Question 6.

Consider the equation

$$x_1 + x_2 + \cdots + x_7 = 82,$$

where $x_1, x_2, \ldots, x_7 \in \mathbb{N}$.

How many solutions are there if:

a) $x_i \le 13$ for all $1 \le i \le 7$?

Consider $x_i = 13 - y_i$

$\therefore\ 13 - y_1 + 13 - y_2 + \ldots + 13 - y_7 = 82$

$\Rightarrow -(y_1 + y_2 + \ldots + y_7) = -9$

$y_1 + y_2 + y_3 + \ldots + y_7 = 9$, $0 \le y_i \le 13$.

↑ This condition is satisfied by $= 9$.

$\therefore\ \binom{15}{7}$.

b) $x_i \le 22$ for all $1 \le i \le 7$?

Following similarly.

$x_i = 22 - y_i$

$y_1 + \ldots + y_7 = 67$

where $y_i \le 22$.

This is Not satisfied.

$\therefore$ Consider $x_1 \ge 23$

$x_1 + x_2 + x_3 + \ldots + x_7 = 59$

$= \binom{65}{6}$

Consider $x_1, x_2 \ge 23$

$x_1 + x_2 + \ldots + x_7 = 36$

$= \binom{42}{6}$

Consider $x_1, x_2, x_3 \ge 23$

$x_1 + x_2 + \ldots + x_7 = 13$

$= \binom{19}{6}$

x_4 not possible.

$\therefore$ Consider:

$|U| - |S_1 \cup S_2 \cup \ldots \cup S_7|$

$$\binom{88}{6} - \binom{7}{1}\binom{65}{6} + \binom{7}{2}\binom{42}{6} - \binom{7}{3}\binom{19}{6}$$

c) $x_1 \geq 11$, and

$x_i \equiv i \pmod{6}$ for all $1 \leq i \leq 7$?

Thanks to Jeff. I forgot the mod conversion thing. Ooops.

Remember that

$$x_i \equiv i \pmod{6} \iff x = 6a + i, \quad a \geq 0.$$

$\therefore$ Combining both restrictions, we get:

$$x_1 = 6a_1 + 1 + 11$$
$$x_2 = 6a_2 + 2$$
$$x_3 = 6a_3 + 3$$
$$\vdots$$
$$x_7 = 6a_7 + 7$$

$$\therefore 6(a_1 + a_2 + \dots + a_7) + 39 = 82$$
$$6(a_1 + a_2 + \dots + a_7) = 43$$
$$a_1 + a_2 + a_3 + \dots + a_7 = 7$$
$$\binom{13}{6}$$

Note for me (and maybe for you):

$$x \equiv b \pmod{c}$$

$$\Rightarrow x = cd + b \quad \text{where } b \in \mathbb{Z}.$$