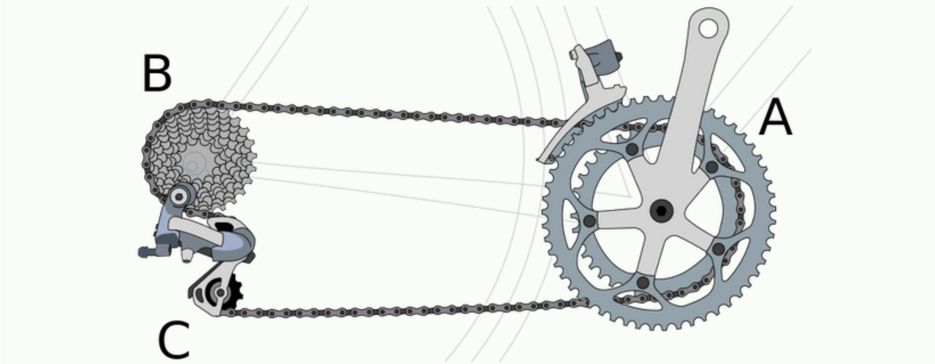
MATH1081 2020 T3

Question 2.

Alexander's favourite bike has a chain with 62 links which is connected to three gears as shown:



The gear A has 33 teeth, gear B has 18 teeth, and gear C has 8 teeth.

These gears wear down over time. Wear is minimised when every link in the chain touches every tooth in the gear. Which gears have minimal wear?

Find the one with the closest remainder:

A : 29 (33) ⇒ A has the least wear.

B : 8 (18)

C : 6 (8)

Question 3.

(a) Your MATH1081 friend gives you the modulus $m = 391$, their public key $\alpha = 123$, and shows you how to encrypt the word MATH as a sequence of numbers.

Letter	M	A	T	H
Number x	14	2	21	9
Encrypted Number $x^{123} \pmod{391}$	333	280	387	219

$\alpha = 123 \qquad m = 391$

$3^{123} \bmod 391 = 262$

$6^{123} \bmod 391 = 343$

$20^{123} \bmod 391 = 313$

$21^{123} \bmod 391 = 387$

(b) Your friend tells you that their public key uses the prime number 17. Use this information to deduce:

1. The other secret prime **23**
2. Their private key $\beta =$ Number .

a) $m = pq$

$391 = 17q$

$q = 23$

b) $(123 \bmod 352)^{-1}$

$= 123x \equiv 1 \pmod{352}$

$352y + 123x = 1$

$352 = 123 \times 2 + 106$
$123 = 106 \times 1 + 17$
$106 = 17 \times 6 + 4$
$17 = 4 \times 4 + 1$

$1 = 17 - 4 \times 4$
$1 = 17 - 4 \times (106 - 17 \times 6)$
$1 = 17 - 4 \times 106 + 24 \times 17$
$1 = 25 \times 17 - 4 \times 106$
$1 = 25 \times (123 - 106) - 4 \times 106$
$1 = 25 \times 123 - 29 \times 106$
$1 = 25 \times 123 - 29 \times (352 - 123 \times 2)$
$1 = 25 \times 123 - 29 \times 352 + 58 \times 123$
$= 83 \times 123 - 29 \times 352$

$x = 83$

c) $360^{83} \bmod 391 = 10 \Rightarrow I$
$281^{83} \bmod 391 = 15 \Rightarrow N$
$387^{83} \bmod 391 = 21 \Rightarrow T$
$118^{83} \bmod 391 = 16 \Rightarrow O$

Question 4.

Let m be an integer such that $2 < m < 15$. Define the set $X = \{0, 1, 2, \cdots, m-1\}$ and the function $f : X \to X$ by

$$f(x) = 15x \pmod{m}.$$

a) is injective.

7 is prime, and hence will not have overlapping factors; → every value being visited.
13 also works.

b) is not surjective.

5. has common factors with multiple values (as multiplied by 15).

Question 5.

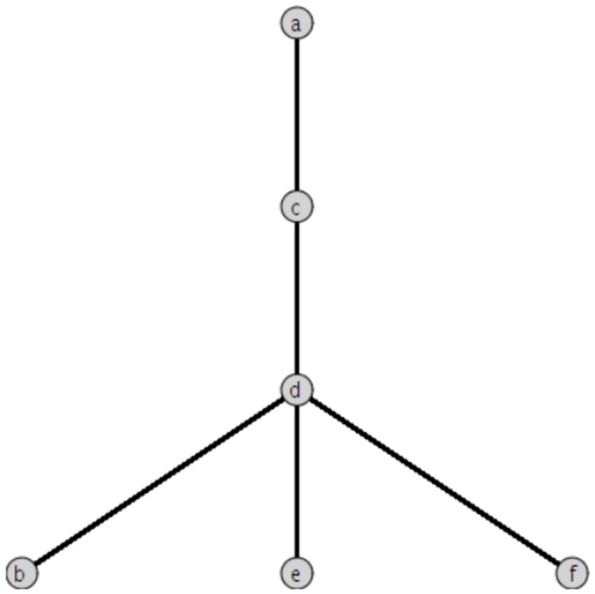
An eccentric professor grades four equally eccentric students Xavier, Yusuf, Zoey and Warwick according to the following rubric.

- Either Warwick or Xavier will pass, but not both.
- If Zoey passes then Yusuf will pass.
- Warwick will pass or Xavier will fail.
- It is certain that Yusuf will pass because they are top of the class.

- [ ] Xavier will pass.
- [x] Xavier will fail.
- [x] Yusuf will pass.
- [ ] Yusuf will fail.
- [ ] Zoey will pass.
- [ ] Zoey will fail.
- [x] Warwick will pass.
- [ ] Warwick will fail.

Yusuf will definitely pass.
We are given an exclusive or for Warwick and Xavier. Hence, only one of them can pass.
We are also given Warwick will pass or Xavier will fail. Consider what happens if Xavier will
If Xavier fails, then Warwick must also fail - but this contradicts our earlier statement.
Hence, Warwick must pass (and Xavier must fail).

Question 6.



a) The eccentricity of a vertex is defined to be the length of the longest path from it to any other vertex. For example, in this graph the eccentricity of a is 3. Find the eccentricity of the vertex b.

The longest path from b is $b \to a$, which has a path length of three. Therefore, the eccentricity of b is 3.

b) The diameter of a graph is the maximum eccentricity of any vertex. Find the diamater of the graph G. 3

This follows by inspection from above.

c) Now consider a simple graph H with n vertices with the property that $\deg(v) \geq n/2$ for every vertex v. Prove that the diamater of H is at most two.

Consider that a simple graph is defined as a graph with no more than one edge between two vertices.

Consider two vertices, x and y, and let the sets X and Y be the set of vertices adjacent to x and y respectively. By the given conditions $|X| \geq n/2$, $|Y| \geq n/2$.
Take that x and y are not adjacent (if they are, then the distance is simply one).
Then:

$$\begin{aligned} |X \text{ int } Y| &= |X| + |Y| - |X \cup Y| \\ &\geq n/2 + n/2 - (n-2) \\ &= 2 \end{aligned}$$

Therefore, the diamater is at most two.

Question 7.

a)

(a) Aboard the spaceship Scarborough, a group of 22 crewmates are fulfilling various tasks. Each crewmate is assigned exactly one task to complete, though more than one crewmate can be assigned to a particular task.

Yesterday, only three tasks had to be carried out. Among them, 9 crewmates were assigned to aligning the engine output, 4 crewmates were assigned to aligning the telescope, and 9 crewmates were assigned to assembling artefacts.

In how many ways could this have happened?

c(22,9)c(13,4)

$$\binom{22}{9}\binom{13}{4}$$

b)

(b) Today there is a much larger selection of tasks that everyone has been assigned to.

The crew is worried some shape-shifting aliens may have infiltrated their ranks. Captain Red and Co-captain Lime each trust their own group of friends, but they do not have exactly the same friend group.

Red says, "7 of the tasks my friends were assigned to are the same as the tasks your friends were assigned to. But only 3 tasks were assigned to our mutual friends."

Lime says, "That's impossible! You must be lying, impostor!"

Not enough info.
We're given just S int T, where S and T are the friend groups of Red and Lime.
Realistically, we need |S| and |T|, to figure out if the allocation of tasks lines up.

c)

(c) To alleviate suspicion, Red proves to Lime that Red's first sentence was accurate.

Red and Lime confirm that 10 different tasks were assigned to Red's group of friends, while 9 different tasks were assigned to Lime's group of friends.

Lime says, "Interestingly, only 12 different tasks were assigned to the combined group of your and my friends."

Red says, "That's impossible! You must be lying, impostor!"

Select the correct statement:

- ○ Lime is lying.
- ○ Lime is telling the truth.
- ○ There is not enough information to decide if Lime is lying or telling the truth.

$|S \wedge T| = 7 \quad |S| = 10 \quad |T| = 9.$

$10 + 9 - 7 = 12. \quad \therefore$ TRUE!

Question 8.

At Gardiner University, there are 39 students. Each student must choose to take their degree in exactly one of the five available subjects: Arts, Business, Computer Science, Design, or Engineering. No student can study more than one subject. Prove that one or more of the following must be true:

- there are at least 2 Arts students,
- there are at least 5 Business students,
- there are at least 16 Computer Science students,
- there are at least 4 Design students, or
- there are at least 16 Engineering students.

The statements are very hard to prove directly - so one must consider contradiction.
To prove that one or more of the following must be true - prove that everything being false is impossible.
Hence, at maximum, there must be:
1 Art student
4 Business Students
15 Computer Science students
3 Design students
15 Engineering Students
Which sums up to be 38. Therefore 38 $\neq$39, and hence one statement must be true.

Question 9.

The year is 2041 and negative interest rates have made debt into a worthwhile investment. You sign a contract with Better Bank allowing you to borrow 145000 dollars at −8% per annum (compounded annually), and then borrow a further 5000 dollars every following year.
Let a_n denote the amount owing after n years have passed.

a) Evaluate $a_0 =$ 145000

b) If $n \in \mathbb{Z}^+$ then which relation does a_n satisfy?

- ○ $\frac{92}{100} \times (a_n - a_{n-1}) = 5000$
- ● $a_n = \frac{92}{100} \times a_{n-1} + 5000$
- ○ $\frac{92}{100} \times a_n + 5000 = a_{n-1}$

$$a_n = \frac{92}{100} \times a_{n-1} + 5000$$

$\frac{92}{100}$: -8%

$+5000$: $+5000$ every year.

c) Evaluate $a_1 =$ 138400

$$a_1 = \frac{92}{100}(145000) + 5000$$

d) $a_n = ca_{n-1} + d$

The limit is given by:

$$L = \frac{d}{1-c}$$

e) A la financial mathematics in HSC

First:

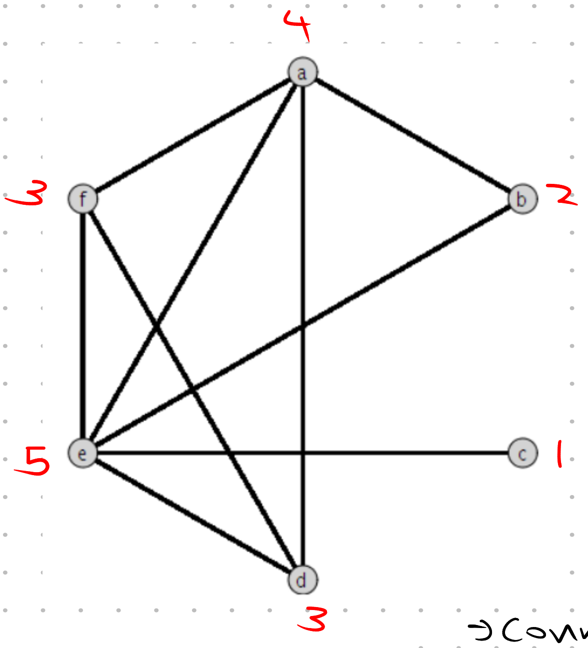
b_n = 145000 * (92/100)^n + borrowing money

borrowing money is reduced to 5000 * (92/100)^n every year, and is borowwed every year, hence:
5000 * n * (92/100)^n

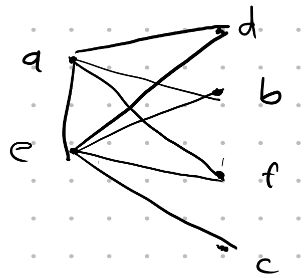
therefore:

b_n = 145000 * (92/100)^n + 5000 * n * (92/100)^n

Question 10.



→ Connected.



- [x] The graph is connected.
- [ ] The graph is bipartite. ×
- [ ] The edges can be coloured with three colours, such that edges incident on the same vertex have different colours. ×
- [ ] The vertices can be coloured with three colours, such that vertices incident on the same edge have different colours. ×
- [x] The graph is planar $e \leq 3v - 6$ $e \leq 12$
- [ ] The graph contains a Hamiltonian circuit × c has degree 1.
- [ ] The graph contains a Euler path that is not a circuit × too many odd degrees.

↳ do exactly two vertices have odd degree?

Question 11.

Let

$$G = \{\text{MATH1081 students in term 2 of 2028}\}$$

and consider the function $f : G \to \mathbb{N}$ defined by

$$f(g) = \text{the number of other students in } G \text{ who make eye-to-eye contact with } g.$$

Prove that the function is not injective.

For a function to not be injective, it must satisfy that f(x1) = f(x2) does not imply x1 = x2.

For the sake of contradiction, first assume that f(g) is indeed injective.

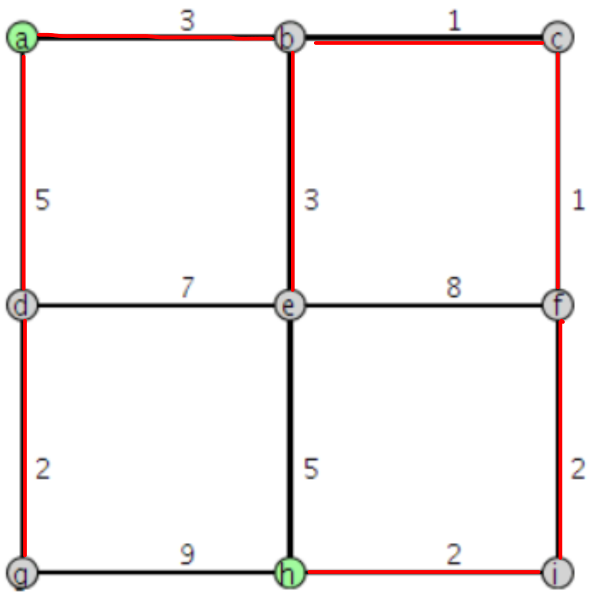
Hence, consider two individuals in G, s and t.

f(s) = f(t) represents the number of individuals both s and t had eye-to-eye contact with.

When considering the forward implication s = t, this does not necessarily have to be true.
For sake of example, consider G to be {a, b, c, d, e, f}. If d and e make eye contact with f, and a and b make eye contact with c, then f(c) = f(f), but c does not equal f.

Hence, the function f(g) is not injective.

Question 12.



a) What is the weight of a shortest path from a to h? 9

b) Suppose you are allowed to change the weight of the edge $\{g, h\}$ so this edge occurs on a shortest path from a to h. What is the largest weight possible for this edge? 2

Look above for reasoning.
$\{g, h\} = 2$ would be taken faster than $h \to i$.

c) Explain your answer to part (b).

The edge {g, h} is not taken due to it's weight, but is checked before the edge {h, i}. Hence, if the accumulative edge weight from a, d, g, h is equivalent to the original path, then that path will be taken, as there is less edges, and it discovers such a route faster.

Question 13.

Suppose $a_1, a_2, \cdots, a_9$ are non-negative integers. How many solutions are there to the equation

$$a_1 + a_2 + \cdots + a_9 = 67$$

a) ... with no further restrictions? $\binom{75}{8}$

b) ... where each $a_i \leq 8$? $a_i = 8 - b_i$

$72 - b_1 - b_2 - b_3 - \cdots - b_9 = 67$

$b_1 + b_2 + b_3 + \cdots + b_9 = 5$. $\leftarrow \binom{13}{8}$ Naive solution holds

c) ... where each $a_i \leq 9$?

$81 - b_1 - b_2 - \cdots - b_9 = 67$

$b_1 + b_2 + \cdots + b_9 = 14$. Naive solution does NOT hold.

$\therefore$ Consider when $a_i \geq 10$

$a_1 \geq 10$

$a_1 + \cdots + a_9 = 57$ $\quad \binom{65}{8}$. Huge inclusion-exclusion

$a_1, a_2 \geq 10$ $\quad \binom{55}{8}$

$a_1, a_2, a_3 \geq 10$ $\quad \binom{45}{8}$

$a_1, a_2, a_3, a_4 \geq 10$ $\quad \binom{35}{8}$

$a_1, a_2, a_3, a_4, a_5 \geq 10$ $\quad \binom{25}{8}$

$a_1, a_2, a_3, a_4, a_5, a_6 \geq 10$ $\binom{15}{8}$.