

2023 T1

Q2)

Proof. Suppose $0 \leq x \leq \sqrt{\frac{36}{17}}$. Notice that the Maclaurin series of f is given by:

$$f(x) = \sum_{k=0}^{\infty} A_k x^{2k+1}$$

Solution:

First, consider that the Maclaurin series of e^x :

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ e^{\frac{-17t^2}{12}+7} &= e^7 \sum_{k=0}^{\infty} \frac{\left(-\frac{17t^2}{12}\right)^k}{k!} \\ &= e^7 \sum \int_0^x \frac{\left(-\frac{17}{12}\right)^k t^{2k}}{k!} dt \\ &= e^7 \sum \frac{x^{2k+1} \left(-\frac{17}{12}\right)^k}{(2k+1)k!} dt \end{aligned}$$

therefore, A_k is given by:

$$\frac{\left(-\frac{17}{12}\right)^k e^7}{(2k+1)}$$

Continuing on with the question:

By Taylor's theorem with the Lagrange formula for the remainder, there exists some $0 \leq c \leq x$ such that:

$$\begin{aligned} |f(x) - P_3(x)| &= |R_4(x)| = \frac{f^{(4)}(c)x^4}{4!} \\ &= Bx^4 e^{-17/12c^2+7} \end{aligned}$$

Find B.

Solution:

We've been given the 4th derivative in the maple output at the beginning of the question:

$$289/12 \cdot \exp(7) \cdot x \cdot \exp(-17/12 \cdot x^2) - 4913/216 \cdot \exp(7) \cdot x^3 \cdot \exp(-17/12 \cdot x^2)$$

We can then remove all the $\exp(7)$ and $\exp(-17/12x^2)$, and the sub in x for c , to find B .

Continuing on with the question:

However, notice that for such c we must have

$$0 \leq \left| 3 - \frac{17}{6}c^2 \right| \leq 3$$

$$0 \leq \exp\left(-\frac{17}{12}c^2 + 7\right) \leq \exp(7)$$

Solution (Thanks to one eight seven on CSESOC discord):

For $3 - \frac{17}{6}c^2$, we see that it is constantly decreasing, and therefore, we simply check the endpoints.

It is seen that 0 produces the maximum value, which is 3.

For $\exp\left(-\frac{17}{12}c^2 + 7\right)$, it is similar in the fact that it is monotonically decreasing. Therefore, check the end points once more, and we find that $\exp(7)$ is the maximum

Note: $0 \leq x \leq \sqrt{\frac{36}{17}}$. Therefore $0 \leq c \leq x$.

Q3)

a) Consider the different trigonometric identities until they fit:

$$\begin{aligned}\cos^2 x + \sin^2 x &= 1 \\ 1 + \tan^2 x &= \sec^2 x \\ \cosh^2 x - \sinh^2 x &= 1 \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x\end{aligned}$$

b) Same as above, but with a side note:

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

Don't be fooled into thinking we can use the negative \tanh as a substitution!

c) Same as above, we instead of $\tan x$, we end up using $\cot x$, as well as a cosech^x identity.

Q4)

a) T_1 is a linear transformation

Consider if the zero vector exists within the linear transformation.
 $\langle 0.a, 0.b \rangle = \langle 0, 0 \rangle$, therefore, true.

Consider if the linear transformation is closed under addition.

Two arbitrary vectors, x and y , in R^3 , are applied to the linear transformation:

$$T(x) + T(y) = \langle x.a, x.b \rangle + \langle y.a, y.b \rangle = \langle x.a + y.a, x.b + y.b \rangle$$

$$= \langle a.(x + y), b.(x + y) \rangle$$

$$= T(x + y)$$

Consider the scalar multiplication of an arbitrary vector within the linear transformation:

$$T(cx) = \langle (cx).a, (cx).b \rangle$$

$$= c.\langle x.a, x.b \rangle$$

b) T_2 is not a linear transformation

Consider the zero element:

$$\text{sqrt}(0 + 0) = 0$$

Consider closure under addition with two arbitrary vectors a and b in R^2 .

$$T(a) + T(b) = \text{sqrt}(8*a1^2 + 6*a2^2) + \text{sqrt}(8*b1^2 + 6*b2^2)$$

$$T(a + b) = \text{sqrt}(8*(a1 + b1)^2 + 6*(a2 + b2)^2)$$

Therefore, not closed under addition

c) T_3 is not a linear transformation

$$T(p(x)) = p(x) + 5x^2 - 2x$$

Consider the existence of zero:

$$p(0) + 5(0)^2 - 2(0)$$

$= p(0)$, not guaranteed to be zero.

Q4)

You must check both the standard eigenvectors of multiple 1, and also, the fact that there

can be a linear combination of the vectors.

A matrix A has eigenvalues and eigenvectors

$$\lambda = 9, \quad \mathbf{v} = s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \lambda = -1, \quad \mathbf{v} = u \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.$$

Which of the following give a correct diagonalisation $A = MDM^{-1}$? [Select all correct answers.]

Multiple selection advice. In a multiple selection question, marks are deducted for incorrect selections (but you cannot get less than zero marks). You are advised to only select options that you are sure about.

☒ $D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 9 \\ 3 & 14 & -3 \end{pmatrix}.$

☒ $D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \quad M = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 3 & 0 \\ 2 & -1 & 3 \end{pmatrix}.$

☐ $D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & -1 \end{pmatrix}.$

☐ $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 3 & 2 & -1 \end{pmatrix}.$

☒ $D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 3 & 2 & -1 \end{pmatrix}.$

☐ none of the above, A is not diagonalisable.

Q6)

Three axioms are contained within vector spaces:

a) Addition: Given two elements x, y in X , one can form the sum $x + y$, which is also an element of X .

b) Inverse: Given an element x in X , one can form the inverse $-x$, which is also an element of X .

c) Scalar multiplication: Given an element x in X and a real number c , one can form the product cx , which is also an element of X .

1. For all vectors u and v and all scalars λ , we have $\lambda(u + v) = \lambda u + \lambda v$
This is axiomatic
2. There exists a vector $\mathbf{0}$ such that for all vectors v , we have $\mathbf{0}v = \mathbf{0}$
This is not true, as it refers to the **vector** $\mathbf{0}$.
3. For all vectors u and v and w , we have $(u + v) + w = v + (w + u)$
This is not axiomatic, but it can be proved through axioms
4. For any vector v , if 0 is the zero scalar and $\mathbf{0}$ is the zero vector, then $0v = \mathbf{0}$.
This is not axiomatic, but it can be proved through axioms
5. For all vectors u and v and all scalars λ and μ , we have $(\lambda + \mu)(u + v) = \lambda u + \mu v$
This is false, and breaks the distributive axiom
6. For all vectors u and v and all scalars λ , if $\lambda u = \lambda v$ then $u = v$.
This is visibly not true.

Q7)

a) Consider that the matrix is in form:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

Consider that $a, b, c \in \{0, 1, 2, \dots, 43\}$. Therefore, there are $44 \cdot 44 \cdot 44$.

b) Give the number of matrices in T that are not diagonalisable

A matrix in the triangular form that this question poses is not diagonalisable when:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

However, it must be taken into consideration that $b \neq 0$ for this to be true. (As the zero matrix is diagonalisable)

Hence, this comes out to be $43 \cdot 44$.

c) Simply put the result of b on a, to find the probability of getting a diagonalisable matrix.