

PROBABILITY AND STATISTICS (SOLUTION)

Question 01:

[5+3+2=10]

- a) From a box containing 4 black balls and 2 green balls, 3 balls are drawn in succession, each ball being replaced in the box before the next draw is made.

- i) Find the probability distribution for the number of green balls.

| | | | | |
|------------|--------|-------|-------|--------|
| x | 0 | 1 | 2 | 3 |
| $P(X = x)$ | $8/27$ | $4/9$ | $2/9$ | $1/27$ |

- ii) Find the expected number of green balls. $E(x = \text{green ball}) = 1$
 iii) Draw the probability histogram.

- b) Suppose the measurement error X of a certain physical quantity is decided by the density function

$$f(x) = \begin{cases} k(3 - x^2), & -1 \leq x \leq 1, \\ 0, & \text{elsewhere} \end{cases}$$

- i) Determine k that renders $f(x)$ a valid density function. $k = \frac{3}{16}$
 ii) Find the probability that a random error in measurement is less than $\frac{1}{2}$.

$$P\left(x < \frac{1}{2}\right) = \int_{-1}^{\frac{1}{2}} f(x) dx = \frac{99}{128}$$

- iii) Find cumulative distribution function $F(x)$?

$$F(x) = \frac{3}{16} \int_{-1}^x (3 - t^2) dt = \left(3t - \frac{1}{3}t^3\right) \Big|_{-1}^x = \frac{1}{2} + \frac{9}{16}x - \frac{x^3}{16}.$$

- c) Let X be a random variable with the following probability distribution

| | | | |
|--------|-------|-------|-------|
| x | -3 | 6 | 9 |
| $f(x)$ | $1/6$ | $1/2$ | $1/3$ |

Find $\mu_{g(X)}$, where $g(X) = (2X + 1)^2$.

$$\mu_{g(X)} = E[(2X + 1)^2] = (25)(1/6) + (169)(1/2) + (361)(1/3) = 209.$$

Question 02:

[3+2+3=08]

- a) A recent study found that 2 out of every 10 houses in a neighbourhood have no insurance. If 5 houses are selected from 10 houses,

- i) Find the probability that exactly 1 will be uninsured

$$P(X) = \frac{{}^2C_1 \cdot {}^8C_4}{{}^{10}C_5} = \frac{2 \cdot 70}{252} = \frac{140}{252} = \frac{5}{9}$$

- ii) What is probability at most 1 houses are uninsured $= P(X \leq 1) = 0.778$

- b) On average, a textbook author makes two-word processing errors per page on the first draft of her textbook. What is the probability that on the next page she will make

- i) 4 or more errors?

$$(a) P(X \geq 4) = 1 - P(X \leq 3) = 0.1429.$$

- ii) no errors?

$$(b) P(X = 0) = p(0; 2) = 0.1353.$$

- c) Heart failure is due to either natural occurrence (87%) or outside factors (13%). Outside factors are related to induced substances or foreign objects. Natural occurrences are caused by arterial blockage, disease, and infection. Suppose that 20 patients will visit an emergency room with heart failure. Assume that causes of heart failure between individuals are independent.
- What is the probability that three individuals have conditions caused by outside factors? $P(x = 3) = 0.235$
 - What is the probability that three or more individuals have conditions caused by outside factors? $P(x \geq 3) = 1 - P(x < 3) = 0.492$

Question 03:

[7+5=12]

- a) Suppose that X and Y have the following joint probability density function where X = automobile policy deductible amount and Y = homeowner policy deductible amount

| $p(x, y)$ | | y | | |
|-----------|-----|-----|-----|-----|
| | | 0 | 100 | 200 |
| x | 100 | .20 | .10 | .20 |
| | 250 | .05 | .15 | .30 |

Evaluate

- i) the marginal distribution of x and y

| x | 100 | 250 | y | 0 | 100 | 200 |
|----------|-----|-----|----------|-----|-----|-----|
| $p_X(x)$ | .5 | .5 | $p_Y(y)$ | .25 | .25 | .5 |

- ii) the mean and variance of x and y

$$\mu_X = \sum x p_X(x) = 175 \text{ and } \mu_Y = 125.$$

$$\sigma_X^2 = 36,250 - (175)^2 = 5625, \quad \sigma_Y^2 = 6875.$$

- iii) the covariance of x and y

$$\text{cov}(x, y) = E(xy) - E(x)E(y) = 23750 - (175)(125) = 1875$$

iv) $P(Y \leq 100 | X = 250) = 0.05 + 0.15 = 0.20$

- b) Let X denote the reaction time, in seconds, to a certain stimulus and Y denote the temperature (°F) at which a certain reaction starts to take place.

$$f(x, y) = \begin{cases} 4xy, & 0 < x < 1, 0 < y < 1, \\ 0, & \text{elsewhere} \end{cases}$$

- Find $P\left(0 \leq X \leq \frac{1}{2} \text{ and } \frac{1}{4} \leq Y \leq \frac{1}{2}\right) = \frac{3}{64}$
- Calculate $\sigma_x^2 = E(x^2) - E(x)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18} = 0.0556$

THE END