44 \cdot 44 \cdot 44$.

b) Give the number of matrices in T that are not diagonalisable
A matrix in the triangular form that this question poses is not diagonalisable when:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

However, it must be taken into consideration that $b \neq 0$ for this to be true. (As the zero matrix is diagonalisable)
Hence, this comes out to be $43 \cdot 44$.

c) Simply put the result of b on a, to find the probability of getting a diagonalisable matrix.