MATH1081 2022 T2+

Question 1.

Let $\mathcal{G}$ be the set of all graphs, and let $\preceq$ be a relation on $\mathcal{G}$ given by

$$G \preceq H \text{ if and only if } G \text{ is a subgraph of } H$$

for all graphs $G, H \in \mathcal{G}$.

(a) Prove that $\preceq$ is a partial order relation.

When typing your answer, you may use the character < instead of the symbol $\preceq$ wherever needed.

Reflexivity:
The relation is reflexive, as every graph is a subgraph of itself.

Transitive:
If A is a subgraph of B, then A composes some part of B. If B is a subgraph of C, then B composes some part of C. Since B exists within C, and A exists within B, then A is also a subgraph of C.

Anti-symmetric:
If A is a subgraph of B, and B is a subgraph of A, then A = B.
Consider the case where we assume that the relation is not antisymmetric.
If B was comprised of some graph with 8 vertices and 6 edges, and A represented 4/8 verticies and 2/6 edges - then B could not be a subgraph of A, as A would by definition be some partition of B.

Hence, the relation is partially ordered.

Let S_n be the set of all (unlabelled) graphs with n vertices.

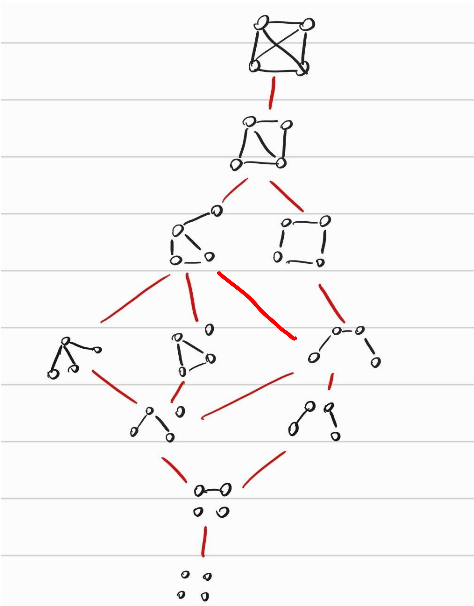
(b) Briefly describe the greatest and least elements of the partially-ordered set $(S_5, \preceq)$, if they exist.

The least element in this case is the null graph - or a graph with no vertices and no edges. Such a subgraph exists within every single graph, and no other subgraph can be composed of the null graph.

The greatest element would be a complete graph with all edges taken.
A graph that had more edges and more vertices than S(5) would no longer be a subgraph, and hence a limit is formed.

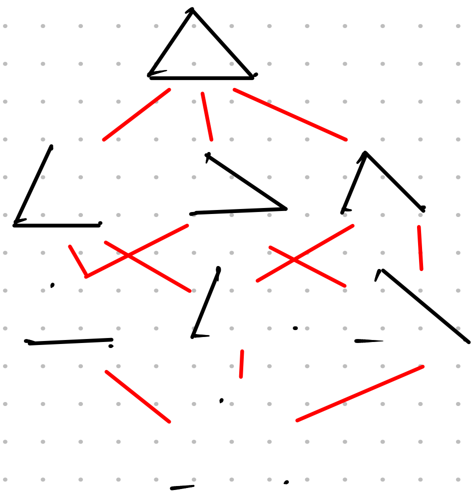
(c) Draw the Hasse diagram for the partially-ordered set $(S_4, \preceq)$. How many connecting lines does this Hasse diagram contain?

(Hint: $|S_4| = 11$.)



14 links.

d) Consider that each "level" has ^nC_k items, where n is edges and k is levels. Consider the following example:



At each stage, for every single edge present within the current graph, there are n amount of edges that you can remove. At each level, there is a combination of nCk items, as each level portrays how many edges have been removed.

In the above example, level one shows when 1 edge has been removed - and hence, it has 3C1 items. Level 2 shows when 2 edges have been removed, hence, there are 3C2 items.

For the edges between these items themselves, you must consider the amount of edges present within the original graph, as an indicator of how many edges can be removed. Hence, for a level, the formula to find the amount of lines is:
Edges that can be removed * Items that represent each removal
= e * nCk.
The e variable will decrement each stage you go down, where k will increment each stage you go down.
n represents the edges in the complete graph of however many variables you originally had.

Question 3.

Lara has invented a function f that takes multigraphs as input values and returns integers as output values, defined as follows:

For any multigraph G, we have $f(G) = |V(G)| \times |E(G)|$.

(a) Complete the following sentences:

The function $f : \{\text{all graphs}\} \to \mathbb{N}$ is neither injective nor surj... .

The function $f : \{\text{all connected graphs}\} \to \mathbb{N}$ is neither injective nor surj... .

Briefly justify your answers below.

By inspection, both are clearly not injective. For example, there are multiple graphs and connected graphs with and vertice * edge value of 40. For example, 10 vertices, 4 edges; 8 vertices, 5 edges, etc. Therefore, this is definitely not injective.

For the consideration of surjectivity, we need to consider if every single value of N is used.

For connected graphs, the simple limitation that the graphs are connected doesn't allow for surjectivity. For example, the construction of certain numbers would be impossible, as at a minimum, there would need to be V - 1 edges. (For example, how could you construct 13?)

(b) Lara thinks that $f(K_{m,n})$ is even for all $m, n \in \mathbb{Z}^+$. Are they correct? Carefully prove your answer.

We have m + n vertices, therefore, lV(K(m, n)l = m + n. This is definitely not guaranteed to be even.
Hence, RTP that the lE(K(m, n))l = 2k

To consider the amount of edges that exist within a K(m, n) function, first define the K(m, n) as a complete bipartite function. Hence, this means that for all every edge on the m side, it must have n edges, and for every edge on the n side, it must have n edges.

Therefore, the total sum of edges comes out to be:
m*n + n*m = 2mn
Hence, the sum of the edges are indeed even. QED.

Question 4.
a)

Jordan is building a tower out of different types of blocks. The blocks come in two different sizes and many different colours. There is an unlimited supply of:

- blocks of height 1 that come in 2 different colours, and
- blocks of height 2 that come in 6 different colours.

Jordan considers two towers different even if they are reflections of one another. For example, the tower formed by placing a red block of height 1 on top of a red block of height 2 is different from the tower formed by placing a red block of height 2 on top of a red block of height 1.

(a) Let a_n be the number of different towers of height n that Jordan can construct. Then we have

$a_1 = 2,$

$a_2 =$ 6 + 2^2 ,

and

(a) Let a_n be the number of different towers of height n that Jordan can construct. Then we have

$a_1 = 2$,

$a_2 =$ 6 + 2^2 ,

and

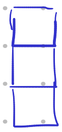
$$a_n = Xa_{n-1} + Ya_{n-2}$$

for all integers $n \geq 3$, where

$X =$ 2 and

$Y =$ 6 .

$a_3 =$



$$6 \times 2 + 6 \times 2 + 2^3$$

$\therefore$ Must be $Y = 6$.

$$= 4 \times 6 + 2^3 - 2 \times 6 - 2^3$$
$$= 2 \times 6$$

(b) Jordan notices that a_2, a_5, and a_8 are all divisible by 5. Does this pattern continue? Carefully prove your answer.

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a(2) = 3*i

a(5) = 3*j

a(8) = 3*k for some integers i, j, k

Prove that for all n >= 2, where n = n + 3*k, for some k as an integer, that

a(n) is divisible by 3.

Test n = 2 (already done)

Assume true for n = k
a(k) = 3*t, for some integer t.

Prove true for n = k + 3

a(k+3) = 2*a(k+2) + 6*a(k+1)

= 2*(2*a(k+1) + 6*a(k)) + 6*a(k+1)
= 10 * a(k+1) + 12 * a(k)
= 10 * (2*a(k) + 6 * a(k - 1)) + 12 * a(k)
= 32 * a(k) + 60 * a(k-1)
= 3 * (3t + 20*a(k-1))

Therefore true, divisible by 3.

(c) Jordan says, "I think the closed form solution for this recurrence relation should be of the form

$$a_n = A\left(\frac{2+\sqrt{28}}{2}\right)^n + B\left(\frac{2-\sqrt{28}}{2}\right)^n$$

for some integers A and B. But there should always be an integer number of towers for any height n, while this formula looks like it will usually produce irrational values. I must have made a mistake."

Has Jordan made a mistake? Briefly justify your answer.

No, not necessarily.

The root values within the polynomial expansion can simply be cancelled out for appropiate values of A and B.

For example, when n = 2:

a(n) = A(8 + sqrt(7)) + B(8 - sqrt(7)
If A = B, then the sqrt(7) is cancelled.

Question 5.

(a) Use mathematical induction to prove that for all $t \in \mathbb{N}$, the set $S = \{1, 2, 3, \ldots, t\}$ with t elements has exactly 2^t subsets.

Test for n = 0.
An empty set obviously only has one subset (itself)

Assume true for n = k.
|Subsets of S(k)| = 2^k

Prove true for n = k + 1
|Subsets of S(k+1)| = 2^(k+1)
= 2 * |Subsets of S(k)|

Therefore, true, as the inductive step hypothesis that there is a k -> 2^k mapping for elements to subsets; the definition of (k+1) -> 2 * 2^k is achieved.

(b) Consider a set S of size 17. Let $T \subseteq \mathcal{P}(S)$ be a set such that the elements of T are pairwise non-disjoint (that is, $A \cap B \neq \varnothing$ for all $A, B \in T$).
Construct an example showing that there exists such a set T whose cardinality is exactly 2^{16}.

Consider some arbitrary set, such that:
S = {1, 2, 3, 4, ..., 25}
The power set P(S) contains all subsets within S. If T = P(S), there would be disjoint elements, for example:
{2}, {3, 4, 5}
Hence, fix one element as a mandatory element within the subsets.
Choose any arbitrary value, such as 1. Therefore, the subsets of this new set, would look something akin to:
{{1}, {1, 2}, {1, 3}, {1, 4}, ..., {1, 2, 3, ..., 25}}

In this set, the number of sets possible is 2^24, as we have fixed one - hence, we can choose to use (or not use) the 24 other elements. QED.

(c) Carefully prove that no such set T as defined in part (b) can have cardinality greater than 2^{16}.

Consider once again, the subsets that are created when there is an arbitrary element in T fixed within the subset.
To find other possibilities, divide P(S) into two distinct subcollections of P(S).
S(1), which contains all subsets that have 1 as an element.
S(~1), which contains all subsetes that don't have 1 as an element.
S(~1), may only contain elements within {2, ..., 100}. Hence, at most, |S(~1)| = 2^24.
Further consider that S(1) cannot contain any complement of S(~1), and hence, at most, contains 2^24 - |S(~1)|.

We may now conclude that at most, the set shown in part b) has at most 2^24 elements.

Question 6.

In the popular monster-collection game Pokumon, monsters can have different types: Acid, Baby, Cosmic, Dark, Evergreen, or Fighting, which for the rest of this question we will refer to as a, b, c, d, e, f. Certain types do bonus damage against certain other types, as summarised in the below "Type Chart". If there is a 1 in the row for Type x and the column for Type y, this means Type x does bonus damage against Type y. Otherwise, a 0 implies no bonus damage is done.

	a	b	c	d	e	f
a	0	1	0	1	1	1
b	1	0	0	1	1	0
c	1	1	1	1	1	1
d	1	0	0	0	0	0
e	1	1	0	0	1	0
f	0	0	1	1	1	1

Let R be the relation on the set of types $\{a, b, c, d, e, f\}$ defined by $x \; R \; y$ if and only if Type x does bonus damage against Type y.

(a) Find a set S of size at least 2 for which the relation R on S is reflexive.

c, e, f

(b) Find a set S of size at least 2 for which the relation R on S is symmetric.

a, b

(c) Find a set S of size at least 3 for which the relation R on S is transitive.

c, e, f

(d) The Type Chart above is converted to a 6×6 matrix M, and the matrix M^2 is calculated to be

$$M^2 = \begin{bmatrix} 3 & 1 & 1 & 2 & 3 & 1 \\ 2 & 2 & 0 & 1 & 2 & 1 \\ 4 & 3 & 2 & 4 & 5 & 3 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 2 & 2 & 0 & 2 & 3 & 1 \\ 3 & 2 & 2 & 2 & 3 & 2 \end{bmatrix}.$$

What can be deduced from the fact the $(3, 4)$th entry of M^2 is 4?

The number of length 2 "walks" from 3 to 4.

(e) For the sequel to Pokumon, the Type Chart is going to be redesigned. There will now be only 5 different monster types total, and the designers want each type to be able to do bonus damage to either $\underline{3}$ or $\underline{4}$ other types. How many possible Type Charts are there?

$$\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & \times & \times & \times & \times & \times \\ b & \times & \times & \times & \times & \times \\ c & \times & \times & \times & \times & \times \\ d & \times & \times & \times & \times & \times \\ e & \times & \times & \times & \times & \times \end{array}$$

$$\underline{\left(\binom{5}{3} + \binom{5}{4}\right)^5}$$

My wrong solution:

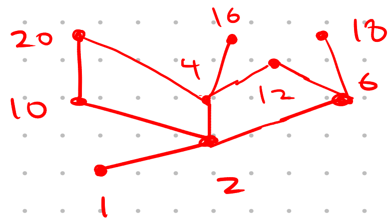
$$\left(\binom{5}{3}\right)^5 + \left(\binom{5}{4}\right)^5$$

This limits to every column having 3 or 4.
However, it should really give the choice of 3 or 4 each row.

Question 7.

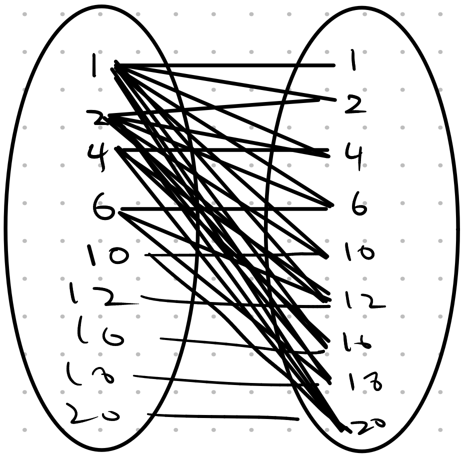
(a) Consider the set $S = \{1, 2, 4, 6, 10, 12, 16, 18, 20\}$.
How many edges are in the standard Hasse diagram for the partially-ordered set $(S, \mid)$?

10



How many edges are in the unsimplified diagram for $(S, \mid)$?

30



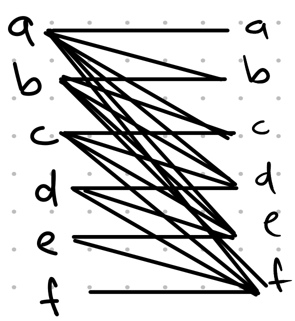
(b) Suppose T is a set with $|T| = 10$, and the standard Hasse diagram for a poset $(T, \preceq)$ has 8 edges, while its unsimplified diagram has 18 edges.
What can be deduced about the poset $(T, \preceq)$?

We could state a couple of things.
Due to the fact that the unsimplified diagram having far more edges than the Hasse diagram, the poset has multiple 'levels' or 'hiearchies' present within the Hasse diagram.

Further more, it is evident that not every single element within the poset is related to another, as the amount of edges within the Hasse diagram do not represent this.

(c) Suppose T is a set with $|T| = 6$. What is the maximum number of edges that can be in the unsimplified diagram for $(T, \preceq)$?

21



Note: Has to be anti symmetric, ∴ you can't have aRb & bRa.

(d) Suppose $|T| = 13$, and that the poset $(T, \preceq)$ has a greatest element. What is the minimum number of edges that can be in the unsimplified diagram for $(T, \preceq)$?

Minimum would be just reflexive.

1 —— 1
2 —— 2
3 —— 3
4 —— 4
⋮ ⋮

∴ Greatest element means that all other elements must point to a single one.

So you have:
$13 + 12 =$ reflexive + greatest

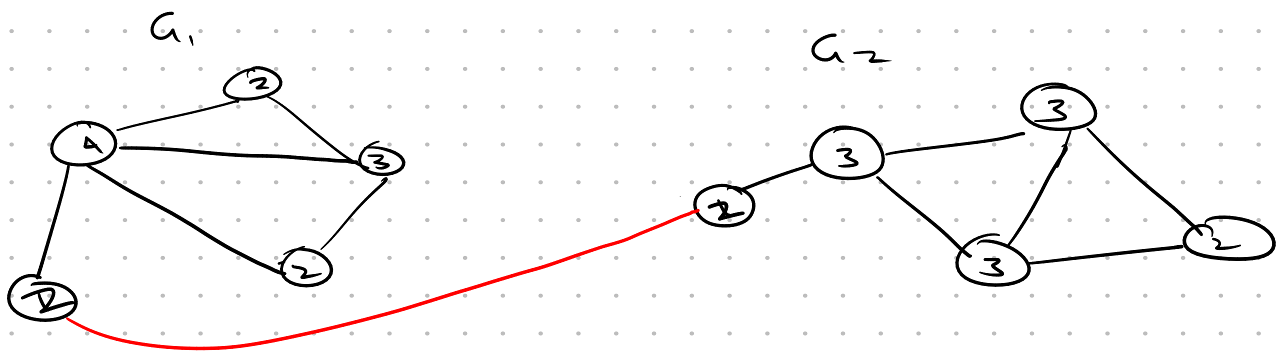
Question 8.

In Algorithm Archipelago, there are two islands that each have several towns connected by roads. These towns and roads form a connected graph on each island, labelled G_1 and G_2 respectively. A bridge is going to be built between two towns, one from each island, effectively introducing a new edge between one vertex in G_1 and one vertex in G_2. This new graph represented the joined islands will be called H.

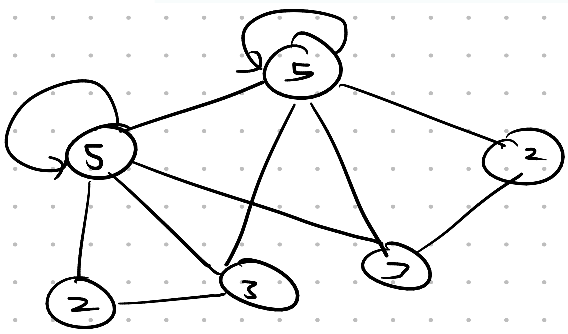
(a) Suppose G_1 has vertex degree sequence $4, 3, 2, 2, 1$ and G_2 has vertex degree sequence $3, 3, 3, 2, 1$. Find a vertex degree sequence for a new graph H which has no pendant vertices.

Enter your answer as a sequence of numbers separated by commas.

4, 2, 3, 2, 2, 2, 3, 3, 3, 2



(b) Explain why no matter what vertex degree sequences G_1 and G_2 have, H cannot have vertex degree sequence $5, 5, 3, 3, 2, 2$.



If the graph H is valid, then it should be possible, by removing one of the edges from the graph H, to create two connected components (or islands). Considering the vertex degree sequence, the two vertexes that could be possibly removed to create a partitioned connected components (the two 2's), are both connected to 5's and 3's. Hence, when removing an edge from one of these vertices, the entire graph stays connnected.

(c) For each statement below, select the correct conclusion.

"If both G_1 and G_2 contain a Hamilton cycle, then H contains a Hamilton cycle."

- ○ This statement is always true.
- ○ This statement is sometimes true and sometimes false. ✔
- ◉ This statement is always false.

Imagine two graphs with a Hamiltonian circuit are connected by a single edge. The single edge part of the problem, presents an impossibility for the hamiltonian circuit, as it must traverse through that edge twice to get back to the starting point.

"If both G_1 and G_2 are trees, then H is a tree."

- ◉ This statement is always true.
- ○ This statement is sometimes true and sometimes false. ✔
- ○ This statement is always false.

Joining up a tree at any node simply just creates a new subtree - hence, this is always true.

"If both G_1 and G_2 contain an Euler trail, then H contains an Euler trail."

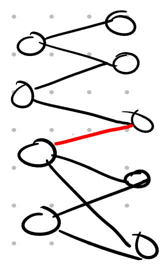
- ○ This statement is always true.
- ◉ This statement is sometimes true and sometimes false. ✔
- ○ This statement is always false.

Depends on the graphs vertex degree sequence, and as well, where the graphs were joined at. They must have an even degree for every single vertex.

"If both G_1 and G_2 are bipartite, then H is bipartite."

- ◉ This statement is always true.
- ○ This statement is sometimes true and sometimes false. ✔
- ○ This statement is always false.

Consider the two bipartite graphs below.
Join any two bi-partite graphs, at any node, and then the arrival node simply becomes the opposing node - and the bipartite graphs have been combined.



(d) Mayor Tin has noticed that both G_1 and G_2 are planar. They are worried that since creating H adds a new edge but does not change the total number of vertices, that this will violate Euler's formula for planar graphs. Is their claim correct? Explain your answer.

Mayor Tin forgot that creating a new edge between G1 and G2 would remove a region from H, as the external regions of G1 and G2 would combine into one.
Hence, this becomes:
$r - 1 - e + 1 + v = 2$
$r - e + v = 2$

Question 8.

Jasper is playing a role-playing game in which they can allocate skill points to any of 6 different abilities. A player always starts with 0 skill points in each ability, and each time they earn 1 skill point, they assign it to their choice of ability. Each ability can have a **maximum** of 100 skill points assigned to it.

(a) Jasper announces that they have assigned a total of 59 skill points. In how many ways could they have done this?

comb(64, 5)

$$a_1 + a_2 + a_3 + \dots + a_6 = 59 \qquad = \binom{64}{5}$$

(b) Jasper's more experienced friend Oliver announces that they have assigned a total of 555 skill points. In how many ways could they have done this?

comb(50, 5)

$a_1 + a_2 + a_3 + \dots + a_6 = 555.$ $\quad a_i \leq 100$ not satisfied

$\therefore 100 - b_1 + 100 - b_2 + \dots + 100 - b_6 = 555$

$\therefore b_1 + b_2 + \dots + b_6 = 45. \quad = \binom{50}{5}$

(c) Jasper's friend Charles has assigned a total of 192 skill points. Charles reports the following facts:

- They have assigned a skill points to the Archery ability.
- The Archery ability has the least amount of skill points assigned to it.
- Each other ability has at most $a + 8$ skill points assigned to it.

What is the minimum possible value for a?

26

What is the maximum possible value for a?

32

$a + b + c + d + e + f = 192.$

$a \leq b, c, d, e, f \leq a + 8.$

If evenly spread, then $a, b, c, d, e, f = \underline{32}$.

$\therefore \max(a) = 32.$

$\therefore \min(a) = 26$

Strategy
Maximum: Find where all the elements are equal, or atleast close to as equal as possible.

Minimum: Find the smallest value of a, such that every other value is equal. I did this by starting at x - 8 + x + ... = 192, and then slowly incremented up to -6, where I found everything was equal.

(d) To complete the game, Jasper must pass two "skill checks". In particular, they need:

- At least 12 skill points in the Archery ability or at least 18 skill points in the Buoyancy ability, **and**
- At least 16 skill points in the Camouflage ability or at least 10 skill points in the Dexterity ability.

Suppose at this stage that Jasper has assigned a total of 63 skill points, and the points have been assigned in such a way that Jasper is able to complete the game. In how many ways could they have assigned the skill points?

I(A U B) and (C U D)I = IA U BI + IC U DI = IA U B U C U DI

IA U BI = IAI + IBI = IA and BI

IAI:
A + B + C + D + E + F = 63
A + B + C + D + E + F = 51 = comb(55, 5)

IBI:
A + B + C + ... = 63 - 18 = 45
A + B + C + ... = comb(50, 5)

ICI:
A + B + C + ... = 63 - 16 = 47
A + B + C + ... = comb(52, 5)

IAI = comb(55, 5)

IDI:
A + B + C + ... = 63 - 10 = 53

IA and BI:
A + B + C + D + E + F = 63 - 30 = 33 => comb(38, 5)

IA and CI:
A + B + C + ... = 63 - 28 = 35 => comb(40, 5)

IA and DI:
A + B + C + ... = 63 - 22 = 41 => comb(46, 5)

IB and CI:
A + B + C + ... = 63 - 34 = 29 => comb(34, 5)

IB and DI:
A + B + C + ... = 63 - 28 = 35 => comb(40, 5)

IC and DI:
A + B + C + ... =

ahhh case bash cringe and unbased noon cares

Question 10.

(a) Ida begins with just one weapon that does 38 damage each turn. Their first foe (i.e. enemy) is the snoring snapdragon. It has 281 health, and never attacks back, but it only perishes (i.e. dies) when its health reaches exactly 0. If it suffers an attack that would leave it with negative health, instead of perishing, it regenerates 95 health. For example, if the snapdragon has 8 health and receives 10 damage, its health will be reduced to -2 and its regeneration spell will regain 95 health, leaving it with 93 health before the next turn.

Is Ida able to defeat the snoring snapdragon? Why or why not?

No;

Consider the following sequence:
281 - 38 * 8 => -23 + 95

72 - 38 * 2 => -4 + 95
91 - 3 => -23 + 95

A loop occurs

The reason for this is, is that the mod of gcd(95 % 38, 38) != 1, and therefore, loops can occur after a certain amounts of steps.

Select all weapons that would allow an adventurer to eventually defeat the snoring snapdragon.

- [x] The denim dagger, which does 66 damage each turn.
- [ ] The silver sword, which does 75 damage each turn.
- [x] The blue blade, which does 33 damage each turn.
- [ ] The mauve mace, which does 25 damage each turn.
- [ ] The crimson cutlass, which does 57 damage each turn.

(b) Ida's second foe is the hale hippogriff. It has 2808 health, and never attacks back, but it also only perishes when its health reaches exactly 0. If it suffers an attack that would leave it with negative health, instead of perishing, its health resets to 2808. This time Ida has two weapons at their disposal: the vorpal sword which does 84 damage in one turn, and the manxsome blade which does 66 damage in one turn. Ida claims that they are able to defeat the hale hippogriff. What is the least number of attack turns required to defeat the hale hippogriff?

$84x + 66y = 2808$

$84 = 66 + 18$
$66 = 18 \times 3 + 12$
$18 = 12 + 6$
$12 = 6 \times 2 + 0$

$6 = 18 - 12$
$= 18 - 66 + 3 \times 18$
$= 4 \times 18 - 66$
$= 4 \times (84 - 66) - 66$
$6 = 4 \times 84 - 5 \times 66$
$2808 = 1872 \times 84 - 2340 \times 66$

$x = 1872 + 66s$ $\quad y = -2340 + 84t$

$x = 24$ $\quad y = 12$

$\therefore x + y = 36$

(c) Ida's third foe is the retaliatory wraith, but it is undefeatable! The retaliatory wraith has three weapons at its disposal, and on each turn will use exactly one of them. The weapons are:

- the small sabre, which does 9 damage in one turn,
- the middling mace, which does 10 damage in one turn, and
- the large lance, which does 11 damage in one turn.

The retaliatory wraith decides to draw up a battle plan, listing which choice of weapon it will use each turn until Ida, who has 32 health, perishes (once Ida's health reaches a non-positive amount). How many such battle plans exist?

Question 11

(a) Your first guess is PQAJI, and the response is that P is in the secret word, but not in the first place, and Q, A, J, I are not in the secret word. How many possibilities remain for the secret word?

21 * 22^4 - 21^5

P = ? P = 0

21	22	22	22	22	−	21	21	21	21	21

Everywhere else, we don't know.

= P21.

(b) While you are still thinking about the puzzle in (a), a friend who has already solved the same puzzle tells you that the secret word has no repeated letters. Now how many possibilities remain for the secret word?

* perm(21, 4) - perm(21, 5)

Same as above, but with permutations.

← no p/q/a/j/i

$$21 \times {}^{21}P_4 - {}^{21}P_5.$$

(c) Next day you are playing the game again, so that the secret word may be the same or may be different. This time your first guess VZSKL is very lucky and you are told that V is in the secret word but not in the first place; Z is in the secret word but not in second place; S is in the secret word but not in third place; K is in the secret word but not in fourth place; and L is not in the secret word. Your friend again tells you that there are no repeated letters. How many possibilities are there for this day's secret word?

Let U be the set of lucky words which contain VZSK but don't contain L.
|U| = 21 * 5! (re-arranging the letters)

Let S_(V, Z, S, K) be the set of lucky words that contain VZSK, doesn't contain J, and has V in first place.
21 * 4!

Let S_V and S_Z, S_V and S_S .... be the set of lucky words that contain VZSK that don't contain J, and have the two letters at the respective
21 * 3!

Let S_V and S_Z and S_S, ... be the set of lucky words that contain VZSK that don't contain j, and have the three letters at their respective conditions
21 * 2!

Let S_V and S_Z and S_S and S_K:
21

Therefore, combinations are

21 * 5! - (4 * 21 * 4! - C(4, 2) * 21 * 3! + C(4, 3) * 21 * 2! - 21)

(d) Finally one more day. Again you are told that there are no repeated letters. From your first two guesses you determine that the secret word contains I, U and E; that it does not contain any of B, M, A, T, P, F; that I is not in the first or the second place, U is not in the third place and E is not in the fourth place. How many possibilities remain for this secret word?

S_I(1), S_I(2), ... = C(17, 2) * 4!
S_I(1) and S(U) = C(17, 2) * 3!
S_I(1) and S(U) and S(E) = 2 * C(17, 2) * 2!

C(17, 2) * 5! = Chose two other alphabetic characters, and arrange characters into positions.

C(4, 1) * 4! * C(17, 2) = 4 different found characters to fix to their positions, 4! different positions for other characters, 2 different other alphabetic characters.

(C(4, 2) - 1) * 3! * C(17, 2) = C(4, 2) - 1 represents every combination of two found positions, besides the double up of I (I cannot exist twice), 3! different positions and two other alphabetic characters

2 * 2 * C(17, 2) = 2 combinations of three in fixed place (C(4, 3) - 2, as we cannot have I double up), 2 different positions for remaining characters, chose two other alphabetic characters.
C(17, 2) * 5! - (C(4, 1) * 4! * C(17, 2) - (C(4, 2) - 1) * 3! * C(17, 2) + 2 * 2 * C(17, 2)) = 6800

Question 12.

For any positive integer n, we define the set

$$S_n = \left\{ k \in \mathbb{Z} \mid 3n + 4 \leq k \leq 3n^2 + 8 \right\}.$$

Find the number of elements in each of the following sets.

Syntax advice: You may give your answers as exact numbers, or as expressions in Maple notation. As well as the standard operations +, -, *, /, ^ you may also write

- `n!` for factorials,
- `c(n,r)` for combination numbers (binomial coefficients), and
- `p(n,r)` for permutation numbers.

Make sure you include any necessary brackets. Always use * to denote multiplication.

(a) If $A = \{ S_n \mid 6 \leq n \leq 26 \}$, then the number of elements in A is

21 .

$S_6, S_7, S_8, \ldots, S_{26} = 21$ sets.

(b) If $B = \{ x \mid x \in X \text{ for some } X \in A \}$, then the number of elements in B is

3*(26)^2 + 8 - 3 * (6) - 4 + 1 .

Since $3n+4 \leq k \leq 3n^2+8$

$\Rightarrow (3 \times (26^2) + 8) - (3 \times (6^2) + 4) + 1$

(c) If $C = \{ x \mid x \in X \text{ for some } X \subseteq A \}$, then the number of elements in C is

21 .

Same idea as above.

(d) If $D = \{\, x \mid x \in X \text{ for all } X \in A \,\}$, then the number of elements in D is

3 * (6)^2 + 8 - 3 * 26 - 4 + 1

(e) If $E = \{\, x \mid x \in X \text{ for all } X \subseteq A \,\}$, then the number of elements in E is

0

such an X does not exist.

(f) If $F = \{\, \text{functions from } S_6 \text{ to } S_7 \,\}$, then the number of elements in F is

$|S_7| = (3\times7^2+8) - (3\times7+4) + 1$

$|S_6| = (3\times6^2+8) - (3\times6+4) + 1$

To find element from $n \to k$, it is $k - n + 1$.

To make functions from S_6 to S_7, we need to find every combination of one S_6 value mapping to one or more S_7 values.

$= 131^{95}$

(g) If

$$G = \{\, \text{one-to-one (injective) functions from } S_6 \text{ to } S_7 \,\},$$

then the number of elements in G is

perm(131, 95)

(h) If

$$H = \{\, \text{onto (surjective) functions from } S_6 \text{ to } S_7 \,\},$$

then the number of elements in H is

0

Question 13.

(i) If R is reflexive or S is reflexive, then T is reflexive.

- [x] True
- [ ] False

If xRx or xSx, then xTx must also exist within the definition of the relation T.

(ii) If T is reflexive, then either R is reflexive or S is reflexive.

- [ ] True
- [x] False

Not necessarily true. T exists when R or S exists, but does not necessarily share common elements. For example, xTx does not imply that in the relation R and S, that xRx and xSx.

(iii) If T is antisymmetric, then both R and S are antisymmetric.

- [x] True
- [ ] False

For xTy and yTx to imply x = y - then for both R and S, this must also be true, as the relations of R and S also both exist.
Since xTy => xRy U xSy, if a relation of T is anti-symmetric, then it must also be anti-symmetric in xRy or xSy, as it dictates a relationship between two values.

(b) Let P be a relation on a set B. We define a relation Q on B by

$$x \; Q \; y \quad \text{if and only if} \quad \text{there exists } z \in B \text{ such that } x \; P \; z \text{ and } z \; P \; y$$

(i) If

$$B = \{3, 14, 15, 20\} \quad \text{and} \quad P = \{\, (20,3),\ (3,3),\ (3,14),\ (14,15) \,\},$$

find the relation Q, giving your answer as a set of ordered pairs.

Syntax advice: You must use Maple notation; specifically, square brackets for ordered pairs whose entries are separated by commas, and curly brackets for the set whose elements are separated by commas. For example, if the answer is

$$Q = \{(1,2), (3,4), (5,6)\},$$

then enter

`{[1,2],[3,4],[5,6]}`

$Q =$), 3], [20, 14], [3, 14], [3, 3],

{[20, 3], [20, 14], [3, 14], [3, 3], [3, 15]} (dont forget reflexive relation!)

(ii) For relations P and Q, we say that $Q \subseteq P$ when the set of ordered pairs comprising Q is a subset of the set of ordered pairs comprising P.

Prove that P is transitive **if and only if** $Q \subseteq P$.

First prove that if Q is a subset of P then P is transitive.
If Q is a subset of P, then every ordered pair within Q exists within P. By definition of Q, it must be true that in every equation, xPz and zPy, for an ordered pair within Q. However, since Q is a subset of P, it must be true that xPy and yPy for some ordered pair (x, y) in Q, which also satisfies the transitivity of P.

Prove that if P is transtivie, then Q exists within P.
If P is transitive, then for every ordered element xRy and yRz, (x, z) exists. Since this is also the innate definition of Q, it follows that Q is indeed a subset of P.

(iii) By considering the example where $B = \mathbb{Z}$ is the set of integers and $x\ P\ y$ means $x < y$, or otherwise, prove that the following statement is **not true**: if P is transitive then $Q = P$.

Assume that if P is transitive, then Q does indeed = P.
Hence, when B = Z, and P is some set, for example = {(1, 2), (2, 3), (1, 3)}, we get Q = {(1, 3)}. However, immediately a contradiction is shown. P is transitive, but Q != P. Hence, the statement is false.

Question 14.

(a) The aim of this part is to find a solution of the congruence

$$x^2 \equiv -568 \pmod{1231^2}.$$

Let $x = 66 + 1231t$, where t is an integer. Then

$$x^2 \equiv -568 \pmod{1231^2} \quad \text{if and only if} \quad at \equiv b \pmod{1231}$$

for certain integers a and b.

Find suitable values of a and b and enter them in the box **in that order**.

Syntax advice: Enter your answer as an **ordered pair** using square brackets and separated by a comma. For example, enter the values $a = 12$ and $b = -34$ using the syntax

[12,-34]

Answer: suitable values of a and b are [132, -4] .

Before attempting to solve the congruence $at \equiv b \pmod{1231}$, it is wise to confirm that it actually does have a solution. This is true because (choose the correct reason)

- a and 1231 are relatively prime
- b and 1231 are relatively prime
- 1231 is a factor of b
- a, b and 1231 are relatively prime
- a and b are relatively prime
- 1231 is not a factor of b
- b is a multiple of a

$$(66+1231t)^2 \equiv -568 \pmod{1231^2}$$

$$(1231)^2t^2 + 2(1231)(66)t + 66^2 \equiv -568 \pmod{1231^2}$$

$$= 0 + 2(1231)(66)t \equiv -4924 \pmod{1231^2}$$

$$= 132t \equiv -4 \pmod{1231}$$

By solving the congruence $at \equiv b \pmod{1231}$, find **one** suitable value of t.

(You do not need to give all values of t: enter your answer as a single number.)

Answer: $t \equiv$ 1604 $\pmod{1231}$.

$132t \equiv -4 \pmod{1231}$

$1231s + 132t = 1$

$1231 = 132 \times 9 + 43$

$132 = 43 \times 3 + 3$

$43 = 3 \times 14 + 1$

$1 = 43 - 3 \times 14$

$1 = 43 - 14 \times (132 - 43 \times 3)$

$= 43 - 14 \times 132 + 42 \times 43$

$= 43 \times (1231 - 132 \times 9) - 14 \times 132$

$= 43 \times 1231 - 387 \times 132 - 14 \times 132$

$1 = 43 \times 1231 - 401 \times 132$

$-4 = -172 \times 1231 + 1604 \times 132$

Hence, find **one** solution of the congruence $x^2 \equiv -568 \pmod{1231^2}$.

(You do not need to give all values of x: enter your answer as a single number.)

Answer: $x \equiv$ (66 + 1231 * 1604) $\pmod{1231^2}$.

(b) By modifying the ideas used in (a), or otherwise, prove that for every positive integer n, the congruence

$$x^2 \equiv -568 \pmod{1231^n}$$

has a solution.

$(66 + 1231t)^2 \equiv -568 \pmod{1231^n}$

$\Rightarrow 66^2 + 2 \times 66 \times 1231t + (1231)^2 t^2 \equiv -568 \pmod{1231}$

$\Rightarrow 132t \equiv -4 \pmod{1231^{n-2}}$

$\therefore$ Always has solution, 1231 is prime, and 132 is not a multiple of 1231, $\therefore$ always coprime, $\therefore$ always solution.

Question 15.

A metafield is an object studied in the field of mathematical stereotopodynamics. A metafield may be doubly-Euclidean, or not, and it may be sporadic, or not.

Let X be a metafield. We wish to prove the statement "if X is doubly-Euclidean, then X is sporadic".

Select all of the following proof outlines which could **not possibly** give a proof of the above statement. That is, if there is no possibility of filling in the dots in an option so as to give a logically correct proof of the statement, then you should select it; if it is possible to fill in the dots so as to obtain a correct proof, then you should not select it.

- [x] Suppose that X is both doubly-Euclidean and sporadic .... This is a contradiction.
- [ ] Suppose that X is not sporadic .... Therefore X is not doubly-Euclidean.
- [ ] Suppose that X is doubly-Euclidean .... Therefore X is not doubly-Euclidean.
- [x] Suppose that X is not doubly-Euclidean .... Therefore X is not sporadic.
- [ ] Suppose that X is doubly-Euclidean .... Therefore X is sporadic.
- [ ] Suppose that either X is doubly-Euclidean or X is sporadic .... This is a contradiction.
- [ ] Suppose that X is doubly-Euclidean and not sporadic .... This is a contradiction.
- [x] Suppose that X is doubly-Euclidean .... Therefore X is not sporadic.

Question 16.

The graph G has 10 vertices, and the degrees of its vertices are

$$6, 5, 5, 5, 5, 3, 2, 2, 2, 1\,.$$

1 2 3 4 5 6 7 8 9 10

Prove that G is **not** bipartite.

Be sure to give a clearly written, detailed and logically accurate answer - full marks will not be given for sketchy work.

Consider a bipartite graph can be partitoned into two distinct sets, such that edges only go between the sets, to their respective adjacent neighbours.

Consider the vertices:
v1v2v3..., and arbitarily partition them into sets A and B.

Putting v1 with degree 6 into A, we would, at the minimum have:
A = {v1} B = {v2, v3, v4, v5, v6, v7}
Putting v2 with degree 5 into A, we would, at the minimum have:
A = {