MATH1081 2022 T1

Question 2.

(a) The first partial truth table looks like this:

p	q	X
T	T	T
T	F	
F	T	
F	F	T

Select all options below that could represent the compound proposition X.

- [ ] $\sim p \wedge q$
- [x] $p \leftrightarrow q$
- [x] $p \vee \sim q$
- [ ] $p \wedge \sim q$

$T \Leftrightarrow T \mid F \Leftrightarrow T$

$T \vee F \mid F \vee T$

b)

p	q	r	Y_1	Y_2
T	T	T	1	2
T	T	F	3	F
T	F	T	4	T
T	F	F	5	F
F	T	T	6	F
F	T	F	T	7
F	F	T	8	9
F	F	F	10	11

$2^{11} \Rightarrow$ number of ways

$2^3 \Rightarrow$ Logically equivalent patterns for three empty rows.

$\underline{2^{11} - 2^3}$

c)

$F \Rightarrow F$

$T \Rightarrow T$ Three different if-then statements

$F \Rightarrow T$

p	q	r	Z_1	Z_2
T	T	T		
T	T	F		
T	F	T		
T	F	F	1	F
F	T	T	F	2
F	T	F	2	T
F	F	T	2	T
F	F	F	2	T

Three empty rows: 3^3

One rows where result is determined: 1

Four rows, where there are 2 valid results: 2^4

$\underline{3^3 \times 2^4}$

Question 3.

Consider the sets $S = \{1,2,3,4,5\}$ and $A \subseteq S \times S$ given by
$$A = \{(1,3),(2,1),(2,2),(2,4),(3,2),(3,5),(4,3),(4,5),(5,2),(5,5)\}.$$

(a) Calculate the number of sets $B \subseteq S \times S$ such that $A \subseteq B$ and B is a reflexive relation.

2^{12}

(b) Calculate the number of sets $B \subseteq S \times S$ such that $A \subseteq B$ and B is a symmetric relation.

2^{5}

(c) Calculate the number of sets $B \subseteq S \times S$ such that $A \subseteq B$ and B is an antisymmetric relation.

$2^{3} \times 3^{2}$

(d) Suppose $B \subseteq S \times S$ is a transitive relation such that $A \subseteq B$. What is the smallest possible cardinality of B?

25

a) 25 sets in S (5×5)

10 mandatory elements $\in A$.

To make reflexive, need to include $\{1,1\}, \{3,3\}, \{4,4\}$.

$\therefore$ 13 mandatory elements $\in A$.

Hence, these 13 mandatory element $\in B$.

2^{12}.

b) To be symmetric:

$$A = \{(1,3),(2,1),(2,2),(2,4),(3,2),(3,5),(4,3),(4,5),(5,2),(5,5)\}.$$

$(3,1), (1,2), (4,2), (2,3), (5,3), (3,4), (5,4), (2,5)$

$8 + 10 = 18$

$25 - 18 = 7$.

7 remain – 3 reflexive. 4 verticies

↑

But must be chosen in pairs, $\therefore$ 2 pairs.

c)
$$A = \{(1,3),(2,1),(2,2),(2,4),(3,2),(3,5),(4,3),(4,5),(5,2),(5,5)\}.$$

Anti-symmetry can be understood as "only one-way street exists".

Definitionally, A is already anti-symmetric.

3 reflexive definitions remain, and all of them can be included, (or excluded) hence 2^3.

There are 8 un-paired elements.

In the 12 elements that remain, choosing any of these would lead to symmetry.

Hence, 4 elements remain.

As soon as you choose 1 element, the other pair becomes mutexe. Hence, each branch has 3 logical children.

With two pairs of element, this becomes 3^2.

d)

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{pmatrix}
\cancel{0} & \cancel{0} & 1 & \cancel{0} & \cancel{0} \\
1 & 1 & \cancel{0} & 1 & \cancel{0} \\
\cancel{0} & 1 & \cancel{0} & \cancel{0} & 1 \\
\cancel{0} & \cancel{0} & 1 & \cancel{0} & 1 \\
\cancel{0} & 1 & \cancel{0} & \cancel{0} & 1
\end{pmatrix}
\end{array}
$$

15 new elements need.

Steps:

1. Add transitive element.
   ↳ Need to check that new destination for transitivity.
2. Mark off transitivity.

Question 4.

(For this question, recall that $\oplus$ is the exclusive disjunction ("xor") logical operator.)

Suppose p and q are simple propositions, and consider the set

$$S = \{p,\ q,\ p \wedge q,\ p \vee q,\ p \wedge {\sim} q,\ {\sim} p \oplus q,\ {\sim} p,\ {\sim} p \leftrightarrow q\}.$$

Let $R \subseteq S \times S$ be the relation given by

$$xRy \text{ if and only if } x \Rightarrow y.$$

(a) Prove that R is a partial order on S.

```
Reflexivity:
x R x implies x ⇒ x. This is trivially maintained, as any proposition can be
a conclusion of it self (for examples, p ⇒ p is true).

Anti-symmetric:
if xRy and yRx then x = x
Consider two arbitrary logical propositions x and y.

x ⇒ y and y ⇒ x are both valid propositions.
Since we are given that no two statements are logically equivalent, it must then
also be true that x = y. (Or the converse would not be true).

Transivity:
x ⇒ y and y ⇒ z imply x ⇒ z by virtue of transitive logic.
```

(b) Select all the ordered pairs below that are elements of R.
(Be sure to select all options that apply.)

1 ☒ $(p \wedge q,\ \sim p \oplus q)$
2 ☐ $(\sim p \oplus q,\ p \vee q)$
3 ☐ $(p \wedge \sim q,\ p \vee q)$
4 ☐ $(\sim p \oplus q,\ p \wedge \sim q)$

(b) Select all the ordered pairs below that are elements of R.
(Be sure to select all options that apply.)

- [x] $(p \wedge q,\ \sim p \oplus q)$
- [ ] $(\sim p \oplus q,\ p \vee q)$
- [x] $(p \wedge \sim q,\ p \vee q)$
- [ ] $(\sim p \oplus q,\ p \wedge \sim q)$

1. $p \wedge q \Rightarrow \sim p \oplus q$.

$p = T$
$q = T$

$\sim p \vee q \wedge p \vee \sim q$.

$T \wedge T \Rightarrow T$.

2. $\sim p \vee q \wedge p \vee \sim q \Rightarrow p \vee q$.

Case $F \Rightarrow F$: √

Case $F \Rightarrow T$: ×

Hence $\sim p \vee q$ or $p \vee \sim q$ is false.

Case $T \Rightarrow T$: ×

This must be $F \vee F$.
Hence both cannot be false at the same time.

3. $p \wedge \sim q \Rightarrow p \vee q$.

$F \Rightarrow T$

p is T q is true $\Rightarrow$ $p \vee q$ is true
p is false q is false $\Rightarrow$ $p \vee q$ is false

$F \Rightarrow F$

$T \Rightarrow T$.

(c) Select the statements below that are true.
(Be sure to select all options that apply.)

- [ ] $p \wedge q$ is an upper bound of p and q under R.
- [ ] $p \wedge q$ is the least upper bound of p and q under R.
- [x] $p \wedge q$ is a lower bound of p and q under R.
- [x] $p \wedge q$ is the greatest lower bound of p and q under R.

$p \wedge q \Rightarrow$ p is T and q is T.

∴ Is a lower bound, and is also the GLB, as it has the most fundamental definition $p \wedge q \Rightarrow p, q$.

(d) The poset (S, R) does not have a least element. Select each option below that would be a least element of (S, R) if it were added to the set S.
(Be sure to select all options that apply.)

- [x] $(p \leftrightarrow q) \wedge (p \oplus q)$
- [ ] $p \vee \sim p$
- [ ] $(p \leftrightarrow q) \vee (p \oplus q)$
- [x] $p \wedge \sim p$

Anything fundamentally true or false would fit here.
(As it can then stem to everything).

Question 5.

Chelsea owes William some money, but neither of them can remember how much! Let $\$x$ be the amount of money owed, where x is some positive integer.

Chelsea remembers that if they wanted to buy some number of $\$52$ items, spending 40 times the amount of money owed would return $\$28$ in change. That is,

$$40x \equiv 28 \pmod{52}.$$

William remembers that if they wanted to buy some number of $\$25$ items, spending 15 times the amount of money owed would return $\$20$ in change. That is,

$$15x \equiv 20 \pmod{25}.$$

(a) Find a single value of x that solves both congruences.

28

$40x \equiv 28 \pmod{52}$

$52x + 40y = 1$

$52 = 40 + 12$
$40 = 12 \times 3 + 4$

$4 = 40 - 12 \times 3$
$4 = 40 - 3 \times (52 - 40)$
$4 = 40 - 3 \times 52 + 3 \times 40$
$= 4 \times 40 - 3 \times 52$
$= 28 \times 40 - 21 \times 52$

$\underline{2, 15, 28, 41}$

$x = 2 + 13a$

(b) Chelsea says, "I found a value of x that solves both congruences, but it took a terribly long time using trial and error." William declares, "I was able to find a value of x without using any trial and error!" Briefly describe how William might have found their answer to part (a).

Solve one of the congruences, and then plug in solutions until both congruences are true (as I have done above).

(c) Suppose x and y are both solutions to the pair of congruences. What is the smallest possible positive value of $x - y$?

$25 = 15 + 10$
$15 = 10 + 5$

$5 = 15 - 10$
$5 = 15 - 25 + 15$
$5 = 15 \times 2 - 25$

$20 = 8 \times 15 - 4 \times 25$

$3, 8, 13, 18, 23$

$y = 3 + 5b$

$13a + 5b = 28$

$13 = 5 \times 2 + 3$

$5 = 3 + 2$

$3 = 2 + 1$

$1 = 3 - 2$

$1 = 3 - 5 + 3$

$= 2 \times (13 - 5 \times 2) - 5$

$1 = 2 \times 13 - 5 \times 5.$

$z = 2 + 5c$

$t = -5 + 13d$

$t = 8 + 13d$

$2 + 5 = 7$. (Next solution after 2?)

$\therefore$ Smallest difference is:

Solution − Solution before.

Question 6.

Jordan and Sean write down their favourite finite nonempty subsets of $\mathbb{N}$ and label them S and T respectively. They also choose a set $A \subseteq S \times T$.

Recall that we write $P(T)$ for the power set of T. Jordan defines a function $f : S \to P(T)$ by

$$f(x) = \{t \in T \mid (x, t) \in A\} \text{ for all } x \in S.$$

(a) Sean declares that the relation A is a function. What does this tell Jordan about $f(S)$?

```
If A is a function, then for every input x in S, it must output a single value y in T.

Hence for each input value x in S, the cardinality of f must = 1, as there
must be only one output value in the ordered pairs of A.
```

(b) Jordan reveals that $\{y\} \notin f(S)$ for some $y \in T$. What does this tell Sean about the function A?

(Select all the correct statements below.)

- [ ] A must be onto.
- [x] A cannot be onto.
- [ ] There is not enough information to determine whether A is onto.
- [ ] A must be one-to-one.
- [ ] A cannot be one-to-one.
- [x] There is not enough information to determine whether A is one-to-one.

```
Definitely can no longer be surjective, as they have removed a required
output for surjectivity. We cannot determine injectivity based alone
from the removal of a possible output, as the other values may still map to
unique outputs that aren't y.
```

(c) Sean chooses a set $B \subseteq S \times S$ and tells Jordan that the relation B is both a function and an equivalence relation. Jordan says, "That means there's only one possible choice for B."

Is Jordan correct?

Yes

Briefly explain.

If B is a equivalence relation, then B must be symmetric, reflexive and transitive.

Since B is a function, it also must produce a single output for a single input.
With this in mind, transitivity only works iff reflexive.

Imagine if it were to be transitive, as we allowed non-reflexive relations,

then x R y and y R z ⇒ x R z, which means that both (x, y) and (x, z). This is not a function.

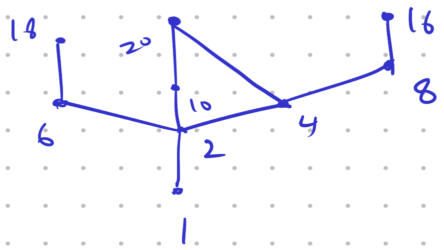
Hence, B is simply the set of reflexive relations from S → S.

Note for self: Should be remembering the injectivity/surjectivity rules based on cardinality of the sets.

Question 7.

Let S be the set $\{1, 2, 4, 6, 8, 10, 16, 18, 20\}$. The divisibility relation $|$ is a partial order on S.

Let H be the undirected graph defined by the Hasse diagram for the poset $(S, |)$.

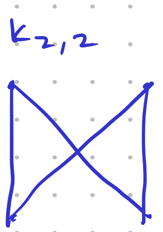


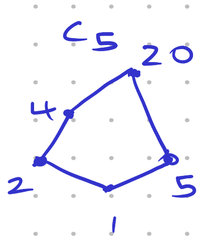
(a) How many edges does H have?

9

(b) What is the degree of the vertex labelled 2 in H?

4



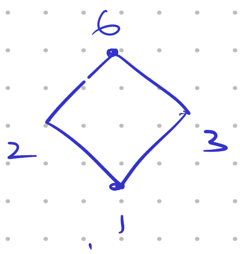


(c) Find some $A \subseteq \mathbb{Z}^+$ such that the Hasse diagram for the poset $(A, |)$ is isomorphic to $K_{2,2}$.

$A =$ {1, 2, 3, 6}

(d) Find some $B \subseteq \mathbb{Z}^+$ such that the Hasse diagram for the poset $(B, |)$ is isomorphic to C_5.

$B =$ {1, 2, 4, 5, 20}



(e) Explain why there is no $C \subseteq \mathbb{Z}^+$ such that the Hasse diagram for the poset $(C, |)$ contains a circuit of length 3.

Suppose for sake of contradiction that there is indeed a possible circuit length of 3.
Consider the circuit a → b → c → a. By definition of the poset, this means that:
b = ak
c = bk
a = ck

Building upwards we get c = ak^2, and then a = ak^3. However, this cannot be true, as k^3 can be any integer.

Question 8.

The *digit sum* of a natural number is the sum of its component digits. For example, the digit sum of 70 is $7+0=7$, and the digit sum of 77 is $7+7=14$.

A 4-digit number is any integer n such that $1000 \le n \le 9999$.

Syntax advice:

- To enter $C(n,k)$, type either `C(n,k)` or `comb(n,k)`.
- To enter $P(n,k)$, type either `P(n,k)` or `perm(n,k)`.
- To enter n^k, type `n^k`.
- *Always* use `*` for multiplication. For example, $2x$ should be entered as `2*x`.

(a) How many 4-digit numbers have a digit sum divisible by 3?

(b) How many 4-digit numbers have a digit sum equal to 35?

(c) How many 4-digit numbers have a digit sum equal to 21?

a) A number is divisible by 3 iff it's digit sum is.

$\therefore \frac{9999}{3}$ (find number divisible by 3).

$\left\lfloor \frac{1000}{3} \right\rfloor$ (remove lower end).

b) $a+b+c+d=35$.

$9-a+9-b+9-c+9-d=35$

$a+b+c+d=1 \quad = \underline{\binom{4}{3}}$

This guarantees atleast one of a,b,c,d are ≥ 1, $\therefore$ no adjustment needed.

c) $a+b+c+d=21$

$9-a+9-b+9-c+9-d=21$

$a+b+c+d=15. \quad \le 9$ not satisfied.

$\therefore$ consider ≥ 10.

$a+b+c+d=5. \quad = \binom{8}{3}$

$\therefore$

$|S_1 \cup S_2 \cup S_3 \cup S_4| = |S_1|+|S_2|+|S_3|+|S_4| - (\text{intersections})$.

$= \binom{18}{3} - \binom{8}{3}$

But we haven't satisfied $a \ge 1$, $\therefore$ consider $a=0$.

$= 9-b+9-c+9-d=21 \quad b+c+d=6 \quad \therefore \binom{8}{2}$

Hence: $\binom{18}{3} - \binom{8}{3} - \binom{8}{2}$

d)

(d) How many 4-digit numbers have a digit sum divisible by 7?
(Hint: Since this question is multiple-choice, you should not need to do any complicated calculations.)

- ○ 1082
- ○ 1182
- ○ 1382 ✔
- ◉ 1282

Honestly not entirely sure how to do this, just choose the closest number to

$$\left\lfloor \frac{9999}{7} \right\rfloor - \left\lfloor \frac{1000}{7} \right\rfloor$$

e)

(e) How many 6-digit numbers have a digit sum divisible by 9 or 4?
(Hint: For this question part, you may write **a** *to represent the correct answer to part (a),* **b** *to represent the correct answer to part (b), and similarly for* **c** *and* **d***. Note in particular that lowercase* **c** *represents the correct answer to part (c), while uppercase* **C** *can be used for the number of combinations function.)*

a + d - c ✔

Fairly simple:

$$|9 \cup 4| = |\underset{d}{9}| + |\underset{9}{4}| - |9 \cap 4|.$$

$|9 \cap 4| = |36|$. ← c.

a+d−c.

a+b+c+d+e+f

f) 4 pigeonholes.

You want one pigeonhole to have 17 at a minimum.

In the worst case scenario, you have 16 pigeons, which distribute themselves evenly across 4 pigeonboxes.

∴ 4×16.

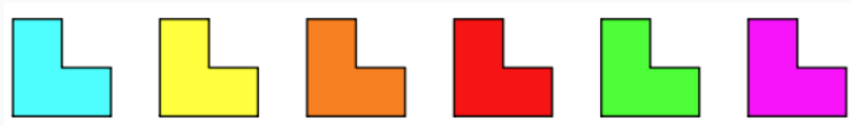
The next pigeon will satisfy the 17.

∴ 4×16+1 = 65.

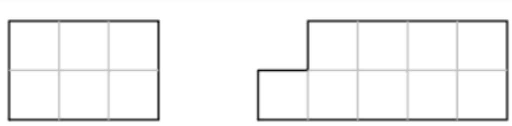
Question 9.

Harriet has an unlimited number of decorative tiles each made of three unit squares. Some tiles are straight 3×1 tiles, which come in 7 different colours.

The other tiles are L-shaped tiles, which come in 6 different colours.



Let a_n be the number of ways to completely tile a $2 \times (3n)$ grid, and let b_n be the number of ways to completely tile a $2 \times (3n-1)$ grid whose top-left corner square has been removed. The corresponding grids for a_1 and b_2 are shown below.



Grid for a_1 Grid for b_2

(a) Complete the following list of initial values:
(Recall that the straight tiles come in 7 different colours, and the L-shaped tiles come in 6 different colours.)

$a_0 =$ 1

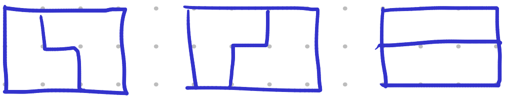
$a_1 =$ 6^2 * 2 + 7^2

$b_0 = 0$

$b_1 =$ 6

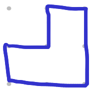
a_0: One way to sort nothing

a_1:



$6^2 \times 2 + 7^2$

b_1:



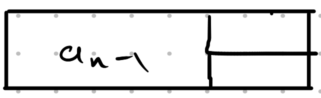
6.

(b) Help Harriet find recurrence relations that relate the two sequences.

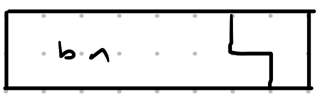
We have $a_n = p\,b_n + q\,a_{n-1}$ for all $n \geq 2$,
where $p =$ 12 and $q =$ 49,

and $b_n = r\,a_{n-1} + s\,b_{n-1}$ for all $n \geq 2$,
where $r =$ 6 and $s =$ 49.

or

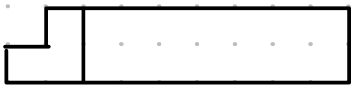


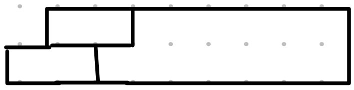
or



$2 \times b_n \times b_1 + 7^2 \times a_{n-1}$

$a_n = 12 \times b_n + 7^2 \times a_{n-1}$





$b_1 \times a_{n-1}$ $\quad\quad$ $7^2 \times b_{n-1}$

$b_n = 6a_{n-1} + 49b_{n-1}$.

$12 \times b_n = a_n - 7^2 a_{n-1}$.

(c) Hence find a recurrence relation that relates terms only in the sequence (a_n).

We have $a_n = t\,a_{n-1} + u\,a_{n-2}$ for all $n \geq 2$,

where $t =$ 170 and $u =$ −2401.

$a_n = 12b_n + 49a_{n-1}$

$12b_n = a_n - 49a_{n-1}$

$b_n = 6a_{n-1} + 49b_{n-1}$

$= 12(6a_{n-1} + 49b_{n-1}) = a_n - 49a_{n-1}$

$= 72a_{n-1} + 588b_{n-1} = a_n - 49a_{n-1}$

$= 588b_{n-1} = a_n - 121a_{n-1}$

$49(a_{n-1} - 49a_{n-2}) = a_n - 121a_{n-1}$

$49a_{n-1} - 2401a_{n-2} = a_n - 121a_{n-1}$

$a_n = 170a_{n-1} - 2401a_{n-2}$

Question 10.

Poppy is studying the degree sequence $a, b, 3, 3, 3, 3, 3, 1, 1$ for some integers a, b such that $a \geq b \geq 3$.

Give possible values for a and b such that:

(a) A multigraph with Poppy's degree sequence cannot exist.

$a, b =$ 4, 3

Degree sum must be even (Handshaking Lemma).

(b) A graph with Poppy's degree sequence exists.

$a, b =$ 4, 4

Degree sum is even. (Handshaking lemma)

(c) A multigraph with Poppy's degree sequence exists, but no graph with this degree sequence exists.

$a, b =$ 7, 7

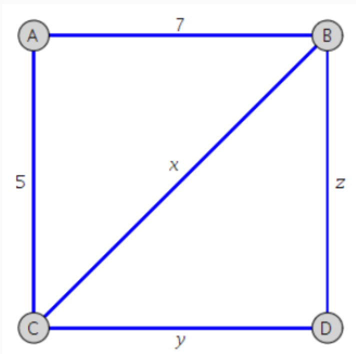
How I find these:
Look at number of vertices/vertex degree sequence:
a, b, 3, 3, 3, 3, 1, 1

There are 8 vertices. A simple graph (or a graph) must only have 1 edge between two vertices. Hence, add (8 - 1) degree vertices to ensure and even spread.

When you try this, you will notice that the 1 vertex degree sequence, cannot stay at degree 1. Hence, multi-graph exists (since handshaking lemma), but simple graph does not.

Question 8.

Sergei has been labelling weights on the graph G below. Currently they have labelled the edges AB and AC, but the remaining edges do not yet have weights given to them. Sergei refers to the weights of BC, CD, and BD as x, y, and z respectively.



(a) Find positive integer values for x, y, z such that both

- the edge BC is not contained in any minimal spanning tree of G, and
- the edge BC is contained in a shortest path from C to D.

$x, y, z =$ 8, 20, 1

MAKE SURE CA → AB → BD isn't shortest.

(b) Suppose $y = 16$. How many possible pairs $(x, z) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ are there such that the conditions given in part (a) hold?

4 + 5 + 6 + 7 + 8

The naive solution is:

$y = 16 \quad x = 15 \quad z = 1$

$x = 14 \quad z = 1, 2$

$x = 13 \quad z = 1, 2, 3$

$x = 12 \quad z = 1, 2, 3, 4$

$x = 11 \quad z = 1, 2, 3, 4, 5$

$x = 10 \quad z = 1, 2, 3, 4, 5, 6$

$x = 9 \quad z = 1, 2, 3, 4, 5, 6, 7$

$x = 8 \quad z = 1, 2, 3, 4, 5, 6, 7, 8$

NOTE: For $x = 15, 14, 13$, the shortest path does NOT use BC, because of the path:

CA → AB → BD.

Question 12.

Let $G(r, s)$ be the graph with vertex set

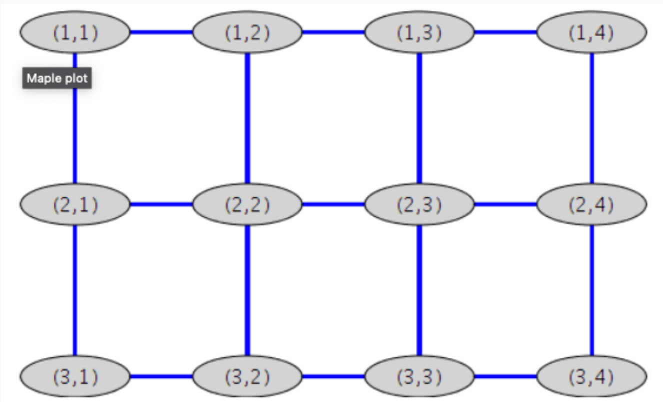
$$V = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq a \leq r, 1 \leq b \leq s\}$$

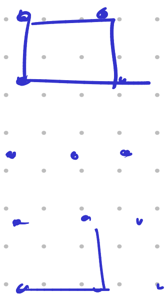
and edge set

$$E = \Big\{\{(a, b), (c, d)\} \mid (a, b), (c, d) \in V, |c - a| + |d - b| = 1\Big\}$$

for any integers $r, s \geq 2$.

For example, the graph $G(3, 4)$ looks like the following.





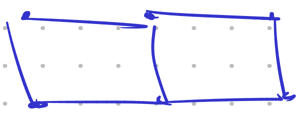
Hazel is interested in studying graphs of this type, and is especially keen to investigate certain properties they can have.

(a) Find all values of r, s for which $G(r, s)$ has an Euler path or an Euler circuit. Explain your answer.

An euler circuit will exist with every single vertex degree being even. This exists at G(2, 2), but cannot exist anywhere else, as every where will have 'intermediary nodes' inbetween sides, which guarantee a odd degree vertex.

An euler path exists when exactly two vertices are odd. This means that we require two 'intermediary' nodes. This could be G(3, 2) or G(2, 3)

This is probably wrong.



(b) For which values of r, s is $G(r, s)$ bipartite? Explain your answer.

(This answer is probably completely wrong. Please let me know if someone has a good answer)

For all r and s, G is bipartite. By the nature of the graph's subdivisions into squares, every graph in G becomes a combination of K(2, 2). Hence, since G is essentially a chain of complete, bipartite graphs, then G must also be bipartite.

(c) Briefly explain why $G(7, 7)$ does not have a Hamilton circuit.

For the graph G to have a Hamiltonian circuit, a predictable pattern appears. The pattern is to leave a column or row untouched all the way until the end, and then pathing in a "sub-rectangle" that does not use that row/column. When r and s are both odd, the pathing pattern (which zig zags throughout the sub-rectangle, and then terminates on the reserved row, but with the only path to the starting node being to backtrack a path.

(d) Complete the following true statement:

$G(r, s)$ has a Hamilton circuit if and only if [at least one of] r and s is divisible by [2].

At least one of row or column must be even, to create an escape for the zig-zag.

(e) What is the minimum number of edges that can be added to $G(2, 3)$ to ensure it is nonplanar?

[2]

G(2, 3) is similar to a hexagon. One of the edges to create K(3, 3) exists within the graph, so you must add 2 more edges to create K(3, 3), which makes it non-planar.