MATH1081 2020 T1 Finals

a) Best done with Maple.

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sum(-4/k + 6/(k+1) - 2/(k+2), k=1..18)
```
(Your question might vary!)

b) $$\sum_{i=1}^{n} \frac{-2i-8}{i^3+3i^2+2i} - \sum_{i=1}^{n} a_i = 2n.$$

$$= \sum_{a_i}^{n} = \sum_{i=1}^{n} \frac{-2i-8}{i^3+3i^2+2i} - 2n.$$

Prove that $a_k = -\frac{4}{k} + \frac{6}{k+1} - \frac{2}{k+2} - 2 \quad \forall k \in \mathbb{Z}^+.$

Test $k=1$

$$a_1 = -4 + 3 - \frac{2}{3} - 2 = -\frac{11}{3} = \underline{f(1) - 2}.$$

Assume true for $n=k$

$a_k = f(k) - 2.$

Prove true for $n = k+1.$

$a_{k+1} = f(k+1) - 2.$

$$\sum_{i=1}^{k+1} (f(k+1) - a_{k+1}) = 2(k+1)$$

$$= f(k+1) - a_{k+1} - \sum_{i=1}^{k} (f(k) - a_k) = 2(k+1)$$

$$= f(k+1) - a_{k+1} - 2k = 2k + 2.$$

$$= \quad a_{k+1} = f(k+1) - 2. \quad \square$$

a) Given that for all k

$$\frac{-2k-8}{k^3+3k^2+2k} = \frac{-4}{k} + \frac{6}{k+1} + \frac{-2}{k+2},$$

find the following:

$$\sum_{i=1}^{18} \frac{-2i-8}{i^3+3i^2+2i} = \boxed{-549/190}$$

b) The sequence (a_k) (defined for all positive integers k) has the property that

$$\sum_{i=1}^{n} \frac{-2i-8}{i^3+3i^2+2i} - \sum_{i=1}^{n} a_i = 2n$$

for all positive integers n.
Prove using mathematical induction that

$$a_k = \frac{-4}{k} + \frac{6}{k+1} + \frac{-2}{k+2} - 2 \quad \forall k \in \mathbb{Z}^+$$

is the only possible solution for the sequence (a_k).

To help with typing up your explanation, you might find it useful to define $f(k) = \frac{-2k-8}{k^3+3k^2+2k}$.

Question 3.

$f(x) = 2x + 3$ $\qquad$ $g(y) = \left\lfloor \frac{y-1}{2} \right\rfloor$

a) Consider $y = 2x+3$.

$2x = y - 3$

$x = \frac{y-3}{2}$.

$\therefore$ Not every x value maps to a valid y value, e.g, 2.

$\therefore$ Not surjective.

b) $f(y_1) = f(y_2) \Rightarrow y_1 = y_2$.

$\left\lfloor \frac{y_1 - 1}{2} \right\rfloor = \left\lfloor \frac{y_2 - 1}{2} \right\rfloor$.

Consider: $y_1 = 0$ & $y_2 = -1$.

Both equate to -1.

Hence, not injective.

c)

i) $(g \circ f)(x) = g(f(x))$

$= \left\lfloor \frac{2x+3-1}{2} \right\rfloor$

$= \lfloor x+1 \rfloor$

$\underline{= x+1}$. $(x \in \mathbb{Z})$.

ii) $2\left\lfloor \frac{x-1}{2} \right\rfloor + 3$

d) $(g \circ f)^{-1}(x)$

$= y = x + 1$

$x = y + 1$

$y = x - 1$

e) A function must be injective to have an inverse.

Due to the floor function, multiply x-values map to the same y-value.

$\therefore$ No inverse.

Question 4.

i) Symmetry.

$aRb \Rightarrow bRa$

$a = b^2 \Rightarrow b = a^2$.

$1 = (-1)^2 \Rightarrow -1 = 1^2$

Not true.

ii) Anti-symmetry.

aRb & bRa then $a = b^2$.

$a = b^2$ & $b = a^2$ then $a = b$.

The situation only holds when $a = 1$ & $b = 1$.

$\therefore$ $a = b$, and antisymmetric

iii) Transitive.

aRb & $bRc \Rightarrow aRc$.

$a = b^2$; $b = c^2$

$b^2 = c^4$

$a = c^4$, $!= a = c^2$.

$\therefore$ Not transitive.

Question 5.

a) $7a + 11b = 5$. OR

```
isolve(7*a + 11*b = 5)
```

$7a + 11b = 1$.

$11 = 7 + 4$
$7 = 4 + 3$
$4 = 3 + 1$

$1 = 4 - 3$
$1 = 4 - 7 + 4$
$= 2 \times 4 - 7$
$= 2 \times (11 - 7) - 7$
$= 2 \times 11 - 3 \times 7$

$= 10 \times 11 - 15 \times 7$.

& $-15 + 11t$
$10 + 7t$

$b = 10$

$10 + 7t$
$3 + 7t$

$-15 + 11t$
$\Rightarrow 7 + 11t$.

b) $9c - 14d = 17$

```
isolve(9*c - 14*d = 17)
```

$14 = 9 + 5$
$9 = 5 + 4$
$5 = 4 + 1$

$1 = 5 - 4$
$1 = 5 - 9 + 5$
$1 = 2 \times 5 - 9$
$= 2 \times (14 - 9) - 9$
$1 = 2 \times 14 - 3 \times 9$.
$17 = 34 \times 14 - 51 \times 9$

$c = -51 + 14t$ $\quad$ $d = -34 + 9t$
$c = 5 + 14t$ $\quad$ $d = 2 + 9t$

Remember the 'multiples' oppose the values. For e.g., c use 14, etc.

c) This question sucks.

$9(7x + 11z) - 14y = 17$

This means:
$7x + 11y = 5 + 14t$
& $y = 2 + 9t$ $\quad$ [as shown in part b)].

Solve $7x + 11y = 1$.

$11 = 7 + 4$
$7 = 4 + 3$
$4 = 3 + 1$

$1 = 4 - 3$
$1 = 2 \times 4 - 7$
$1 = 2 \times (11 - 7) - 7$
$= 2 \times 11 - 3 \times 7$

Particular solutions:

$x = -3 \times (5 + 14t) + 7s$
$z = 2 \times (5 + 14t) + 11s$
$y = 2 + 9t$.

Note: Apparently there was a problem with Maple TA...

Question 6.

Lots of different variations on this one.

p	q	r	$r \wedge \neg q$	$p \wedge \neg r$	$(r \wedge \neg q) \rightarrow (p \wedge \neg r)$
False	False	False	False	False	True
False	False	True	True	False	False
False	True	False	False	False	True
False	True	True	False	False	True
True	False	False	False	True	True
True	False	True	True	False	False

True	True	False	False	True	True
True	True	True	False	False	True

Should be easy.

Question 7.

Let r, s, h, b and f be the following statements:

r	I am rich
s	The sun is shining
h	The harbour is inviting
b	My boat is at the mooring
f	My friends are happy

Select the logical notation that matches the sentences below.

a) "If the sun is shining and my boat is at the mooring then the harbour is inviting and my friends are happy."

$(s \wedge b) \to (h \wedge f)$

b) "My boat is not at the mooring only if the sun is shining and the harbour is inviting."

$\sim b \to (s \wedge b)$.

Question 8.

Consider the statement:

Given a positive integer x, if x is a perfect square, then for all prime integers p, there exists an even integer a such that p^a divides x and p^{a+1} does not divide x.

The contrapositive of the above statement is:

x [is not] a perfect square if [$\exists$ primes] $p \in \mathbb{Z}$ [$\forall$ even] $a \in \mathbb{Z}$ p^a [does not divide] x [or] p^{a+1} [divides] x.

Contrapositive of:

$p \to q \Rightarrow \sim q \to \sim p$.

Question 9.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be "eventually increasing" if

$$\exists x \in \mathbb{R}\ \forall y_1 \in \mathbb{R}\ \forall y_2 \in \mathbb{R}\Big[((y_1 > x) \wedge (y_2 > y_1)) \to (f(y_2) > f(y_1))\Big].$$

Explain in words in the box below why choosing $y_1 = x + \pi$ and $y_2 = y_1 + 2\pi$, for any $x \in \mathbb{R}$, proves that the function $f(x) = 4 + 7\cos x$ is not eventually increasing.

Sub in the values...

$\exists x \in \mathbb{R}\ \forall y_1 \in \mathbb{R}\ \forall y_2 \in \mathbb{R}$

$((x + \pi > x) \wedge (x + 3\pi > x + \pi)) \to (f(x+3\pi) > f(x+\pi))$.

$\Rightarrow (\pi > 0 \wedge 2\pi > 0) \to (f(x+3\pi) > f(x+\pi))$

Assume this to be true.

If $x = \pi$:

$f(\pi + 4\pi) > f(2\pi)$

$= f(5\pi) > f(2\pi)$

$= 4 + 7(-1) > 4 + 7\cos(2\pi)$ $= -3 > 11$. Contradiction.

Question 10.

Inhabitants of planet Britz are telepathic and have a language with 33 distinct letters made up of 17 sounds, 4 gestures and 12 thoughts. They make words by putting these letters together in a sequence. Repetitions of letters are allowed.

a) How many 14-letter words can be constructed from the 33 letters of the Britzian alphabet?

Repetitions allowed, so:

$\underbrace{33 \times 33 \times \cdots \times 33}_{14 \text{ times}} = 33^{14}$

b) How many of the 14-letter words in part (a) contain exactly one laugh sound, one wink gesture and one happy thought?

$14 \times 13 \times 12 \times 30^{11}$.

14 places to put first.
13 places to put next.
12 places to put last.
30 letters left!

c) How many of the 14-letter words in part (a) contain at least one laugh sound, one thumbs up gesture and one surprised thought?

Principle of Inclusion-Exclusion.

$|U| = 33^{14}$.

$|\text{One missing}| = \binom{3}{1} 32^{14}$

$|\text{Two missing}| = \binom{3}{2} 31^{14}$

$|\text{All missing}| = 30^{14}$

$= 33^{14} - \binom{3}{1} 32^{14} + \binom{3}{2} 31^{14} - 30^{14}$

d) How many of the 14-letter words in part (a) contain at exactly 6 sounds, 3 gestures and 5 thoughts? (In each of the categories, the letters need not be distinct.)

Choose 6 sounds:

17^6

Choose 3 gest.:

4^3

Choose 5 thoughts:

12^5

Arrange:

$17^6 \times \binom{14}{6} \times 4^3 \times \binom{8}{3} \times 12^5 \times \binom{5}{5}$

Question 11.

a) The recurrence relation

$$a_n - 25\,a_{n-2} = 4^n$$

has general solution to the corresponding homogeneous relation of the form

$$h_n = A\lambda_1^n + B\lambda_2^n$$

where A and B are arbitrary constants, and

$r^2 - 25 = 0$

$r = \pm 5$.

$\lambda_1 = 5 \qquad \lambda_2 = -5$

b) A particular solution to

$$a_n - 25\,a_{n-2} = 4^n$$

is

$c4^n - 25c4^{n-2} = 4^n$

$-\dfrac{9c4^n}{16} = 4^n$

$c = -\dfrac{16}{9}$

c) Solve the recurrence relation

$$a_n - 25\,a_{n-2} = 4^n, \;\; a_0 = 0, a_1 = 0$$

and enter your answer in the box below using Maple syntax.

```
rsolve({a(n) - 25*a(n-2) = 4^n, a(0) = 0, a(1) = 0}, a(n))
```

You could do this manually... but ceebs.

Question 12.

The simple connected planar graph G has degree sequence $[1, 3, 4, 4, 4, 4, 2]$. What is:

a) The number of edges?

Handshaking lemma.

$2|E(G)| = \sum \deg(s)$

$2|E(G)| = 22$

$|E(G)| = 11$

b) The number of regions?

$r - e + v = 2$.

$r - 11 + 7 = 2$

$r - 4 = 2 \qquad r = 6$

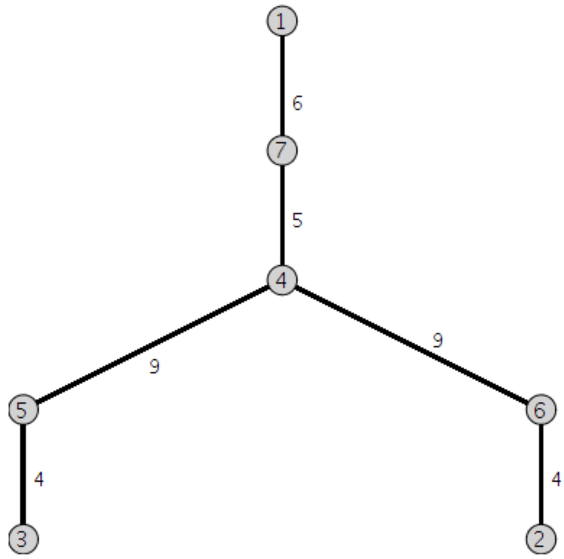
c) What properties of G were necessary for you to be able to calculate the number of regions? (Select all that apply.)

- [ ] G had to have at least 3 edges.
- [x] G had to be planar.
- [x] G had to be connected.
- [ ] G had to be simple.

EULER'S FORMULA :)

Question 15. (OOPS Q13 BELOW)

Consider the weighted graph G



a) Pick one edge that can be chosen as the first edge in Kruskal's algorithm and enter the number labels on the vertices at each end of the edge. Separate these labels by a comma. 2, 6

Kruskal's PRIO's lowest edge weight.

b) How many choices of first edge are there (including the one you chose)? 2

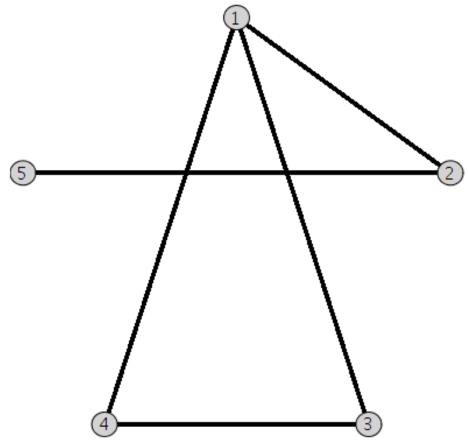
Two edges with weight 4.

c) What is the weight of a minimal spanning tree for this graph? 37

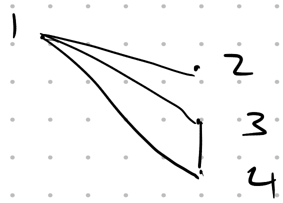
Graph is already a tree - so count every edge.

Question 13.

Consider the graph G



Bipartite?



3 ← No.

Hamiltonian Circuit?

No. (Try constructing). ← Could probably use Dirac's or something.

Euler Walk?

EC Every vertex has even degree? ✗

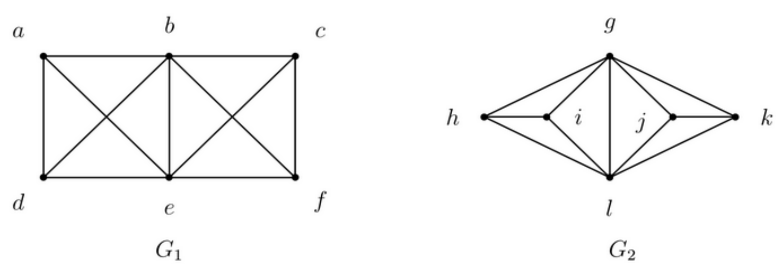
ET Two vertices have odd degrees? ✓ (So, Yes!)

- The graph G is bipartite? ○ true ◉ false
- The graph G contains a Hamiltonian circut? ○ true ◉ false
- The graph G contains an Eulerian walk. ◉ true ○ false

Question 14.

This is done visually.

The graphs G_1 and G_2 are isomorphic.

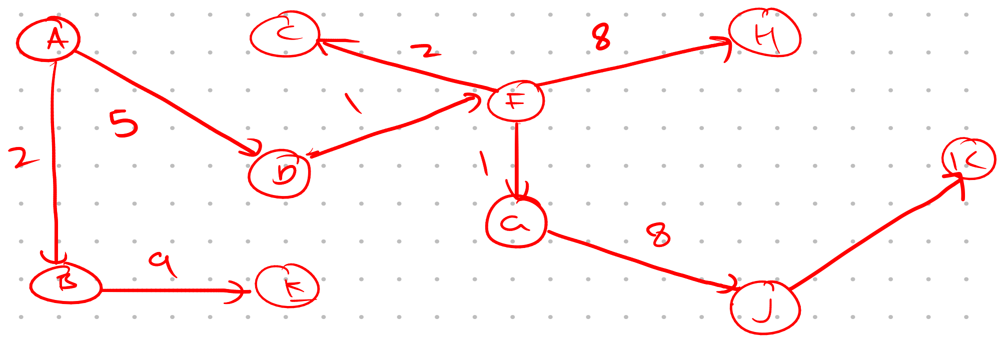


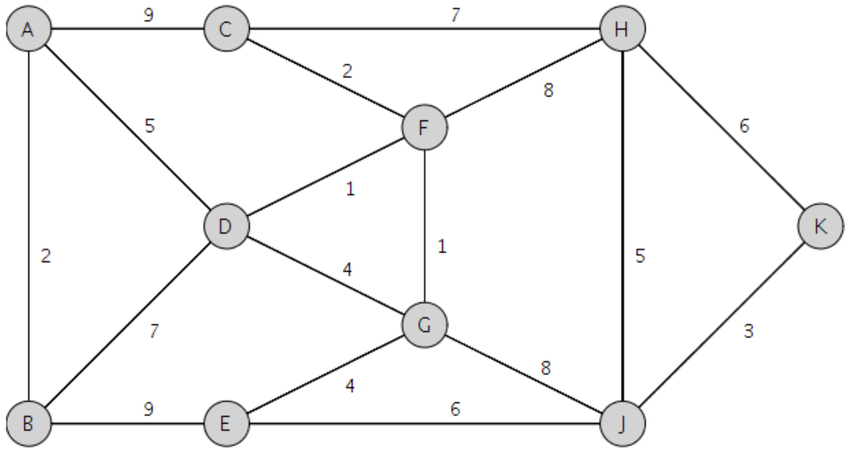
Give an isomorphism that proves this by completing the isomorphism table below.

vertex in G_1	vertex in G_2
a	h
b	g
c	k
d	i
e	l
f	j

Question 16.

COMP2521 Question 101





Next edge	Next vertex	Total distance to this vertex
AB	B	2
AD	D	5
DF	F	6
FG	G	7
CF	C	8
BE	E	11
FH	H	14
GJ	J	15
JK	K	18