Pham and Park's Proof of the Kahn-Kalai Conjecture

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Outline

1. Preliminaries: Random Graphs
2. Expectation Threshold
3. Kahn-Kalai Conjecture Statement
4. Outline of the Proof

$G_{n,p}$ and $G_{n,m}$ Models

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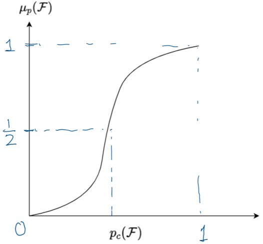
- For large n, $G_{n,p} \sim G_{n,m}$ when $m = \binom{n}{2}p$, i.e. when m is close to the expected number of edges of $G_{n,p}$

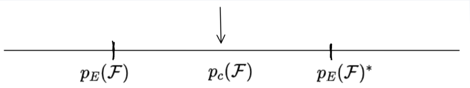
In an abstract setting

- X a finite set. 2^X the power set of X.
- $X_p \sim \mu_p$ where $\mu_p(A) = p^{|A|}(1-p)^{|X \setminus A|} \quad A \subseteq X$
- X_m is the uniformly random subset of X of size m
- $\mathcal{F} \subseteq 2^X$ is an increasing property if $B \supseteq A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$
- $\mu_p(\mathcal{F}) \left(:= \sum_{A \in \mathcal{F}} \mu_p(A)\right)$ is a strictly increasing in p, so that we have:



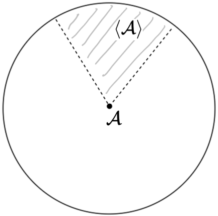
The *threshold* $p_c(\mathcal{F})$ is the unique p for which $\mu_p(\mathcal{F}) = 1/2$

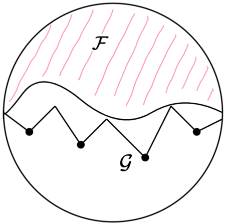
Locating the Threshold



Our main aim is to provide bounds for the threshold $p_c(\mathcal{F})$.

- $\langle A \rangle := \{B : B \supseteq A\}$,
- $\mathcal{G}$ *covers* $\mathcal{F}$ if $\mathcal{F} \subseteq \langle \mathcal{G} \rangle := \bigcup_{S \in \mathcal{G}} \{U : U \supseteq S\}$





Expectation Threshold

Observation

$p_c(\mathcal{F}) \geqslant p$ if $\exists\, \mathcal{U} \subseteq 2^X$ such that

- $\mathcal{U}$ covers $\mathcal{F}$: $\mathcal{F} \subseteq \langle \mathcal{U} \rangle := \bigcup_{u \in \mathcal{U}} \langle u \rangle$
- $\mathcal{U}$ is p-cheap: $\sum_{u \in \mathcal{U}} p^{|u|} \leqslant \frac{1}{2}$

$$\implies q(\mathcal{F}) := \max\{p : \exists\, \mathcal{U}\}$$

Expectation Threshold is the maximum value of probability that allows a cheap cover of $\mathcal{F}$

Kahn-Kalai and the Reformulation

Write $I(\mathcal{F})$ for the size of the largest minimal element of $\mathcal{F}$.

Kahn-Kalai Conjecture:

There exists $K > 0$ a universal constant such that for any finite set X and increasing property $\mathcal{F}$

$$q(\mathcal{F}) < p_c\mathcal{F} < K.q(\mathcal{F}).\log I(\mathcal{F})$$

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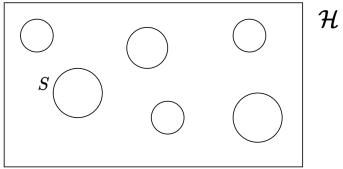
Reformulation

There exists $L > 0$ a universal constant such that for all l-bounded $\mathcal{H} \subseteq 2^X$ if $p > q(\langle\mathcal{H}\rangle)$ then with $m = (Lp\log l).|X|$

$$\mathbb{P}\left(X_m \in \langle H\rangle\right) = 1 - o_{l\to\infty}(1)$$

Idea of the Proof

With the reformulation, we have access to the elements of the cover, which we could try to manipulate. Recall $\mathcal{H} = \{$ minimal elements of $\mathcal{F}\}$



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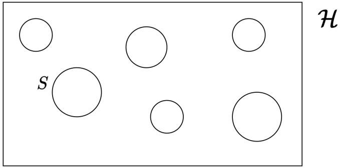
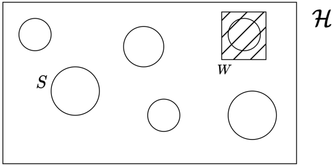


We would like to capture some $S \in \mathcal{H}$ in a random $m-$subset X_m, which we will call W. If we capture it whp, then we would be done.



Dreamy Situation - log l steps

Recall $\mathcal{H}$ is $l-$bounded; each $|S|$ is at most l.
A dreamy situation: we sprinkle our random set W little by little in some steps $W_1 \cup W_2 \cup \ldots W_n$, where in the $i-$th step $|W_i| = Lp|X|$

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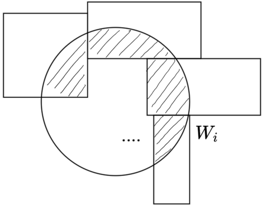

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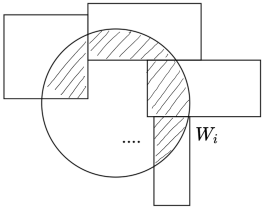

- Each W_i is such that it covers, for example, 10% of the remaining area. Since the size of S is at most l, we cover S in at most $\log l$ steps.
- $\implies |W| = (Lp \log l)|X|$. Would be done.

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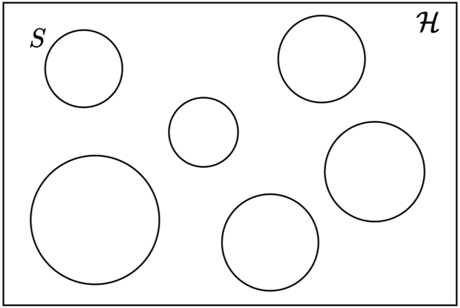
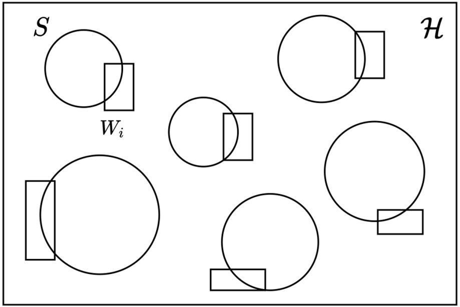
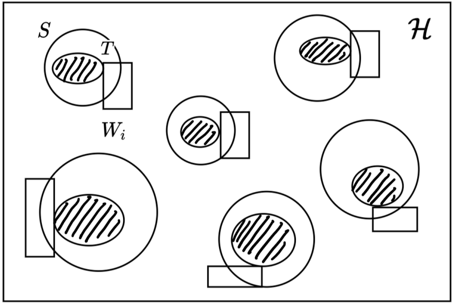
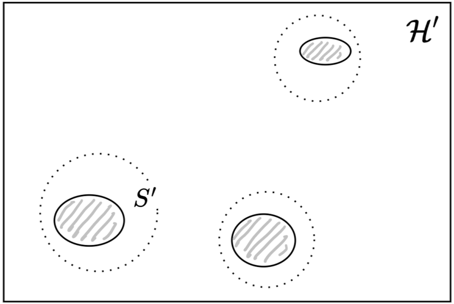




Iterative Algorithm

- We look at the subsets S' of $W_i \cup S$ and take $T = T(S, W)$ such that $|T(S, W)| = |S' \setminus W|$ is the minimum.





- If $|T| > 0.9l$, we put T in $\mathcal{G}$. These constitute the elements which will cover $\mathcal{H}$. For T's such that $|T| \leqslant 0.9l$, we put them in our next hypergraph $\mathcal{H}'$

Iterative Algorithm

- Important: A cover of the new $\mathcal{H}'$ also covers the leftovers from the previous $\mathcal{H}$
- Since T is almost the size of S, it is cheap to specify it
- In this way, we carefully we construct a cheap cover of $\mathcal{H}$
- The iteration terminates if:
    - either we capture some S in W
    - or we fully cover $\mathcal{H}$
- For p that is above the expectation threshold $q(\mathcal{F})$, $\mathcal{F}$ doesn't admit a cheap cover *implies* the latter case happens with a low probability.
- $\implies S \subseteq W$ with high probability.
- terminates in $\leqslant \log l$ steps.

A few applications

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- The conjecture trivially locates the threshold for increasing properties such as the presence of Hamiltonian cycles and Perfect matchings.
- Also resolves many other related conjectures, such as those that appeared in the original conjecture paper
- The authors are working on classifying the logarithmic gap, but as of now, no broad mechanism to do so.

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