1081 2020 T2 Final.

Question 2

Let $B = \{\text{Bijections from } \mathbb{R} \text{ to } \mathbb{R}\}$
and let $b : \mathbb{R} \to \mathbb{R}$ be defined by $b(x) = 18x^{27} + 9x^{15} + 13x - 15$.

(a) Show that $b \in B$.

Prove b is bijective.

$b(x) = 18x^{27} + 9x^{15} + 13x - 15$

$b'(x) = 486x^{26} + 135x^{14} + 13$.

$\therefore$ strictly increasing $\Rightarrow$ injective.

$\lim_{x \to \infty} b(x) = \infty$

$\lim_{x \to -\infty} b(x) = -\infty$ $\therefore$ surjective, and hence, a bijection.

$b \in B$ $\square$.

(b) We define a function $F : B \to B$ by $F(f) = b \circ f$. Prove that F is a bijection.

b) $F : B \to B$ $\quad F(f) = b \circ f$

$F(f_1) = F(f_2) \Rightarrow f_1 = f_2$.

$b \circ f_1 = b \circ f_2$

b has an inverse function, and hence:

$b^{-1} \circ b \circ f_1 = b^{-1} \circ b \circ f_2$

$f_1 = f_2$ (Surjective).

For some $x \in B$:

$F(x) = y, \quad y \in B$.

Consider $b^{-1} \circ x \in F$.

$F(b^{-1} \circ x) = b \circ b^{-1} \circ g = g$.

$\therefore$ Surjective.

Note: Here the inverse, and specifically, the proof that b is bijective plays a big role. Remember to use the existence of inverse to simplify problems!

QUESTION 3:

i) In this first part, we are interested in the congruence equation

$$23x \equiv 10 \pmod{53}.$$

a) Beau claims the first step to solving this congruence equation is to apply the Extended Euclidean Algorithm. Show all the lines of working that Beau would have written down when applying the Extended Euclidean Algorithm in this way.

$23x + 53y = 10$

$53 = 23 \times 2 + 7$

$23 = 7 \times 3 + 2$

$7 = 2 \times 3 + 1$

$1 = 7 - 2 \times 3$

$1 = 7 - 3 \times (23 - 7 \times 3)$

$1 = 7 - 3 \times 23 + 9 \times 7$

$= 10 \times (53 - 23 \times 2) - 3 \times 23$

$= 10 \times 53 - 23 \times 23$

$x = -230$

b)

b) In order to solve this equation, Beau should multiply both sides of $23x \equiv 10 \pmod{53}$ by which number(s)? (Select **all** values that apply.)

- [ ] The congruence equation cannot be solved by multiplying both sides by any value.
- [ ] 16
- [x] -23
- [ ] 10
- [ ] $\frac{1}{23}$
- [ ] 23
- [ ] -37
- [ ] -23
- [ ] $\frac{1}{10}$
- [x] 30

$ax \equiv b \pmod{m}$

$m - a \quad \cancel{d} - a.$

c) Use a)

d) In a modular congruence

$ax \equiv c \pmod{m}$

If $\gcd(a,m) \nmid c$, then no solutions.

Else, $\gcd(a,m) =$ no. of sols.

ii)

d) Beau next considers the more general congruence equation

$$ax \equiv 30 \pmod{96}$$

for all integers $76 \leq a \leq 81$.

For which values of a will there be no solution?

{76, 80}

For which values of a will there be exactly one solution modulo 96?

{77, 79}

For which values of a will there be more than one solution modulo 96?

{78, 81}

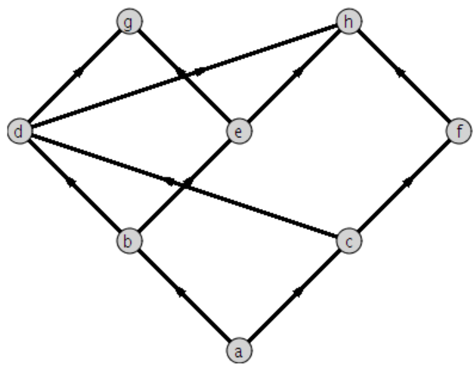
e) Describe a general rule for determining exactly how many solutions

$$ax \equiv b \pmod{m}$$

will have modulo m.

The number of solutions is defined by the relationship between gcd(a, m) and the divisibility of gcd(a, m) with b.
If we define some integer gcd(a, m) = n, then the amount of solutions will be dictated by n | b; where if n does not divide b, there is no solutions.

Aaliyah's favourite partial order $\preceq$ on the set $S = \{a, b, c, d, e, f, g, h\}$ happens to have the Hasse diagram shown below:



The edges in this graph are: {[a, b], [a, c], [b, d], [b, e], [c, d], [c, f], [d, g], [d, h], [e, g], [e, h], [f, h]}

a) Tick all statements that are true:

- [ ] $h \preceq c$
- [ ] $c \preceq a$
- [x] $a \preceq h$

Fairly self-explanatory.
$h \npreceq c$; no arrow from $h \to c$.
$c \npreceq a$, but $a \preceq c$.
$a \preceq h$ has an arrow, $\therefore$ true.

b) Find all maximal elements of S.

- [ ] a
- [ ] b
- [ ] c
- [ ] d
- [ ] e
- [ ] f
- [x] g
- [x] h

Nodes with no nodes connecting upwards.

c) Find or explain why they do not exist: the least upper bound of a and e, and the greatest lower bound of h and c.

No element z exists, such that $a \preceq z$, $e \preceq z$, $\therefore$ no least upper bound. Similarly for h and c.
Both don't exist.

For a sequence of integers $\{a_n\}_{n=1}^{\infty}$, we say that a_n is **eventually even** if

$$\exists N \in \mathbb{Z} \quad \forall n > N \quad a_n \text{ is even.}$$

a) Write symbolically the statement "a_n is not eventually even", simplifying your answer so that the negation symbols are not used.
(You can write the symbols using words or symbols using the equation editor. See the blue help box below.)

$\forall N \in \mathbb{Z}, \exists n > N,$ a_n is odd.

negate final statement.

b) Prove that the sequence given by

$$a_n = 851 + \left\lfloor \frac{36n + 493}{n} \right\rfloor$$

is **not** eventually even.

For any integer n, let $n = N + 493$.

$$a_N = 851 + \left\lfloor \frac{36(N+493) + 493}{N+493} \right\rfloor$$

$$= 851 + 36 + \left\lfloor \frac{493}{N+493} \right\rfloor$$

$$= 887 + \left\lfloor \frac{493}{N+493} \right\rfloor \Rightarrow \text{Always} = 0$$

$\therefore$ Always 887. $(N \geq 1)$

Question 7.

$$f_n - 10f_{n-1} + 24f_{n-2} = 4^n.$$

$$r^2 - 10r + 24$$
$$(r-6)(r-4) = 0$$
$$r = 6, 4.$$

$\begin{matrix} r & -6 \\ r & -4 \end{matrix}$

$\therefore f_n = A(6)^n + B(4)^n + ps. \ldots$

Please just use maple.

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rsolve({f(n) - 10*f(n-1) + 24*f(n-2) = 4^n, f(0) = 9, f(1) = 1}, f(n))
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(b) Explain the procedure Henry used to find the general solution for

$$f(n) - 10f(n-1) + 24f(n-2) = 4^n$$

and hence explain the procedure to solve the initial value problem in part a.

Find the homogenous solution, but recognising the common solution f(n) = Cr^n. Divide through by Cr^{n-2}, to get r^2 - 10r + 24. The roots of this 'homogenous equation' are your values that are to the power of n. Hence, this becomes f(n) = A(-6)^n + B(-4)^n + p(s), where p(s) is the particular solution. And then find the particular solution. I leave the rest to you.

Question 8.

Using the 26 uppercase letters of the English alphabet:

(a) How many 18-letter words contain the subword "BEDTIME" twice? An example of such a word is V<u>BEDTIME</u>DY<u>BEDTIME</u>G.

(BEDTIME)(BEDTIME) X X X X .

$\binom{6}{2} \times 26^4$

6 diffeent spots for two BEDTIME's.

26^4 for remaining letters.

b) (BEDTIME) X X X X X X X X X X X

Comb. of BEDTIME appearing:

$12 \times 26^{11} - \binom{6}{2} \times 26^4$

single - double.

Now, words that don't contain BEDTIME at all:

$26^{18} - \left(12 \times 26^{11} - \binom{6}{2} \times 26^4\right)$

(c) Explain how you obtained your answers to parts (a) and (b).

We want to know when the subword 'BEDTIME' appears, so that we can complement it with the universal set. If we just found the times one BEDTIME occured, we'd be including the times BEDTIME occurs twice. Therefore, we must use the principle of inclusion and exclusion.
|U| - |S1| + |S1 and S2|

Question 9.

Maddison is a MATH1081 student who has particularly enjoyed studying relations and combinatorics. They want to see what they can find out about the number of relations between certain sets. Maddison starts by declaring A to be a non-empty set with k elements.

a) How many binary relations are there from A to A? Explain your answer.

Consider the complete graph of relations - which has k * k edges. To consider the amount of binary relations from A to A (where A has k elements), you can choose to to take or not to take elements, and hence, becomes 2^(k^2)

b) How many reflexive relations are there from A to A? Explain your answer.

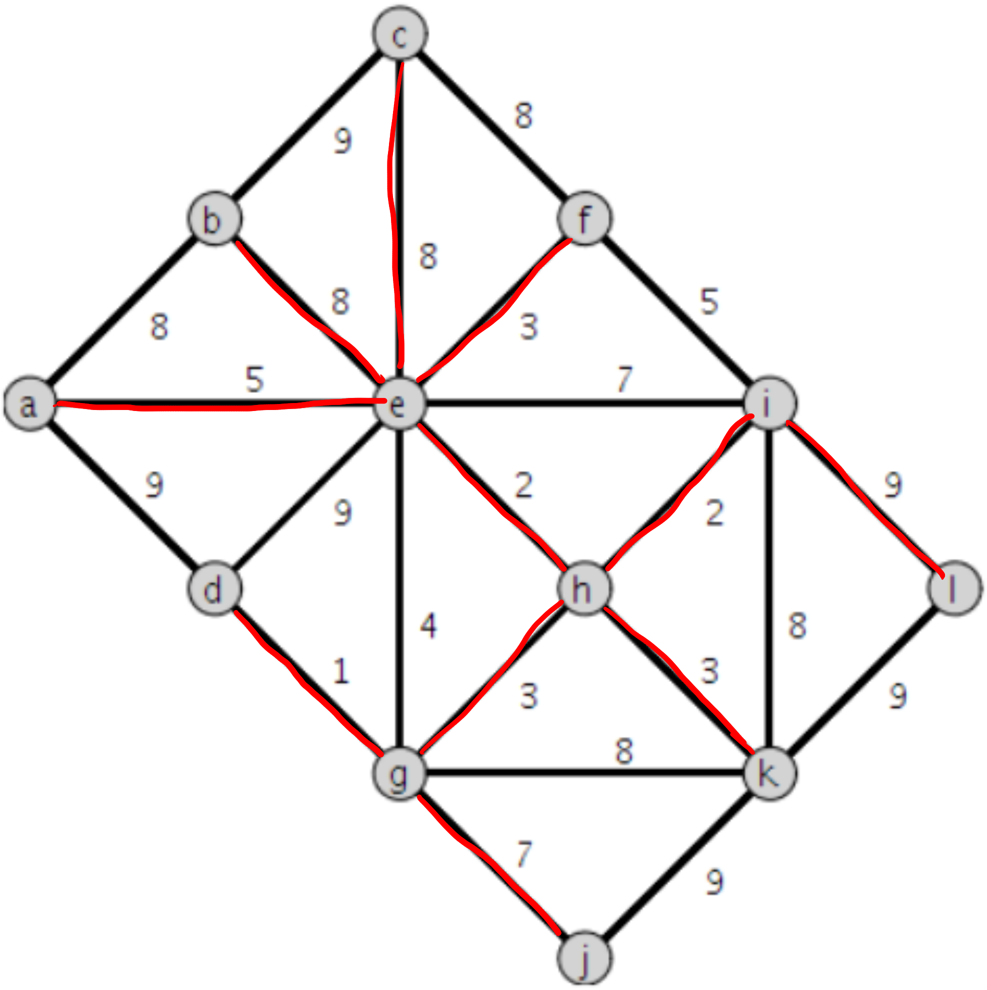
Consider a complete graph of k^2 edges. Out of the k^2 edges, k of them are compulsory in a reflexive relation.
Hence, the optional edges are k^2 - k.
Hence, the number of reflextive relations will become:
2^(k^2 - k)

c) How many antisymmetric relations are there from A to A? Explain your answer.

Consider an adjacency matrix of the relations between A to A. An anti-symmetric relation, forces a relation to choose either side of the reflexive relations - hence, there is 2 options at either point.
The amount of elements in one side of a adjacency matrix triangle is (k^2 - k) / 2, and hence this becomes 2^((k^2 - k)/2) * 2^k, where 2^k represents some optional reflexive relations.

Question 10

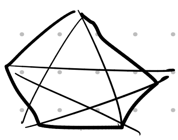
Use Kruskal's I guess?



Question 11.

a) Two odd degrees.

a) For which values of n does G have an Euler path?



- [x] 3 Trivial
- [x] 4 2 even, 2 odd
- [x] 5 3 even, 2 odd.
- [ ] 6 2 even, 4 odd.
- [x] 7 2 odd, 5 even.
- [ ] 8 6 odd, 2 even.
- [x] 9 7 even, 2 odd
- [ ] 10 8 even, 2 odd

b)

b) For which values of n does G have a Hamiltonian circut?

If $n \geq 3$ verts, and has degree atleast $\frac{n}{2}$.

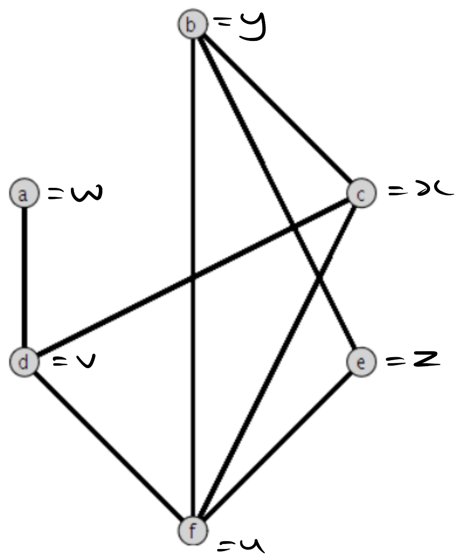
- [ ] 3
- [x] 4
- [x] 5
- [x] 6
- [x] 7
- [x] 8
- [x] 9
- [x] 10

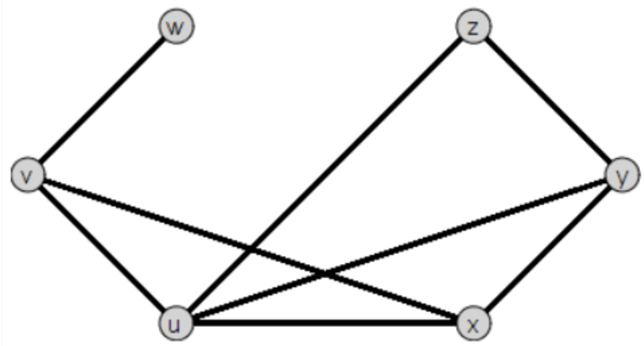
c)

c) For which values of n is G planar?

- [x] 3
- [x] 4
- [x] 5 ← one edge removed, $\therefore$ Not K_5.
- [ ] 6 All contains K_5.
- [ ] 7
- [ ] 8
- [ ] 9
- [ ] 10

Recall if a graph contains K_5 / $K_{3,3}$, or a subdivision of these graphs, then they are not planar.





Just start from the most distinguishable edge (for me, this was (a,d), and then build the isomorphism from there.

QUESTION 13.