2020 T2

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Most of these solutions have been derived entirely by me, so there may
be imperfections and wrong answers/reasonings. Please contact me at my
student email for any corrections.

These solutions were made to aid the preparation of MATH1231 students,
not replace it. Good luck :)
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Q2)
We can find this fairly trivially using conditional probability theory. First consider $P(Z < 2.42 \cap Z > 1.42)$. It is evident to understand that this is derived by $P(Z < 2.42) - P(Z > 1.42)$. The final step, is to divide the previous result by $P(Z > 1.42)$, which can be found by doing $1 - P(Z < 1.42)$.

Q3)
S:
a)
Consider that the nullity of a linear map can be found by the *dimension of it's kernel.* Hence, consider the kernel, p'(x) $=$ 0. This will be all the constant polynomials, and for a basis polynomial, the dimension of the kernel will be **1.** Hence, the nullity(T) $=$ 1. Thereby rank-nullity theorem, the rank is 64.

b)
Consider that the nullity(R) $=$ 0. This means that no constant polynomials exist with the linear mapping R. Therefore, the remaining basis polynomials are x, x^2, x^{65}, to create a vector space of dimension 65.

Q4)
a)
Consider that the rank is 1 (since there was one image basis vector left), and the nullity is 3. Therefore, the dimension of the mapping's domain must be 4; due to rank-nullity theorem.
b)
The matrix can be all empty, besides on column vector which includes the image, ergo:

$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Q5)

Use the closure axioms to do this question. It can easily be seen that 1, 2, and 3 are correct. 4 is not correct - as $v_1 \neq T(v_1)$. 6 is also correct. 8 is deceiving; however consider that V and W represent the **domain and co-domain**, hence they are not integer values, and therefore, they are not correct.

Note on the subtraction condition:

$$T(v_1 - v_2) = T(v_1) - T(v_2) = T(v_1) + (-T(v_2))$$

Therefore, true by addition and scalar condition.

Q6)

Proof:

The proof can be done with simple sub-space proofing methods. Remember that we want to start with two vectors in the set, and prove that those two vectors are also in the vector space - proving that it is a sub-space. Here, since it is an intersection, we have to prove this twice to show closure under addition and scalar multiplication.

The greatest possible dimension is the dimension of the smaller intersection.
The least possible dimension is found by doing $dom(U) + dom(W) - dom(V)$.

The reason the last possible dimension is found this way, is that we are finding the **minimal intersection** between U and W. Therefore, we consider what would happen if U and W were a union, and then consider what is the minimum amount of intersecting vectors between the two sets.

For it to be the maximum possible dimension, it must be true that $U \subseteq W$; as that means all of U is contained within W, and the intersection allows for all elements of U to be within W (thus, allowing the $dim(S) = dim(W)$)

Q7)

This can be trivially found using eigenvector diagonalisation, such that $A = MDM^{-1}$.

Q8)

a) This is given in the question - just read it

b) This is found using the fact given: $A\mathbf{v} = \mathbf{v}$. This clues us in that the eigenvalue of this transformation is 1 - and hence, we can use this eigenvalue to figure out a relationship between the two.

$$\lambda = 1$$
$$\begin{pmatrix} \frac{1}{10} - 1 & \frac{1}{5} \\ \frac{9}{10} & \frac{4}{5} - 1 \end{pmatrix} = 0$$
$$-\frac{9}{10}p_1 + \frac{1}{5}p_2 = 0$$
$$p_2 = \frac{9}{2}p_1$$

Then, use the fact that $p_1 + p_2 = 1$ to solve the question.

c) I hate this question
Charlotte is currently in the garden, and we want to know what the probability is she was previous in the garden.
So, the probability will look something like:

$$P(\text{Previously in garden} \mid \text{In garden now}) = \frac{P(\text{Previously in garden} \cap \text{In garden now})}{P(\text{In garden now})}$$
$$= \frac{\frac{9}{11} \cdot \frac{4}{5}}{\frac{9}{11} \cdot \frac{4}{5} + \frac{2}{11} \cdot \frac{9}{10}}$$

Why can't we use $9/11$ for the bottom, and instead have to consider what happened before?
Well, since the question itself intrinsically asks us to consider what happened before, we can't just use $9/11$ as that gives us the probability of the cat being in sunroom/garden as A $\to \infty$. Hence, we need to consider the cat being in the sunroom before, and then moving, as well as being in the garden before, and staying there.

Hammy's explanation:

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P(where it is now) is only dependent on where it was before, and on the bottom we have P(in garden now, in sunroom before) + P(in garden now, in garden before) since we assume what happened in the previous hour, we use the conditional probability
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Q9)

a) The sign test considers the 'middle' case, and then considers the signs. Here, the 'middle' case is 'zero curvature', therefore, we use $B(20, 1/2)$.

b) The expected value is found using the formula $E(X) = np$, $Var(X) = np(1-p)$. Where we use n and p from the sign test binomial distribution we found above.

c) The 'normal approximation' for the binomial distribution takes E(X) and Var(X) form part b), and then uses the 'continuity correction'. Essentially, all we have to do is add 0.5 to the value we are trying to approximate for.
Hence, we are asked to find $P(X \leq 5) \approx P(Y \leq 5.5)$.
Then, use the z-score with $x = 5.5$ to find the probability.

Q10)

a) Using z-score formula, $z = \frac{x-\mu}{\sigma}$, we can find the proportion of packets under 410 grams.

$$z = \frac{410 - 419}{7} = -\frac{9}{7}$$

Now, use the z-score table.

b) A packet is consider *underweight* if it is below 410 grams. To consider what to make μ such that the proportion of underweight packets are at most $33/10000$, we must again use the z-score formula (this time, finding the z-score which represents $33/10000$):

$$-1.84 = \frac{410 - \mu}{7}$$

Then solve for μ.

c) Consider that continuity corrections are used in *discrete* random variables, to estimate them as continuous random variables. Since grams are indeed *continuous*, we are not required to use a continuity correction.

Q11)

This one is fairly easy. Note that since $\exp(7x)$ is recursive in nature, we can just use x^n as our derivative. Doing this ends up with:

$$a_n = \frac{x^n e^{7x}}{7}$$
$$b_n = \frac{-n}{7}$$

Q12)

a) $\phi(x) = e^{\int \frac{-3}{x+1}}$, $\therefore \phi(x) = e^{-3\ln x+1}$.

b) Solve differential equation.

Q13)

a)

We must consider that the general solution has form $Ae^{\lambda_1 x} + Be^{\lambda_2 x}$, the values of the homogenous equation must be unique and *not complex.*

Therefore, let us consider the homogenous equation $\lambda^2 + a\lambda + 5 = 0$. Using the quadratic equation:

$$\frac{-a \pm \sqrt{a^2 - 20}}{2}$$

Therefore, $a > 20$, to be a unique and *not complex* solution.

b) Using polynomial identities: $\lambda_1 + \lambda_2 = -a$ and $\lambda_1\lambda_2 = 5$. Given that λ_1=-3. Therefore, a can be found by first finding λ_2; which answers both parts of the question.

or

You can sub in the derivatives for the general solution, and then equation coefficients.

$$\lambda_1^2 Ae^{\lambda_1 x} + \lambda_2 Be^{\lambda x} + a(\lambda_1 Ae^{\lambda_1 x} + \lambda_2 Be^{\lambda_2 x}) = -7Ae^{\lambda_1 x} - 7Be^{\lambda_2 x}$$

c) Solve the differential equation with all the information above.

Q14) (This question is fairly difficult)

a)

$$\begin{aligned}
\frac{dC}{dt} &= 12\pi r(562 - C) \\
A &= 24C \\
\frac{dA}{dC} &= 24 \\
\frac{dA}{dr} &= 2\pi r \\
\frac{dA}{dt} &= 288\pi r(562 - C) \\
\frac{dr}{dt} &= \frac{dA}{dt} \cdot \frac{dr}{dA}
\end{aligned}$$

b) Seperable, non-linear, first-order ODE.

c) Solve for ODE on maple:

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ode := diff(r(t), t) = 144*(562-Pi*r(t)^2/24)
ics := r(0) = 0
dsolve({ode, ics})
limit(4*tanh(24*t*sqrt(Pi)*sqrt(843))*sqrt(843)/sqrt(Pi), t=infinity)
Output: 4*sqrt(843)/sqrt(Pi)
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Q15) Fairly common sense divergence stuff; know your comparison tests

Q16) Use limits to find the sequence; and then realise that if $\lim_{n\to\infty} a_n$ is > 0, then it converges.

Q17)
Consider that:

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{(n+1)^{n+1}(x-5)^{n+1}6^n n!}{6^{n+1}(n+1)!n^n(x-5)^n}\\ &= \frac{(n+1)^n(x-5)}{6n^n}\\ &= \lim_{n\to\infty}(\frac{n+1}{n})^n \cdot 1/6\\ &= e/6\end{aligned}$$

Therefore, the radius of convergence becomes 6/e.

Q18) Straight forward calculus question.
b)
Just get 0 -> 8's surface area, and then divided by 12 to find 1/12th of that surface area.

c) involves using the total differential approximation to find absolute error. You are given the derivatives.

Q19)
a) Given our function $r = 5e^{-\theta/4}$, apply the formula:

$$\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

b) Use maple to use the formula:

$$\int_{0}^{\infty} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

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int(function, theta=0...infinity)
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c) Consider that due to the polar nature of the function, that the function repeats every 2π radians.

Therefore, applying this *inside* of the function, we get:

$5\exp\left(-\frac{1}{4}(\theta + 2\pi n)\right)$

and hence, our ϕ becomes:

$-\frac{\pi n}{2}$