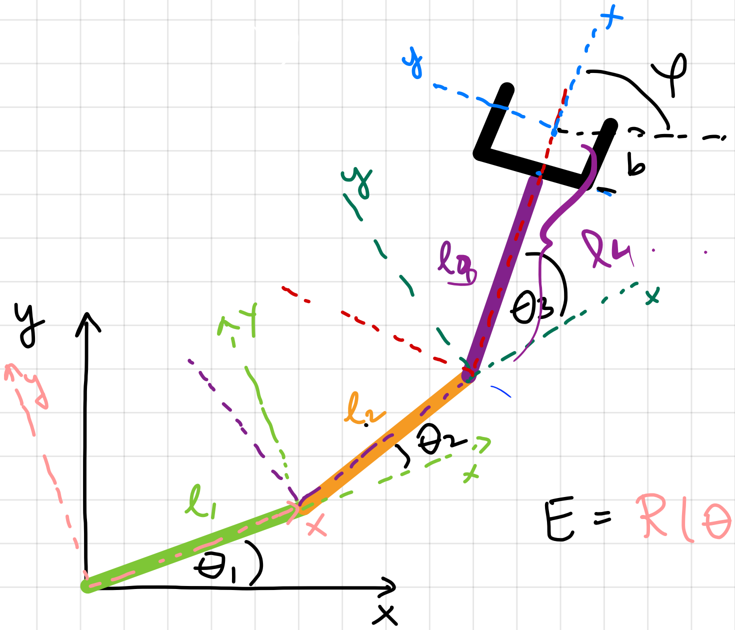


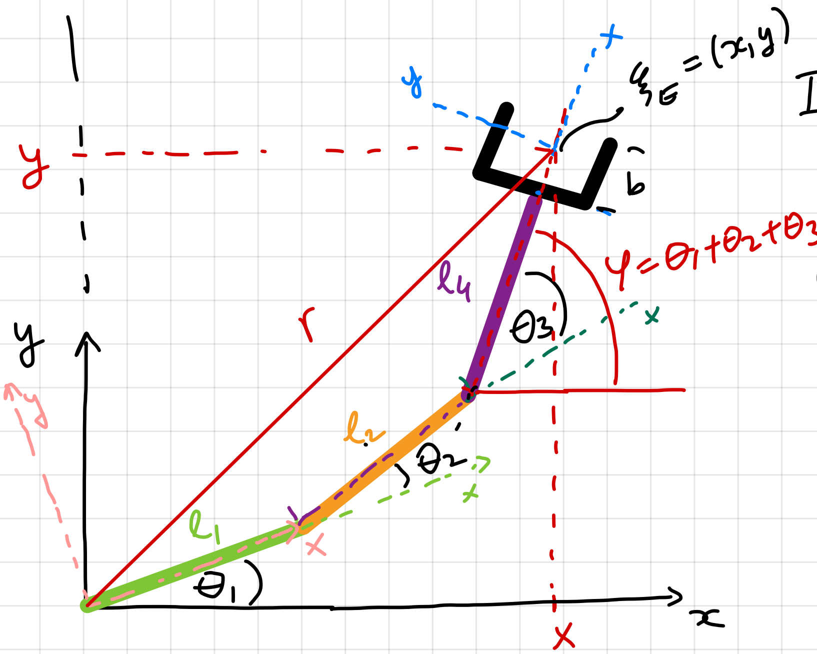
Forward Kinematics

$l_4 = l_3 + \frac{b}{2}$

E : end efector

$$E = R(\theta_1) T(l_1) \cdot R(\theta_2) T(l_2) R(\theta_3) T(l_4)$$

$$R(\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta \end{bmatrix} \qquad T(l_4) = \begin{bmatrix} 1 & 0 & l_4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Inverse kinematics

$r^2 = x^2 + y^2$

$a^2 = b^2 + c^2 - 2bc\,\measuredangle(b, c)$

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_4 \cos(\theta_1 + \theta_2 + \theta_3) \qquad (1)$$

$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_4 \sin(\theta_1 + \theta_2 + \theta_3) \qquad (2)$$

$$\varphi = \theta_1 + \theta_2 + \theta_3 \qquad (3)$$

Given x, y, φ

sub. (3) into (1) and (2):

$$\Rightarrow \quad x - l_4 \cos\varphi = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \qquad (4)$$

$$y - l_4 \sin\varphi = l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \qquad (5)$$

let $x_1 = x - l_4 \cos\varphi$

$y_1 = y - l_4 \sin\varphi$

$$x_1 - l_1 \cos\theta_1 = l_2 \cos(\theta_1+\theta_2) \qquad /^2$$

$$y_1 - l_1 \sin\theta_1 = l_2 \sin(\theta_1+\theta_2) \qquad /^2$$

$$\left.\begin{aligned} x_1^2 - 2x_1 l_1 \cos\theta_1 + l_1^2 \cos^2\theta_1 &= l_2^2 \cos^2(\theta_1+\theta_2) \\ y_1^2 - 2y_1 l_1 \sin\theta_1 + l_1^2 \sin^2\theta_1 &= l_2^2 \sin^2(\theta_1+\theta_2) \end{aligned}\right\} +$$

$$1 \qquad\qquad 1$$

$$x_1^2 + y_1^2 - 2x_1 l_1 \cos\theta_1 - 2y_1 l_1 \sin\theta_1 + l_1^2 = l_2^2$$

$$\Rightarrow \quad \underbrace{-2x_1 l_1}_{A} \cos\theta_1 \underbrace{- 2y_1 l_1}_{B} \sin\theta_1 + \underbrace{(x^2+y^2+l_1^2-l_2^2)}_{C} = 0$$

$$A\cos\theta_1 + B\sin\theta_1 + C = 0$$

define α:

$$\cos\alpha = \frac{A}{\sqrt{A^2+B^2}} \qquad \sin\alpha = \frac{B}{\sqrt{A^2+B^2}}$$

$$\alpha = \tan^{-1} \frac{\frac{B}{\sqrt{A^2+B^2}}}{\frac{A}{\sqrt{A^2+B^2}}}$$

⋮

Solution :

$$\theta_1 = \alpha \pm \cos^{-1}\left(\frac{-C}{\sqrt{A^2+B^2}}\right)$$

$$\alpha = \tan^{-1}\frac{B}{A} = \tan^{-1}\frac{y_1 l_1}{x_1 l_1} = \tan^{-1}\frac{y_1}{x_1}$$

$$\theta_1 = \tan^{-1}\frac{y_1}{x_1} \pm \cos\left(\frac{-(x^2+y^2+l_1^2-l_2^2)}{2l_1\sqrt{x_1^2+y_1^2}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{y_1 - l_1\sin\theta_1}{x_1 - l_1\cos\theta_1}\right) - \theta_1$$

$$\theta_3 = \phi - \theta_1 - \theta_2$$

$$x = l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_4\cos(\theta_1+\theta_2+\theta_3) \quad (1)$$

$$y = l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) + l_4\sin(\theta_1+\theta_2+\theta_3) \quad (2)$$

$$\varphi = \theta_1+\theta_2+\theta_3 \quad (3)$$

Velocities: differentiate (1), (2), (3) with respect to time;

$$\dot{x} = -l_1\dot{\theta}_1\sin\theta_1 - l_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) - l_4(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\sin(\theta_1+\theta_2+\theta_3)$$

$$\dot{y} = l_1\dot{\theta}_1\cos\theta_1 + l_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) + l_4(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)\cos(\theta_1+\theta_2+\theta_3)$$

$$\dot{\varphi} = \dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{bmatrix} = \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{bmatrix}}_{\text{Jacobian matrix}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial \varphi}{\partial \theta_1} & \frac{\partial \varphi}{\partial \theta_2} & \frac{\partial \varphi}{\partial \theta_3} \end{bmatrix}$$

$$a = -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) - l_4 \sin(\theta_1+\theta_2+\theta_3)$$

$$b = -l_2 \sin(\theta_1+\theta_2) - l_4 \sin(\theta_1+\theta_2+\theta_3)$$

$$c = -l_4 \sin(\theta_1+\theta_2+\theta_3)$$

$$d = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_4 \cos(\theta_1+\theta_2+\theta_3)$$

$$e = l_2 \cos(\theta_1+\theta_2) + l_4 \cos(\theta_1+\theta_2+\theta_3)$$

$$f = l_4 \cos(\theta_1+\theta_2+\theta_3)$$

$$\Rightarrow \underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{bmatrix}}_{\dot{e}} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial \varphi}{\partial \theta_1} & \frac{\partial \varphi}{\partial \theta_2} & \frac{\partial \varphi}{\partial \theta_3} \end{bmatrix}}_{J} \cdot \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}}_{\dot{\theta}}$$

$$\Rightarrow \boxed{\dot{\theta} = J^{-1} \dot{e}}$$

state equation