2021 T1

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Most of these solutions have been derived entirely by me, so
there may be imperfections and wrong answers/reasonings. Please
contact me at my student email for any corrections.

These solutions were made to aid the preparation of MATH1231
students, not replace it. Good luck :)
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Q2)

```
a) {v1, v2, v3} is a linearly independent set
S: We cannot know if this is true; what if v3 is a scalar
multiple?

b) {v1, v2, v3} spans R^4
S: For a set of basis vectors that span R^n, we need n vectors.

c) dim(S) = 2
S: We have zero clue what v3 is, so we are not sure if it is
exactly 2.

d) dim(S) = 3
S: Same as above

e) dim(S) >= 2
S: We know this is true. We already have two lin. ind. vectors,
so it is either 2 or greater than 2.

f) dim(S) < 2
S: Definitely false, we already have two lin. ind. vectors
```

```
g) there exists a vector v4 such that {v1, v2, v3, v4} is a
basis for R^4
S: Impossible to know without knowing what v3 is.

h) there exists vectors v4, v5 such that {v1, v2, v3, v4, v5}
spans R^4
S: Definitely true, as we already have enough vectors (4)
excluding v3 to make a basis.

i) one of the vectors v1, v2, v3 is a linear combination of the
other two
S: We don't know what v3 is

j) R^4 has a basis including at least two of the vectors v1, v2,
v3:
S: Definitely true - we have v1, v2 already and they are
linearly independent, so we can construct a basis out of them.
```

Q3)

a)

For a matrix to be diagonalisable, the algebraic multiplicities must $=$ the geometric multiplicities. Hence, we must reduce the algebraic multiplicities to below 2.

Hence, find the characteristic polynomial:

$$\begin{pmatrix} 82 & x \\ -61 & -73 \end{pmatrix}$$

Our goal is to turn the characteristic polynomial into the form $(x-c)^2$.

$$\lambda^2 - 9\lambda + 61x - 5986$$

Therefore, our equation will look like $(\lambda - \frac{9}{2})^2$, and hence, we must make $61x - 5986 = \frac{81}{4}$.

b) Look above for reasoning

Q4)

a) You can construct the matrix using matrix representation theorem for linear maps.

$$\frac{dy_1}{dt} = y_1(-k_1 - k_2) + k_2 y_2 \quad \frac{dy_2}{dt} = k_2 y_1 + y_2(-k_2 - k_3)$$

This can be then represented trivially as:

$$\begin{pmatrix} (-k_1 - k_2) & k_2 \\ k_2 & (-k_2 - k_3) \end{pmatrix}$$

b)
Input the values of k_1, k_2, k_3 given to create a matrix and use:

```
Eigenvectors(A)
and
Eigenvalues(A)
```

to solve the question

c)
To solve a differential equations from a linear first-order differential equation, we create a matrix with the differential equations and then find eigenvalues λ_1, λ_2 and eigenvectors $\vec{a}, \vec{b}$:

$$solution = e^{\lambda_1 t}\vec{a} + e^{\lambda_2 t}\vec{b}$$

Therefore, our solution is:

```
C1*exp(-0.02*t)*<0.707106781186548, 0.707106781186547> +
C2*exp(-0.32*t)*<-0.707106781186547, 0.707106781186548>
```

d)
With the initial conditions of 7 degrees in the storage area, and 26 degrees in the living area, create a matrix equation and use linear solve:

```
M := <<0.707106781186548, 0.707106781186547>|
<-0.707106781186547, 0.707106781186548>|<7, 26>>
LinearSolve(M)
Output: <23.3345237791561, 13.4350288425444>
```

Q5)

a) Consider if the zero element exists within T(p).

$T(0) = (x^4 + 2x + 6)\left(\frac{d(0)}{dx}\right) = 0$ therefore, the zero element exists.

$\vec{a}$ and $\vec{b}$ are two arbitrary vectors in the space $\mathbb{P}$.

Now consider:

$$\begin{aligned}
T(\vec{a}) + T(\vec{b}) &= (x^4 + 2x + 6)\left(\frac{da}{dx}\right) + (x^4 + 2x + 6)\left(\frac{db}{dx}\right) \\
T(\vec{a}) + T(\vec{b}) &= \frac{d(a+b)}{dx}(x^4 + 2x + 6) \\
&= T(a+b) \\
T(\vec{\lambda} a) &= \lambda(x^4 + 2x + 6)\left(\frac{da}{dx}\right) \\
&= \lambda T(\vec{a})
\end{aligned}$$

Hence, T(x) is a linear map.

b) For $T(x) = 0$, p must equal a constant as $q(x)$ is a non-zero polynomial for all p. So choose all the constant values.

c) The only choice that is not valid within the choices is $(x^4 + 2x + 6) + 99$. This can be proven by:

$$\begin{aligned}
T(p) &= q(x) + 99 \\
q(x)\left(\frac{dp}{dx}\right) &= q(x) + 99 \\
q(x)\left(\frac{dp}{dx} - 1\right) &= 99
\end{aligned}$$

Therefore, this is impossible. The degree of the LHS is at minimum 4. The RHS has a degree of zero.

d) This can be trivially proven

Let $\vec{a}$ and $\vec{b}$ be two arbitrary vectors in polynomial space $\mathbb{P}$.

The zero element trivially exists.

Consider closure under addition:

$$\begin{aligned} T(\vec{a}) + T(\vec{b}) &= a^2\left(\frac{da}{dx}\right) + b^2\left(\frac{db}{dx}\right) \\ &= \frac{d(a+b)}{dx}(a^2+b^2) \\ &\neq \frac{d(a+b)}{dx}(a+b)^2 \end{aligned}$$

Therefore, not closed under addition, and hence, not linear.

Q6)

Consider that $E(X) = nP(X)$.

For a probability density function, we must consider the net total probability.

Therefore, $\frac{a\cos x + b\sin x}{x^2}$ becomes $\frac{a\cos x + b\sin x}{x}$.

Sticking this in maple, we learn that:

$0.0855769059a + 0.65932990066b = 1.3$

Also, consider that the total probability of the CDF must $= 1$;

therefore:

$0.0890458176a + 0.4723991777b = 1$

Solving this as a system of linear equations, using `LinearSolve()`, we achieve the answer.

Note: The formula for the expected value given a probability density function $f_x(X)$ is:

$$E(X) = \int_{-\infty}^{\infty} x f_x(X)\, dx$$

b) See above

c) Variance can be found using $E(X^2) - E(X)^2$

d) Use z score and the formula $z = \frac{x-\mu}{\sigma}$

Q7)

a) The total differential approximation for a point differs from the *change of a function.*

The formula here used is:

$$f(x, y) \sim f(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

b) This is fairly trivial; realise that this formula above is the same as the formula we use to find the tangent plane at point x_0, y_0.

Q8)

a)

$\cosh^2(7x) - \sinh^2(7x) = 1$

$\sinh^2(7x) = \cosh^2(7x) - 1$

Therefore, the denominator can be simplified to:

$$3\cosh^3(7x) - 3\cosh(7x))$$

b) Sub in 0 and 7 into $\cosh(7x) = t$ for the values of a and b.

Integrate by substitution by using the sub $dx = \frac{1}{7\sinh(x)}dt$

$P(t) = 21(t^3 - t)$

c) $P(t)$ can be factorised into $t(t^2 - 1)$ which can then be exapnded into $t(t-1)(t+1)$. Therefore, we expect 3 terms.

Q9)

Consider the options:

```
The series diverges for every x > 2
S: This is not true - it actually converges for every x > 2.

The series diverges for multiple values of x
S: This is definitely true.

The series converges for at most one value of x.
S: This is not true; refer to the solution of the first option.
```

```
The series converges for at least one value of x
S: This is definitely true, as we know x converges such that x >
2

The series converges for every x
S: This is not true
```

Q10)

a)

```
f(0) = 1/7 (by inspection)
f'(0) = 1/14 (since d/dx = 1)
f''(0) = 1/14 (sinde d/dx = 2, 1/28 / 2)
f'''(0) = 6/56 (sinde d/dx = 6, 1/ 56 / 6)
```

b)

$$a_n = \frac{1}{7 \cdot 2^n}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{7 \cdot 2^n} x^n \\
&= \lim_{n\to\infty} \frac{x^{n+1}}{7 \cdot 2^{n+1}} \frac{7 \cdot 2^n}{x^n} \\
&= \frac{|x|}{2}
\end{aligned}$$

Therefore, the radius of converge is 2.

c)

Consider that the series is contained by:

$$\sum_{n=0}^{\infty} \frac{1}{7 \cdot 2^n} x^n$$

Using this, and the GP formula:

$$
\begin{aligned}
&= \frac{1}{7 \cdot 2^n} \cdot \frac{1}{4^n} \\
&= \frac{1}{7} \sum \frac{1}{8^n} \\
&= \frac{1}{7} \left(\frac{1}{1 - \frac{1}{8}} \right) \\
&= \frac{8}{49}
\end{aligned}
$$

d) Repeat above steps

Q11)

a) Note that for a question to be a first-order linear ODE, it mustbe in the form:

$$
\frac{dy}{dx} + g(x)y = f(x)
$$

Therefore, to find the integrating factor, we must divide $\sqrt{x^2 - 25}$ out of the derivative (or whatever your questions multiplying factor is).

b) Standard theory - check out the notes if your confused.

c) Just do this in maple:

```
int(x^3/(sqrt(x^2-25)), x)
g := x -> ((x - 5)*(x + 5)*(x^2 + 50))/(3*sqrt(x^2 - 25))
h := x -> exp(g(x))
h(sqrt(26))
Output: exp(76*(sqrt(26) - 5)*(sqrt(26) + 5)/3)
```

Q12)

a)

Let's consider the options

```
1. The differential equation is exact
This is false - it's not in the right form
2. The different equation is of second order
```

```
True; we have y''
3. This is a partial differential equation
False: There are no partial derivatives within this equation
4. The differential equation is of first order
False; see 2
5. The equation is homogenous linear
False; it does not = 0, therefore not homogenous
6. This is a linear ODE
True; linear. (There are no powers on the derivatives)
```

b) Since the $ODE = \frac{x^3}{3}$, the solution should be representative of the polynomial of order 3, which is $ax^3 + bx^2 + cx + d$.

Q13)

a) Find the homogenous equation:
$\lambda^2 + a\lambda + 6$
Consider that the homogenous equation oscillates when it passes the x-axis; therefore, we wish the discriminant > 0. (As when the homogenous equation is floating, $\sin(x)$ and $\cos(x)$ will occur).
Therefore:
$a^2 - 24 > 0$
$a > \sqrt{24}$

b) Just means finds a particular solution.
The maple command for this is:

```
dsolve(diff(y(x), x, x) + 12*diff(y(x), x) + 6*y(x) = 15*cos(x))
Output:
y(x) = exp((-6 + sqrt(30))*x)*_C2 + exp(-(6 + sqrt(30))*x)*_C1 +
(180*sin(x))/169 + (75*cos(x))/169
```

And we take `(180*sin(x))/169 + (75*cos(x))/169` as the particular solution.