MATH1081: Lab Test 1

Haeohreum Kim (z5480978)

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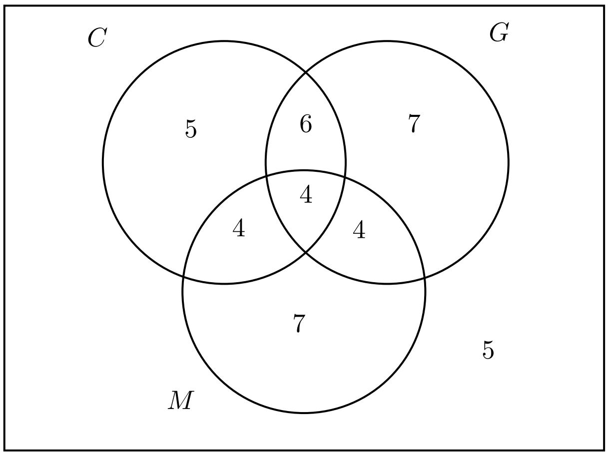
Question 1

This question is fairly simple, and has a lot of repeats. So, I will only include my example. For further questions, I will go through each permutation and explain any patterns that arise.
Question:
In a class of 42 students:

- 21 study German
- 19 study Maths
- 10 study both Chemistry and German
- 8 study both Chemistry and Maths
- 8 study both German and Maths
- 4 study all three subjects, and
- 5 study none of these subjects.

a) How many students study Chemistry?
I like to begin with all three, which can be represented as $C \cap G \cap M$ as well as the complement, which is none of them. From there, you can construct a venn-diagram, like below.



and hence, the amount of Chemistry students is $5 + 6 + 4 + 4 = 19$.

b) Writing C, G and M for the sets of students study Chemistry, German and Maths respectively, evaluate $|(G^c \cap C)^c \cup M|$.
Using the laws of set algebra, we can simplify the expression down to:

$$
\begin{aligned}
&= (G^c \cap C)^c \cup M) \\
&= (G \cup C^c) \cup M \\
&= G \cup C^c \cup M
\end{aligned}
$$

So, essentially, this is $G \cup M$, which is $7 + 6 + 4 + 4 + 4 + 7 + C^c = 5$, as the G and M have already been counted. This sums to 37.

Question 2

There are different permutations of the last question, which is $|\mathcal{P}(S_x) - \mathcal{P}(S_y)|$, $|\mathcal{P}(S_x) \cap \mathcal{P}(S_y)|$ and $|\mathcal{P}(S_x) \cup \mathcal{P}(S_y)|$. For efficiency sake, I'll go through all three examples with one question, but in the actual lab test, you will only receive one of them.

Question: For any integer k, let S_k be the set defined by:

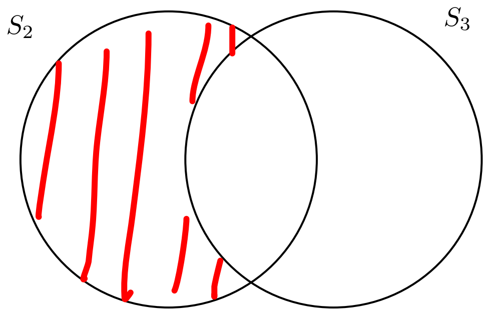
$$S_k = \{n \in \mathbb{Z} \mid 2k \leq n \leq 3k + 2\}$$

a) What is $S_3 - S_2$?
   $S_3 = \{6, 7, 8, 9, 10, 11\}$
   $S_2 = \{4, 5, 6, 7, 8\}$
   $S_3 - S_2 =$ `set(9, 10, 11)`

b) Find $|P(S_2) \times P(S_3)|$
   $|P(S_2)| = 2^5$, $|P(S_3)| = 2^6$, $\therefore 2^5 \times 2^6$

c) The three permutations are:

   (a) Find $|P(S_2) - P(S_3)|$



   This can be found by doing:

$$|P(S_2)| - |P(S_2 \cap S_3)| = 2^5 - 2^3$$

   (b) Find $|P(S_2) \cup P(S_3)|$

$$\begin{aligned} P(S_2) \cup P(S_3) &= P(S_2) + P(S_3) - P(S_2 \cap S_3) \\ &= 2^6 + 2^5 - 2^3 \end{aligned}$$

   (c) Find $|P(S_2) \cap P(S_3)|$

$$\begin{aligned} P(S_2) \cap P(S_3) &= P(S_2 \cap S_3) \\ &= 2^3 \end{aligned}$$

Question 3

It seems as though the general gist of this question is proving that it's not injective and proving that it's not surjective. If you have a good understanding of both, finding a counter-example will be fairly easy.

a) Consider the function:

$$f : \mathbb{R}_0^+ \to \mathbb{R}_0^+, f(x) = (x-1)^2$$

Complete the following to make a true statement:
Since the equation f(x) = **1** has **more than one solution**, we can conclude that f is **not injective**.
Some other functions:

$$x(x+3)(x+4), \mathbb{R}_0^+ \to \mathbb{R}$$

Graphing the functions is also a good idea. Here, it's easy to assume a counter example for injectivity like $x = 0$, but keep close attention to the domain. Here, it is actually not *surjective*, as there is no solution for, for example -5.

b) Consider the function:

$$g : \mathbb{R}_0^+ \to \mathbb{R}_0^+, g(x) = x(x-1)^2$$

Here, graphing the function allows you to easily see that there are two solutions for $g(x) = 0$. Therefore, not injective.

Question 4

Suppose $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and that the function $f : S \to S$ is given by:
$f(x) = (6x^2 + 4x + 1) \mod 8$
Let $T = \{1, 3\}$

a) What is f(T)?
Just go through both values and see what their mod values are. In this case, it's 3. Although, remember that it's mapping from a set to a set, therefore, your answer will be `set(3)`

b) What is $f^{-1}(T)$?
This basically means - `what values of S would map to T in the function f?`. You can check this fairly easily by just cycling through the values from 0 to the modulo you have. Here, every value $0 \leq n \leq 8$ n maps to the set T.

c) Complete the sentence: f is **neither injective nor sujective**.
First consider injectivity:
Does the function have a one-for-one relationship? No, as multiple values map to T.

Consider surjectivity:
Is every value of S represented through the function? No (only T is), so not surjective.

Question 5

Two positive integers x and y are chosen, and their GCD and LCM are found to be the following:

$$\gcd(x,y) = 70 = 2 \times 5 \times 7$$
$$\text{lcm}(x,y) = 20751500 = 2^2 \times 5^3 \times 7^3 \times 11^2$$

a) You are told that $x \neq \gcd(x,y)$. What is the smallest possible value of x?

   Consider that the gcd and lcm are basically bitwise maximum and minimum functions - ergo, the lcm considers the maximum power of 2, 5, 7, 11 and the gcd considers the minimum power of 2, 5, 7, 11 present within the two integers x and y. Using this, you can find the smallest possible value, and the largest possible value.

b) You are now told that $x = 3500 = 2^2 \times 5^3 \times 7$. What is the value of y?

   Consider that the definition of the lowest common multiple is:

$$\text{lcm}(x,y) = \frac{|x \cdot y|}{\gcd(x,y)}$$

   Therefore, we have x. All we must do is divided out the lcm by x, and then multiply it by the gcd to find y, which in this case, is: 415030.

Question 6

There are two different types to question 6 - one where the mod is prime, and one where it isn't. In the example where it is prime, use Fermat's Little Theorem to break it up. In the case it is not, use binary representation to break it up. See the following examples:

a) Evaluate $3^{382} \mod 19$

   First notice that the mod is prime. So let's use Fermat's Little theorem and postulate that: $3^{18} \equiv 1 \mod 19$. Now we can break up the power of $382 = 18 \times 21 + 4$, and hence, re-represent the congruence as:

$$3^{18(21)} \times 3^4 \equiv 1 \times 3^4 \mod 19$$

   This then simply becomes our answer, 5.

b) Evaluate $3^{50} \mod 24$.
   Let's quickly represent the power in binary: $50 = 32 + 16 + 2$. Now, let us assemble our modulo in binary powers:
   $3 \equiv 3 \mod 24$
   $3^2 \equiv 9 \mod 24$
   $3^4 \equiv 9 \mod 24$
   $3^{16} \equiv 9 \mod 24$
   $3^{32} \equiv 9 \mod 24$
   Now, we can resemble our original equation as: $3^{32} \times 3^{16} \times 3^2 = 9 \times 9 \times 9 \equiv 9 \mod 24$

Question 7

Solving these questions are pretty straight forward, and involve the same method. I'll just go through one example - if you have any problems, refer to the lecture slides or try and re-investigate my method.
Note: Remember that when there is GCD in the original modular congruence, you must resolve it first. Infact, the GCD will be the amount of solutions you have at the end.

$$\begin{aligned} 141x &\equiv 9 \mod 396 \\ 141x + 396y &= 9 \\ 396 &= 141 \times 2 + 114 \\ 141 &= 114 \times 1 + 27 \\ 114 &= 27 \times 4 + 6 \\ 27 &= 6 \times 4 + 3 \\ 6 &= 3 \times 2 + 0 \end{aligned}$$

Now, reversing, as you have learnt in lectures:

$$\begin{aligned} 3 &= 27 - 6 \times 4 \\ &= 27 - 4 \times (114 - 4 \times 27) \\ &= 17 \times 27 - 4 \times 114 \\ &= 17 \times (141 - 114) - 4 \times 114 \\ &= 17 \times 141 - 21 \times 114 \\ &= 17 \times 141 - 21 \times (396 - 2 \times 141) \\ &= 59 \times 141 - 21 \times 396 \\ 9 &= 177 \times 141 - 63 \times 396 \end{aligned}$$

Note that in our last step, we multiplied the result by 3, therefore, we must divide our modulus by 3, hence, becoming mod 132. Making sure you get the lowest positive value to zero:

$$45x \equiv 9 \mod 132$$

or, as the question wants: `set(45, 174, 306)`
Consider another question:
Before we start, note that gcd(978, 1284) = 6, therefore, we will have 6 solutions at the end.

$$\begin{aligned} 1284 &= 978 + 306 \\ 978 &= 306 \times 3 + 60 \\ 306 &= 60 \times 5 + 660 & &= 6 \times 10 + 0 \end{aligned}$$

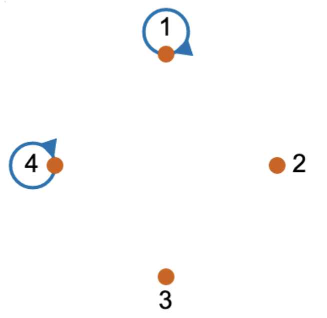
now reversing

$$\begin{aligned} 6 &= 306 - 60 \times 5 \\ &= 306 - 5 \times (978 - 306 \times 3) \\ &= 306 - 5 \times 978 + 15 \times 306 \\ &= 16 \times 306 - 5 \times 978 \\ &= 16 \times (1284 - 978) - 5 \times 978 \\ &= 16 \times 1284 - 21 \times 978 \end{aligned}$$

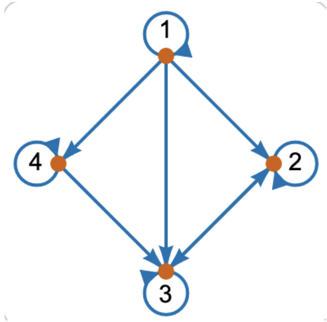
Therefore, we have a particular solution: $-21 \times 5 + 1284 = 1179$. However, we must find the other 5 solutions by decrementing down $\frac{1284}{6}$, therefore we have: `set(109, 323, 537, 751, 965, 1179)`

Question 8

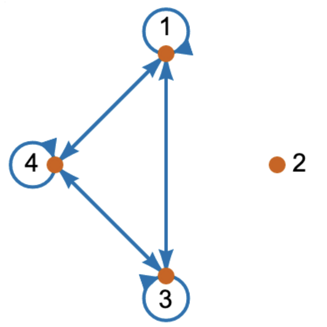
I won't do an exhaustive case, but I'll go through some which helped me.



Definitely not reflexive, as not every point has a loop. However, it *is* symmetic, as every path that exists, can be represented as it's converse. For example $1 \to 1 = 1 \to 1$. Also, it's transitive, as every path can be represented in a similar way: $1 \to 1 \to 1 = 1 \to 1$.



It is reflexive, as every single point has a loop. The path $1 \to 4 \to 3$ and $1 \to 2 \to 3$ has $1 \to 3$, however, $4 \to 3 \to 2$ does not have a transitive equivalent. Further, not every path has a double head. Therefore, only reflexive.



Definitely not transitive, as 2 does not have a loop. It is symmetric, as every path is double headed. It is transitive, as: $1 \to 3 \to 4 = 1 \to 4$, etc. etc. Therefore symmetric and transitive.