2021 T2

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Written by Haeohreum Kim (z5480978)

Most of these solutions have been derived entirely by me, so there may
be imperfections and wrong answers/reasonings. Please contact me at my
student email for any corrections.

These solutions were made to aid the preparation of MATH1231 students,
not replace it. Good luck :)
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Q2)

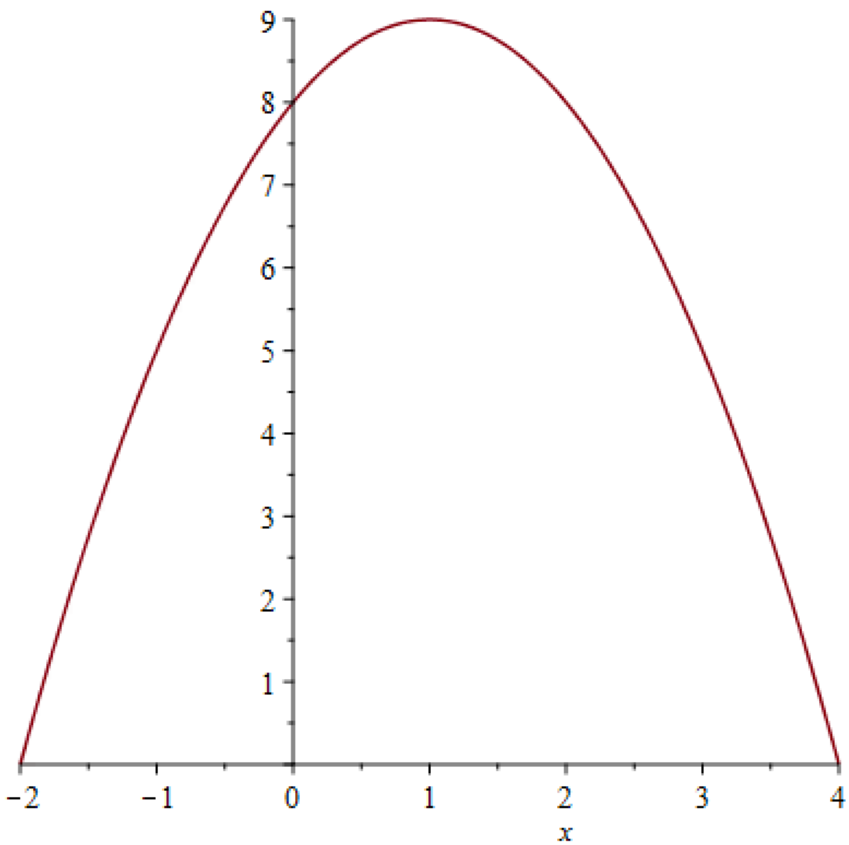
a) The value of α should be dictated by the fact that the $\int CDF\, dx = 1$.
Therefore, integrating $\alpha x^2 - 10\alpha x + 21$ becomes $\frac{\alpha x^3}{3} - 5\alpha x^2 + 21x$.
Further:
$[\frac{\alpha x^3}{3} - 5\alpha x^2 + 21x]_3^7 = 1$.
This turns out to be $a = -\frac{3}{32}$.

b) The expected value is found by doing the calculation $\int xp(X)\, dx$ where p(X) represents the probability density function.

```
f := -3/32*x^3 - 10*(-3/32)*x^2 + 21*(-3/32)*x
int(f, x=3..7)
Output: 5
```

Q3)

You can do this question graphically, fairly easily.




```
Consider the options:
1. P(X=1) = 0.5
S: This doesn't have to be true, so no
2. P(2 <= X <= 4) < P(2 <= X <= 5)
S: You can graphically see this is not true. (Infact, it'd be equal)
3. P(X <= 1) = P(X >= 1)
S: This is true because it's symmetric
4. P(X >= 1) > P(X >= 0)
S: This is graphically false (symmetric around x = 1)
5. P(X = 1) >= P(X = 0)
S: This is true, since X = 1 will be the peak of our PDF.
```

Q4)

```
Super easy maple question:
a) Eigenvectors(<<16, 6>|<24, 34>>)


b) Eigenvectors(<<-4*a + 8, a + 8>|<4*a + 32, -a + 32>>)


c) Use above
```

```
d) Use above
```

Q5)

a) `plot3d(sqrt(16-(x^2+y^2))`

b) Use the formula:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ -1 \end{pmatrix}$$

i) And then use point normal form

```
f := (x, y) -> sqrt(16-(x^2+y^2))
a := (x, y) -> -x/sqrt(-x^2 - y^2 + 16) // This is diff(f(x, y), x);
b := (x, y) -> -y/sqrt(-x^2 - y^2 + 16)
c := <a(3, sqrt(6)), b(3, sqrt(6)), -1>
DotProduct(c, <x-3, y-sqrt(6), z-1>) = 0
```

ii) Using the same maple definitions as above

```
g := f(3, sqrt(6)) + a(3, sqrt(6))*(0.1) + b(3, sqrt(6))*(-0.1)
```

c) Consider the symmetry of the function.
Therefore, if we apply negatives to either x and/or y; the tangent plane will intercept the same point.

d) We are tasked to find **f(t)**
Consider that this is the surface area around the ***z-axis.***
Therefore, we will find the equation with formula:

$$x(t)\sqrt{x'(t)^2 + z'(t)^2}$$

In my case, the equation became:

$$16\cos(t)\sqrt{\sin(t)^2 + \cos(t)^2}$$

Q6)

a) `diff(sqrt(x^2+6/x^2), x)`

b) `convert(x^4/((x^4+6)*(x^4+49x^2+6)), parfrac)

Now, using the knowledge that one side will keep the 1/49, and the other will multiply:

$$-\frac{x^2}{49(x^4+49x^2+6)}+\frac{x^2}{49(x^4+6)}$$

We can use the right fraction much easier, by substituting $\frac{1}{y^2}=\frac{x^2}{x^2+6}$

Q7)

a)

The nullity is 2.

Reasoning:

We have three vectors, sitting somewhere on $\mathbb{R}^3$; and they sit on the same *plane*. They are also non-zero.

If we consider some direction vectors, we can see that they are linearly independent.

Since it is given that T is a linear map, and hence:

$T(\vec{b}-\vec{a})=T(\vec{b})-T(\vec{a})=0$

$T(\vec{c}-\vec{a})=T(\vec{c})-T(\vec{a})=0$

Therefore, the nullity$(T)\geq 2$. (as seen above)

Now consider $\vec{c}-\vec{b}$:

Note that $\vec{c}-\vec{b}=(\vec{b}-\vec{a})-(\vec{c}-\vec{a})$

and therefore, $\vec{c}-\vec{b}$ is linearly dependent, and further, the nullity is two.

b) As seen before, finding two direction vectors gives us a basis for the kernel.

Therefore, do $\vec{b}-\vec{a},\vec{c}-\vec{a}$ will provide you with the two basis vectors.

c)

Consider that $17v_1+20v_2-4v_3=<9,-3,7>$

We recognise that the linear map is transferring the basis vectors $\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}$ to their new locations, and hence solving these for v_1,v_2,v_3 gives us the scalar multiples of the basis vectors in the transition.

Hence, $v_1=Ae_1,v_2=Ae_2,v_3=Ae_3$.

Further simplifying this for the other two linear maps:

$17v_1+20v_2-4v_3=<9,-3,7>$

$14v_1+16v_2-3v_3=v$

$-4v_1 - 5v_2 + v_3 = v$

Therefore, we can now construct the matrix.

Q8)

a) Using the ratio test, reduce to terms of x

$$\lim_{n\to\infty} \frac{(x-4)^{n+1}}{(n+1)6^{n+1}} \cdot \frac{n6^n}{(x-4)^n}$$

This becomes:

$$-1 \leq \frac{x-4}{6} \leq 1$$

I leave the rest to the reader :)

b)

Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(x-4)^n}{n6^n}$$

Differentiate with respects to x on both sides:

$$f'(x) = \sum_{n=1}^{\infty} \frac{(x-4)^{n-1}}{6^n}$$

Therefore, the argument iterates.

c)

Using the infinite GP formula $\frac{a}{1-r}$, we find:

$$\begin{aligned} S &= \frac{\frac{1}{6}}{1-\frac{x-4}{6}} \\ &= \frac{\frac{1}{6}}{\frac{6-x+4}{6}} \\ &= \frac{1}{10-x} \end{aligned}$$

d) Use f(4), such that it $=$ 0. This would mean that this is the constant.

$$f(x) = \int \frac{1}{10-x}\,dx$$

$$= -\ln(10-x)$$

Let x = 4

$$\begin{aligned} f(4) &= \sum \frac{(4-4)}{n3^n} \\ &= -\ln(10-4) + C \\ &= -\ln(6) + C = 0 \\ &= \ln(6) = C \end{aligned}$$

Q9)

a) Note that $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$ for the ODE to be exact.

b) Multiply both sides by $x^p y^q$ and differentiate in order to create a system of linear equations such that the coefficients of the equations are equal.

c) `dsolve(2*y(x)^4*x + 4*x^3*y(x)^3 + (4*x^2*y(x)^3 + 3*x^4*y(x)^2)*diff(y(x), x))`

Q10)

a)
Consider that the nth Taylor polynomial at x = 0 is represented by:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Therefore, using this, we can find the coefficients of the Taylor polynomial.

b)
Consider that the *lower* and *upper* bounds of a Taylor polynomial are represented by the Lagrange reminder.
The Legrange reminder is defined as:

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

The *upper* bound is $+R_{n+1}(x)$, where as the lower bound $-R_{n+1}(x)$.

$$R_3(x) = 2x^3 = \frac{1}{256}$$

Hence, you must consider:
$f(x)_{\frac{lower}{upper}} = 2 \cdot \left(\frac{1}{8}\right) + \frac{5}{2} \cdot \left(\frac{1}{8}\right) \pm \frac{1}{256}$.

Q11)

a)

Consider the probability 'tree':

```
Positive -> True Positive | False Negative


Negative -> True Negative | False Positive


True Pos = 0.99
False Neg = 0.01
True Negative = 0.92
False Positive = 0.08
Therefore, the probability that a person receives at least one can be
defined by:
P(TN & FP) + P(FP & TN) + P(FP & FP)
= 0.1536
```

b)

Alternative solution:

Let p be the probability that people have the alpha gene.

$$0.06 = p(0.99)^2 + (1-p)(0.06)^2$$

Then solve for p

REMEMBER EXACT FORM!

MATH1231 Staff's Solution:

From now T_1 denotes the first tests, T_2 denotes the second test, and $\bar{T}_1$ denotes it's complement.

P(A) is the proportion of people who actually have the alpha gene. This is using the total probability theorem.

$$P(T_1 \cap T_2) = P(T_1 \cap T_2|A)P(A) + P(T_1 \cap T_2|A^c)P(A^c)$$
$$P(T_1 \cap T_2|A)P(A) = P(T_1 \cap T_2) - P(T_1 \cap T_2|A^C)P(A^C)$$
$$= P(T_1 \cap T_2) - P(T_1 \cap T_2|(1 - P(A)))(1 - P(A))$$

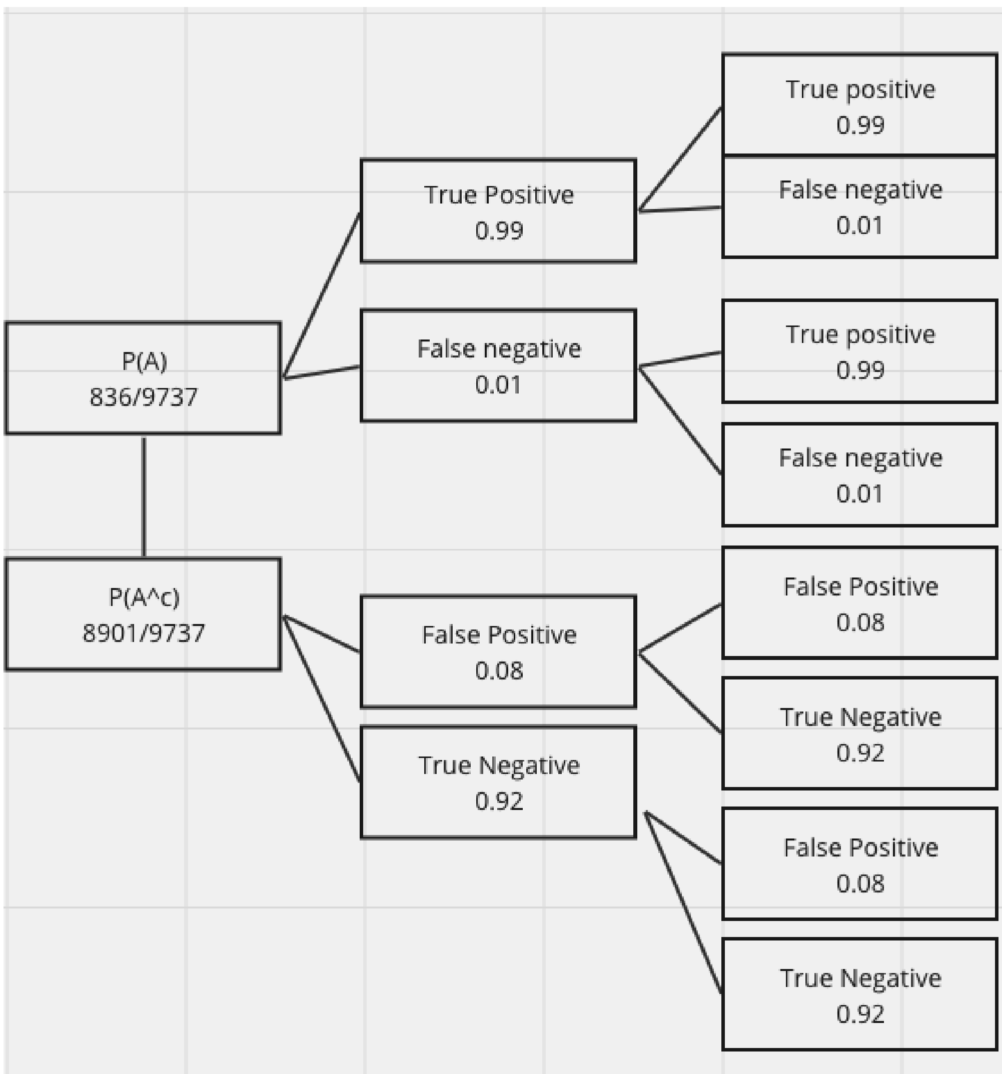
I leave the rest to you (ceebs)

In words:

$$\frac{\text{Probability of two positives - (Probability of false positive)(Probability of false positive)}}{(\text{Probability of true positive})^2 - (\text{Probability of false positive})^2}$$

c)

For c), we can use the fact of b) and create a tree diagram fairly easily to understand the



situation.

We only care about the routes where they end up with different results, and still has the alpha gene.

Alternative solution:

Q12)

a)

$$\left(\frac{1}{38}\right) \cdot \left(\frac{37}{38}\right)^{11}$$

b)

$$\frac{1}{38} \cdot \left(\frac{37}{38}\right)^{51} + \ldots$$

$$\frac{1}{38} \sum_{n=51}^{\infty} \left(\frac{37}{38}\right)^{n}$$

c) Consider that the sunken cost is 50. $To recoup this, Jesse must win at a * minimum * twice. So, we can simply do $1 - (P(0) + P(1))$ to find this easily.

$$P(Earnings) = 1 - \left(\frac{37}{38}\right)^{50} - 50 \cdot \left(\frac{37}{38}\right)^{49} \cdot \left(\frac{1}{38}\right)$$

d)
Approx. B(n, p) by $X \sim N(\mu, \sigma^2)$ and convert to $Z \sim N(0, 1)$
n = 126
m = 10
p = $\frac{1}{38}$
$\mu = \frac{126}{38}$
$\sigma^2 = np(1 - p) = 126 \cdot \frac{1}{38} \cdot \left(\frac{37}{38}\right)$
$z = \frac{m - \frac{1}{2} - \mu}{\sigma}$

Q13)
a)

$$\begin{aligned} Note : M_1 + M_2 &= M \\ \therefore M_1 + M_2 &= V_2 G(t) + V_1 F(t) \\ F(t) &= \frac{M}{V_1} - \frac{V_2}{V_1} G(t) \end{aligned}$$

$= -\frac{20790}{2100}$

b) Trivial ODE solve

c)
Just by inspection, exp(-t) tends to zero, so:

$\frac{M}{22890}$

d) By conservation of mass and the equal distribution within the differential equation:

$$\frac{M}{22890}$$

The total concentration, as time tends to infinity, is simply derived by dividing M by the total volume:

$$\frac{M}{V_1 + V_2}$$

Furthermore, the intrinsic definition of F and G ensures that they must be equal, as if G $>$ F, then F is negative, etc.

Q14)

To prove the set is linearly independent, we must prove that all scalars $a_1, a_2, \ldots, a_n$ are equal to zero.
Hence we need to show that *the only solution to the equation is where all the scalars* a_i *are zero.*
Note that the fact that $S^{28}(u) = 0$ ensures the same is true of $S^{29}(u), \ldots$.
To both sides of equations, apply the linear transformation S^{27}.
Then using the fact that S preservers linear combinations and zero, we can conclude at this point that $a_0 = 0$.

We must continue to repeat this with $S^{26}, S^{25}, \ldots, S$ in order to prove that $a_1, a_2, \ldots, a_{27}$ are $= 0$.

Q15)

a) Yes, (1) can be replaced with (A).
Reasoning:

```
Suppose that S satisfies (1), (2), and (3). Since (1) is already true,
and contains the zero vector, we can already satisfy (A). Therefore, (A)
holds.
Next, suppose that (A), (2) and (3) hold. Since S is non-empty by (A),
there exists some vector in S. However, since (3) holds, 0v = 0 also
holds. Hence (1) holds.
```

b)

Reasoning:

```
Suppose that S satisfies (B). If u belongs to S, then it is implied
that:

u + u = 2u belongs to S.

Suppose k**u** belongs to S. Consider that u* + ku = (k+1)u, also
belongs to S. Therefore, this is proved by induction. Therefore, (B) ->
(X).

Consider a counter example:

Let (nv : n is a positive integer), then -v is not contained in S.
However, S satisfies X, and thereofre, not closed under scalar
multiplication.
```