MATH1081 2021 T1 Final

Question 2.

Clara's favourite set is $A = \{x^2 + 2 : x \in \mathbb{Z},\ -2 \le x \le 3\}$, and Oscar's favourite set is just $B = \{3, \dots, 8\} \subseteq \mathbb{N}$.

$A = \{6, 3, 2, 11\}$ $B = \{3, 4, 5, 6, 7, 8\}$

$|A - B| =$ 2

$|P(A - B)| =$ 2^2

$|P(B) - P(A)| =$ 2^6 - 2^2

$|P(A \times B)| =$ 2^(4 * 6).

$A - B = \{2, 11\}$

$P(A - B) = 2^{|A-B|} = 2^2$

$|P(B) - P(A)| = |P(B)| - |P(A - B)|$

$|P(A \times B)| = 2^{|A| \times |B|}$

Question 3.

At Unique University, there are four courses available to students this term: Arts, Business, Computer Science, and Design. At the annual faculty meeting, the course leaders are evaluating information about which students study which subjects.

Writing A, B, and C as the sets of students studying Arts, Business, and Computer Science respectively, it is known that they satisfy the following identities:

$$\begin{array}{lcllcllcl} |A| & = & 35 & |A \cup B| & = & 46 & |A \cap B \cap C| & = & 8 \\ |B| & = & 29 & |B \cup C| & = & 45 & & & \\ |C| & = & 32 & |C \cup A| & = & 51 & & & \end{array}$$

Calculate the total number of students studying Arts, Business, or Computer Science:

$|A \cup B \cup C| =$ ____.

$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |B \cap C| + |A \cap C|) + |A \cap B \cap C|$

$|A \cup B| = |A| + |B| - |A \cap B|$

$= |A \cap B| = 18.$

$|B \cap C| = 16.$

$|C \cap A| = 16$

$\therefore |A \cup B \cup C| = 54$

Question 4.

(a)

Matthew is trying to solve a modular congruence that Dr Aritz has written up on the blackboard:

$$24x \equiv 99 \pmod{213}.$$

First, Matthew writes the solution as an integer x with respect to a smallest possible modulus k :

$x \equiv$ 13 $\pmod{k}$.

Next, Matthew writes the solution as a set of integers $\{x_1, x_2, \ldots\}$ with respect to the original modulus 213 :

$x \in$ 13, 84, 155 $\pmod{213}$.

$24x \equiv 99 \pmod{213}$

$= 213x + 24y = 1$

$213 = 24 \times 8 + 21$
$24 = 21 + 3$
$21 = 3 \times 7 + 0$

$3 = 24 - 21$
$3 = 24 - 213 + 24 \times 8$
$3 = 9 \times 24 - 213$
$99 = 297 \times 24 - 213$
$x = 297 \pmod{213}$

$x \equiv 13 \pmod{7}$

(b)

Matthew looks up at the board to copy down the next question, but Dr Aritz has already started cleaning the board! Matthew copies down what they can, putting question marks ($\boxed{?}$) where they were not able to copy down certain numbers. The series of question marks ($\boxed{???}$) could represent a list of zero, one, or more numbers:

$$85x \equiv \boxed{?} \pmod{250}$$

$$\text{has the solution } x \in \{220, \boxed{???}\} \pmod{250}.$$

Help Matthew find the complete solution set $\{x_1, x_2, \ldots\}$ with respect to the modulus 250 :

$x \in$ $\pmod{250}$.

$85x \equiv 200 \pmod{250}$

$250 = 85 \times 2 + 80$
$85 = 80 + 5$
$80 = 5 \times 16 + 0$

$5 = 85 - 80$
$5 = 85 - 250 + 2 \times 85$
$5 = 3 \times 85 - 250$
$200 \equiv 120 \times 85 - 250 \times 40$

$\{20, 70, 120, 170, 220\}$

c)

Dr Aritz turns to his lecture notes for the next example, but realises he has accidentally spilled homebrand Coke on his notes and cannot read all the numbers. The example looks like this, where again a question mark ($\boxed{?}$) represents an unknown number, and a series of question marks ($\boxed{???}$) could represent a list of zero, one, or more numbers:

$$27x \equiv \boxed{?} \pmod{\boxed{?}}$$

$$\text{has the solution } x \in \{\boxed{???}\} \pmod{\boxed{?}},$$

so there are $n = \boxed{?}$ different solutions in the original modulus.

What are the possible values for the size of the solution set? That is, taking n as the number of solutions for x in the original modulus, write the set of all possible values for n as a set of integers $\{n_1, n_2, \ldots\}$:

$n \in$ set(0, 1, 3, 9, 27)

gcd(27, mod) = solutions iff.
gcd(27, mod) | $\boxed{?}$

This is a theory question.

Question 5.

Consider the partially ordered set $P = (S, |)$ of integers

$$S = \{n \in \mathbb{N} \ : \ 11 \leq n \leq 97 \ \text{ and } \ n = 3^i 5^j \ \text{ for some integers } \ i, j \in \mathbb{N}\}$$

ordered by divisibility $|$.

(a)
Writing your answer as a set of integers $\{x_1, x_2, \ldots\}$, state the minimal elements of P :

{15, 25, 27}

(b)
Writing your answer as a set of integers $\{x_1, x_2, \ldots\}$, state the maximal elements of P :

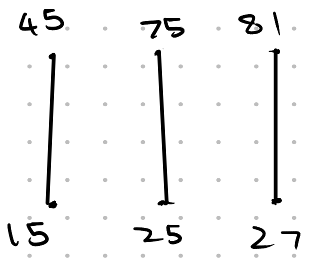
{45, 75, 81}

$S = \{n \in \mathbb{N} : 11 \leq n \leq 97$ and $n = 3^i 5^j$ for $i, j \in \mathbb{N}\}$

$S = \{15, 27, 81, 45, 75, 25\}$.

$\uparrow$
Apparently $0 \in \mathbb{N}$

Hasse Diagram:



Question 6.

p	q	r	$r \wedge {\sim}q$	$p \vee {\sim}q$	$(r \wedge {\sim}q) \rightarrow (p \vee {\sim}q)$
False	False	False	○ True ◉ False ✔	◉ True ○ False ✔	◉ True ○ False ✔
False	False	True	◉ True ○ False ✔	◉ True ○ False ✔	◉ True ○ False ✔
False	True	False	○ True ◉ False ✔	○ True ◉ False ✔	◉ True ○ False ✔
False	True	True	○ True ◉ False ✔	○ True ◉ False ✔	◉ True ○ False ✔
True	False	False	○ True ◉ False ✔	◉ True ○ False ✔	◉ True ○ False ✔
True	False	True	◉ True ○ False ✔	◉ True ○ False ✔	◉ True ○ False ✔
True	True	False	○ True ◉ False ✔	◉ True ○ False ✔	◉ True ○ False ✔
True	True	True	○ True ◉ False ✔	◉ True ○ False ✔	◉ True ○ False ✔

Question 7.

Eleanor wants to completely fill a 2×18 grid with non-overlapping tiles chosen from 1×2 tiles numbered from 1 to 4 and from unnumbered 2×2 tiles:

Eleanor may choose more than one of each type of tile, and does not want to rotate the 1×2 tiles, since the numbers on the tiles should be upright and readable.

(a)

In how many ways can Eleanor tile the 2×18 grid in this way?

$(4^2+1)^9$

Take the biggest block size (2x2), and then partition the grid into this size.
This is useful, as we can figure out the biggest block taking up that entire grid, else, figure out the different combinations of smaller blocks separately.
Hence, we have 9 blocks of (2x2). Consider the different combinations of a single block.

We have 4 distinct small blocks, and two slots to put them in. This just becomes 4^2.
Then, we have another extra case where the 2x2 block takes up the whole grid. So, for the block, we have 4^2 + 1 combinations.
For the entire grid, we now have (4^2 + 1)^9.

(b)

Eleanor accidentally only ordered three of the 2×2 tiles. Therefore, Eleanor can now only use at most 3 of the 2×2 tiles in any tiling.

In how many ways can Eleanor tile the 2×18 grid with this new restriction?

$(4^2)^9 + 9 \times (4^2)^8 + \binom{9}{2}(4^2)^7 + \binom{9}{3}(4^2)^6$

Case bash:
We want to find when we use 0, 1, 2 and 3 blocks.
This becomes:

Case 0: $(4^2)^9$

Case 1: $9 \times (4^2)^8$

Case 2: $\binom{9}{2} \times (4^2)^7$

Case 3: $\binom{9}{3} \times (4^2)^6$

(c)

Eleanor has decided it's bad luck if two square tiles are adjacent.

In how many ways can Eleanor tile the 2×18 grid so that at most 3 of the 2×2 tiles are used, and no two 2×2 tiles touch each other?

This is quite similar to the previous question, but we now need to make sure to remove the cases where there is a double up of adjacent tiles.

So, start with (4^2)^9 + comb(9, 1) * (4^2)^8 + comb(9, 2) * (4^2)^7 + comb(9, 3) * (4^2)^6

Case 1: 1 tiles
There is no way to have adjacent tiles with one tile.

Case 2: 2 tiles
Partition the grid into 9 subdivisions of 2 x 2 grids.
X X X X X X X X X
Putting in two adjacent tiles, we get:
(S S) X X X X X X X
Hence, there are 8 different ways to place two blocks.
Hence, your new multiplier for two tiles will be:
(comb(9, 2) - 8)

Case 3: 3 tiles
Partition the grid into 9 subdivision of 2 x 2 grids.
X X X X X X X X X
Consider the case of putting two adjacent tiles in, we get:
(S S) X X X X X X X
There is 8 different ways to do this, and then 7 different spots to put the 3rd tile in.

However, there is a double count of the triple case. The triple case has 7 different cases. Hence the new multiplier for three tiles is: 8 * 7 - 7
Therefore:
(4^2)^9 + 9 * (4^2)^8 + (comb(9, 2) - 8) * (4^2)^7 + (comb(9, 3) - (8 * 7 - 7)) * (4^2)^7

Question 8.

Consider the following 1×2 tiles marked with numbers from 1 to 3,

① ② ③ ,

and the following 2×1 tiles marked with numbers from 1 to 11,

① ② ••• ⑪ .

Let a_n be the number of ways in which to entirely fill a horizontal $2 \times n$ grid with non-overlapping tiles chosen from a selection of the tiles above .

There are an unlimited number of each tile available, and we are allowed to freely use as many tiles of each type as we want, but we are not allowed to rotate any tile, since we would like the marked numbers on every tile to be readable.

(a)

Calculate $a_1 =$ 11 .

Calculate $a_2 =$ $3^2 + 11^2$.

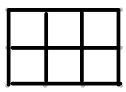
Calculate $a_3 =$ $2 \times 11 \times 3^2 + 11^3$.

a_1) Only verticals fit, $\therefore 11$

a_2) Either verticals or horizontals can fit, $\therefore$

$3^2 + 11^2$

a_3) Cases:



1 Vertical:

2 Valid Positions

11 Vertical Tiles

3^2 Combs of horizontal.

$2 \times 11 \times 3^2$

2 Verticals: (Impossible)

3 Verticals:

11^3.

$\therefore 2 \times 11 \times 3^2 + 11^3$

(b)

For $n \geq 2$, the numbers a_n satisfy the integer recurrence relation

$$a_{n+1} = C_1\, a_n + C_2\, a_{n-1}$$

where $C_1 =$ and $C_2 =$.

$a_{n+1} = C_1 a_n + C_2 a_{n-1}$

$22 \times 3^2 + 11^3 = C_1(3^2 + 11^2) + C_2(11)$

$22 \times 3^2 + 11^3 = C_1 \times 3^2 + C_1 \times 11^2 + 3^2 \times 11$

$= 11 \times 3^2 + 11^3 + 3^2 \times 11$

$= 22 \times 3^2 + 11^3$

(c)

Use the recurrence relation in part **(b)** to compute

$a_5 =$ 211640 .

$a_{n+1} = 11a_n + 9a_{n-1}$

$a_4 = 11 \times a_3 + 9 \times a_2$

$= 11 \times (22 \times 3^2 + 11^3) + 9(3^2 + 11^2)$

$= 17989$

$a_5 = 11 \times 17989 + 9 \times (22 \times 3^2 + 11^3)$

$a_5 = 211640$

Question 9.

a)

(a)

$7, 6, 5, 5, 5, 4, 2$

- ( ) A graph with this vertex degree sequence does not exist.
- (●) A graph with this vertex degree sequence exists but cannot be simple.
- ( ) A graph with this vertex degree sequence exists and can be simple.

Note that the vertex degree is even - hence some graph exists.
Further, consider that there is a vertex of degree 7, but not 7 other vertices. Hence, there must be a self loop. Therefore, not simple.

(b)

$6, 6, 5, 5, 4, 4, 2$

- ( ) A graph with this vertex degree sequence does not exist.
- ( ) A graph with this vertex degree sequence exists but cannot be simple.
- (●) A graph with this vertex degree sequence exists and can be simple.

First, note that the degree sum is even, and hence some graph exists.
First make the assumption that the graph is simple. Then, it must be true that each vertex has a single edge.

6 6 5 5 4 4 2
6 1 1 1 1 1 1

6 6 5 5 4 4 2
6 6 2 2 2 2 2

6 6 5 5 4 4 2
6 6 5 5 3 3 2

6 6 5 5 4 4 2
6 6 5 5 4 4 2 ∴ SIMPLE!

(c)

$5, 4, 3, 2, 2, 2, 1$

- (●) A graph with this vertex degree sequence does not exist.
- ( ) A graph with this vertex degree sequence exists but cannot be simple.
- ( ) A graph with this vertex degree sequence exists and can be simple.

Note that the degree sum is odd, and hence a graph cannot exist.

(d)

5, 5, 5, 3, 2, 2

- ○ A graph with this vertex degree sequence does not exist.
- ◉ A graph with this vertex degree sequence exists but cannot be simple.
- ○ A graph with this vertex degree sequence exists and can be simple.

First note that the degree sum is even, and hence a graph exists.
Now, check the graph distribution.

5 5 5 3 2 2
5 1 1 1 1 1

5 5 5 3 2 2
5 5 2 2 2 2

5 5 5 3 2 2
5 5 5 3 3 3

Uh oh! NOT SIMPLE.

Question 10.

(a)

What is the smallest number of edges that we can delete from the complete graph K_{96}, without deleting any vertices, so that the resulting graph has an Euler circuit?

48

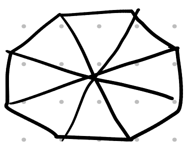
A graph has an euler circuit if and only if every single vertex has an even degree. Consider the implication of the graph K(96). A complete graph ensures a node is connected to every single other node. Hence, a vertex x in K, will have 95 edges. Thus, we must remove 1 edge from x, for it to become even. However, consider that the removed edges' adjacent vertex, will now alsso be even. Hence, we must only remove 96/2 edges, which is 48.

(b)

What is the smallest number of edges that we can add to the cycle graph C_{27} , without adding any vertices, so that the resulting graph is non-planar?

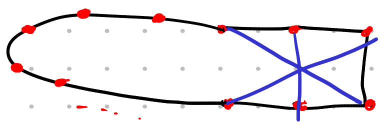
3

Kuratowski's Theorem:
Consider $K_{3,3}$



A cycle graph is similar to $K_{3,3}$

Consider the following subdivision:



21 other subdividing nodes.

∴ $K_{3,3}$ subdiv, Hence, Kuratowski's Theorem.

(c)

Graph G has 32 vertices, is simple, connected, and planar, and does not have a circuit of length 4.

Furthermore, the dual of G has an Euler circuit.

Prove that G has at most 45 edges.

Thanks Gerald! @ CSESOC

We will denote G^* to be the dual of G. Each vertex of G associates itself with a region of G^*. Therefore, the degree of a particular vertex in G is equivalent to the number of sides of the region in G^*. Since G^* has an Eulerian circuit, it follows that every region of G must have an even number of sides.

Let r_n denote the number of regions with n sides in G. Since the graph is simple, no region can have two sides. Since G has no circuit of length 4, no region can have four sides. Therefore, each region must have at least six sides. Each edge is associated with exactly two regions; therefore, the sum of the number of sides is twice the number of edges, which gives

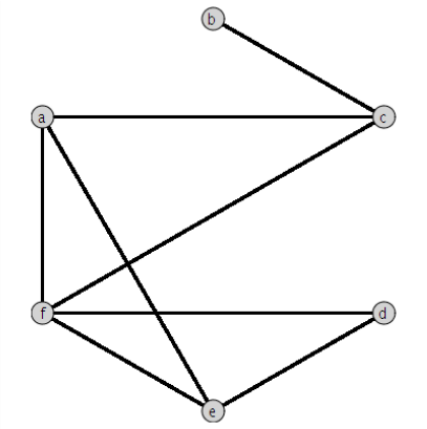
$$2e = \sum_{n\geq 6} n r_n \geq 6 \sum_{n \geq 6} r_n = 6r.$$

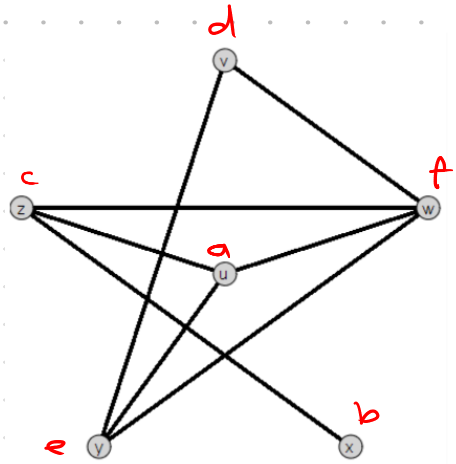
This implies that

$$r \leq \frac{e}{3}.$$

Plug this into Euler's formula.

Question 11.

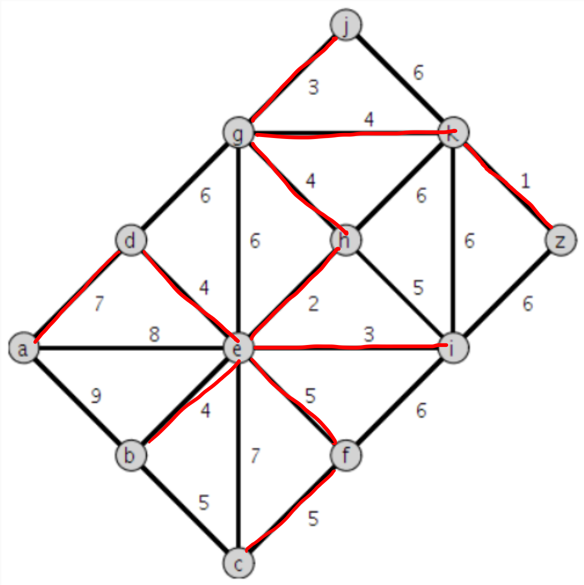




Vertex in G	Vertex in H
a	u
b	x
c	z
d	v
e	y
f	w

Just find a distinguishable vertex (like b) and then draw it out.

Question 12.



Find the weight of a minimal spanning tree of G.

42

Find the shortest path distance from vertex a to vertex z.

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QUESTION 13.

Let A be a finite set containing $n \geq 1$ elements and consider any surjective function $f : A \to A$.

(a)

Prove that f is bijective.

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Since the set sizes of the domain and co-domain are equal, and the function
is given to be surjective, then f must also be injective by definition.
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(b)

For each positive integer k, define f^k to be the kth composition of f:

$$f^k = \overbrace{f \circ \cdots \circ f}^{k}.$$

Also, define $f^0 = \iota_A$, the identity function on A.

Prove that, for each element $a \in A$, there exists some positive integer $k \geq 1$ such that $f^k(a) = a$.

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Let m be the cardinality of A. Assume that there exists an x in S, such that there does
not exist a non-negative integer k, such that f^k(x) = x.

Consider the set of numbers:
T = {f^(0)(x), f^(1)(x), f^(2)(x),  ..., f^(m)(x)}, with a cardinality of (m + 1).
All members of this set are in A.

By the PHP, there are atleast two members in S that are the same value.
In other words, there exists 0 ⩽ i < j ⩽ m, such that f^(i)(x) = f^(j)(x)

Because f is a bijective function such that f^(j-i)(x) = f^(i - i)(x) = x
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(c)

Define the relation $\sim$ on A by

$$a \sim b \text{ if and only if } b = f^k(a) \text{ for some non-negative integer } k.$$

Using part **(b)** or otherwise, prove that $\sim$ is an equivalence relation on A.

Reflexive
a ~ a must be true. We have already proven that a = f^{k}(a) exists.

Symmetric
The relation ~ is defined as symmetric iff:
If a ~ b exists, then b ~ a exists.
Hence, the implication is such that if b = f^(k)(a) ⇒ a = f^(j)(b), for some non-negative integers k and j.
Recall from part b) that there is some integer j such that a = f^(k)(a).
Furthermore, the cyclical property of part b) leads to f^(mq)(a) = f^(k)(a), for some large enough m and q.
Therefore, we have:
f^(mq)(a) = b
f^(mq - k)(f^(k)(a)) = f^(mq-k)(b)
f^(mq-k)(a) = f^(mq-k)(b)
a = f^(mq-k)(b),
Therefore, we have some n = mq-k, such that a = f^(n)(b) is true.

Hence, we have proven that b ~ a exists if a ~ b exists, and hence, is symmetric.

Transitive
a ~ b and b ~ c then a ~ c.
a ~ b implies b = f^{k}(a) for some integer k
b ~ c implies c = f^{j}(b) for some integer j
By way of substituting,
c = f^{j}(f^{k}(a))
And since there will always be some k such that a = f^{k}(a);
c = f^{j}(a), and therefore, transitive.