Question 1.

Pre-requisite:
Understanding what ∨, ∧ & → entail.
Understanding contingency, tautology and cont.

p	q	$p \vee q$	$\sim p \to q$
T	T	T ✓	T ✓
T	F	T ✓	T ✓
F	T	T ✓	T ✓
F	F	F ✓	F ✓

The expression

$(\sim p \to q) \to \sim(p \vee q)$

T → F
T → F
T → F
F → T

Not true or false ∀
∴ contingent.

p	q	$p \wedge \sim q$	$p \to q$
T	T	F ✓	T ✓
T	F	T ✓	F ✓
F	T	F ✓	T ✓
F	F	F ✓	T ✓

The expression

$\sim(p \to q) \to (p \wedge \sim q)$

F → F
T → T
F → F
F → F

∴ Always true.
Tautology.

p	q	$p \to q$	$\sim p \wedge q$
T	T	T ✓	F ✓
T	F	F ✓	F ✓
F	T	T ✓	T ✓
F	F	T ✓	F ✓

$\sim(\sim p \wedge q) \vee (p \to q)$

T ∨ T ⇒ T
T ∨ F ⇒ T
F ∨ T ⇒ T
T ∨ T ⇒ T

∴ Tautology

p	q	$\sim p \vee q$	$p \wedge q$
T	T	T ✓	T ✓
T	F	F ✓	F ✓
F	T	T ✓	F ✓
F	F	T ✓	F ✓

$(p \wedge q) \wedge \sim(\sim p \vee q)$

T ∧ F ⇒ F
F ∧ T ⇒ F
F ∧ F ⇒ F
F ∧ F ⇒ F

∴ Contradiction.

Question 2.

Use the drop-down menus to select a correct statement.

p	q	r
T	T	F
T	F	T
F	T	F
F	F	F

Check every combo:

$r \Leftrightarrow (p \to q)$ (wrong).

$\sim r \Leftrightarrow (p \to q)$ (true).

The converse of the right-hand side of the above statement is equivalent to:

p ∨ ~q

p	q	r
T	T	T
T	F	F
F	T	T
F	F	T

= $r \Leftrightarrow p \to q$

$T \Leftrightarrow T \to T$

$F \Leftrightarrow T \to F$

$T \Leftrightarrow F \to T$

$T \Leftrightarrow F \to F$

By logical equivalence:

$p \to q$ converse is

$q \to p \Rightarrow p \vee \sim q$.

Must check every perm.

p	q	r
T	T	T
T	F	F
F	T	F
F	F	F

Check $r \Leftrightarrow p \to q$

$T \Leftrightarrow T \to T$

$F \Leftrightarrow T \to F$

$F \Leftrightarrow F \to T$ ×

$F \Leftrightarrow F \to F$.

$\sim r \Leftrightarrow p \to q$

$F \Leftrightarrow T \to T$. ×

$r \Leftrightarrow \sim p \to q$

$T \Leftrightarrow F \to T$

$F \Leftrightarrow F \to F$ ×

$r \Leftrightarrow \sim p \to \sim q$

$T \Leftrightarrow F \to F$

$F \Leftrightarrow F \to T$ ×

$\sim r \Rightarrow \sim p \to \sim q$

$F \Rightarrow F \to F$ ×

$\sim r \Leftrightarrow p \to \sim q$

$F \Leftrightarrow T \to F$

$T \Leftrightarrow T \to T$

$T \Leftrightarrow T \to T$

$T \Leftrightarrow F \to T$ ✓

$\sim r \Leftrightarrow p \to \sim q$

b) The negation of the right-hand side of the above statement is equivalent to:

$\sim (p \to \sim q)$

$\Rightarrow p \wedge q$.

Note: $\sim (a \to b) = a \wedge \sim b$.

a)

p	q	r
T	T	F
T	F	F
F	T	F
F	F	T

$r \Leftrightarrow p \to q$ ×

$\sim r \Leftrightarrow p \to q$ ×

$r \Leftrightarrow \sim p \to q$ ×

$r \Leftrightarrow \sim p \to \sim q$ ×

$\sim r \Leftrightarrow \sim p \to \sim q$ ×

$\sim r \Leftrightarrow \sim p \to q$

b) The negation of the right-hand side of the above statement is equivalent to:

$\sim (\sim p \to q) \Rightarrow \sim p \wedge \sim q$

Question 3 Solutions

For the production of a local play, 19 people auditioned and 7 of them joined the cast. Of these, 5 had named roles.

In how many ways could this have happened?

Auditioning done by $19 \to 7$, $\therefore$

$$\binom{19}{7}.$$

5 "named roles" $\therefore$ ${}^{7}P_{5}$.

$\therefore$ Final answer: $\binom{19}{7}\,{}^{7}P_{5}$

In how many ways can 5 boys and 6 girls be arranged in a line so that the boys and girls are in separate groups?

(BBBBB)(GGGGGG)

$2 \times 5! \times 6!$ ($5!$ & $6!$ different ways to order the groups).

How many 4-digit numbers have the property that the first and last digits are different, and all the digits are odd?

0 1 2 3 4 5 6 7 8 9

5×4, first & last digit.

5^2 (two other digits).

$\therefore 5^3 \times 4$.

b)

In how many ways can 4 boys and 8 girls be arranged in a line so that two particular students are not next to each other?

Consider case when they ARE together:

(P_1P_2)RRRRRRRRRR

$= 11! \times 2$

$\therefore |U| - |\text{Together}| = |\sim\text{Together}|$

$= 12! - 11! \times 2$

How many 5-digit numbers have the property that the first and last digits are different, and all the digits are even?

0 1 2 3 4 5 6 7 8 9

$4 \times 4 \times 5^3$ (first term can't be zero).

In how many ways can 7 boys and 7 girls be arranged in a line so that the boys and girls are alternating?

(BG)(BG)(BG)(BG)(BG)(BG)(BG)

Line up 7 boys: 7!

Line up 7 girls: 7!

Either boy or girls first, ∴

$2 \times (7!)^2$

In how many ways can 6 boys and 7 girls be arranged in a circle so that two particular students are next to each other?

(BG)PPPPPPPPPPP

$(n-1)!$

12 group.

11! ways in a circle.

Two ways to change BG group.

∴ $2 \times 11!$

In how many ways can 6 boys and 3 girls be arranged in a circle so that the boys and girls are in separate groups?

(BBBBBB)(GGG)

$= 1! \times 6! \times 3!$

One way on circle; 6 ways to order boys, 3 ways to order girls.

In how many ways can 8 boys and 4 girls be arranged in a circle so that two particular students are not next to each other?

Consider the two students sitting together.

$(P_1P_2)RRRRRRRRRR$

$= (10!)\times 2.$

Deduct from universal case:

$11! - 2\times 10!$

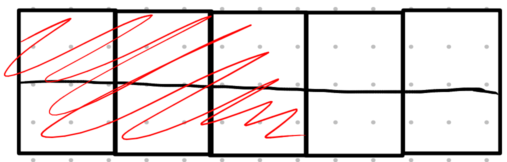
Question 4.

A firm works for the same 5 days each week.

Every employee must work exactly 3 full days and 2 half-days each week.

A half-day can be either morning or afternoon, and two half-days cannot be held on the same day.

a) How many possible different weekly schedules are there?



Choose 3 full days to work.

$^5C_3$

Now, four different ways to choose half days.

(mm, mn, nn, nm).

$\therefore$ $^5C_3 \times 4 = 210$.

b) If the firm has 186 employees, how many people must have the same work schedule for a particular week?

By PHP, $\text{ceil}\left(\frac{186}{40}\right) = 5$

c) What is the smallest number of employees needed to guarantee at least 4 workers have exactly the same schedule?

Find smallest number of employees x : $\frac{x}{40} > 3$ $\underline{x = 121}$

Another example:

A firm works for the same 6 days each week.

Every employee must work exactly 2 full days and 4 half-days each week.

A half-day can be either morning or afternoon, and two half-days cannot be held on the same day.

a) How many possible different weekly schedules are there?

To find the general formula:

$\binom{n}{n-k} 2^k$. k: half days, n: days

$\binom{n}{n-k}$ = full days

2^k = two sessions (morning / arvo).

2 gives a "decision tree" type division of "choices".

$\therefore \binom{6}{2} 2^4 = 240$

Question 5.

a) A 12-sided die has each of the integers 1 through to 12 written on its faces.

If 7 different people each roll their own 12-sided die, in how many ways can exactly 5 people roll a 9?

$$\binom{7}{5} \times 11^2$$

$7C5 \times 1 \Rightarrow$ Choose 5 players out of 7 to receive a 9.

$11^2 \Rightarrow$ other players receive any other value.

b) Suppose 30 different people each roll their own 10-sided die.

In how many ways can exactly 6 people roll a 6, exactly 7 people roll a 7, …, and exactly 9 people roll a 9?

$$\binom{30}{6}\binom{24}{7}\binom{17}{8}$$

6 people out of 30

$\Rightarrow$ 7 people out of 24

$\Rightarrow$ 17 people out of 8.

c) How many 6-digit numbers less than 361146 do not contain any digits greater than 7?

First digit:

0, 1, 2.

Remaining Digit:

0, 1, 2, 3, 4, 5, 6, 7.

$\therefore\ 2 \times 8^5 + 6 \times 8^4 + 1 \times 8^3 + 1 \times 8^2 + 4 \times 8^1 + 6 \times 8^0.$

How many 5-digit numbers less than 43784 do not contain any digits greater than 8?

First digit:

1, 2, 3

cannot be zero $\nearrow$ $3 \times 9^4 + \underbrace{3 \times 9^3}_{\text{if equal}} + \overset{\text{if equal}}{7 \times 9^2} + 8 \times 9 + 4 \times 9^0$

How many 6-digit numbers less than 562244 do not contain any digits greater than 7?

First Digit:

1, 2, 3, 4

$\therefore\ 4 \times 8^5 + 6 \times 8^4 + 2 \times 8^3 + 2 \times 8^2 + 4 \times 8^1 + 4$

Question 6.

Consider the equation

$$x_1 + x_2 + \cdots + x_7 = 82,$$

where $x_1, x_2, \ldots, x_7 \in \mathbb{N}$.

How many solutions are there if:

a) $x_i \leq 13$ for all $1 \leq i \leq 7$?

Consider $x_i = 13 - y_i$

$\therefore 13 - y_1 + 13 - y_2 + \ldots + 13 - y_7 = 82$

$\Rightarrow -(y_1 + y_2 + \ldots + y_7) = -9$

$y_1 + y_2 + y_3 + \ldots + y_7 = 9$, $0 \leq y_i \leq 13$.

↑ This condition is satisfied by $= 9$.

$\therefore \binom{15}{7}$.

b) $x_i \leq 22$ for all $1 \leq i \leq 7$?

Following similarly.

$x_i = 22 - y_i$

$y_1 + \ldots + y_7 = 67$

where $y_i \leq 22$.

This is NOT satisfied.

$\therefore$ Consider $x_1 \geq 23$

$x_1 + x_2 + x_3 + \ldots + x_7 = 59$

$= \binom{65}{6}$

Consider $x_1, x_2 \geq 23$

$x_1 + x_2 + \ldots + x_7 = 36$

$= \binom{42}{6}$

Consider $x_1, x_2, x_3 \geq 23$

$x_1 + x_2 + \ldots + x_7 = 13$

$= \binom{19}{6}$

x_4 not possible.

$\therefore$ Consider:

$|U| - |S_1 \cup S_2 \cup \ldots \cup S_7|$.

$$\binom{88}{6} - \binom{7}{1}\binom{65}{6} + \binom{7}{2}\binom{42}{6} - \binom{7}{3}\binom{19}{6}$$

c) $x_1 \geq 11$, and

$x_i \equiv i \pmod{6}$ for all $1 \leq i \leq 7$?

Thanks to Jeff. I forgot the mod conversion thing. Ooups.

Remember that

$$x_i \equiv i \pmod{6} \iff x = 6a + i, \quad a \geq 0.$$

$\therefore$ Combining both restrictions, we get:

$$x_1 = 6a_1 + 1 + 11$$
$$x_2 = 6a_2 + 2$$
$$x_3 = 6a_3 + 3$$
$$\vdots$$
$$x_7 = 6a_7 + 7$$

$$\therefore 6(a_1 + a_2 + \dots + a_7) + 39 = 82$$
$$6(a_1 + a_2 + \dots + a_7) = 43$$
$$a_1 + a_2 + a_3 + \dots + a_7 = 7$$
$$\binom{13}{6}$$

Note for me (and maybe for you):

$$x \equiv b \pmod{c}$$

$$\Rightarrow x = cd + b \quad \text{where } b \in \mathbb{Z}.$$

Question 7.

a)

Find the solution to the recurrence relation

$a_n = 14a_{n-1} - 45a_{n-2}$ for all $n \geq 2$

which satisfies the initial conditions $a_0 = 11$ and $a_1 = 83$.

$a_n - 14a_{n-1} + 45a_{n-2} = 0$

$r^2 - 14r + 45 = 0$

$\begin{array}{ccc} r & & -9 \\ & \times & \\ r & & -5 \end{array}$ $\quad (r-9)(r-5) = 0$

$r = 9, 5.$

$\therefore a_n = A(9)^n + B(5)^n$

$11 = A + B \Rightarrow B = 11 - A$

$83 = 9A + 5B \Rightarrow 83 = 9A + 5(11 - A)$

$\Rightarrow 83 = 9A + 55 - 5A$

$\Rightarrow 83 = 4A + 55$

$\Rightarrow 4A = 28$

$A = 7 \quad B = 4 \qquad \therefore a_n = 7(9)^n + 4(5)^n$

b)

Find the general solution to the recurrence relation

$b_n = 7b_{n-1} + 8b_{n-2} - 14n + 37$ for all $n \geq 2$

which satisfies the initial conditions $b_0 = -5$ and $b_1 = 13$.

hs:

$b_n - 7b_{n-1} - 8b_{n-2} = 0$

$r^2 - 7r - 8 = 0$

$\begin{array}{ccc} r & & -8 \\ & \times & \\ r & & 1 \end{array}$ $\quad (r-8)(r+1) = 0$

$r = 8, -1$

$an + b$

$b_n = A(8)^n + B(-1)^n + ps.$

$ps = an + b - 7(a(n-1) + b) - 8(a(n-2) + b) = -14n + 37$

$\Rightarrow an + b - 7an + 7a - 7b - 8an + 16a - 8b = -14n + 37$

$\Rightarrow -14an + 23a - 14b = -14n + 37$

$\Rightarrow a = 1$

$23 - 14b = 37$

$-14b = 14$

$b = -1$ $\quad \therefore p_s = n - 1$

$b_n = A(3)^n + B(-1)^n + n - 1$

$-5 = A + B - 1$

$A + B = -4 \Rightarrow B = -4 - A$

$13 = 3A - B$

$13 = 3A + 4 + A$

$9A = 9 \quad \underline{A = 1.} \quad \underline{B = -5}$

$b_n = 3^n - 5(-1)^n + n - 1$

c)

Find the general solution to the recurrence relation

$c_n = 9c_{n-1} - 20c_{n-2} + 2 \times 4^n$ for all $n \geq 2$

which satisfies the initial conditions $c_0 = 2$ and $c_1 = -15$.

h_s:

$c_n - 9c_{n-1} + 20c_{n-2} = 0$

$r^2 - 9r + 20 = 0$

$(r-5)(r-4) = 0$

$r = 5, 4$

$c_n = A(5)^n + B(4)^n + p_s$.

$p_s = cn4^n$, as 4^n is alr. sol.

$cn4^n - 9c(n-1)4^{n-1} + 20c(n-2)4^{n-2}$

$= cn4^n - 9cn4^{n-1} + 9c4^{n-1} + 20cn4^{n-2} - 40c4^{n-2}$

$= 9c4^{n-1} - 40c4^{n-2} = 2 \times 4^n$

$c = -8$

$$c_n = A(5)^n + B(4)^n - 8n(4)^n$$

$$2 = A + B$$

$$-15 = 5A + 4B - 32$$

$$B = 2 - A$$

$$5A + 4B = 17$$

$$5A + 8 - 4A = 17$$

$$A = 9 \qquad B = -7.$$

$$\therefore \; c_n = 9(5)^n - 7(4)^n - 8n(4)^n$$

Note:

Be careful with Ps. Make sure your solution is unique.