2020 T2

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Most of these solutions have been derived entirely by me, so
there may be imperfections and wrong answers/reasonings. Please
contact me at my student email for any corrections.

These solutions were made to aid the preparation of MATH1231
students, not replace it. Good luck :)
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Q2)
We can find this fairly trivially using conditional probability theory. First consider $P(Z < 2.42 \cap Z > 1.42)$. It is evident to understand that this is derived by $P(Z < 2.42) - P(Z > 1.42)$. The final step, is to divide the previous result by $P(Z > 1.42)$, which can be found by doing $1 - P(Z < 1.42)$.

Q3)
S:
a)
Consider that the nullity of a linear map can be found by the *dimension of it's kernel.* Hence, consider the kernel, p'(x) = 0. This will be all the constant polynomials, and for a basis polynomial, the dimension of the kernel will be **1.** Hence, the nullity(T) = 1. Thereby rank-nullity theorem, the rank is 64.

b)
Consider that the nullity(R) = 0. This means that no constant polynomials exist with the linear mapping R. Therefore, the remaining basis polynomials are x, x^2, x^{65}, to create a vector space of dimension 65.

Q4)
a)
Consider that the rank is 1 (since there was one image basis vector left), and the nullity is 3. Therefore, the dimension of the mapping's domain must be 4; due to

rank-nullity theorem.

b)

The matrix can be all empty, besides on column vector which includes the image, ergo:

$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Q5)

Use the closure axioms to do this question. It can easily be seen that 1, 2, and 3 are correct. 4 is not correct - as $v_1 \neq T(v_1)$. 6 is also correct. 8 is deceiving; however consider that V and W represent the **domain and co-domain**, hence they are not integer values, and therefore, they are not correct.

Q6)

Proof:

The proof can be done with simple sub-space proofing methods. Remember that we want to start with two vectors in the set, and prove that those two vectors are also in the vector space - proving that it is a sub-space. Here, since it is an intersection, we have to prove this twice to show closure under addition and scalar multiplication.

The greatest possible dimension is the dimension of the smaller intersection. The least possible dimension is found by doing $dom(U) + dom(W) - dom(V)$.

For it to be the maximum possible dimension, it must be true that $U \subseteq W$; as that means all of U is contained within W, and the intersection allows for all elements of U to be within W (thus, allowing the $dim(S) = dim(W)$)

Q7)

This can be trivially found using eigenvector diagonalisation, such that $A = MDM^{-1}$.

Q8) Question 8 is a Markov Chain question, and is out of the syllabus for 1231. a) and b) can be found trivially, where b) uses eigenvectors and the process of finding them in order to create a relationship between p_1 and p_2.

Q9)

a) The sign test considers the 'middle' case, and then considers the signs. Here, the 'middle' case is 'zero curvature', therefore, we use $B(20, 1/2)$.

b) The expected value is found using the formula $E(X) = np$, $Var(X) = np(1-p)$.

Q10)

a) Using z-score formula, $z = \frac{x-\mu}{\sigma}$, we can find the proportion of packets under 410 grams.

$$z = \frac{410 - 419}{7} = -\frac{9}{7}$$

Now, use the z-score table.

b) A packet is consider *underweight* if it is below 410 grams. To consider what to make μ such that the proportion of underweight packets are at most $33/10000$, we must again use the z-score formula (this time, finding the z-score which represents $33/10000$):

$$-1.84 = \frac{410 - \mu}{7}$$

Then solve for μ.

Q11)

This one is fairly easy. Note that since $\exp(7x)$ is recursive in nature, we can just use x^n as our derivative. Doing this ends up with:

$$a_n = \frac{x^n e^{7x}}{7}$$
$$b_n = \frac{-n}{7}$$

Q12)

a) $\phi(x) = e^{\int \frac{-3}{x+1}}$, $\therefore \phi(x) = e^{-3 \ln x + 1}$.

b) Solve differential equation.

Q13)

a)

We must consider that the general solution has form $Ae^{\lambda_1 x} + Be^{\lambda_2 x}$, the values of the homogenous equation must be unique and *not complex.*
Therefore, let us consider the homogenous equation $\lambda^2 + a\lambda + 5 = 0$. Using the quadratic equation:

$$\frac{-a \pm \sqrt{a^2 - 20}}{2}$$

Therefore, $a > 20$, to be a unique and *not complex* solution.

b) Using polynomial identities: $\lambda_1 + \lambda_2 = -a$ and $\lambda_1\lambda_2 = 5$. Given that λ_1=-3. Therefore, a can be found by first finding λ_2; which answers both parts of the question.

c) Solve the differential equation with all the information above.

Q14) (This question is fairly difficult)
a)
We must first dissect all the different information they give us.
The cell is surrounded by a growth medium of concentration 449mg/l.
The area of the cell is proportional to C, such that $A = 58C$.
The change of the concentration, can be modelled as $\frac{dC}{dt} = 2\pi r(502 - C)$.
$A = \pi r^2$
$dA/dr = 2\pi r$
$\therefore dA/dt = 2\pi r \cdot \frac{dr}{dt} = 58 \cdot 2\pi r(502 - C)$
$\therefore dr/dt = 57 \cdot 502 - \pi r^2$

b) It is separable, non-linear, first-order and an ordinary differential equation

c) Upside down parabola; so consider the intercept.

Q15) Fairly common sense divergence stuff; know your comparison tests

Q16) Use limits to find the sequence; and then realise that if $\lim_{n\to\infty} a_n$ is > 0, then it converges.

Q17)

Consider that:

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{(n+1)^{n+1}(x-5)^{n+1}6^n n!}{6^{n+1}(n+1)!n^n(x-5)^n} \\ &= \frac{(n+1)^n(x-5)}{6n^n} \\ &= \lim_{n\to\infty}(\frac{n+1}{n})^n \cdot 1/6 \\ &= e/6\end{aligned}$$

Therefore, the radius of convergence becomes 6/e.

Q18) Straight forward calculus question.

b)

Just get 0 -> 8's surface area, and then divided by 12 to find 1/12th of that surface area.

c) involves using the total differential approximation to find absolute error. You are given the derivatives.

Q19)

a) Use arc length formula

b) Create a limit as theta becomes infinitely large