2022 T1

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Most of these solutions have been derived entirely by me, so there may
be imperfections and wrong answers/reasonings. Please contact me at my
student email for any corrections.

These solutions were made to aid the preparation of MATH1231 students,
not replace it. Good luck :)
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Q2)

a) $T(x, y) = 9x^3y + 7xy^3$

We are given the equation starts at $(x, y) = (1, 1)$

So, sub in these values:

$T(1, 1) = 16$

Therefore, a point is:

$$\begin{pmatrix} 1 \\ 1 \\ 16 \end{pmatrix}$$

The normal vector of a tangent plane can be found using the formula:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ -1 \end{pmatrix}$$

Therefore:

$$\vec{n} = \begin{pmatrix} 27x^2y + 7y^3 \\ 9x^3 + 21xy^2 \\ -1 \end{pmatrix}$$
$$\vec{n} = \begin{pmatrix} 34 \\ 30 \\ -1 \end{pmatrix}$$

b)

$$
\begin{aligned}
\frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} \\
&= (27x^2y + 7y^3)(-5\sin(5t)) \\
&= -7(-5\sin(5t)) \\
&= -7\left(-5\sin\left(\frac{5\pi}{2}\right)\right) \\
&= 35
\end{aligned}
$$

Look above for reasoning :)

Q3)

a) Consider each option:

1. ... is a linear independent set, because U has one or more leading columns
   Well, we know this isn't true, because linear independence means there is no non-leading columns.
2. ... is a linearly dependent set, because A has more columns than rows
   This is true - even if we could row reduce it, there would always be a non-leading column somewhere.
3. ... is a linearly independent set, because U has one or more non-leading columns
   This is the negation of the actual definition of a linearly independent set
4. ... is a linearly dependent set, because U has one or more non-leading columns
   Yep.
5. ... is a linearly independent set, because U has no zero rows
   ... This doesn't do anything for us
6. ... is a linearly dependent set, because U has no zero rows
   Same as above
7. ... is a linearly dependent set, because A has one or more non-leading columns
   A isn't row-reduced, so it doesn't matter whether A has non-leading columns

b) Only {p1, p2, p3, p4} spans $\mathbb{P}_2$, because U has no zero rows is true - as this means that there is only one non-leading column, which further insinuates that there are 3 linearly independent vectors

c) 3 linearly independent vectors, 3 dimensions.

d) Take the first three linearly independent vectors, so for me, that was:
{p1, p2, p4}

Q4)

a) The question reads: *It is reasonable to assume that the rate of diffusion of the joke in the town is proportional to the number of people that have already heard the joke* (P) *as well as to the difference between the number of residents and the number of people that already heard the joke* (800 - P)

Therefore, our differential equation becomes:

$$\frac{dP}{dt} = cP(800 - P)$$

b) By either integrating (separable ODE) or by knowledge of ODE characteristics: 800

c)

$$\begin{aligned} \frac{dP}{dt} &= cP(800 - P) \\ \frac{1}{P(800 - P)} dP &= cdt \\ \frac{1}{800P} - \frac{1}{800(P - 800)} dP &= ct + C \\ \frac{1}{800}\ln(P) - \frac{1}{800}\ln(P - 800) &= ct + C \\ \frac{1}{800}\left(\ln\left(\frac{P}{P - 800}\right)\right) &= ct + C \\ \ln\left(\frac{P}{P - 800}\right) &= 800ct + C \\ \frac{P}{P - 800} &= Ae^{ct} \\ P &= \frac{800}{1 - Be^{-800ct}} \end{aligned}$$

Then equation for P(0) = 2. My question's solution was that B = -399, therefore q = 399.

d) For my question, the maple looked like this:

```
a := 800/(1+399*exp(-800*1/300*t)) = 800*0.9
solve(a)
Output: 3.069819748
```

Therefore, answer = 4.

Q5)

a) Let's walk through these options:

1. ker(T) is a subspace of V
   This is axiomatically true - or at the very least, defined as true. (im(T) is the subspace of W, or whatever the co-domain is)
2. ker(T) is a subspace of W
   No, but im(T) is
3. nullity(T) $\leq$ 2
   We already have two nullities (v_1, v_2), so we can only have more
4. nullity(T) $=$ 2
   Not sure about this yet
5. nullity(T) $\geq$ 2
   This is true, as we have 2 already, and can have mroe
6. $7v_2 - 4v_3$ is **not** an element of ker(T)
   Then it must be true, such that:

$$T(7v_2 - 4v_3) = 0$$
$$T(7v_2) - T(4v_3) = 0$$

$$0 - 4d = 0$$

   Therefore, contradiction

b) Let's walk through these options:

1. im(T) is a subspace of V
2. im(T) is a subspace of W
   Yep, since T is mapping out from $V \to W$, it is true that the image (the range of solutions) is $\subseteq$ W.

aside) Let's go through the proofs of $\boldsymbol{ker(T)} \subseteq V$ and $\boldsymbol{range(T)} \subseteq W$:

1. ker(T) $\subseteq V$
   T(0) $=$ 0 (as given)
   Consider two arbitrary vectors, such that T(a) $=$ 0 and T(b) $=$ 0.
   Therefore, it must be true such that T(a $+$ b) $=$ 0, for it to be closed under addition.
   T(a $+$ b) $=$ T(a) $+$ T(b) $=$ 0.
   The scalar multiplication condition is trivial.

2. range(T) $\subseteq W$

Consider two vectors, $\vec{a}, \vec{b}$, such that, T($\vec{a}$) $= \vec{x} \in V$ and $T(\vec{b}) = \vec{y} \in V$.

For the set to be closed under addition, it must be true that T($\vec{a} + \vec{b}$) $= \vec{c} \in V$.

Therefore:

$$T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b}) = \vec{x} + \vec{y}$$

and be definition, $\vec{x} + \vec{y} \in V$, therefore closed under addition.

The scalar multiplication is trivial.

c) `either nullity(T) = 3 or rank(T) = 3, but we do not have enough information to tell which`

Consider the rank-nullity theorem rank(T) + nullity(T) = dim(T). With the information that rank(T) ≥ 2 and nullity(T) ≥ 2, it must be true that **one** of them must $= 3$, to satisfy the rank-nullity theorem. However, there is not enough information to dictate which one is $= 3$.

Q6)

a) y(t) $= 4t^2 + 8$

sub in x = t

$3 \leq t \leq 4$

b) Find x in terms of y:

$x = \frac{\sqrt{y-8}}{2}$

Then use the formula:

$$\int 2\pi f(t) \sqrt{1 + [f'(t)]^2}\, dt$$

with your specific y range to find the surface area (this question is misleading)

Q7)

a) Use `Eigenvectors(A)` to find the eigenvectors, then apply scalar multiples to the z-axis to see if they fit any of the eigenvectors.

b) A square matrix A is said to be a diagonalisable matrix if there exists an invertible matrix M and diagonal matrix D such that $M^{-1}AM = D$.

Hence, in maple:

```
M := <<3, -3, 1>|<3/4, -4, 1>|<7/2, -11/4, 1>> (The eigenvector matrix)
A := <<179, -196, 64>|<-402, 424, -140>|<-1746, 1863, -613>> (The given matrix)
M.A.MatrixInverse(M) should = D (The diagonal matrix of the eigenvalues)
```

c) *Theorem*: An n $\times$ n matrix A is diagonalisable iff A has n linearly independent vectors. Therefore, since this matrix has 3 linearly independent vectors (which can be checked by `GaussianElimination()`)

Q8)

The radius of the convergence should stay the same.
The two changed terms are $(x-1)^{n-1}$ and n. The addition of n is trivial, as it does not affect the calculation of the limit. The limit will still tend to zero for $n \to \infty$. Hence, since there are no addition added terms or constants that are affecting the radius of convergence, it can be seen that the radius of convergence is still R.

Q9)

a) P(First no repeats) $\times$ P(Second no repeats) $\times$ P(Third no repeats)

b) We either can either repeat the first, second or third tests.
Therefore:
P(Fail first) • P(Pass repeat) • P(Pass second no repeats) • P(Pass third no repeats)
Repeat for the permutations

c) (P(Fail first) • P(Pass repeat) • P(Pass second no repeats) • P(Pass third no repeats)) / (P(Take four or fewer tests))

d)
Consider k trials, such that the candidate passes on the kth trial.
Failing the first time, $\frac{7}{10}$
Then failing each subsequent time, will be k - 2 times (since we already had one trial), $\left(\frac{4}{10}\right)^{k-2}$
Then passing on the kth time, is $\frac{6}{10}$
Therefore, together:

$$\left(\frac{7}{10}\right)\left(\frac{4}{10}\right)^{k-2}\left(\frac{6}{10}\right)$$

e) Consider that P(X) represent the probability of passing after X times.
Therefore P(1) $= \frac{3}{10}$
P(2) $= \frac{7}{10} \cdot \frac{6}{10}$
and so on.
Consider that E(X):

$$XP(X) = X\sum_{k=1}^{\infty} P(X)$$

Therefore, a $=$ **3/10**, as it is alone within the sequence.
Then, the expected value for k $=$ **2** $\rightarrow \infty$ is $\boldsymbol{k \cdot p_k}$

f)

$$\frac{3}{10} + \sum_{k=2}^{\infty} kp_k$$

On Maple:

```
3/10 + sum(k*(4/10)^(k-2), k=2..infinity)
Output: 13/6
```

Q10)
a) You can derive this for yourself, or check out the course notes:
$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

b) Visually inspect for substitution, $\frac{d}{dx}(\tan(x)) = sec^2(x)$

c)

$$\begin{aligned}
\bar{f} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{2}{\pi}\left(\int_{-\frac{\pi}{4}}^{0} f(x) + \int_0^{\frac{\pi}{4}} f(x)\,dx\right) \\
&= \frac{2}{\pi}\left(\frac{13}{20} - \frac{2}{10}\sqrt{2}\right)
\end{aligned}$$

d) Yes; mean value theorem of integrals

Suppose that f is continuous on [a, b]. Then there is a number c in (a, b) such that

$$\int_a^b f(t)dt = f(c)(b-a)$$

Proof:

Define F: [a, b] $\rightarrow$ $\boldsymbol{R}$ by the formula

$$F(x) = \int_a^x f(t)dt\,dx$$

By the fundamental theorem of calculus, F is continuous on [a, b], differentiable on (a, b) and F'(x) = f(x). By the MVT, there exists $c \in [a, b]$:

$$\frac{F(b)-F(a)}{b-a} = F'(c)$$

But

$$F(a) = 0, F(b) = \int_a^b f(t)dt\,dx, F'(c) = f(c)$$

Therefore:

$$\frac{1}{b-a}\int_a^b f(t)dt \;= f(c)$$

Q11)

a)

```
Considering the derivative of zero in all variables is zero, the zero
element exists within set S.

Further, consider closure under addition.

For some arbitrary vector **a** and **b**, we are RTP that S(**a**) +
S(**b**) = S(**a** + **b**) = 0.

Therefore:

a1Fx + a2Fy + a3Fz = 0 - 1

b1Fx + b2Fy + b3Fz = 0 - 2
```

```
1 + 2:

Fx(a1 + b) + Fy(a2 + b2) + Fz(a3 + b3) = 0 = S(a + b)

Therefore, closed under addition.

Further, consider the scalar s in R.

S(s**a**) = sa1Fx + sa2Fy + sa3Fz = 0
= s(a1Fx + a2Fy + a3Fz) = 0

= sS(a)

Therefore, all axioms are met, and S is a subspace.
```

b)
$5\alpha + 2\beta + 2\gamma = 0$
Just find three integers such that it $= 0$

$$
\begin{gathered}
\frac{\partial F}{\partial x} = \frac{dF}{du} \cdot \frac{du}{dx} \\
\frac{\partial F}{\partial y} = \frac{dF}{du} \cdot \frac{du}{dy} \\
\frac{\partial F}{\partial z} = \frac{dF}{du} \cdot \frac{du}{dz}
\end{gathered}
$$

c)
Same as above, but doesn't $= 0$.

d)

$$
\begin{aligned}
& 5\alpha \frac{dF}{du} + 2\beta \frac{dF}{du} + 2\gamma \frac{dF}{du} \\
& = x_1 \frac{dF}{dx} + x_2 \frac{dF}{dy} + x_3 \frac{dF}{dz} \\
& = f'(u)[5x_1 + 2x_2 + 2x_3] = 0 \\
& = 5x_1 + 2x_2 + 2x_3 = 0
\end{aligned}
$$

Q12)

a)

```
0 <= tan(x) <= 4x/Pi

0 <= tan^n(x) <= (4x/Pi)^n

int(0, x=0..Pi/4) <= int(tan^n(x), x=0..Pi/4) <= int((4x/pi)^n,
x=0..Pi/4)

0 <= int(tan^n(x), x=0..Pi/4) <= 0

0 <= 0 <= 0
```

b)

Consider that:

$I_\infty = I_1 + I_3 + I_5 + I_7 + \cdots + I_\infty$

First, to build this series:

$$I_\infty = I_1 + \left(\frac{1}{2} - I_1\right) + \left(\frac{1}{4} - \frac{1}{2} + I_1\right) + \left(\frac{1}{6} - \frac{1}{4} + \frac{1}{2} - I_1\right) + \ldots$$

$$I_\infty = \pm\left(I_1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} + \ldots\right)$$

$$I_1 = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots$$

$$-\ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8}$$

$$\frac{1}{2}\ln(2) = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8}$$

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

$A = 2, a_1 = -\frac{1}{2}, a_2 = \frac{1}{3}, a_3 = -\frac{1}{4}$