2021 T3

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Most of these solutions have been derived entirely by me, so there may
be imperfections and wrong answers/reasonings. Please contact me at my
student email for any corrections.

These solutions were made to aid the preparation of MATH1231 students,
not replace it. Good luck :)
```

Q2)

a)

i)

$$P(2) = \binom{12}{2}\left(\frac{42}{100}\right)^2\left(\frac{58}{100}\right)^{10}$$

ii)

$$P(0) + P(1) + P(2)$$

iii)

$$1 - (P(0) + P(1) + P(2))$$

b)

$$\text{i) } \mu_x = np = 12 \cdot 0.42 = 5.04$$

$$\text{ii) } \sigma^2 = np(1-p) = 12 \cdot 0.42(0.58)$$

Q3)

a)

Solve the systems of linear equations:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1 & -1\\-1 & -1\end{pmatrix}$$

b)

$$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$

By considering where the basis vectors <1, 0> and <0, 1> go, this can be found out visually from the diagram.

Q4)

a)

Consider that the nth Taylor Polynomial about a is calculated by:

$$f_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots$$

Therefore, equate the coefficients with your given Taylor polynomial.

b)

Radius of convergence can be found using the ratio test:

$$\sum_{k=0}^{\infty} \frac{(x-3)^{k+1}((k+1)^2+1)}{3^{k+1}} \cdot \frac{3^k}{(x-3)^k(k^2+1)}$$

With my case, the radius of convergence was 3, with an open interval of (0, 6)

c) This is evidently divergent at x = 3 - R

$$f(x) = \sum a_k(x-3)^k$$
$$f(3-R) = \sum a_k(-R)^k$$

d) kth term test

Q5)

a) $\frac{\partial}{\partial x}(x^2 + xy + 2y^2) = x + 4y$

b) $\frac{\partial}{\partial y}(x^2 + xy + 2y^2) = 2x + y$

c)

$$
\begin{aligned}
F(x,x) &= -5x^2 \\
f(4x^2) &= -5x^2 \\
t &= 4x^2 \\
x^2 &= \frac{t}{4} \\
f(t) &= -\frac{5t}{4}
\end{aligned}
$$

And for F(x, y); simply sub in $t = x^2 + xy + 2y^2$.

$$
\begin{aligned}
F(x,y) &= f(x^2 + xy + 2y^2) \\
&= -\frac{5(x^2 + xy + 2y^2)}{4}
\end{aligned}
$$

Q6)

a)

$$
\begin{aligned}
&= 3\cos^{97}\left(\frac{2\pi t - 15}{7}\right)\sin^{m}\left(\frac{2\pi t - 15}{7}\right) \\
&= \frac{97+1}{2} = 49
\end{aligned}
$$

b)

$$
\begin{aligned}
I_3 &= \int \cos^3\left(\frac{2\pi t - 15}{7}\right)\sin^{6+m}\left(\frac{2\pi t - 15}{7}\right)dx \\
&= \int \sin^{6+m}\left(\frac{2\pi t - 15}{7}\right)\cos\left(\frac{2\pi t - 15}{7}\right)\left(1 - \sin^2\left(\frac{2\pi t - 15}{7}\right)\right)dx \\
&= \int \sin^{6+m}\left(\frac{2\pi t - 15}{7}\right)\cos\left(\frac{2\pi t - 15}{7}\right) - \sin^{8+m}\left(\frac{2\pi t - 15}{7}\right)\cos\left(\frac{2\pi t - 15}{7}\right)dx
\end{aligned}
$$

I leave the rest as an exercise to you. Just take the sin term as a substitution.

Q7)

a)

```
Consider the options:
1. rank(A) = 3
We can check this by creating the matrix:
```

```
B := <<-5, 9, 8>|<-3, 8, 9>|<9, 7, 1>>
l := <<8, 0, 0>|<0, 20, 0>|<0, 0, x>>
GaussianElimination(B.l.MatrixInverse(B))

2. If lambda is an eigenvalue of A, then (lambda - 8)(lambda - 2)(lambda
- x) = 0
This is true, visually check the diagonal matrix

3. A = B^-1DB
A = BDB^-1

4. <6, 0, 0>, <0, -6, 0>, <0, 0, 4> are eigenvectors of D
Yep, just scalar multiples of the existing ones

5. The columns of A are linearly dependent
Since 1 is true, 5 can't be :(
```

b) The corresponding eigenvector of the eigenvalue 20. Make it a unit vector:

$$\left(\frac{1}{\sqrt{154}}\begin{pmatrix}-3\\8\\9\end{pmatrix}\right)$$

c) With the above maple constants and declarations:

```
B.<<8^n, 0, 0>|<0, 20^n, 0>|<0, 0, x^n>>.MatrixInverse(B).B.<8, 0, 9>
```

d) Get the matrix A using the diagonalisation $A = BDB^{-1}$. Gaussian eliminate, using `GaussianElimination(A)`.

$$\begin{pmatrix} -\frac{620}{287}+\frac{153x}{286} & \frac{180}{41}+\frac{27x}{41} & -\frac{3240}{287}-\frac{117x}{287} \\ 0 & -\frac{32(65+204x)}{-620+153x} & \frac{4(3640+2329x)}{-620+153x} \\ 0 & 0 & -\frac{1435x}{65+204x} \end{pmatrix}$$

It is evident that we can create $nullity(A) \neq 0$ by letting x $= 0$, which would leave the last row empty.

Q8)

Check the x-y plane diagram.

Consider that the intercept shows that y = -2 when x, z = 0. Therefore, E = -2.

Use the other diagrams to deduce other properties of the functions.

Q9)

Within my question, the following information was given:

- $f(0) = 3.4 \leq f(x) \leq 9.8 = f(10)$
- $m = 0.4 \leq f'(x) \leq 0.73 = M$

  a) Were tasked with finding the upper and lower bounds using rough estimates with the information above.

  We can do this by subbing in $f(x) = 9.8 \cap m = 0.73$ for the upper bound, and $f(x) = 3.4 \cap m = 0.4$ for the lower bound.

b)

$f(0) + mx = 3.4 + 0.4x$

$f(0) + Mx = 3.4 + 0.73x$

Plug these back into the formula:

$$\int_0^{10} 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx$$

And then the rest is fairly trivial.

Q10)

a)

Create a matrix with the spanning vectors:

$$\begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -3 \end{pmatrix}$$

Row reduce using `GaussianElimination()`:

$$\begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, this is a plane through the origin - as an infinite solution through `<0, 0, 0>` exists.

b) This question is quite convoluted so lets examine each part of it.

- **Any nonzero vector in H is an eigenvector of T**
  Understanding what an eigenvector is *crucial* for these types of questions. Eigenvectors are vectors within a vector space, that after a linear transformation is applied, do not change. Hence, we can see that since T is a transformation which represents the projection of a vector **x** onto H - any vector on H would not be affected.
- **There exists a nonzero vector in H that is not a eigenvector of T**
  Same reason as above. Any vector in H will stay the same, as it is the projection of itself.
- **The linear transformation T has at least one zero eigenvalue**
  This is true. A zero eigenvalue of zero denotes that the null space is non-trivial (non-zero). We know that the zero vector exists in H, and therefore, it also exists in T. Since the null space of H itself is non-trivial, it is therefore found that T will have a non-trivial null space, and therefore, atleast one zero eigen value.
- **The linear transformation T has at least one eigenvalue that equals 1**
  This means that there is at least one linearly dependent vector. Since H $\subseteq$ T, it can be seen that H is not linearly independent, and therefore, T has at least one linearly dependent eigenvector.
- **The nullity of T is 1**
  nullity(T) = dim(ker(T)) = 1
- **The rank of T is 2**
  nullity(T + rank(T) = dim(T)
  rank(T) = 3 - 1 = 2

c)
<-1, 0, 1> is a part of H, so is intrinsically an eigenvector.
The eigenvectors of T are H, and the vectors that are perpendicular to H.
<2, 0, 0> do not fit either criteria.
<1, 0, 1> is an eigenvector, as the perpendicular vector to H is <-1, 0, -1>, and <1, 0, 1> is a scalar multiple.

Q11)
a)

Consider the probabilities and their winning values:

2 | 1/4

5 | 1/4

7 | 1/4

9 | 1/4

So, the expected value would be:

$$2 \cdot \frac{1}{4} + 5 \cdot \frac{1}{4} + 7 \cdot \frac{1}{4} + 9 \cdot \frac{1}{4}$$

$E(X) = \frac{23}{4}$

b)

i) For this question, just draw a probability tree diagram, and it should be easy to figure out the probabilities. Then $E(X)$ is found using the normal formula you've known since Year 9.

ii) The Monty Hall problem :)

Q12)

a) Spanning set just means that for every vector in a vector space, the set has some linear combination such that it equals the vector.

b) For it to not be a spanning set, there must be an arbitrary vector **y** such that there is **no** linear combination that equals **y.**

c) Linear independence means that for a linear combination of the set, the row-reduced null space equation has no non-leading columns.

d) Consider the negation of above.

e) For a set to consist of a basis of a real vector space of dimension n, there must be ≥ n vectors within the set.

Q13)

a)

$P(Y = 0) = P(Y \leq 0) - P(Y < 0) = \frac{4}{23} - \frac{3}{23} = \frac{1}{23}$

$P\left(Y = -\frac{1}{2}\right) = P\left(Y \leq -\frac{1}{2}\right) - P\left(Y < -\frac{1}{2}\right) = \frac{3}{23} - \frac{3}{23} = 0$

$P(-1 < Y < 1) = P(Y < 1) - P(Y < -1) = \frac{4}{23} - \frac{3}{23} = \frac{1}{23}$

$P(X = -1) = P(X \leq -1) - P(X < 1) = 0$
$P(-1 \leq X \leq 1) = P(X < 1) - P(X < -1) = P(-1 < Y < 1) = \frac{1}{23}$

b)
We can find the original distribution X.
Since we have probabilities in terms of X now, we can apply the z-score table in order to find different properties of the original distribution, such as standard deviation, mean, etc.

Q14)
a) 'The forces acting on the rocket are a constant gravitational force with magnitude mg acting downwards and atmospheric resistance which is proportional to velocity'
Therefore:

$$\begin{aligned} F &= ma \\ &= mx'' \\ mx'' &= -kx' \end{aligned}$$

as the force is 'proportional the the velocity' - which is also proportional to atmospheric resistance.

b)
We have an equation of the form:

$$\begin{aligned} my'' + ky' &= -mg \\ m\left(\frac{d^2y}{dx}\right) + k\left(\frac{dy}{dx}\right) &= -mg \end{aligned}$$

Therefore, linear, inhomogeneous and second order.

e) $m\lambda^2 + k\lambda = 0$
$\lambda(m\lambda + k) = 0$
$\lambda = 0, -\frac{k}{m}$

d)

```
ode := 46*diff(y(t), t, t) + 1.5*diff(y(t), t) = -46*g
ics := y(0) = 0, D(y)(0) = V
dsolve({ics, ode})
```

e)
We derive for $v_y(t)$, and then let $v_y(t) = 0$, as that is when the maximum will be.

```
A = diff(y(t), t)
m := t -> A
m((46*T)/1.5)
```

and then solve manually - I forgot how to make maple give you an exact solution... so it doesn't give you the log.

Q15)
a) $1/8$

b)
The Maclaurin series of $\frac{1}{1-x}$ is given by:
$1 + x + x^2 + \cdots + x^n$
Hence, consider the following working out:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots + x^n \\ \frac{1}{1+x^2} &= 1 - x^2 + x^4 + \ldots \\ \int \frac{1}{1+x^2}\,dx &= x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \\ \tan^{-1}(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots \end{aligned}$$

Now, we were given that the interval for $\frac{1}{1-x}$ is (-1, 1)
Now, if we apply 8x:
$-1 \leq x \leq 1$
$-1 \leq 8x \leq 1$
$-\frac{1}{8} \leq x \leq \frac{1}{8}$
Hence, proven

Q16)
a)
Consider the options:
$\{1, z, z^2\}$ is linearly independent, as this is the basis vectors for $\mathbb{P}_2$.
$\{1, i, z, iz, z^2, iz^2\}$ are also linearly independent, as none of them are a linear combination

of each other

$\{i, iz, iz^2\}$ are also all linearly independent

b)

You can do this in maple by constructing a matrix:

$$A = \begin{pmatrix} -1 & z \\ i & -1 \\ 1-i & -1-i \\ i-1 & -i \end{pmatrix}$$

```
GaussianElimination(A) // Should lead to no non-leading columns
```

c)

This can be done by inspection fairly easily.

$p_5(z) = iz - 1$ is trivially in the set, by using Maple:

```
A := <<-1, i, 1 - i, i -1, -1> |<z, -1, -1 - i, -i, i>>
GaussianElimination(A)
```

for $p_6(z), p_7(z)$, there is a z^2. Looking at our original set, there is no possible way to linearly construct z^2. Therefore, they both do not belong to the span S.