Chapter 13

Evolutionary Games

When interacting with other agents, it is crucial to identify the best possible strategy leading to the best expected outcome. This is the goal of *game theory*. Examples include card games, political science, market pricing, and cooperation decisions [1]. One of the most famous game-theory examples is the *prisoner's dilemma* (Fig. 13.1). In its simplest version, two partners in crime have been apprehended and, now prisoners, are being interrogated separately. Let us call them prisoner A and prisoner B. If both remain silent, they will get just one year each. However, each of them can get a better deal by betraying their partner. For example, if A betrays B, A will get away free, but B will get 15 years. If they both betray each other, they will both get 5 years. Therefore, it would be best for them as a group to remain silent and do their one year in prison. However, here is the catch. Since the two prisoners are being interrogated independently, they do not have any means to know or control what their partner decides to do and, therefore, it is always convenient for them to betray each other: If their partner remain silent, they get away free; if their partner also betrays them, they get 5 years instead of 15. The dilemma arises because the choices that are best for individuals lead to a worse outcome for the group.

In most cases, agents need to make similar decisions time after time and can take advantage of their past experience [2, 3, 4]. Therefore, it is interesting to investigate how the behavior of individual agents changes during a series of similar interactions.

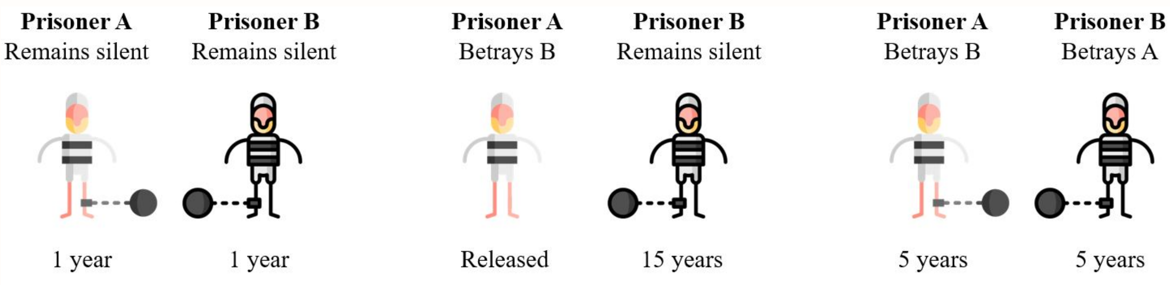

Figure 13.1: **The prisoner's dilemma.** The most famous game-theory example is the *prisoner's dilemma*. Although it is more profitable for prisoners as a group to cooperate (remain silent), it is always better from a single prisoner's perspective to defect (confess).

In this chapter, we first investigate the prisoner's dilemma between two players interacting for multiple rounds. Then, we consider a large group of players on a lattice playing the prisoner's dilemma with their nearest neighbors and updating their strategy according to their experience, observing the emergence of *cooperative behavior*. Then, we consider how these interactions can be affected by *randomness*.

Example codes: Example Python scripts related to this chapter can be found on:
https://github.com/softmatterlab/SOCS/tree/main/Chapter%5F13%5FEvolutionary%5FGames
Readers are welcome to participate in the discussions related to this chapter on:
https://github.com/softmatterlab/SOCS/discussions/22

13.1 The prisoner's dilemma

The original prisoner's dilemma is a two-player game that is often used as an example of the cooperation problem in game theory. As we have seen in the introduction, the dilemma arises from the fact that, from an individual's perspective, it is always more beneficial to defect against their partner in crime (betray them), even if the ideal solution for both as a group would be to cooperate (remain silent).

In its simplest form, the prisoner's dilemma has two partners in crime being interrogated separately, so that they cannot communicate. Both prisoners get an offer from the police (see Fig. 13.1): If they confess and witness against their partner, they will be punished by a mere T years (e.g., 0 years) in prison, while their partner will be punished by S years in prison (e.g., 15 years). However, if they both betray one another, both will be punished by

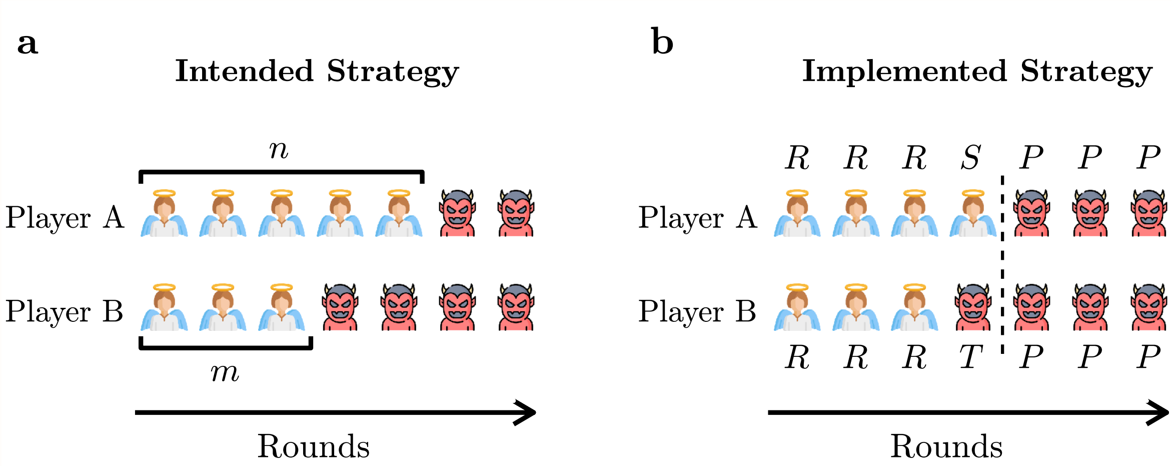

Figure 13.2: **The prisoner's dilemma with multiple rounds.** **a** Initially, player A and player B have strategies of cooperating (angel icon) $n = 5$ and $m = 3$ times, respectively. **b** After player A gets betrayed while cooperating (black dashed line), it starts to defect (demon icon) for the rest of the game. At each round, if both players cooperate, they get R years in prison each; if only one cooperates, the cooperator gets S and the defector gets T; if they both defect, they both get P years ($T < R < P < S$).

P years in prison (e.g., 5 years). If they cooperate and both keep silent, they will only be punished by R years in prison (e.g., 1 year). Importantly, these punishments are arranged so that $T < R < P < S$. Depending on the specific conditions, T maybe zero, or even an award. To reduce the parameter space, we can assume that $T = 0$ and $P = 1$ with $R \in [0,1]$ and $S \in [1,2]$.

Playing the game only once, the best strategy is to defect. In fact, for each individual, it is clearly best to defect, as they will be receiving a smaller punishment regardless of what their partner does.

In a game with $N > 1$ rounds, the considerations change. In each round, both prisoners are going to either defect or cooperate. A strategy is to plan to cooperate for a certain number of rounds and defect afterwards. We denote this strategy by the number of intended cooperation rounds $n \in [0,N]$: Cooperate until round n (included), or until the other player defects, and then defect for the rest of the game. This indicates that $n = 0$ is a strategy that always defects and $n = N$ is a strategy that always cooperates. A schematic representation of two players with strategies $n = 5$ and $m = 3$ is shown in Fig. 13.2.

The multiple-round prisoner's dilemma becomes interesting because even though it is more beneficial for an individual prisoner to defect for a single round, it is also important to keep their partner cooperative for as many rounds as possible. Therefore, the best strategy is for a player to defect one round before their opponent is planning to. For example, let us consider a game with $N = 10$ rounds against a player with strategy $m = 6$. Then, it is best to

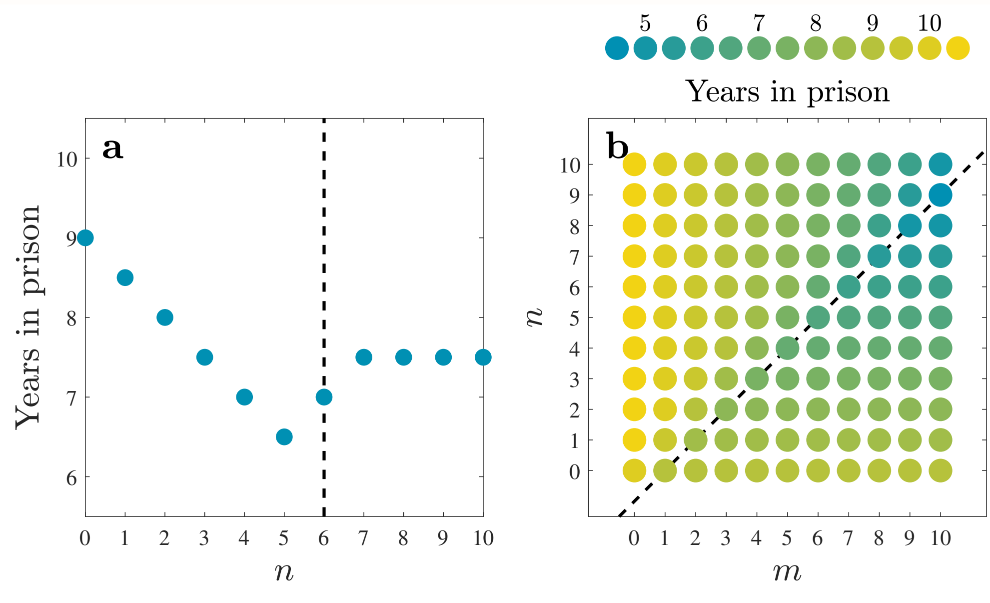

Figure 13.3: **Optimal strategy for a multiple-round prisoner's dilemma.** **a** Years in prison as a function of strategy n playing against a player with strategy $m = 6$ (black dashed line). The best strategy is $n = 5$. This makes the opponent keep cooperating until a round before they defect. **b** 2D map of years of prison as a function of n and m. As shown here, the optimal strategy is always $n = m - 1$ (black dashed line). This is a game with $N = 10$ rounds with parameters $T = 0$, $R = 0.5$, $P = 1$, $S = 1.5$.

cooperate until the 5th round and defect in the 6th round (this is the $n = 5$ strategy), as shown in Fig. 13.3a. A two-dimensional map of the resulting years in prison for prisoner A as a function of both n and m is shown in Fig. 13.3b.

Exercise 13.1: Prisoner's dilemma with multiple rounds. Simulate the prisoner's dilemma for a number of rounds and calculate the accumulated years in prison. Initially, choose the parameters $N = 10$, $T = 0$, $R = 0.5$, $P = 1$ and $S = 1.5$.

a. Fixing the opponent's strategy to m, show that the best strategy is $n = m - 1$, as shown in Fig. 13.3a.

b. Generate a 2D map of years in prison as a function of the player's strategy n and the opponent's strategy m, as shown in Fig. 13.3b.

c. While keeping $T = 0$ and $P = 1$, play with R and S to see how these parameters affect the results. Show that, as long as the essential condition in a prisoner's dilemma ($T < R < P < S$) is satisfied, it is always best to have the strategy $m - 1$ against a player with strategy m.

13.2 Evolutionary games on a lattice

Although the underlying theory and the results are straightforward even with multiple rounds, the prisoner's dilemma becomes complex when it is played many times with multiple players who have a memory of the past games [2].

To study an ensemble of players that interact with each other, we consider a two-dimensional $L \times L$ array of players. At each time step, each player plays the prisoner's dilemma with its four closest neighbors, as shown in Fig. 13.4a, and, if any of its neighbors achieves a better score, it updates its strategy, as shown in Fig. 13.4b [3, 2]. This evolutionary game can be simulated by the following steps:

1. Initialize the $L \times L$ lattice with random strategies for each site ranging, so that $n_{ij} \in [0, N]$ (see previous section for details about the strategy).

2. **Competition**: At each time step, each agent plays the prisoner's dilemma with its four nearest von Neumann neighbors (top, bottom, left, right), as shown in Fig. 13.4a.

3. **Revision**: After the games have been simulated, each player updates its strategy to that with the best score (lowest punishment) amongst those of its neighbors and itself, as shown in Fig. 13.4b. If multiple strategies tie, the choice is random between those.

4. **Mutation**: At the end of each time step, there is a small probability μ for each player to mutate its strategy to a random strategy, as shown in Fig. 13.4c.

To understand the evolution of cooperation and defection, we start with players that can use only two strategies: The players can either cooperate in all rounds ($n = N$) or defect from the first round ($n = 0$). We first look at the case with no mutation ($\mu = 0$). For example, for $N = 7$, $R = 0.9$, and $S = 1.5$, if a single defector is placed in a uniform lattice of collaborators, it moves along the lattice and creates a pattern of defectors, as shown in

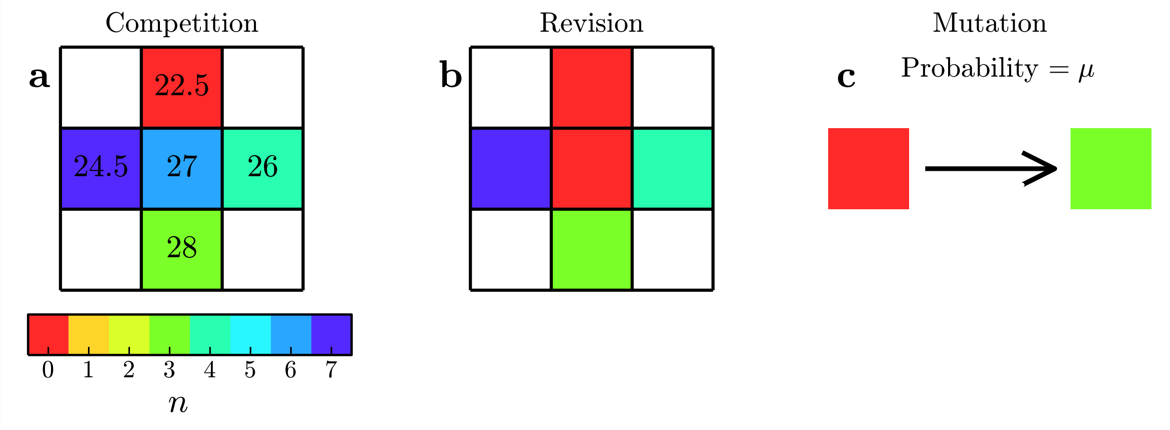

Figure 13.4: **Prisoner's dilemma on a lattice.** **a** At each time step, every player plays simple prisoner's dilemmas (N rounds) with its closest neighbors. Each player's score (lower is better) is shown by the numbers written inside each cell. **b** Then, every player adopts the strategy with the best score (lowest punishment, red square) amongst those played by its neighbors and itself. **c** Finally, some players mutate to a random strategy (probability μ).

Figs. 13.5a-b. Different patterns can be obtained by initially inserting two (Figs. 13.5c-d), three (Figs. 13.5e-f), or four defectors (Figs. 13.5g-h). Yet different (and beautiful) patterns

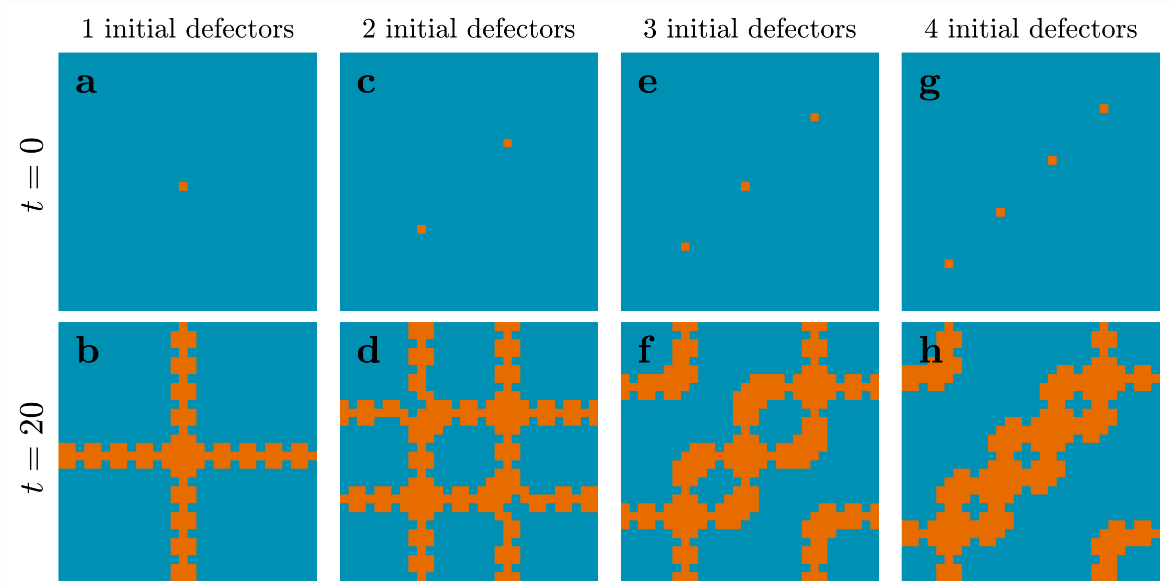

Figure 13.5: **Spreading of defectors amongst cooperators.** **a** A single defector (orange) placed on a lattice of cooperators (blue) spreads, eventually forming the pattern of defectors in **b**. Different patterns are emerge with different initial numbers of defectors: **c-d** two initial defectors, **e-f** three initial defectors, and **g-h** four initial defectors. Parameters used for these simulations are $N = 7$, $R = 0.9$, $S = 1.5$, $L = 30$, and $\mu = 0$.

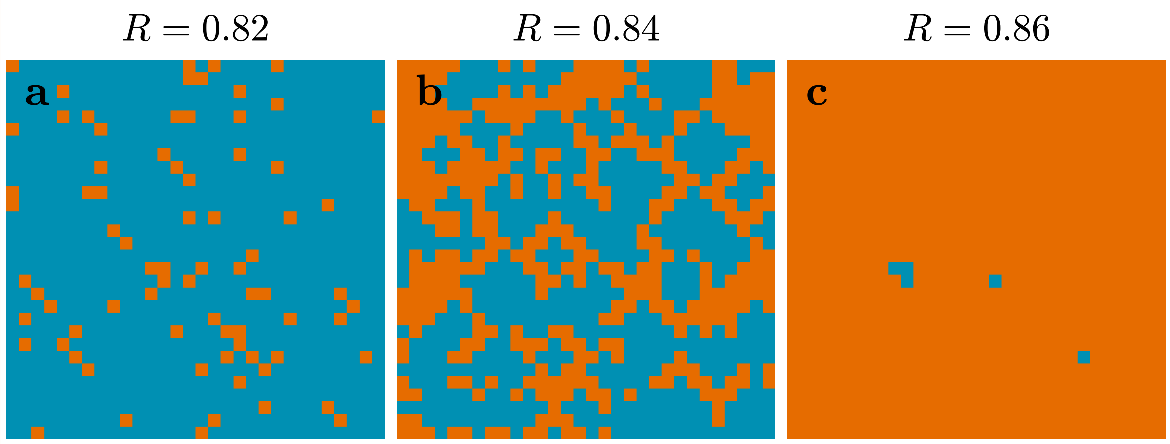

Figure 13.6: **Defectors vs cooperators. a** When the punishment for cooperating prisoners is low ($R = 0.82$), the cooperation strategy (blue) dominates, although small clusters of defectors (orange) can still exist. **b** When the punishment for cooperating prisoners is higher ($R = 0.84$), cooperation (blue) coexists with defection (orange). **c** When the punishment for cooperating prisoners is even higher ($R = 0.86$), the cooperation strategy (blue) disappears, eventually leading to the total domination of defectors (orange). Snapshots taken after 100 time steps. Parameters: $N = 7$, $S = 1.5$, $L = 30$, and $\mu = 0.01$.

emerge by changing the parameters of the prisoner's dilemma or the number of neighbors (e.g., using the Moore neighborhood with 8 closest neighbors). It is even possible to obtain gliding and oscillating patterns, similar to those in the Game of Life (Chapter 4) [5].

Exercise 13.2: Patterns in evolutionary games. Simulate the prisoner's dilemma on a $L \times L$ lattice. Allow only two strategies: always cooperate ($n = N$) and always defect ($n = 0$). Assume $\mu = 0$. Make sure that you choose a very small L at first to try your code and visualize its results. Then, increase L to a larger value (between 20 and 100). As usual, use $T = 0$ and $P = 1$. In addition, you can fix $S = 1.5$ and play only with the parameter R.

a. Initialize a lattice full of cooperators and place a single defector in the middle. Find out the range of R where the defecting behavior would only spread along a line pattern in all directions, as shown in Fig. 13.5b. What happens for other values of R?

b. Play with the number of initial defectors and simulate the system. Observe different pattern formations similar to those that are shown in Figs. 13.5c-h.

c. What happens if you place a single cooperator in a lattice of defectors?

d. Find the range of R when a cluster of cooperators in a background of defectors vanish, stay stable, or spread.

With a non-zero mutation rate ($\mu \neq 0$), three regimes emerge depending on the parameter R (or S, whichever we decide to change). The regime where the cooperation dominates (as shown in Fig. 13.6a), the regime where they coexist (as shown in Fig. 13.6b), and the regime where defection dominates (as shown in Fig. 13.6c).

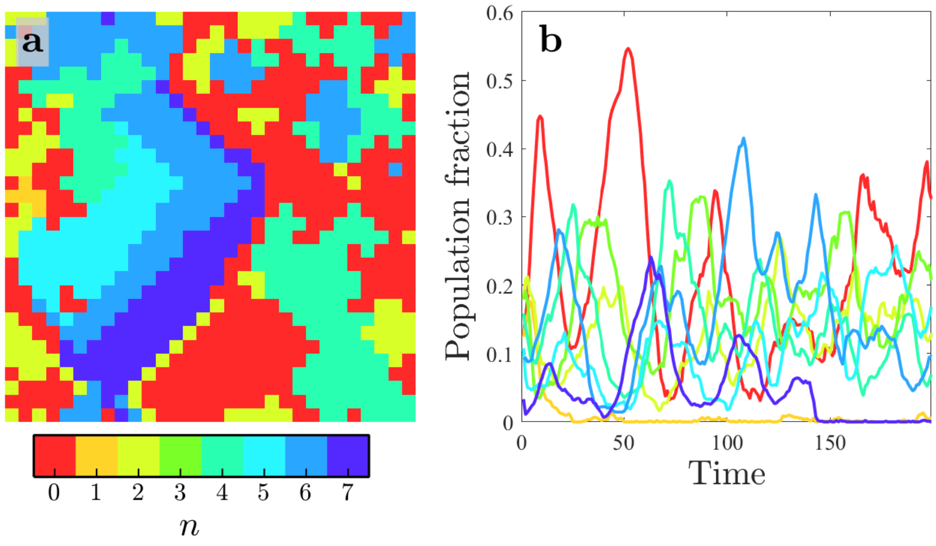

Figure 13.7: **Evolutionary games on a lattice with multiple strategies.** **a** Snapshot of the strategies obtained after 200 steps of the simulation. Different strategies coexist, often as clusters with diagonal borders (matching the neighborhood borders). **b** The population fraction with different strategies during the simulation. Different strategies emerge, grow, and vanish continuously. In general, a large population of defectors (red tones) favors the growth of cooperative behavior (blue tones), and vice versa. Parameters: $N = 7$, $R = 0.72$, $S = 1.5$, $L = 30$, and $\mu = 0.01$.

Exercise 13.3: Defectors vs cooperators. Simulate the prisoner's dilemma on a $L \times L$ lattice with only two strategies: always cooperate ($n = N$) and always defect ($n = 0$). Use a small but non-zero mutation rate, such as $\mu = 0.01$. As usual, use $T = 0$ and $P = 1$. In addition, you can fix $S = 1.5$ and play only with parameter R.

a. With $R = 0.82$, show that the cooperation dominates, as shown in Fig. 13.6a.

b. With $R = 0.84$, show that cooperation coexists with defection, as shown in Fig. 13.6b.

c. With $R = 0.86$, show that the cooperation vanishes (Fig. 13.6c).

d. Repeat your simulations for a range of R to show that this behavior is critical. Determine the critical values of R that lead to each regime.

e. Repeat the same analysis fixing the value of R and varying S.

13.3 Multiple strategies

So far, we have considered evolutionary games with only two strategies: always cooperate ($n = N$) and always defect ($n = 0$). However, one can consider evolutionary games where a player cooperates for an arbitrary number of rounds and then defects. In the case where an agent plays against a single agent, as shown in Fig. 13.3, the outcome is straightforward. However, when the players play this game on a lattice with multiple agents that have different strategies, the simulations get more complex and interesting. In this section, we will generalize our simulations to multiple strategies with $n \in [0, N]$.

In these conditions, small clusters of players with the same strategy may emerge, propagate, and disappear depending on the choice of parameters. With $S = 1.5$, cooperative strategies prevail for small R ($R \lessapprox 0.50$), while all strategies coexist together for intermediate values of R ($0.50 \lessapprox R \lessapprox 0.75$), and defecting strategies dominate for large R ($R \gtrapprox 0.75$). Fig. 13.7 shows the case with $R = 0.72$, where different strategies appear, grow, and propagate. Importantly, there are large fluctuations, as shown in Fig. 13.7b.

Exercise 13.4: Evolutionary games on a lattice with multiple strategies. Simulate a seven-round prisoner's dilemma on a $L \times L$ lattice. Allow all strategies ($0 \leq n \leq 7$). Use a small but non-zero mutation rate, such as $\mu = 0.01$. As usual, use $T = 0$ and $P = 1$. Depending on the parameters of R and S, multiple regimes can be observed.

a. Fix $S = 1.5$ and play only with parameter R. Observe what happens within the lattice as time evolves. Show that different strategies may emerge and propagate, as shown in Fig. 13.7a.

b. Show that there are three main regimes similar to those identified in Exercise 13.3. However, this time the regime where different strategies coexist can have different dominant populations, depending on the value of R. Find out the parameters and the population distribution.

c. Discuss the results of your simulation. What do these numerical outcomes tell us about the evolution of cooperation? What strategies are *evolutionary stable strategies*?

Exercise 13.5: Two-dimensional phase map of evolutionary games. For the case of multiple allowed strategies, the dynamics does not depend on only one parameter, or a simple mathematical relation between the two parameters R and S. Instead, the output behavior is rather complex and depends on both parameters.

a. Simulate the evolutionary games with different values of R and S. For every simulation, record at least 500 steps and omit the first 100 steps in the analysis. Calculate the variance of the population for each strategy σ_n^2.

b. $\sum_n \sigma_n^2$ can be used to determine if there is an active competition between the strategies. Determine a reasonable threshold for this parameter and create a phase diagram for the output behavior as a function of R and S.

13.4 Further Readings

Ref. [1] is an excellent book for going through different models of cooperative game theory; it includes also methodological principles, solution concepts, and model analyses.

Ref. [4] is a great book for theoretical information on evolutionary games; it deals also with the implications and connections between evolutionary biology and cooperative strategies in economy.

Ref. [2] is the pioneering work on evolutionary game theory with players that have memory of past games. Ref. [3] is an excellent source for the evolution of cooperation with numerical simulations.

Ref. [5] demonstrates that the rules in evolutionary games can be tuned so that the output behaviors resemble those encountered in the Game of Life.

Ref. [6] is a recent study that demonstrates the emergence of exploitative behavior when players use reinforcement learning to develop their probabilistic strategies. Interestingly, it is shown that this occurs even when the rules of the game and the strategy dynamics are symmetric.

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