MATH1081 2021 T3

Consider the following relations on the sets $A = \{6, 7, 8, 9, 10\}$ and $B = \{6, 7, 8, 9\}$.

**(a)** Suppose $f_1 = \{(6,6), (7,8), (8,8), (9,7), (10,9)\}$, $f_1 : A \to B$ is a function. Which of the following are true? Indicate all correct answers.

- ☒ $f_1 \subseteq A \times B$. ← All are valid.
- ☐ $f_1 \subseteq B \times A$. ← No, $(10, 9)$ doesn't exist.
- ☒ $f_1$ is surjective. Every value of $B$ has an output.
- ☐ $f_1$ is injective. ← $8 = 7$ & $8$, ∴ not injective.

**(b)** Suppose $f_2 = \{(6,8), (7,7), (8,9), (9,6)\}$, $f_2 : B \to A$ is a function. Which of the following are true? Indicate all correct answers.

- ☒ $f_2 \subseteq A \times B$. } Both exist (visual check).
- ☒ $f_2 \subseteq B \times A$.
- ☒ $f_2$ is injective. ← $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- ☐ $f_2$ is surjective. ← No, doesn't include 10.

**(c)** Suppose $f_3 \subseteq B \times A$ is defined by

$$f_3 = \{(6,7), (7,6), (8,9), (x,8)\}.$$

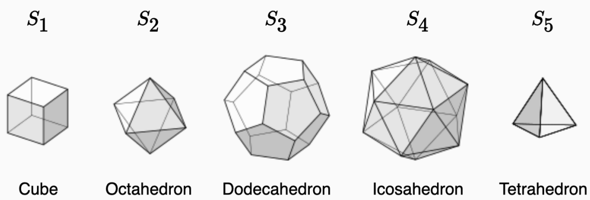
Indicate all values of $x$ for which $f_3 \subseteq B \times A$ but $f_3$ is **not** a function.

- ☒ 6
- ☒ 7
- ☒ 8
- ☐ 9
- ☐ 10

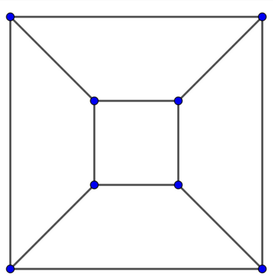
These three create a one to many relation, ∴ relation.

Question 3.

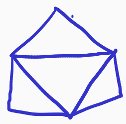
A **platonic solid** is a three dimensional shape where each face is a polygon with $n$ equal sides, and where $m$ faces meet at each corner. In this question you will prove there are exactly five platonic solids.




**(a)** If we puncture a hole in one of the faces and peek through the hole into the shape, then each platonic solid appears as a planar graph. For instance, the cube $S_1$ looks like a planar graph where each face has $n = 4$ sides and where $m = 3$ faces meet at each corner.



$S_2 =$



$S_3 =$

Fill out the table below.

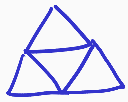
| Platonic Solid | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
|---|---|---|---|---|---|
| $n$ | 4 | Number 3 | Number 5 | Number 3 | Number 3 |
| $m$ | 3 | Number 4 | Number 3 | Number 5 | Number 3 |

Visual Inspection:
1. Check sides
2. Check number of faces adjacent to a corner.

**(b)** Suppose we have a platonic solid where each face has $n$ sides, and each corner meets $m$ faces. The corresponding planar graph will have $e$ edges, $r$ regions and $v$ vertices, where

$$v + r = e + 2.$$

We can compute that $2e$ equals (select all that apply):



- ☐ $nv$
- ☒ $nr$
- ☐ $mv$
- ☐ $mr$

Hence, we can express $\frac{1}{e}$ in terms of $m$ and $n$ as:

$$\frac{1}{e} = \underline{\qquad\qquad}$$

> **Syntax advice:** Enter your answer as an expression in terms of $m$ and $n$. Remember to use * for every multiplication. For example, to enter $\frac{4m}{n} - 2$, use the syntax `4*m/n - 2`

The only integer solutions to this equation, together with the restriction that $m$ and $n$ are at least three, appear in the table for part **(a)**.

$2|E| = \sum_r \deg(r).$ Recall that the degree of a planar graph can be found: at

$nr$ & $mv = 2e$

$r = \frac{2e}{n}$ $\qquad v = \frac{2e}{m}$

$\frac{2e}{n} + \frac{2e}{m} = e + 2$

$\frac{2em + 2en}{nm} = e + 2$

$2em + 2en = enm + 2nm$

$2em + 2en - enm = 2nm$

$e(2m + 2n - enm) = 2nm$

$\frac{1}{e} = \frac{2m + 2n - n}{2nm}$

$= \frac{1}{e} = \frac{1}{n} + \frac{1}{m} - \frac{1}{2}$

**(c)** Two platonic solids are said to be **dual** iff their planar graphs are dual. What are the possible dual(s) of a platonic solid with $m = 4$? Tick all possible answers.

- [ ] Tetrahedron
- [ ] Octahedron
- [x] Cube
- [ ] Icosahedron
- [ ] Dodecahedron

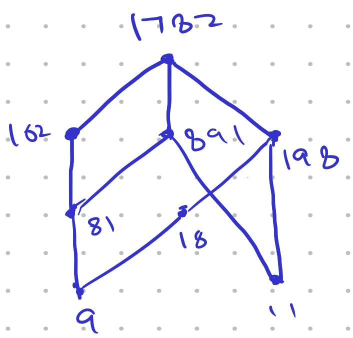
The Cube's planar is the same as $m=4$.
As $m$ and $n$ are of equal but exchanged value, they will have the same planar.
[Look for where $n=4$ :)]

Question 4.

Consider the poset

$$(\{9, 11, 18, 81, 162, 198, 891, 1782\}, |)$$

where $|$ is the divisibility relation.




**(a)** Enter the maximal elements of this poset: {1782}

**Syntax advice:** Enter your answer as a set. For example, enter the set $\{3, 5, 7\}$ using the syntax

```
{3, 5, 7}
```

Is there a greatest element?

- ( ) No
- (•) Yes

**(b)** Enter the minimal elements of this poset: {9, 11}

**Syntax advice:** Enter your answer as a set. For example, enter the set $\{3, 5, 7\}$ using the syntax

```
{3, 5, 7}
```

Is there a least element?

- (•) No
- ( ) Yes

**(c)** Enter all the upper bounds of $11$ and $9$: {198, 891, 1782}

**Syntax advice:** Enter your answer as a set. For example, enter the set $\{3, 5, 7\}$ using the syntax `{3, 5, 7}`. If there is no upper bound, enter the empty set $\{\}$ using the syntax `{}`

Is there a least upper bound of $11$ and $9$?

- (•) No
- ( ) Yes

**(d)** Enter all the lower bounds of $198$ and $891$: {9, 11}

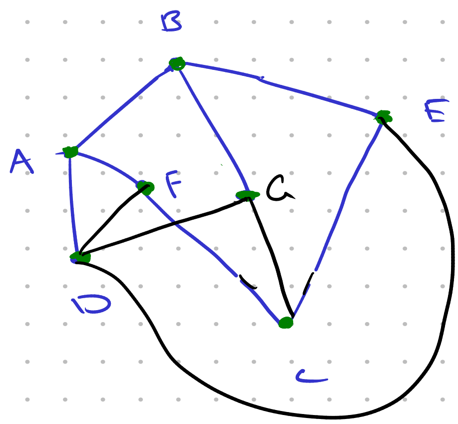
**Syntax advice:** Enter your answer as a set. For example, enter the set $\{3, 5, 7\}$ using the syntax `{3, 5, 7}`. If there is no lower bound, enter the empty set $\{\}$ using the syntax `{}`

Is there a greatest lower bound of $198$ and $891$?

- (•) No
- ( ) Yes

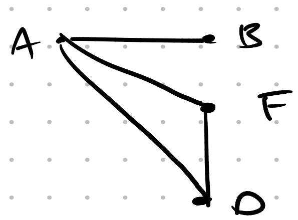
Question 5.

$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ \end{array}$$
$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$




- [x] $G$ is connected.
- [x] $G$ is simple.
- [ ] $G$ is a tree.
- [ ] $G$ does not contain a triangle (three mutually-adjacent vertices).
- [x] $G$ is planar.
- [ ] $G$ has an Euler circuit.
- [ ] $G$ is bipartite.

Bipartite Check.

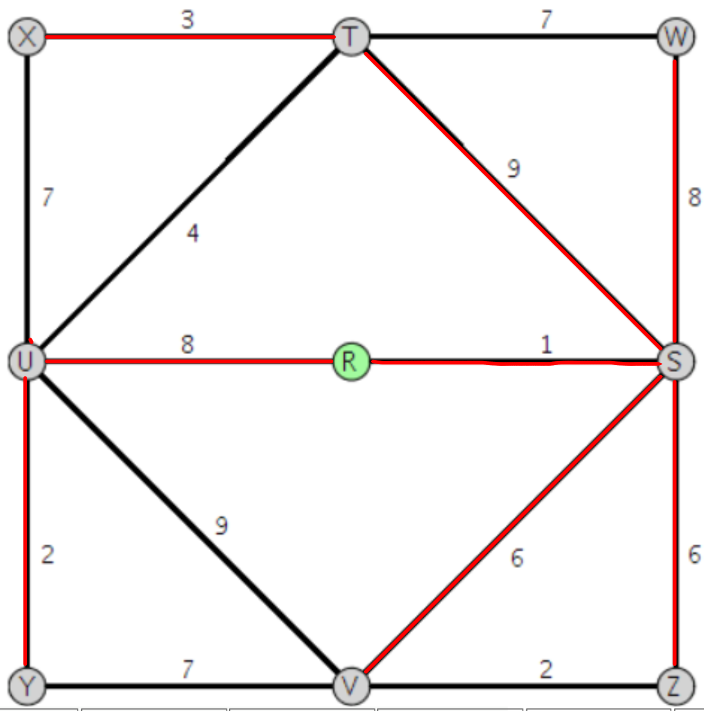



Question 6

You work for the food delivery service Food Bring. Deliveries are made from the trendy restaurant "Vertex $R$" to the luxurious apartment blocks known as vertices $S, T, U, V, W, X, Y$ and $Z$. Food Bring pays you a fixed price per delivery based on the chart below:

| Destination vertex | $S$ | $T$ | $U$ | $V$ | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|---|---|---|---|
| Dollars per delivery | 2 | 3 | 2 | 3 | 9 | 9 | 9 | 9 |

(a) Traffic conditions are terrible! You can't drive directly to some locations without driving past others, and your navigation app estimates that the journey between each pair of adjacent locations takes a certain amount of time (in minutes) indicated by the edge weights below




| | $S$ | $T$ | $U$ | $V$ | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|---|---|---|---|
| | 1 | 10 | 8 | 7 | 9 | 13 | 10 | 7 |
| \$/hr | 2 | 0.3 | 0.25 | 0.43 | 1 | 0.69 | 0.9 | 1.29 |

**(b)** Assuming that the shortest path is always taken, which delivery option offers the best hourly wage if we consider only the time it takes for a one-way trip from Vertex $R$ to the destination?

- (•) $S$
- ( ) $T$
- ( ) $U$
- ( ) $V$
- ( ) $W$ ✓
- ( ) $X$
- ( ) $Y$
- ( ) $Z$

calculate hourly wage as above.

**(c)** Food Bring does not pay you for your return trip to Vertex $R$. Which delivery option offers the best hourly wage if we consider the time it takes to make the delivery and then return to Vertex $R$ for the next delivery? You may assume the shortest path is always taken. Justify your answer.

Still S. The hourly wage simply gets halfed, as the return would take the same time as the time it takes to get there.

Question 7.

**(a)** Any integer with factorisation $2^a \cdot 11^b \cdot 13^c$ where $a, b, c \in \mathbb{N}$, is said to be 8-**regular** if and only if it satisfies the condition

$$a + b + c = 8.$$

How many 8-regular numbers are there?

comb(10,2)

Stars and bars

**(b)** Suppose $x, y \in \mathbb{Z}$. We shall write $x \sim y$, or say that $x$ is equivalent to $y$, if and only if $x = y \times (572)^n$ for some $n \in \mathbb{Z}$. Let $S$ be the set of 8-regular integers such that no two elements of this set are equivalent to each other. Then $S$ can be written as

$$S = \left\{ 2^a \cdot 11^b \cdot 13^c | a + b + c = 8 \text{ where } a, b, c \in \mathbb{N} \text{ and } A \right\}$$

where $A$ is the condition

- ( ) $a < 2$ and $b < 1$ and $c < 1$.
- ( ) $a \leq 2$ and $b \leq 1$ and $c \leq 1$. ✓
- ( ) $a \leq 2$ or $b \leq 1$ or $c \leq 1$.
- (•) $a < 2$ or $b < 1$ or $c < 1$.

$\underline{572 = 2^2 \times 13 \times 11}$

$a + b + c = 8.$

The first two statements are impossible.

$x = y \times (572)^n$

Further, we want it to NOT be some square of 572. So guarantee this with:

$a < 2 \cup b < 1 \cup c < 1$

$\uparrow$ ($a \leq 2$ $b \leq 1$ $b \leq$) does not guarantee this).

$|a<2| + |b<1| + |c<1| - (\text{intersections})$

$|a \cap b \cap c|$

$\uparrow$

(not valid since $1 + 0 + 0 \neq 8$).

a cases:

$1 + b + c = 8$

$b + c = 7$

$\binom{8}{1}$

$0 + b + c = 8 = \binom{9}{1}$

b case:

$\binom{9}{1}$

c case: $\binom{9}{1}$

$\binom{8}{1} + 3 \times \binom{9}{1}$

Now check intersections:

(a<2 & b<1) (b<1 c<1) (a<2 b<1)

2 cases 1 case 2 cases

2 + 1 + 2

= 2+1+2 = 5.

$\binom{8}{1} + 3\binom{9}{1} - 5$

**(c)** Hence, or otherwise, compute $|S|$

mb(8, 1) + 3 * comb(9, 1) - 5

Question 8.

The Warlpiri are a people whose country is found in Central Australia around the Tanami Desert. The following kin system is inspired by theirs and has been simplified for the purposes of this question.

Suppose each person in a tribe belongs to exactly one of the kin groups $1, 2, 3, 4, 5, 6, 7$ or $8$.

- Marriage is recommended between groups $1$ and $5$, $2$ and $6$, $3$ and $7$, or $4$ and $8$.
- The group of the mother determines the group of her children as per the table below:

| Kin group of mother | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Kin group of children | 4 | 3 | 1 | 2 | 7 | 8 | 6 | 5 |

For example:

- A mother in group $3$ will have children in group $1$.
- Anyone in group $3$ will have a mother from group $2$ and a recommended father from group $6$.

For simplicity, suppose everyone in this tribe always follows this tradition in the following questions.

**(a)** Suppose $x, y$ are two kin groups. We write $x \sim_1 y$ when it is possible that every decendent from $x$ to $y$ is female. What are the equivalence classes of $2$ and $6$?

$[2] =$ {3, 1, 4, 2},

$[6] =$ {8, 5, 7, 6}.

First, consider an equivalence relation 2. We want to try to find a scenario such that the descendents could all be female. The only guaranteed female that exists is their mother. Hence, we can go from 2 to 4 (their mother). But we can also continue from 4, to 1 (4's mother), and finally, to 3 (1's mother).

**(b)** Suppose $x, y$ are two kin groups. We write $x \sim_2 y$ when it is possible that every decendent from $x$ to $y$ is male. What are the equivalence classes of $4$ and $1$?

$[4] =$ {4, 5},

$[1] =$ {1, 7}.

For every child to be able to be a male, we must now consider the father of x, and then take the descendent from that group. The way this works is such that:

Take a descendent from kin group 4. The only guaranteed male we can find, is their father. So, the equivalence is 4 ~ 5 (and the only one that exists).

Question 9.

(Jacq. Shares w/ George → George S Tara) and (~(George studies with Tara) U ~(Tara W Jacqueline)

**(a)** Fill out the following truth table:

| Jacquelene shares with George | George studies with Tara | Tara works with Jacquelene | $A$ | |
|---|---|---|---|---|
| False | False | False | ○ False ◉ True ✔ | T ∧ T |
| False | False | True | ○ False ◉ True ✔ | T ∧ T |
| False | True | False | ○ False ◉ True ✔ | T ∧ T |
| False | True | True | ◉ False ○ True ✔ | T ∧ F |
| True | False | False | ◉ False ○ True ✔ | F ∧ T |
| True | False | True | ◉ False ○ True ✔ | F ∧ T |
| True | True | False | ○ False ◉ True ✔ | T ∧ T |
| True | True | True | ◉ False ○ True ✔ | T ∧ F |

**(b)** Hence, or otherwise, deduce whether the following argument is valid:

Suppose we know the following:

1. If Jacquelene shares with George then George studies with Tara, and
2. George does not study with Tara or Tara does not work with Jacquelene, and
3. Jacquelene shares with George.

Therefore George studies with Tara.

◉ This is a valid argument.
○ The argument is invalid. ✔

**(a)** Prove that there is **no** simple graph with vertex degree sequence

$$10, 6, 4, 3, 3, 3, 1, 1, 1, 1, 1.$$

Distribute edges over vertices.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| V | 10 | 6 | 4 | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 1 |
| D | 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 6 | 4 | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 1 |
| 10 | 6 | 2 | 2 | 2 | 2 | 2 | | | | |

Problem! Therefore, not simple.

**(b)** Prove that there is **no** simple graph with $61$ vertices where

- $26$ vertices have degree $60$ and
- $5$ vertices have degree $5$.

```
Consider the 26 vertices. For a simple graph, they must only have one edge between
two nodes, and hence, from vertex {a, (24 nodes), z}, they have to evenly distribute
the edges.

Hence, walk through the distribution of edges.
Consider a set of vertices {v0, ..., v60}. {v0, ..., v25} is the set of vertices with
degree 60.
First assign v0 60 edges to every single other vertex in the graph. This will leave
v0 with a degree of 60, and every other vertex with a degree of 1.
Do the same with v1, leaving every other vertex with a degree of 2.
v3; degree 3.
v4; degree 4.
v5; degree 5.
At this point, if we tried to create a simple graph with v6, we would find that
the 5 vertices with degree 5 would now become degree 6, and hence, some overlapping
is required (hence, not making it simple).
```

# Question 11.

**(a)** A **palindrome** is a word that is spelled the same forwards as it is backwards. How many palindromes of length $2n-1$ can be made using $n-1$ copies of the letter A and $n$ copies of the letter B if

**(i)** $n$ is odd? `comb(n-1, (n-1)/2)`

**(ii)** $n$ is even? `comb(n-1, (n-2)/2)`

```
If n is odd, then |A| is even and |B| is odd. We need to ensure both are even, as
when we use a letter on the left side, we must use it on the right side.

So fix B in the middle. Now both are even. Hence, we have 2n - 2 slots left, and
(n-1) letters each. Now, since we are only filling half of the slots, we actually
have:
n - 1 valid slots
and (n-1)/2 letters for A and B.

Therefore, we have comb(n-1, (n-1)/2)


If n is even, then A is odd and B is even. Hence, we now must affix A to the middle,
and now we will have n - 2 copies of A, and n copies of B.
The number of valid slots are still the same, (n-1), and we can first affix
the A's into the slots (and then the B's will fill the remaining):
comb(n-1, (n-2)/2)
```

**(b)** Suppose $4$ B's, $3$ A's and $1$ O are arranged in a circle. How many different arrangements can be made

**(i)** if the letters are always read clockwise? 7!/(4! * 3!)

**(ii)** if the letters can be read clockwise or anticlockwise? For example "OBBBABAA" is the same as "OAABABBB". (7!/(4! * 3!) - 3) / 2 + 3

i) Imagine the combination of strings:

B B B B A A A O.

There are $\dfrac{8!}{4!3!}$ ways of arranging this.

A table allows for a rotational reading of a combination.
In essence; how many bit rotations are possible with 8 characters before returning? (8).
Hence, a combination is equivalent to 7 different bit rotated combinations.

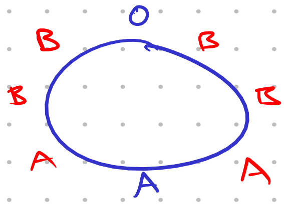
$$\therefore \quad \frac{8!}{4!3!\,8} = \frac{7!}{4!3!}$$

ii) [This question is not advisable]

With the addition of two way reading, a new problem occurs. Now, 'palindromic' tables are an issue.

Consider 4 B's, 3 A's and an O.

To create a palindromic table, we need to try to remove the odd element. Hence, affix O and A.




← 3 ways of distributing a side.

Now every combination that is not symmetric will have two equivalent combinations.

Hence, consider just these combinations:

$$\frac{\left(\frac{7!}{3!2!} - 3\right)}{2}.$$

Now add back the symmetric cases:

$$\frac{\left(\frac{7!}{3!2!} - 3\right)}{2} + 3$$

c) i)

(i) The symmetry of the board means a single game state can be described in equivalent ways by starting from a different position and moving clockwise or anti-clockwise. Check all strings that describe the same game state as A B BABBAOAA.

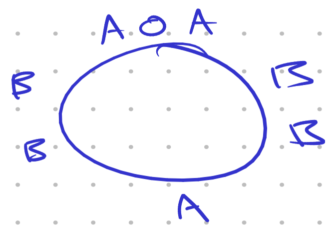
- [ ] A B OABABBAB → = OABX ← OBAX
- [x] A B AAOABBAB → = OABX ← OAAJ
- [x] A B ABBAOAAB etc. etc.
- [ ] B A OABABBAB

Check forwards and backwards, for equality of state.

4 B's, 1 A

ii) (AOA) X X X X X

5 ways of placing A. 1 symmetric way.




$$\frac{5-1}{2} = 2+1 = 3 \text{ ways.}$$

iii) Now, consider the scenario where we have 5 tokens for A and 5 tokens for B. The win condition stays the same - A in the center, and A's encapsulating O. So, we will always need something like this:

(AOA)XXXXXXX

Note that here, we only have 2 A's to use, because one is in the centre, but we won't represent it in our combinations.
Therefore, consider the "normal" combinations, which is just:
7!/(5! * 3!)
And consider the "palindromic" combinations - we must affix an odd numbered token (here, B), and then consider the combinations of either side. There is 3!/2! ways (2 B's and 1A on either palindromic side, therefore): 3.
Hence to find the total combinations, we must first find the amount of repeated 'equivalence relations' as before:
(7!/(5! * 3!) - 3) / 2 + 3 (and then add back on the palindromic combinations)

## Question 12.

Fill in the answers below to find the solutions to the equation

$$3^a + 4^b = 5^c$$

where $a, b, c \in \mathbb{N}$. This question has five parts **(a)**,**(b)**,**(c)**,**(d)**, and **(e)**.

---

**(a)** Suppose $a > 0$ and consider the equation $(\text{mod } 3)$. That is,

$$3^a + 4^b \equiv 5^c \pmod{3}.$$

Then

- $3^a \equiv$ 0 $\pmod{3}$
- $4^b \equiv$ 1 $\pmod{3}$.

Since $5 \equiv -1 \pmod{3}$, this means that $c$ must be Even .

(as min = 0+1)

want positive, e.g $5^2 \equiv 1 \pmod{3}$

---

**(b)** Suppose $b > 0$ and consider the equation $(\text{mod } 4)$. That is,

$$3^a + 4^b \equiv 5^c \pmod{4}.$$

Then

- $4^b \equiv$ 0 $\pmod{4}$,
- $5^c \equiv$ 1 $\pmod{4}$.

Since $3 \equiv -1 \pmod{4}$, this means that $a$ must be Even .

same idea as above.

---

**(c)** Hence, if $a, b > 0$ then $5^c - 3^a$ is the difference of two squares, so

$$4^b = 5^c - 3^a = \left(5^{c/2} - 3^{a/2}\right)\left(5^{c/2} + 3^{a/2}\right).$$

Hence, there are two integers $u, v$ such that $u + v = 2b$ and

$$5^{c/2} - 3^{a/2} = 2^v \text{ and } 5^{c/2} + 3^{a/2} = 2^u.$$

Taking the average sum and average difference of these two equations gives

$$5^{c/2} = 2^{v-1}(2^{u-v} + 1) \text{ and } 3^{a/2} = 2^{v-1}(2^{u-v} - 1).$$

What is the value of $v$ in the above? Briefly explain your method.

```
Consider the outcome postulates some powers of 5 and 3 are divisible by some power
of 2. Therefore, v must equal 1, for the equation to be valid, as an odd number
cannot be divided by 2.
```

**(d)** Hence, we have that

$$3^{a/2} = 2^{u-v} - 1.$$

By considering this equation $(\bmod 3)$, or otherwise, show the RHS is the difference of two squares and then find all positive values for $a, b$ and $c$. Explain your method.

By considering the equation mod 3, get:

3^(a/2) = 2^(u-v) - 1 (mod 3)

0 eqiv 2^(u-v) - 1 (mod 3)

2^(u-v) equiv 1 (mod 3)

(-1)^(u-v) equiv 1 (mod 3)

Therefore, u-v must be even. Let u-v = 2k, for some integer k.

3^(a/2) = 2^(2k) - 1

3^(a/2) = (2^(k) - 1)(2^(k) + 1)

Since a is given to be even (as before), we can deduce the following:

Consider a > 2

Therefore, it must be such that:

Both factors (2^(k)-1)(2^(k)+1) are factors of 3, or one is a factor of three and one is 1.

Hence we have a = 2 and m = 1 or:

2^(k) - 1 (mod 3) - [1]

2^(k) + 1 (mod 3) - [2]

2^(k) = 1 (mod 3)

2^(k) = -1 (mod 3)

These then imply that k is simulatenously odd and even, which is impossible.

Hence we have that a = 2 and k = 1.

Therefore, returning to our previous definitions, we get:

k = (u - v)/2

u - v = 2 (Recalling that v = 1)

u = 3

(Recalling that u + v = 2b)

1 + 3 = 2b

b = 2

And recalling our original equation, 3^a + 4^b = 5^c, c = 2.

Hence, a = 2, b = 2, c = 2.

**(e)** The previous parts were based on the assumption that $a$ and $b$ are both positive integers. Are there any other solutions to $3^a + 4^b = 5^c$? Explain your answer below.

a=b=c=0