2022 T2 and T3

Q2)

Let's go through each of these two reinforce our ability to understand matrix and transformation properties:
i) has 3 linearly independent columns
1. Definitely has a 3 linearly independent columns, by inspection
2. Definitely not, as we are only in $\mathbb{R}$, and hence, can only have a maximum of 1 linearly independent columns.
3. By inspection, this also has 3 linearly independent columns
4. Again, in $\mathbf{R}$, therefore, only one linearly independent column.
5. Only has one linearly independent column
6. Has 4 linearly independent columns, as $\mathbf{P_3} \to \mathbf{R_4}$.
7. The span encompasses $\mathbf{R_4}$, therefore, does not have 3 linearly independent columns
8. Changes into a constant, therefore 3 linearly independent columns can't exist
9. Only 1 linearly independent column
10. $\mathbf{P_4} \to R_5$, therefore, 5 linearly independent columns
11. A span, which encompasses a whole lot more then 3 linearly independent columns

---

ii) is rank 1

1. Has rank 3, with 3 linearly independent columns
   2. Is in the space R, therefore, has rank 1
2. Rank 3, with 3 linearly independent columns
3. Rank 1, as the result becomes a constant
4. Rank 1, as 1 linearly independent column
5. Rank 4, 4 linearly independent column
6. Rank 3
7. Rank 1, as constant
8. Technically, rank 2
9. $\mathbf{P_4}$, therefore, 5 linearly independent columns
10. 4 linearly independent columns, rank 4

iii) has 3 linearly independent columns
For a 3-dimensional eigenspace, we need a 3 dimensional matrix. Therefore, only 1 fits this.

iv) is a four-dimensional vector space

1. Not a vector space
2. 1 dimensional
3. Not a vector space
4. 1 dimensional
5. Not a vector space

   6. 4 dimensional ($\mathbf{P_3} \to \mathbf{R_4}$)

   7. 4 dimensional vectors, but definitely not encompassing, as not all linearly independent

6. 1 dimensional
7. Not a vector space
8. 5 dimensional vector space

   11. 4 dimensional

---

v) has a positive eigenvalue

1. Check using maple command `Eigenvectors()`
2. Eigenvectors only exist within linear transformations from $\mathbf{V} \to \mathbf{V}$.
3. Eigenvectors only exist within **n x n** matrix
4. Eigenvectors only exist within linear transformations from $\mathbf{V} \to \mathbf{V}$.
5. Check using maple command `Eigenvectors()`
6. Eigenvectors are defined as vectors that stay constant throughout a linear transformation. Therefore, $\mathbf{P_3}$ does not make sense.
7. Spans do not have eigenvectors, see above
8. Consider tha $p_1 = ax + b, p_1' = a$, therefore the only vector that stays constant is 0.
9. Check using maple command `Eigenvectors()`
10. Eigenvectors are defined as vectors that stay constant throughout a linear transformation. Therefore, $\mathbf{P_4}$ does not make sense.
11. Spans do not have eigenvectors, see above

---

vi) has positive nullity

1. `NullSpace(<<1, 0, 0>|<0, 1, 0>|<0, 0, 1>>)`

2. Consider functions that follow the pattern $F(1) - F(0) = 0$. Therefore, there is the positive nullity.

3. Row reduced, and has nullities, therefore positive nullity

4. There are multiple variations $\mathbf{v} \cdot \begin{pmatrix} -7 \\ 6 \\ 1 \end{pmatrix}$ such that it $= 0$. Therefore, nullity is positive

5. Nullity is positive, as two equal columns.

2. Vector space, therefore nullity is zero.
3. Span, therefore, nullity is zero.
4. Linearly independent
5. `NullSpace(<<0, 1>|<1, 0>>)`
6. Vector space, therefore nullity is zero
7. Span, therefore nullity is zero

vii) is a subspace of P_{17}

8. Not a set, therefore cannot be a subspace
9. Not a set, therefore cannot be a subspace
10. Not a set, therefore cannot be a subspace
11. Not a set, therefore cannot be a subspace
12. Not a set, therefore cannot be a subspace

6. A lower dimensional polynomial set, therefore is a subspace

13. Span of $\mathbf{R}_4$, therefore not a subspace
14. Not a set, therefore cannot be a subspace
15. Not a set, therefore cannot be a subspace,

10. A lower dimensional polynomial set, therefore is a subspace

11. Span of polynomials, linearly independent, therefore is a subspace

viii) is diagnolisable

1. 3 linearly independent eigenvectors, therefore, diagonlisable

16. Nope, check above
17. Nope, check above
18. Constant term, therefore not diagonlisable

5. Diagonalisable, as two linearly independent eigenvectors

19. Not diagonalisable, vector space therefore no eigenvectors
20. Not diagonalisable
21. Not diagonalisable

9. Diagonalisable, two linearly independent vectors

22. Not diagonalisable, vector space threfore no eigenvectors
23. Set, therefore not diagonalisable

    ix) has linearly dependent columns
24. Not linearly dependent
25. Constant

    3. Yep, linearly dependent
26. Constant

    5. Yep, linearly dependent
27. Vector space, so linearly independent
28. Span, so a distinct set
29. Constant
30. Linearly independent
31. Vector space, so linearly independent
32. Span, so a distinct set

Q3)

Consider all of the statements

1. Let **u, v** be vectors in $\mathbf{R^6}$. The set span{u, v} is always a plane through the origin.
   They must be linearly independent to be a plane.
2. The columns of the matrix A are linearly independent iff A$\mathbf{x} = \mathbf{0}$ has only the trivial solution.
   Yep, the trivial solution is just such that all scalars $\lambda_n = 0$. This is the definition of linear independence.
3. If two rows of a matrix A are the same, then $\boldsymbol{rank(A) > 0}$.
4. If two rows of a matrix A are the same, then nullity(a) $>$ 0.
   We can check this using some arbitrary matrix A:

   $$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$$

   This reduces to:

   $$\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$$

   Therefore, nullity(A) $>$ 0. rank(A) $>$ 0 cannot be guaranteed in these conditions.
5. If A is a matrix with the property that the sum of the entires in each row is 13, then A has a non-zero eigenvalue

Consider AA^T. This is a subspace spanned by A, and a non-zero eigenvector lies in this space. Since we know that the value of these rows and columns are 13, we know that the subspace must also exist.

6. Let A be an n x n matrix. If x is a nontrivial solution of Ax $=$ 0, then every entry in x is non-zero.

   This by knowledge of the non-trivial solutions, can be found to be untrue.

7. If A is a m x n matrix, then T(x) $=$ Ax defines a linear transformation from $\mathbf{R^n} \to \mathbf{R^m}$

   Consider we want to move from a matrix with properties $n : a,\ a \in R$ to $m : b\ ,b \in R$.
   Therefore, we require a matrix defined as m : n to transfer the matrix.
   m : n : n : a $=$ m : a.

Q4)

1. $V = \mathbb{C}^2, H = \{\begin{pmatrix} x \\ y \end{pmatrix} \in C^2 : xy \leq 0\}$

   Evidently not closed under scalar multiplication.

2. $V =$ integrable functions on $[2, 5], H = \left\{f \in V : \int_2^5 f(x)\,dx\right\}$

   Consider that:

$$\begin{aligned} F(a) + F(b) &= \int f(a)\,dx + \int f(b)\,dx \\ &= \int f(a) + f(b)\,dx \\ &= F(a+b) \\ F(\lambda a) &= \int f(\lambda a)\,dx \\ &= \int \lambda f(a)\,dx \\ &= \lambda F(a) \end{aligned}$$

3. $V = \mathbf{P_{46}}, H = \{p \in \mathbf{P_{46}} : p(15) = 0\}$

   Zero exists. $q(15) + r(15) = 0 = p(15)$. $\lambda q(15) = 0$

4. $V = \mathbf{C^2}, H = \{< x, y >: x, y \in \mathbf{Z}$

   This doesn't pass scalar multiplication $i(2) = 2i \neq \mathbf{Z}$

5. $V = \mathbf{P_3}, H = \{p \in P_3 :$ the degree of p is $3\}$

   Passes scalar addition, and scalar multiplication, but the zero element does not exist.

6. $V = M_{3,6}(\mathbf{C}), H = \{$big matrix $: a, b, c \in \mathbf{C}\}$

   Visually passes scalar multiplication and addition condition - zero exists such that $a, b, c = 0$.

7. $V = M_{6,6}(\mathbf{C}), H = \{A \in M_{6,6}(\mathbf{C}) : A^T = -A\}$

Consider two arbitrary vectors $\mathbf{X}, \mathbf{Y} \in \mathbf{M_{6,6}(C)}$

The scalar condition holds trivially, so lets consider the addition condition.

$\mathbf{X}^T + \mathbf{Y}^T = -X - Y$

$(\mathbf{X} + \mathbf{Y})^T = -(\mathbf{X} + \mathbf{Y})$

Therefore, closed under addition.

The zero element too exists, as the transpose of zero is zero, and the negative of zero is also zero.

8. $V = \mathbf{P_{26}}, H = \{p \in \mathbf{P_{26}} : p(x) \leq 0 \text{ for all } x \in \mathbf{C}\}$

Evidently wouldn't hold under scalar multiplication of -1.

9. $V = \mathbf{C^2}, H = \text{span}(\begin{pmatrix} 8 \\ -5 \end{pmatrix})$

Just a line through $\mathbf{R_2}$, therefore, also a line through $\mathbf{C_2}$, and a subspace for both.

10. $V = \mathbb{P}_{77}, H = \{p \in \mathbb{P}_{77} : p(0) = 75\}$

Evidently does not contain the zero element.

Q5)

Note that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denote the basis vectors of the given domain.

a)

$$T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

b) We can't calculate this yet.

c) $T(\mathbf{e}_1) + T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -9 \end{pmatrix}$

d) Computed above for 1st and 3rd.

For T<0, 1, 0>:

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Q6)

a) Specifically stipulates *necessary* to check:

Therefore, we check the two conditions as outlined within the notes:

The closure condition:

$\mathrm{T(p + q) = T(p) + T(q)}$

The scalar multiplication condition:

$\mathrm{T(\lambda q) = \lambda T(q)}$

b) Consider that the ker(T) is the nullspace of the linear transformation.
Therefore, the nullspace exists within the original vector space, here: $\mathbf{P_4}$.

c) Consider that im(T) is the set of all solutions which exists within the linear transformation.
Therefore, the image exists within the new vector space, here: $\mathbf{R^2}$.

d) We know that the ker(T), or the null space, is in the vector space $\mathbf{P}_4$. Therefore, we should look out for any polynomial within the space $\mathbf{P_4}$ such that substituting $x = 0$ would lead it to equal 0.
In my case, the polynomials were:

$x^3 - 3x^2 + 2x$

$p(x) = 0$

$x^2 - 2x$

e) We know that the im(T), or the range, is in the vector space $\mathbf{R_2}$. So, choose any of the vectors that exist within $\mathbf{R_2}$.
In my case, these vectors were:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

f)
The basis for im(T) would be made up of at least two linearly independent vectors from the image (excluding the zero vector).
If the zero vector exists, it can no longer be considered a basis - as the span would be pulled to zero.
In my case, the basis were:

$$\{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}\}, \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}, \{\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix}\}$$

h) nullity(T) = 3
Reasoning (thanks to gerald on mathsoc/CSESoc):

```
T(p) = <p(0), p(2)>
Consider that for a polynomial to be within the kernel, it must have
roots 0 and 2. Therefore, the polynomial p(x) would look akin to: p(x) =
x(x-2)q(x). It's clear to see that q(x) is a polynomial of degree 2, as
p(x) is a polynomial of degree 4. p(x)'s only stipulation is that it
maps to zero to be considered a nullity, hence, q(x) is free to be any
vector. Therefore, since q(x) is of polynomial space 2, it is of
dimension three, and therefore, nullity 3 (nullity(T) = dim(ker(T)))
```

Q7)

a)

$$\begin{aligned} P(2H|H) &= \frac{P(2H \cap H)}{P(H)} \\ &= \frac{\frac{14}{19}}{\frac{4}{19} + \frac{14}{19} \cdot \frac{1}{2}} \end{aligned}$$

b)

$$P(\text{Ordinary} \mid \text{Tails}) = \frac{\frac{14}{19} \cdot \frac{1}{2}}{\frac{14}{19} \cdot \frac{1}{2} + \frac{1}{19}}$$

c)

Consider that the probability $P(X \geq 7)$ can be expressed as $1 - P(X < 7)$.
The new situation has the foundation of 4 heads being guaranteed. Therefore, the $P(X < 7)$ will include: $P(4), P(5), P(6)$.
Calculate these probabilities:

$$P(4) + P(5) + P(6) = \binom{14}{0}\left(\frac{1}{2}\right)^{14} + \binom{14}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{13} + \binom{14}{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{12}$$

Now, minus this from one to find $P(X \geq 7)$.

Q8)

a)

$$\begin{aligned} E(X) &= 0 \cdot 0.1 + 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.2 \\ Var(X) &= E(X^2) - [E(X)]^2 \end{aligned}$$

b) $P(X=1) = 0.1 \therefore (0.4)^3$

c)

Three different ways to make a combination of 4

(3, 1) (2, 2) (1, 3)

Just calculate these using the given probabilities

Q9)

i) We're given that nullity(A) = 2, so rank(A) = 2 by rank-nullity theorem

ii) (Thanks to one eight seven on CSESoc)

The first condition 'ker(A) has a basis consisting of two vectors' gives us that there are two eigenvalues that = 0, as $A\mathbf{v} = \mathbf{0}$.

The second condition 'A<5, 4, 3, 2> = <5, 4, 3, 2>' gives us that theres an eigenvalue 1, as $A\mathbf{v} = \mathbf{v}$.

The third condition 'det(A-2I) = 0', gives us that there is an eigenvalue two by the characteristic polynomial equation.

iii) Yep, it's possible, since we have all 4 eigenvalues.

Therefore, since are eigenvalues are 2, 0, 0 1:

$x^2(x-1)(x-2)$

b) Let B be a matrix such that $B^2 = 0$.

i) Consider that for a matrix to be in powers, it can be put in the form such that:

$B^n = AD^nA^{-1}$.

Therefore, consider that when $B^2 = AD^2A^{-1} = 0$, the eigenvalues must be zero, which are contained in D.

ii) The nullity of a matrix is equal to the eigenspace of a matrix. Therefore, nullity(B) = 3.

iii) By rank-nullity theorem, rank is 3.

iv) No, number of eigenvectors $\neq$ dimension of matrix.

iv)

There is a simple way to check if a matrix is diagonalisable:

1. If the eigenvalues all have multiplicity one, it is diagonalisable
2. If not, check that each eigenvalue has a specific linearly independent vector.

```
It can be seen that there are already 4 linearly independent
eigenvectors. ker(A) has a basis of two vectors, therefore, there are
two linearly independent vectors with eigenvalue zero. Then, the
eigenvalue one has another eigenvector <5, 4, 3, 2>. Then, the
eigenvalue also represents another linearly independent vector,
therefore, diagonalisable.
```

Q10)

a) By inspection, a first order differential equation. It is *not* separable, as you cannot factor out each variables to their own sides.
Note: An *ordinary differential equation* is a DE which only relies on one variable.

b) Use $y_0(x) = C$. $y_0(x) = 0$, and then consider that the differential equation holds.

c) By inspection, this is a first order differential equation (no powers on the dx), and it is separable (terms can be factored into each side).

d) `convert(1/(v*(v^2+1)), parfrac)`

e) Integrate the differential equation from 3.

Q11)

a) $u = e^{4x}$
Sub in $\frac{dx}{du} = \frac{1}{4e^{4x}}$

b) The denominator becomes $4u^3 - 8u^2 = 4u^2(u-2)$.

c)
`int(1/(exp(8*x) - 2*exp(4*x)), x)`
Then factorise into proper form to get variables.

Q12)

a)

```
v := t -> sqrt(1/4*Pi^2*sin(1/2*t*Pi)^2 + Pi^2*sqrt(3)^2/sqrt(t)^2)
v(3)
```

b)

$$
\begin{aligned}
\frac{dT}{dt} &= \frac{\partial T}{dx} \cdot \frac{dx}{dt} + \frac{\partial T}{dy} \cdot \frac{dy}{dt} \\
&= 4x\left(-\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)\right) + 10y\left(\frac{\pi\sqrt{3}}{\sqrt{t}}\right) \\
&= 4\left(\cos\frac{\pi t}{2} - 1\right)\left(-\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)\right) + 20\pi\sqrt{3t}\left(\frac{\pi\sqrt{3}}{\sqrt{t}}\right)
\end{aligned}
$$

$M = -2\pi, N = 60\pi^2$

Q13)

a) Ali then concludes that $\lim_{n\to\infty} a_n$ exists. Explain briefly why this limit exists.

```
a_1 = 4 - sqrt(16-4) = 4 - sqrt(12)

a_1 / a_0 < 1, therefore limit exists.
```

b) As above, $4 - \sqrt{16-0} = 0$

c) Since it tends to zero, the sequence itself tends to $4 - \sqrt{16-0} = 0$.

d) $\beta = 8$. It is convergent, as the ratio test is $-1 < x < 1$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \frac{4 - \sqrt{16 - a_n}}{a_n} \\
&= \frac{4 - \sqrt{16 - a_n}}{a_n} \cdot \frac{4 + \sqrt{16 - a_n}}{4 + \sqrt{16 - a_n}} \\
&= \frac{1}{4 + \sqrt{16 - a_n}} \\
&= \frac{1}{4 + 4} \\
&= \frac{1}{8}
\end{aligned}
$$

Q14)

a) Just add the power to the sequence:

$$\left(\frac{n}{n+4}\right)^{n^2\cdot\frac{1}{n}}$$

```
limit((n/(n + 4))^n, n = infinity) Output: exp(-4)
```

b)

Thank you to Tas for pointing out this mistake!

We have established in part a) that $a_n^{1/n}$ tends to $\exp(-4)$.

Hence, consider the following proof of the convergence of a_n:

$$\begin{aligned} a_n^{1/n} &\to e^{-4} \\ a_n &\to e^{-4n} < 2^n e^{-4n} \\ \sum_n^{\infty} a_n &< \sum_n^{\infty} 2^n e^{-4n} \end{aligned}$$

$$\text{By the ratio test, we get:} \\ = \frac{2^{n+1}e^{-4(n+1)}}{2^n e^{-4}} < 1$$

Therefore, RHS converges, and by comparison, LHS converges.

c)

```
It can be shown by the kth term divergence test, that the series does not tend to zero as k -> infinity, and rather tends to exp(-4). Hence, the series is divergent.
```

Q15)

a)

$$\begin{aligned} \mathcal{L} &= \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, dx \\ &= \sqrt{16\cos^4\left(\frac{\theta}{3}\right)\left(\cos^2\left(\frac{\theta}{3}\right) + \sin^2\left(\frac{\theta}{3}\right)\right)} \\ &= 4\cos^2\left(\frac{\theta}{3}\right) \end{aligned}$$

therefore, $n = 2$.

b)

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{K}{r^2}$$
$$= 3\cos(2\theta) + \frac{3\sin^2(2\theta)}{\cos(2\theta)}$$
$$= \frac{3\cos(2\theta)^2 + 3\sin(2\theta)^2}{\cos(2\theta)}$$
$$= \frac{9}{r^2}$$

therefore, $K = 9$.

c)

Note that for a surface area about the y-axis in polar curves, it is defined by:

$$A = \int_{\theta_0}^{\theta_1} 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,dx$$

Q16)

a) `taylor(exp(x^2), x=0, 9)`

b) Use `diff(exp(x^2), x, x)`

Reasoning: Compare the taylor series coefficients

c)

$$\int e^{x^2}\,dx = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)k!}\,dx$$

$$= \int\sum_{k=0}^{\infty}\frac{f^{(k)}(e^{x^2})}{k!}x^k\,dx$$
$$= \sum\frac{x^{2k+1}}{(2k+1)k!}\,dx$$

d)

$$\int y^2 + 1\, dy = \frac{y^3}{3} + y$$

$D = \frac{4}{3}$

Reasoning:

$$\frac{4}{3} = \sum_{k=0}^{\infty} \frac{1^{2k+1}}{(2k+1)k!} + D$$

$$= D \text{ (As the series tends to zero)}$$

Q17)

a)

$$\frac{\partial S}{\partial x} = 2y - 7z$$
$$\frac{\partial S}{\partial y} = 2x - 7z$$
$$\frac{\partial S}{\partial z} = 7y - 7x$$

then use chain rule.

b)

$$\begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \\ -1 \end{pmatrix}$$

c)

```
DotProduct(<-7/2, 81/14, -1>, <-2, 0, 7>)
Output: 0
```

Therefore, perpendicular normals, therefore, perpendicular tangent planes, as tangent planes are defined by their normal vectors.

Q18)

a)

$$\frac{dV}{dt} = -15$$
$$V = -15t + C$$
$$V = 400 - 15t$$

b)

$$0 = 400 - 15t$$
$$400 = 15t$$
$$\frac{400}{15} = t$$

c)

$$\frac{dM_{in}}{dV} \cdot \frac{dV_{in}}{dt} = 11 \cdot 5$$
$$\frac{dM_{out}}{dV} \cdot \frac{dV_{out}}{dt} = \frac{M(t)}{400 - 15t} \cdot 20$$

d)

All water is drained out, so the concentration of salt should be zero at the time t*.