Question 2.

Consider the sets

$$A = \{\,(x,y) \in \mathbb{R}^2 \mid 7x - 8y \geq -7\,\}\,,$$
$$B = \{\,(x,y) \in \mathbb{R}^2 \mid 8x + 5y \geq -1\,\}\,,$$
$$C = \{\,(x,y) \in \mathbb{R}^2 \mid 15\,x - 3\,y \geq -8\,\}\,.$$

(a) Prove that if $(x,y) \in A$ and $(x,y) \in B$ then $(x,y) \in C$. Be sure to give a clearly written, detailed and logically accurate answer - full marks will not be given for sketchy work.

$7x - 8y \geq -7$ — ① holds true
$8x + 5y \geq -1$ — ② holds true.
$\therefore$ ① + ② must hold true.

$15x - 3y \geq -8$ $\therefore$ true, as, $= C$.

Question 3.

Consider the set

$$A = \{\,\{1,2,3,\ldots,m^2\} \mid m = 23, 24, 25, \ldots, 42\,\}\,.$$

Note that it's a set within a set.

Which of the following statements are true? Select all true statements.

- [ ] $29m^2$ is an element of A
- [ ] 1444 is an element of A
- [x] $\{1,2,3,\ldots,1444\}$ is an element of A
- [ ] $\{1,2,3,\ldots,1444\}$ is a subset of A
- [ ] 1391 is an element of S for some $S \subseteq A$
- [x] 1391 is an element of S for some $S \in A$
- [ ] $\{1,2,3,\ldots,519\}$ is a subset of A
- [x] $\{1,2,3,\ldots,519\}$ is a subset of T for all $T \in A$

Question 4.

You and your friend Aravinden are considering the congruence

$$ax \equiv b \pmod{m}\,,$$

where

$$m = 6 \times 17 \times 67^3\,, \qquad b = 67^2 \times 3209$$

$2 \times 3 \times 17 \times 67^3$

and a is an unspecified integer in the range $22445 \leq a \leq 22465$. In this question you may assume that $17, 67$ and 3209 are prime, and you may use the table of prime factorisations given at the end of the question.

(a) Aravinden askes you to find all a in the range $22445 \leq a \leq 22465$ for which the congruence $ax \equiv b \pmod{m}$ has a unique solution modulo m. Provide your answer in the box below and make sure you read the syntax advice.

Answer: the congruence has a unique solution for a in the set .

a) $ax \equiv 67^2 \times 3209 \pmod{6 \times 17 \times 67^3}$

a needs to have 0 common factors (besides 1) with m, in order to have 1 solution.

$\{22447, 22451, 22453, 22459, 22463, 22465\}$

(b) Another friend Ellie is challenging you to find all a in the range $22445 \leq a \leq 22465$ for which the congruence has more than one solution modulo m. Enter your answer as a set of numbers, as above.

Answer: the congruence has more than one solution for a in the set $\{22445\}$.

Everything else left in the set, such that $\gcd(a, 6 \times 17 \times 67^3) \mid 67^2 \times 3209$.

$\{22445\}$. Only 22445 fits this

Question 5.

Your classmate Luna is interested in the topic of equivalence relations and equivalence classes. Let

$$f(x) = x^3 - 9x + 57\,,$$

and define a relation $\sim$ on $\mathbb{R}$, the set of real numbers, by

$$x \sim y \qquad \text{if and only if} \qquad f(x) = f(y)\,.$$

(a) Luna wishes to prove that $\sim$ is an equivalence relation. Help Luna by writing your proof in the essay box below.

```
Reflexive:
f(x) = f(x), trivially proven.

Symmetric:
f(x) = f(y) denotes f(y) = f(x).
Since f(x) = f(y), the values are equal. Hence, f(y) = f(x) is trivially confirmed.

Transitive:
x ~ y and y ~ z implies x ~ z.
f(x) = f(y) and f(y) = f(z).
Trivially, we can do f(x) = f(z) [By substituting f(y)]
```

(b) Luna also wishes to find the equivalence class of 0 with respect to $\sim$. Help Luna by entering your answer in the box below and make sure you read the syntax advice

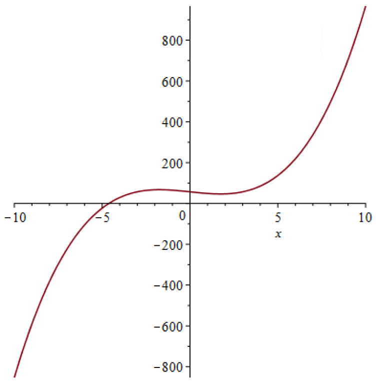
Answer: the equivalence class of 0 is $\{0, 3, -3\}$.

```
The equivalence class of 0 means that when f(0), what other values of x mean that
f(0) = f(x)?
f(0) = 57.
Then, solve the equating x^3 - 9x + 57 = 57
which just comes out to be 3 and -3.
```

(c) Your tutor Dylan is challenging you to find a value of x such that the equivalence class of x contains only one element. Give your answer to Dylan as a single number, for example, 1081.

Answer: the equivalence class of x contains only one element if $x =$

-7



Graphically, anything beyond the minima and maxima are clear to have one solution.

(d) Your tutor Dylan also wants you to answer true or false and then give a brief reason for the following statement.

There exists x such that the equivalence class of x contains exactly 2 elements:

- (selected) True
- False

Would just be the x-value of minima or maxima.

(e) Finally, your tutor Dylan wants you to help your friend Luna by answering true or false and then giving a brief reason for the statement below.

There exists x such that the equivalence class of x contains more than 8 elements:

- True
- (selected) False

This is.. common sense. A polynomial of degree 3 can produce the same value for 8 different inputs.

Question 6.

A tetragroup is an object studied in the field of mathematical stereotopodynamics. A tetragroup may be complexified, or not, and it may be doubly-Euclidean, or not. You are given a list of statements concerning a tetragroup X; some of these statements are logically equivalent, that is, they are just different ways of saying the same thing.

Group the statements into logically equivalent sets and enter your answer below as a list of sets separated by commas.

> Syntax advice: For example, if you think that statements 1,2,3 are logically equivalent; and statements 4,5,6,7 are logically equivalent (but different from 1,2,3); and statement 8 is different from all the others; then your answer should be
>
> `{1,2,3}, {4,5,6,7}, {8}`
>
> The order of your sets, and the order of the elements in each set, are not important.

(1) if X is complexified, then X is doubly-Euclidean $c \to e$ 1

(2) X is complexified, or X is not doubly-Euclidean $c \vee \neg e \Rightarrow \neg c \to \neg e, e \to c$ 2

(3) if X is not doubly-Euclidean, then X is complexified $\neg e \to c, e \vee c, \neg c \to e$ 3

(4) X is complexified only if X is doubly-Euclidean $c \to e$ 1

(5) X is complexified and X is not doubly-Euclidean $c \wedge \neg e, \neg(\neg c \vee e), \neg(c \to e) \Rightarrow c \to \neg e$ 4 $\hookrightarrow e \to \neg c$

(6) X is complexified, or X is doubly-Euclidean $c \vee e, \neg c \to e, \neg e \to c$ 3

(7) X is doubly-Euclidean if X is complexified $c \to e$ 1

(8) if X is doubly-Euclidean, then X is complexified $e \to c$ 2

(1) if X is complexified, then X is doubly-Euclidean

(2) X is complexified, or X is not doubly-Euclidean

(3) if X is not doubly-Euclidean, then X is complexified

(4) X is complexified only if X is doubly-Euclidean

(5) X is complexified and X is not doubly-Euclidean

(6) X is complexified, or X is doubly-Euclidean

(7) X is doubly-Euclidean if X is complexified

(8) if X is doubly-Euclidean, then X is complexified

Answer: {1, 4, 7}, {2, 8}, {3, 6}, {5}

Question 7.

This question involves determining whether it is possible to deduce the truth or falsity of a given statement from other statements. Given that the three propositional formulae

1. $p \vee q$
2. $q \to (p \vee r)$
3. $r \to (p \wedge q)$

are true, is it possible to deduce with certainty whether p is true or false? Select one option, then give reasons.

- (●) p is true
- ( ) p is false
- ( ) it is impossible to say with certainty

p	q	r	$p \vee r$	$p \wedge q$	$q \to p \vee r$	$r \to p \wedge q$
T	T	T	T	T	T	T
F	T	T	T	F	T	F
F	F	T	T	F	T	F
F	F	F	F	F	T	T
T	F	F	T	F	T	T
T	T	F	T	T	T	T

∴ whenever all three are true, then p is also true.

The above question seems to be easy enough for you and your classmates. Now one of your classmates, Manav, asks a similar question.

Given that the same three statements are true, is it possible to deduce with certainty whether q is true or false? Could you please help Manav by selecting one option, then giving reasons?

○ q is true ○ q is false ◉ it is impossible to say with certainty ✔

In the three cases where the three logical statements are true, q is a contingency.

Question 8.

Let x be a real number. Prove that

$$x \text{ is an integer} \qquad \text{if and only if} \qquad \lfloor 0.665x \rfloor + \lceil 0.335x \rceil = x\,.$$

```
Consider both directions.
Forwards
floor(0.665x) < 0.665x + 1
0.665x ⩽ floor(0.665x)

ceil(0.335x) ⩽ 0.335x
0.335x - 1 < ceil(0.335x)

Adding the two equations together, we arrive to:
x - 1 < floor(0.665x) + ceil(0.335x) < x + 1
Since x is an integer,
floor(0.665x) + ceil(0.335x) = x

Backwards
If floor(0.665x) + ceil(0.335x) = x, then x is an integer.
Since floor and ceil always produce an integer, x must always be an integer.
QED.
```

Question 9.

Your classmate Hudson is interested in counting the number of solutions of an equation under various conditions. Consider the equation

$$x_1 + x_2 + x_3 + \cdots + x_{17} = 118\,.$$

Hudson wishes to count the number of solutions of this equation, where $x_1, \ldots, x_{17}$ are non-negative integers, and various other conditions may hold.

(a) When no other conditions are imposed, the total number of solutions is

C(134,16).

$\binom{118+16}{16}$ Stars and bars! $\binom{n+k-1}{k-1}$

(b) The number of solutions in which every x_k is congruent to 0 modulo 9 is

0.

This just means "divisible by 9".

$9x_1 + 9x_2 + 9x_3 + 9x_4 + \ldots + 9x_{17} = 118$

Invalid, as 118 is not divisible by 9.

(c) The number of solutions in which every x_k is either congruent to 0 modulo 9 or congruent to 5 modulo 9 is C(17,11)xC(23,16)+C(28,16)x C(17,2).

Give a detailed explanation for your answer to (c).

$\equiv 0 \pmod 9$ has zero solution.

The MAXMUM you can deduct is $118 - 5\times 17 = 33$

This does not work. Increment till divisible by 9.

At 63, when 11 4's have been used, it is divisible. Hence,

$\binom{17}{11} \times \binom{23}{16}$

position of 4's.

At 108, when 2 4's have been used, it is divisible. Hence,

$\binom{17}{2} \times \binom{28}{16}$

$\therefore \binom{17}{11}\binom{23}{16} + \binom{17}{2}\binom{28}{16}$

Question 10.

(a) How many 12-letter words are there which contain the letter M exactly 5 times?

Answer: C(12, 5) * 25^7

(b) How many 12-letter words are there which contain M exactly 5 times and S exactly 5 times?

Answer: C(12, 5) * C(7, 5) * 24^2

(c) How many 12-letter words are there which contain **at least one consonant** exactly 5 times?

Answer: 21 * C(12, 5) * 25^7 - C(21, 2) * C(12, 5) * C(7, 5) * 24^2

```
Let S be the set, where a specific consonant appears 5 times.

Hence, to find the amount of 5-tuple consonants, we will have:

|S1 U S2 U ... U 21| = |S1| + |S2| + ... + |S21| - (|S1 intersection S2| + ... )
The intersection of all of them combined does not exist.
```

21 consonants.
Choose 5 places
With 25 remaining character.

$21\binom{12}{5}25^7$.

But there are repeats!

$\binom{21}{2}\binom{12}{5}\binom{7}{5}24^2$

$21\binom{12}{5}25^7 - \binom{21}{2}\binom{12}{5}\binom{7}{5}24^2$

(e) How many 12-letter words are there which have no repeated letters, and include YAK as a subword? Note that "include ... as a subword" means that these letters must occur consecutively and in that order in the 12-letter word.

${}^{23}P_9$ (YAK) x x x x x x x x x

10 different positions,

$10 \times {}^{23}P_9$

(f) We will say that a word w_1 is a "spread subword" of w_2 if the letters of w_1 occur **in the same order but not necessarily consecutively** in w_2. For example **MATHS** is a spread subword of BX**MAT**Y**H**E**S**, and also of PF**MATHS**OE. How many 12-letter words are there which have no repeated letters, and include YAK as a spread subword?

Twelve different positions for the sequence of characters YAK.

order DOESN'T matter. We want purely combinations of Y→A→K.

$\therefore \binom{12}{3}$.

Then, fill rest with ORDER: ${}^{23}P_9$

Question 11.

In this question you are to consider the recurrence

$$a_n + 11\,a_{n-1} + 24\,a_{n-2} = -7n4^n\ .$$

(a) Select all statements from the following list which correctly describe the above recurrence.

- [ ] The recurrence has order 1. $\rightarrow n-(n-2) \Rightarrow 2.$ X
- [x] The recurrence has order 2.
- [ ] The recurrence has order greater than 2.
- [x] The recurrence is linear. ← Nothing like $a_n a_{n-1}$.
- [x] The recurrence has constant coefficients. ← Yes.
- [ ] The recurrence is homogeneous. ← No ($\neq 0$).

(b) First consider the associated recurrence

$$a_n + 11\,a_{n-1} + 24\,a_{n-2} = 0\ .$$

This recurrence has a solution $a_n = r^n$ for two values of r. Find these two values and enter them in the box, separated by a comma.

Characteristic Equation:

$r^2 + 11r + 24 = 0$

$(r+8)(r+3) = 0$

$r = -8, -3$

(c) Find a solution of the initial value problem

$$a_n + 11\,a_{n-1} + 24\,a_{n-2} = 0\ , \quad a_0 = -4\ , \quad a_1 = 37\ .$$

Enter your answer as a formula in terms of n.

$a_n = A(-8)^n + B(-3)^n$

$-4 = A + B$

$37 = -8A - 3B$

$B = -4 - A$

$37 = -8A - 3(-4 - A)$

$= -8A + 12 + 3A$

$25 = -5A$

$A = -5 \quad B = 1$

(d) Now we return to the recurrence

$$a_n + 11\,a_{n-1} + 24\,a_{n-2} = -7n4^n\ .$$

Which of the following would be a correct form of a particular solution to this recurrence? Select one answer only.

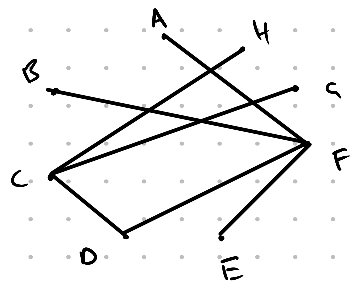
- ( ) $(cn^2 + dn)4^n$
- ( ) $n4^n + d4^n$
- ( ) $cn4^n$
- (•) $(cn + d)4^n$ ✓
- ( ) $cn4^n + d$
- ( ) $(n^2 + cn + d)4^n$

Since the 4^n root doesn't exist yet - you must also check it.

Question 12.

A weighted graph has eight vertices A, B, C, D, E, F, G, H.

(a) You are asked by your friend Adam to find a minimal spanning tree in this graph. The edges of the graph, and their weights, are listed below. Select all the edges which make up your minimal spanning tree.



- [x] edge DF has weight 2
- [x] edge FA has weight 3
- [x] edge CH has weight 4
- [x] edge BF has weight 2
- [x] edge FE has weight 3
- [ ] edge DG has weight 11
- [ ] edge GA has weight 11 ✓
- [x] edge DC has weight 8
- [ ] edge BD has weight 3
- [ ] edge AH has weight 9
- [x] edge GC has weight 9
- [ ] edge DA has weight 4
- [ ] edge EA has weight 8

(b) From the above list, find the edge **of smallest weight** that you **did not** choose. Name this edge in the essay box below, and briefly explain (maximum 50 words) to Adam why you did not choose it.

BD, because it creates a cycle.

(c) Does your answer to (a) give a tree of shortest paths in the original graph from H to all other vertices? Answer yes or no and give a brief reason (maximum 50 words).

Answer:

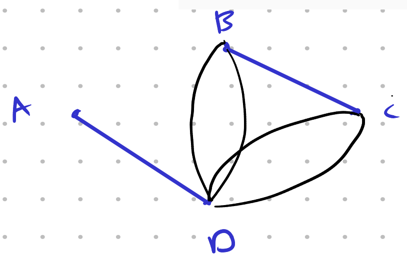
( ) yes (•) no ✓

Simply choosing the shortest vertices from adjacent vertices does not guarantee the shortest path between non-adjacent vertices. The naive approach of creating a minimal spanning tree aims to reduce the total weight of the tree, rather than the distance between two vertices.

Question 13.

A graph G has four vertices A, B, C, D. Taking the vertices in that order, the graph has adjacency matrix

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix} \end{array} .$$



(a) Does the graph contain any loops? If so, which vertices have loops incident on them? Select all correct answers.

- [ ] vertex A has one or more loops
- [ ] vertex B has one or more loops
- [ ] vertex C has one or more loops ✓
- [ ] vertex D has one or more loops
- [x] there are no loops

(b) Does the graph contain any parallel edges (multiple edges) between different vertices? If so, give **one example** of two different vertices which are joined by parallel edges.

Part (b) Syntax advice: Type your two vertices as **capital letters** separated by a comma, for example, **A,B**.
If you think there are no parallel edges in this graph, type the answer **NO in capital letters**.

Answer: B, D

c)

(c) Find the number of walks of length 3 from vertex B to vertex D. Then explain briefly (without detailed calculations - maximum 30 words) how the matrix M can be used to answer this question.

Answer: the number of walks is 20 .

– Consider the implications of matrix multiplication.

For a matrix A and B, it's product at the ith and jth position can be represented as:

$$(A \times B)_{i,j} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

In an adjacency matrix, a self multiplication of the matrix by the nth power, gives us at each (i,j), the amount of n-length walks from $i \to j$.

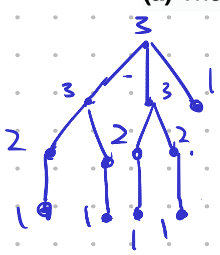
```
M := <<0,0,0,1>|<0, 0, 1, 2>|<0, 1, 0, 2>|<1, 2, 2, 0>>
M^3
```

$$\begin{bmatrix} 0 & 2 & 2 & 9 \\ 2 & 8 & 9 & 20 \\ 2 & 9 & 8 & 20 \\ 9 & 20 & 20 & 8 \end{bmatrix}$$

Question 14.

There are three problems below concerning simple connected graphs G whose degrees are given. For each graph you are to answer two questions. (i) Is the graph a tree? (ii) If it is possible for a graph with the given degrees to be planar, how many regions would there be in a planar representation of G? If it is impossible for the graph to be planar, enter 0 for the number of regions.

(a) The graph G_1 has vertices with degrees $1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3$.



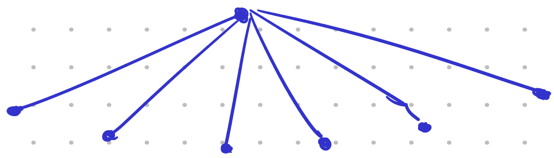
So, yes, a tree.

All trees have planar representation.

Evidently one region as there are no enclosures ("tree").

Not connected, so can't use Euler's.

(b) The graph G_2 has vertices with degrees $5, 5, 5, 5, 6, 6, 6, 6$.



Already there exists a problem. A tree should have children, but the vertex degree has no 1's.

$e \leq 3v - 6$

22 edges (Handshaking Lemma)

$22 \leq 15$. No planar rep.

(c) The graph G_3 has vertices with degrees $2, 3, 4, 4, 4, 4, 4, 4, 5$.

Again, no children nodes (degree 1) so not a tree.

$e \leq 3v - 6$

$= 17 \leq 21$.

$e = 17$

$v = 9$

so yes, there is a planar representation.

$r - e + v = 2$.

$r = 2 + e - v$

$= 10$.

Yipeee Congrats :)