2020 T3

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Most of these solutions have been derived entirely by me, so
there may be imperfections and wrong answers/reasonings. Please
contact me at my student email for any corrections.

These solutions were made to aid the preparation of MATH1231
students, not replace it. Good luck :)
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Q2)
a) and b) are trivial questions.
All you are tasked to do is derive $x = 7\sin t$, which occurs to be $dx = 7\cos t$.
Then, you sub this in, and find $g(t)$.

c) For c, you need to understand some trig identity usage. The trick here is that to find I in terms of x, we have to come from $x = 7sint$. This becomes $t = \arcsin x/7$. Considering a triangle, the hypotenuse is 7, the opposite is x, and the adjacent would be $\sqrt{49 - x^2}$. Therefore, the constant term in the triangle is **7.**

Q3)
a) The left hand size of an exact ODE is always $\partial/\partial x$, whereas the right side is $\partial/\partial y$.

b) To reverse this exact ode method - we now have to integrate LHS by y and integrate RHS by x. So, the LHS actually represent $\partial S/\partial y$ and the RHS is actually $\partial T/\partial x$. (The test which gives us if the ODE is exact or not.)

c) If Dominic has already found an equation which satisfies the conditions of a), then it is already **exact**. Therefore, testing again would be *not necessary*.

d) Exact ODE solving:

$$\begin{aligned}
\partial Q/\partial x &= 27x^2y + 9y^2 + 3 \\
\partial Q/\partial y &= 9x^3 + 18xy + 8 \\
\therefore Q &= 9x^3 + 9xy^2 + 3x + C_1(y) \\
Q &= 9x^3 + 9xy^2 + 8y + C_2(x) \\
\therefore Q &= 9x^3 + 9xy^2 + 3x + 8y + K
\end{aligned}$$

Then, solve for K.

e) To find the slope of the level set, we must find $\partial Q/\partial x$, where $x = 1, y = -1$.
Hence, $Q(x, y) = 9x^3y + 9xy^2 + 3x + 8y + 11$.
$\therefore Q_x(x, y) = 27x^2y + 9y^2 + 3$.
Hence, substituting x = 1 and y = -1, the slope is -15.

Q4)

a)

Since the left picture is intersected at y = b, the equation of the cross section will be $z = F(x, b)$. For the right picture, since the equation is intersected at x = a, the equation of the cross section will be $z = F(a, y)$.

b)

i) Fairly simple, just note that this is the rate of change of z proportional to x; as y has already been intersected.
So, since the gradient of the tangent line is -2.9 (as given), we can simply state that as x increases by 1, z increases by -2.9. In vector form:

$$\begin{pmatrix} 1 \\ 0 \\ -2.9 \end{pmatrix}$$

ii) Similar to above, however this time x is intercepted, and we are finding a *perpendicular* tangent. So, since it's 2D, we can simply use the reciprocal, $m_p = -1/m$. In vector form:

$$\begin{pmatrix} 0 \\ 1 \\ -1/3.1 \end{pmatrix}$$

c) Find the cross product of the two vectors, using:

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CrossProduct(<1, 0, -2.9>, <0, 1, 3.1>)
```

Convert to integer by multiplying the vector by 10.

d)
Consider the following:

$$\begin{pmatrix} a - 0.5 - a \\ b + 0.9 - b \\ F(a - 0.5, b + 0.9) - F(a, b) \end{pmatrix} \cdot \vec{c}$$

By doing this dot product, one can isolate the bottom row to find the *linear approximation.*
This ends up to be $|c| \cdot -0.5 + |c| \cdot 0.9$ divided by -10.

Q5)
a) The ball starts at rest, so $c = 0$, and the ball is suspended 16 metres in the air, so $d = 16$.

b) u^2 is fairly obvious here - since v becomes u^3.

c)

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convert(1/(R^3-64*u^3), parfrac, u)
```

d) Integrate both sides of (1), substitute x = H/2, and then solve for v is just the most **sensible.** This is how we would do it if we were tasked to do it.

Q6)
a)

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polarplot(2*theta, theta = 0..2*Pi)
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b)

Just derive $\sinh u$, which $= \cosh u$.

Then, do the substitution, remembering that β, can be found by solving for u in $2\pi = \sinh(u)$.

Then, L is simply expressed by:

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sinh(2*arcsinh(2*Pi))/2
```

by substituting β.

Q7)

a)

You are given that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad x \in (-1, 1].$$

One can then show that

$$\ln(x+9) = \frac{d}{dx}[b_1(x+c)^2 + b_2(x+c)^3 + b_3(x+c)^4 + b_4(x+c)^5 + \ldots], \quad x \in (e_1, e_2].$$

a) Find $b_1, b_2, b_3, b_4, c, e_1$ and e_2 for which the above equation is true, and type them in the boxes below.

$b_1 =$ 1/2

$b_2 =$ -1/6

$b_3 =$ 1/12

$b_4 =$ -1/20

$c =$ 8

$e_1 =$ -9

$e_2 =$ -7

I found this by just considering when the constant wouldn't exist (it's pretty clear that c = 8, as $\ln 1 = 0$). I then used:

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taylor(ln(x+9), x=0, 9)
```

and then found the constants required to equal that.
-9 is clear as **ln** is undefined beyond that, and then take the normal upper domain.

b)
For be, all we have to do is integrate both sides of the $\ln x + 9$ identity, which ends up with

$$\ln x + 9(x + 9) - x - 9 + C$$

then use the initial value $f(-8) = 8$ to find C.

c)
Consider the case of x = -5.
ln(4) < 2
1.386... < 2
Deriving both sides with respects to x
1/(x+9) < 1/(2*sqrt(x+9).

It can be trivially seen that 2*sqrt(x+9) < x + 9 for all x.
Hence, the rate of change of sqrt(x+9) is larger, and the initial value of the domain is also larger.

This can be applied to lim_(n->infinity) ln(n+9)/(n+9), such that since sqrt(n+9) > ln(x+9), then n + 9 > ln(x+9) for all x. Therefore, the limit will tend to **zero.**

d) Consider the base case of the inequality, x = -6.
$\ln 3 > 1$
1.0986... > 1
Differentiating both sides, we arrive at:
$1/(x + 9) > 0$1 > 0$
Therefore, the initial case and rate of change is larger, and therefore it can be confirmed that ln(x+9) > 1.

To consider the series, we can begin to understand the limit as n approaches infinity.

Use this property and divide both sides by $n^{1/9}$.

$$\ln(n+9)/n^{1/9} > 1/n^{1/9}$$

Consider that the function $1/n^{1/9}$ axiomatically diverges. Hence, by comparison, LHS must also diverge.

Q8)

a) $E(X) = np$ and $Var(X) = np(1-p)$ and $\sigma = \sqrt{Var(x)}$.
Here, n = 500, p = 1/4.

b) It has been revealed that bots can only win if they make 120 or less *sus* activities.
We can use the z-score formula to find the probability of this occuring, finding:

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{120 - 125}{\sqrt{\frac{375}{4}}}$$
$$z = -0.52$$

Using the standard normal table, find the probability and the multiply it by the sample size, 1000.

Q9)

a)

$$p = P(P)$$
$$q = P(L|P) = \frac{P(L \cap P)}{P(P)}$$

Therefore, the probability of $P(L \cap P) = pq$.

b)
Consider that $r = P(L|T)$. As well, $p = P(P)$, therefore $P(T) = 1 - p$. Since a supporter can either be a liar or not a liar, $P(\bar{L}|T) = 1 - r$.
Hence, $P(\bar{L} \cap T) = (1-r) \cdot (1-q)$.

c) Trivial to calculate with above information.

Q10)

Very year 11 extension 1-esque question.

All you have to do is sub both into the formula, and then solve the linear equations simultaneously (after finding the corresponding z-scores for the probabilities).

$$\begin{aligned} 1.90 &= \frac{62.6 - \mu}{\sigma} \\ -1.20 &= \frac{19.2 - \mu}{\sigma} \end{aligned}$$

and then solve.

Q11)

a) All that must be done in the first part is that you input the respect x values 3, -1, -1 (or whatever values you are given), and ensure that the vectors = 0 to be considered within ker(T).

b) Take an arbitrary polynomial for the dimension $ax^3 + bx^2 + cx + d$.

Apply the mapping to this polynomial, to find the matrix of the map:

$$\begin{aligned} (1) &= 27a + 9b + 3c + d \\ (2) &= -a + b - c + d \\ (3) &= -a + b - c + d \end{aligned}$$

Now, represent them as a matrix

$$\begin{pmatrix} 27 & 9 & 3 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

Row reducing

$$\begin{pmatrix} 27 & 9 & 3 & 1 \\ 0 & \frac{4}{3} & -\frac{8}{9} & \frac{28}{27} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the rank is 2 and the nullity is two.

c) No values are multiples of each other, and each mapping consists of a unique x-value.
With this in mind, it can be easily considered that the only nullity that will occur is during the transition between $\mathbf{Re}^4 \to \mathbf{Re}^3$. Hence, the nullity will be 1, and the rank will be 3.

Q12)
a) Matrices can have infinitely many eigenvectors, as they are represented as lines across the vector space. A matrix M will have n linearly independent eigenvectors depending on the dim(M) = n.

b)

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LinearSolve(<<v1>|<v2>|<x>>)
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This will give you the answer fairly easily.

c) This is an interesting question.
The trick here is to use eigenvector diagonalisation in order to find the matrix in terms of the eigen vectors.
Recall that $A = MDM^{-1}$, where M is comprised of the eigenvectors stacked horizontally, and D is a diagonal vector consisted of the eigenvectors. We can find this in terms of a, b and c, by doing the process on maple.

Then, multiplying A by x - we can create a system of linear equations, and then solve it.

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Example working out:
M := <<4, 4, −4>|<2, −4, 3>|<−4, 2, 4>>.<<a, 0, 0>|<0, b, 0>|<0,
0, c>>.MatrixInverse(<<4, 4, −4>|<2, −4, 3>|<−4, 2, 4>>)
A := M.<0, 12, −10>
```

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LinearSolve(<M|<-20, -8, 10>>)
Output: <-4, 1>
```

Q13)

a) This is pretty straight forward CDF knowledge, starts at zero, ends at 1; and hence β must be the value x such that sin(3x) = 1.

b) P(Y=0) = 0, however for P(Y = 6), you must get the inclusive probability 1, and then minus the cumulative probability from it which is $\frac{12}{15}$. The overall answer then becomes $3/15$.

Q14)

a) V is obviously a vector space, so it should be closed under scalar multiplication and addition. The addition operation should be associative and commutative, as these are addition properties. The scalar multiplication should be distributive, as this is a multiplication property.

b) The scalar identity is trivially one, as when we multiply the multiplication by 1, it still equals the same value.
Consider the addition operation $(a, b) + (c, d) = (3 + a + c, -5 + b + d)$. It can be seen that the set operation adds the integers (3, -5). Hence, to reverse this action, the additive identity is (-3, 5).

c) $k = 3 - 5$
$k = -2$
For a mapping to be considered linear, zero must exist, and the closure axioms must be true.
To include the zero, the additive identity must be carried through the mapping. (Hence, 3 - 5).

Jiefu Lu's explanation:

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Since additive identity = - additive identity, applying T to
both sides gives T(additive identity) = T(- additive identity).
Further, since T is linear you can take the negative sign out
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giving T(additive identity) = −T(additive identity) meaning that
T(additive identity) = 0.
Since the additive identity is (−3,5), that means that
T((−3,5))=0 therefore `−3 + 5 + k = 0
```