Chapter 10 One- and Two-Sample Tests of Hypotheses

10.1 Statistical Hypotheses: General Concepts

10.2 Testing a Statistical Hypothesis

One- and Two-Tailed Tests

How Are the Null and Alternative Hypotheses Chosen?

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Paired Observations

Tests Concerning a Single Mean

1. State the null and alternative hypotheses.

2. Choose a fixed significance level α.

3. Choose an appropriate test statistic and establish the critical region based on α.

4. Reject H_0 if the computed test statistic is in the critical region. Otherwise, do not reject.

5. Draw scientific or engineering conclusions.

Hypothesis: A hypothesis is a statement that something is true.

For example, the statement “the mean weight of all bags of pretzels packaged differs from the advertised weight of 454 g” is a hypothesis.

In a typical study, the hypothesis test involves two types of hypotheses called “Null Hypothesis” and “Alternate Hypothesis”

Null hypothesis: A hypothesis to be tested. The symbol H_o to represent the null hypothesis.
H_o: $\mu = \mu_o$

Alternative hypothesis: A hypothesis to be considered as an alternative to the null hypothesis. The symbol H_a to represent the alternative hypothesis.
H_a: $\mu \neq \mu_o$ H_a: $\mu < \mu_o$ H_a: $\mu > \mu_o$

A hypothesis test is called a one-tailed test if it is either left tailed or right tailed.

The **critical value** separates the critical region from the noncritical region. The symbol for critical value is C.V.

The **critical** or **rejection region** is the range of values of the test value that indicates that there is a significant difference and that the null hypothesis should be rejected.

The **noncritical** or **nonrejection region** is the range of values of the test value that indicates that the difference was probably due to chance and that the null hypothesis should not be rejected.

A **one-tailed test** indicates that the null hypothesis should be rejected when the test value is in the critical region on one side of the mean. A one-tailed test is either a **right-tailed test** or **left-tailed test,** depending on the direction of the inequality of the alternative hypothesis.

In a **two-tailed test,** the null hypothesis should be rejected when the test value is in either of the two critical regions.

The **level of significance** is the maximum probability of committing a type I error. This probability is symbolized by α (Greek letter **alpha**). That is, $P(\text{type I error}) = \alpha$.

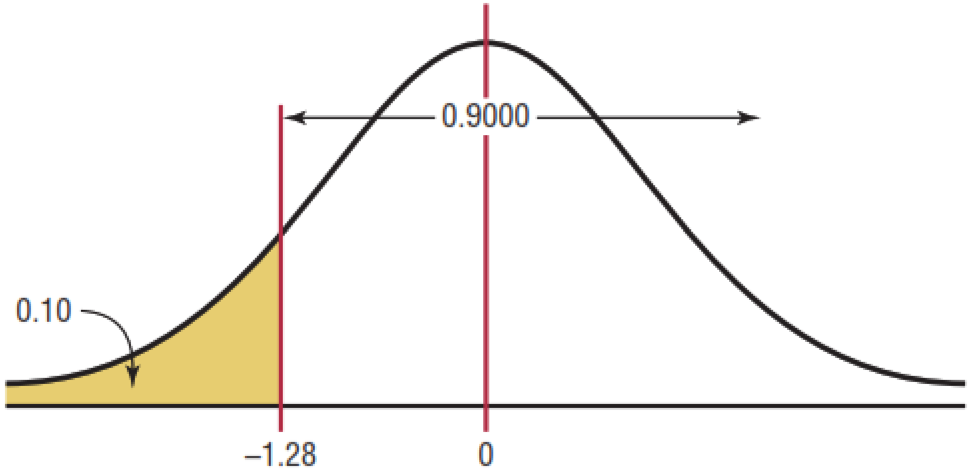
Two-tailed test	Right-tailed test	Left-tailed test
$H_0: \mu = k$	$H_0: \mu = k$	$H_0: \mu = k$
$H_1: \mu \neq k$	$H_1: \mu > k$	$H_1: \mu < k$

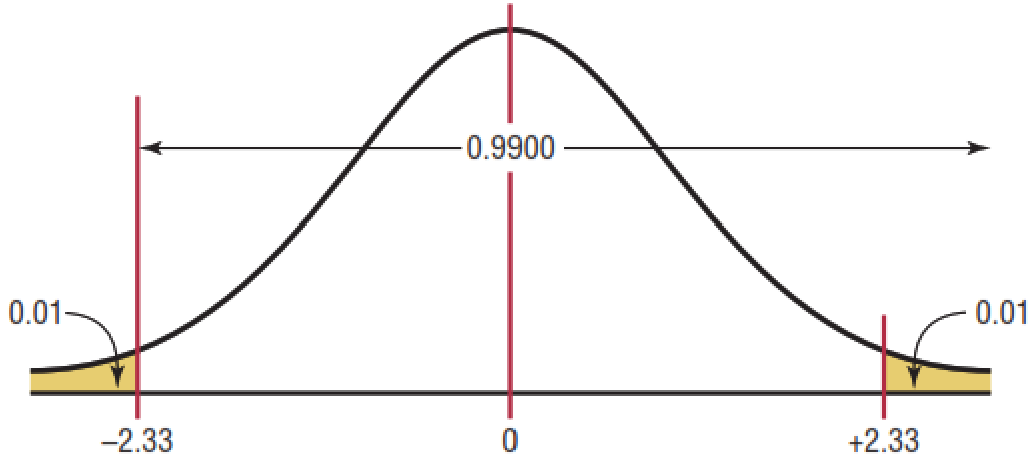
Using Table in Appendix , find the critical value(s) for each situation and draw the appropriate figure, showing the critical region.

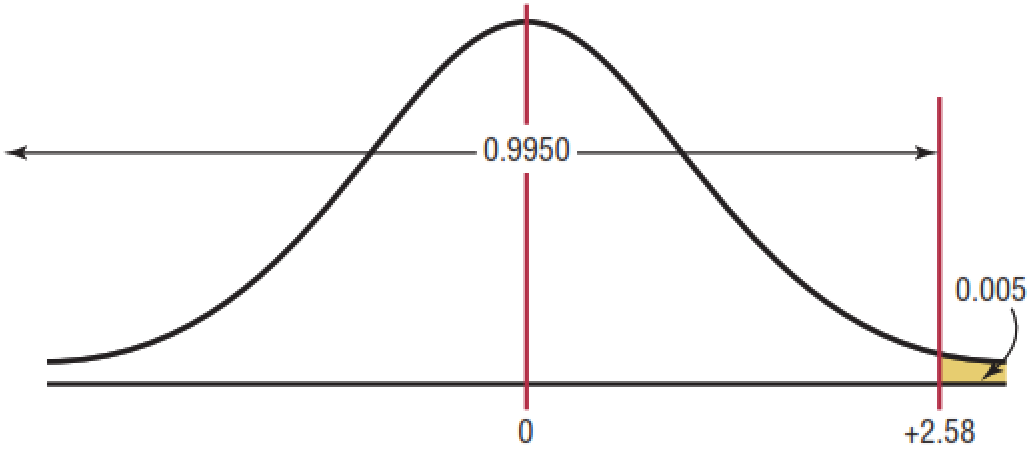
a. A left-tailed test with $\alpha = 0.10$.

b. A two-tailed test with $\alpha = 0.02$.

c. A right-tailed test with $\alpha = 0.005$.







Days on Dealers' Lots

A researcher wishes to see if the mean number of days that a basic, low-price, small automobile sits on a dealer's lot is 29. A sample of 30 automobile dealers has a mean of 30.1 days for basic, low-price, small automobiles. At $\alpha = 0.05$, test the claim that the mean time is greater than 29 days. The standard deviation of the population is 3.8 days.

Source: Based on information from *Power Information Network*.

Solution

Step 1 State the hypotheses and identify the claim.

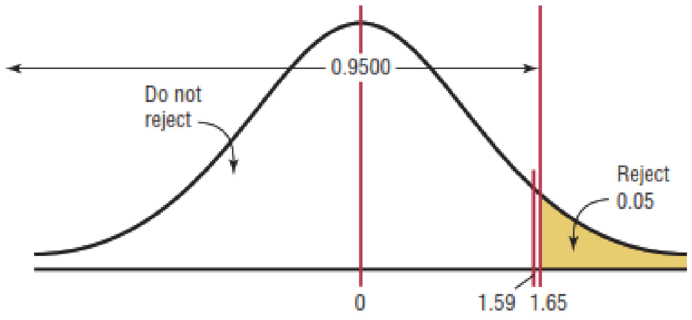
$H_0: \mu = 29$ and $H_1: \mu > 29$ (claim)

Step 2 Find the critical value. Since $\alpha = 0.05$ and the test is a right-tailed test, the critical value is $z = +1.65$.

Step 3 Compute the test value.

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{30.1 - 29}{3.8/\sqrt{30}} = 1.59$$

Step 4 Make the decision. Since the test value, +1.59, is less than the critical value, +1.65, and is not in the critical region, the decision is to not reject the null



Step 5 Summarize the results. There is not enough evidence to support the claim that the mean time is greater than 29 days.

Procedure Table

Solving Hypothesis-Testing Problems (Traditional Method)

Step 1 State the hypotheses and identify the claim.

Step 2 Find the critical value(s) from the appropriate table in Appendix C.

Step 3 Compute the test value.

Step 4 Make the decision to reject or not reject the null hypothesis.

Step 5 Summarize the results.

Cost of College Tuition

A researcher wishes to test the claim that the average cost of tuition and fees at a four-year public college is greater than \$5700. She selects a random sample of 36 four-year public colleges and finds the mean to be \$5950. The population standard deviation is \$659. Is there evidence to support the claim at $\alpha = 0.05$? Use the P-value method.

Source: Based on information from the College Board.

Solution

Step 1 State the hypotheses and identify the claim. H_0: $\mu = \$5700$ and H_1: $\mu > \$5700$ (claim).

Step 2 Compute the test value.

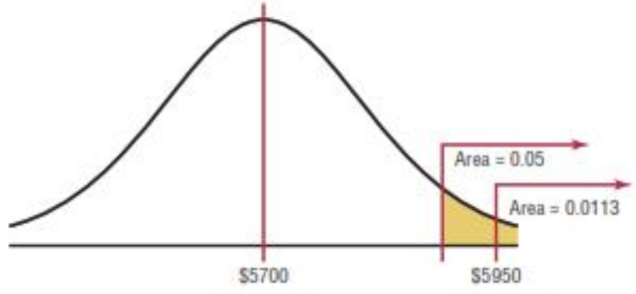
$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{5950 - 5700}{659/\sqrt{36}} = 2.28$$

Step 3 Find the P-value. Using Table E in Appendix C, find the corresponding area under the normal distribution for $z = 2.28$. It is 0.9887. Subtract this value for the area from 1.0000 to find the area in the right tail.

$1.0000 - 0.9887 = 0.0113$

Hence the P-value is 0.0113.

Step 4 Make the decision. Since the P-value is less than 0.05, the decision is to reject the null hypothesis. See Figure 8–19.



Step 5 Summarize the results. There is enough evidence to support the claim that the tuition and fees at four-year public colleges are greater than \$5700.

Note: Had the researcher chosen $\alpha = 0.01$, the null hypothesis would not have been rejected since the P-value (0.0113) is greater than 0.01.

Substitute Teachers' Salaries

An educator claims that the average salary of substitute teachers in school districts in Allegheny County, Pennsylvania, is less than \$60 per day. A random sample of eight school districts is selected, and the daily salaries (in dollars) are shown. Is there enough evidence to support the educator's claim at $\alpha = 0.10$?

60 56 60 55 70 55 60 55

Source: *Pittsburgh Tribune-Review.*

Solution

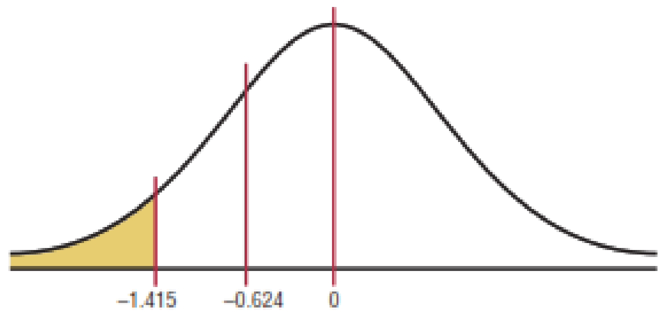
Step 1 H_0: $\mu = \$60$ and H_1: $\mu < \$60$ (claim).

Step 2 At $\alpha = 0.10$ and d.f. = 7, the critical value is -1.415.

Step 3 To compute the test value, the mean and standard deviation must be found. Using either the formulas in Chapter 3 or your calculator, $\overline{X} = \$58.88$, and $s = 5.08$, you find

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{58.88 - 60}{5.08/\sqrt{8}} = -0.624$$

Step 4 Do not reject the null hypothesis since -0.624 falls in the noncritical region. See Figure 8–23.



Step 5 There is not enough evidence to support the educator's claim that the average salary of substitute teachers in Allegheny County is less than \$60 per day.

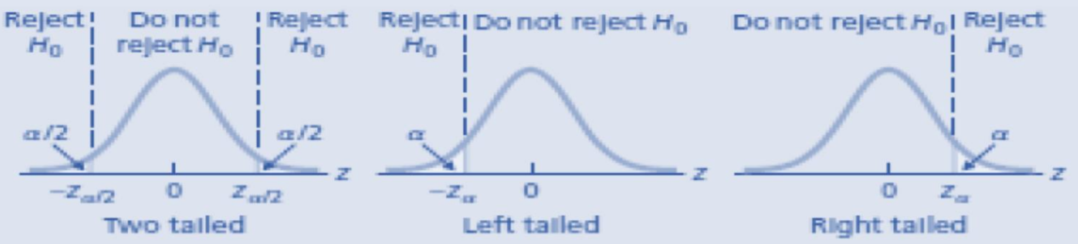
Stepwise Procedure for Performing One Mean Z-test:

Step 1: The null hypothesis is H_0: $\mu = \mu_0$, and the alternative hypothesis is,
H_a: $\mu \neq \mu_0$ (Two tailed) or H_a: $\mu < \mu_0$ (Left tailed) or H_a: $\mu > \mu_0$ (Right tailed)
Step 2: Decide on the significance level, α.
Step 3: Compute the value of the test statistic,

$$z = \frac{\bar{x} - \mu_o}{\sigma/\sqrt{n}}$$

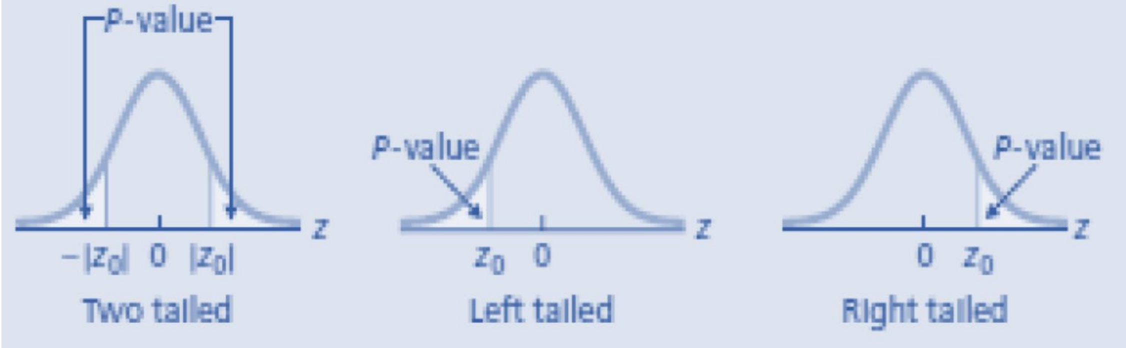
and denote that value $\mathbf{z_0}$.
Step 4: Analyze the value
Approach 1: Critical Value Approach

$\pm z_{\alpha/2}$ (Two tailed) or $-z_{\alpha}$ (Left tailed) or z_{α} (Right tailed)



Use Z-Table to find the critical value(s).

Approach 2: P-Value Approach



Step 5: If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

OR

If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6: Interpret the results of the hypothesis test.

Type I & Type II Error

In hypothesis testing, there are also two possible errors,

Type I error: Rejecting the null hypothesis when it is in fact true.

Type II error: Not rejecting the null hypothesis when it is in fact false.

		Truth	
Decision		H_o is true	H_o is false
	Do not reject H_o	Correct Decision	Type II Error
	Reject H_o	Type I Error	Correct Decision

Prices of History Books The R. R. Bowker Company collects information on the retail prices of books and publishes the data in *The Bowker Annual Library and Book Trade Almanac*. In 2005, the mean retail price of history books was \$78.01. Suppose that we want to perform a hypothesis test to decide whether this year's mean retail price of history books has increased from the 2005 mean.

a. Determine the null hypothesis for the hypothesis test.

b. Determine the alternative hypothesis for the hypothesis test.

c. Classify the hypothesis test as two tailed, left tailed, or right tailed.

Solution Let μ denote this year's mean retail price of history books.

a. The null hypothesis is that this year's mean retail price of history books *equals* the 2005 mean of \$78.01; that is, $H0{:}\mu = \$78.01$.

b. The alternative hypothesis is that this year's mean retail price of history books is *greater than* the 2005 mean of \$78.01; that is, $Ha{:}\ \mu > \$78.01$.

c. This hypothesis test is right tailed because a greater-than sign (>) appears in the alternative hypothesis.

Hypothesis Test with One Population Mean When σ is Known

Test Procedure for a Single Mean (Variance Known)

Assumptions
1. Simple random sample
2. Normal population or large sample
3. σ known

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha/2} \quad \text{or} \quad z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} < -z_{\alpha/2}$$

If $-z_{\alpha/2} < z < z_{\alpha/2}$, do not reject H_0. Rejection of H_0, of course, implies acceptance of the alternative hypothesis $\mu \neq \mu_0$. With this definition of the critical region, it should be clear that there will be probability α of rejecting H_0 (falling into the critical region) when, indeed, $\mu = \mu_0$.

10.3: A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.

Solution:

1. H_0: $\mu = 70$ years.
2. H_1: $\mu > 70$ years.
3. $\alpha = 0.05$.
4. Critical region: $z > 1.645$, where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.
5. Computations: $\bar{x} = 71.8$ years, $\sigma = 8.9$ years, and hence $z = \frac{71.8-70}{8.9/\sqrt{100}} = 2.02$.
6. Decision: Reject H_0 and conclude that the mean life span today is greater than 70 years.

$$P = P(Z > 2.02) = 0.0217.$$

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

10.4: A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilogram. Test the hypothesis that $\mu = 8$ kilograms against the alternative that $\mu \neq 8$ kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a 0.01 level of significance.

Solution:

1. H_0: $\mu = 8$ kilograms.
2. H_1: $\mu \neq 8$ kilograms.
3. $\alpha = 0.01$.
4. Critical region: $z < -2.575$ and $z > 2.575$, where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.
5. Computations: $\bar{x} = 7.8$ kilograms, $n = 50$, and hence $z = \frac{7.8-8}{0.5/\sqrt{50}} = -2.83$.
6. Decision: Reject H_0 and conclude that the average breaking strength is not equal to 8 but is, in fact, less than 8 kilograms.

$$P = P(|Z| > 2.83) = 2P(Z < -2.83) = 0.0046,$$

The t-Statistic for a Test on a Single Mean (Variance Unknown)

Hypothesis Tests for One Population Mean When σ Is Unknown

For the two-sided hypothesis

$$H_0\colon\ \mu = \mu_0,$$
$$H_1\colon\ \mu \neq \mu_0,$$

we reject H_0 at significance level α when the computed t-statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Example:

The Table shows the pH levels obtained by the researchers for 15 lakes. At the 5% significance level, do the data provide sufficient evidence to conclude that, on average, high mountain lakes in the Southern Alps are nonacidic? A lake is classified as nonacidic if it has a pH greater than 6.

pH levels for 15 lakes

7.2	7.3	6.1	6.9	6.6
7.3	6.3	5.5	6.3	6.5
5.7	6.9	6.7	7.9	5.8

Solution:

Step 1 State the null and alternative hypotheses

$H0$: $\mu = 6$ (on average, the lakes are acidic)

Ha: $\mu > 6$ (on average, the lakes are nonacidic).

Step 2 Decide on the significance level, α.

We are to perform the test at the 5% significance level, so $\alpha = 0.05$.

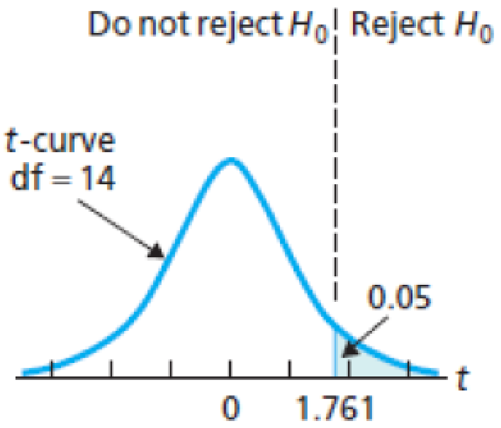
Step 3 Compute the value of the test statistic

We have $\mu_0 = 6$ and $n = 15$ and calculate the mean and standard deviation of the sample data in given Table as 6.6 and 0.672, respectively. Hence the value of the test statistic is

$$t = \frac{6.6 - 6}{0.672/\sqrt{15}} = 3.458.$$

Step 4 The critical value for a right-tailed test is t_{α} with df = $n - 1$. Use T Table to find the critical value.

We have $n = 15$ and $\alpha = 0.05$. T Table shows that for df = 15 − 1 = 14, $t_{0.05} = 1.761$.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

The value of the test statistic, found in Step 3, is $t = 3.458$. It falls in the rejection region. Consequently, we reject $H0$. The test results are statistically significant at the 5% level.

Step 6 Interpret the results of the hypothesis test.

Interpretation At the 5% significance level, the data provide sufficient evidence to conclude that, on average, high mountain lakes in the Southern Alps are nonacidic.

10.5: The Edison Electric Institute has published figures on the number of kilowatt hours used annually by various home appliances. It is claimed that a vacuum cleaner uses an average of 46 kilowatt hours per year. If a random sample of 12 homes included in a planned study indicates that vacuum cleaners use an average of 42 kilowatt hours per year with a standard deviation of 11.9 kilowatt hours, does this suggest at the 0.05 level of significance that vacuum cleaners use, on average, less than 46 kilowatt hours annually? Assume the population of kilowatt hours to be normal.

Solution:

1. H_0: $\mu = 46$ kilowatt hours.
2. H_1: $\mu < 46$ kilowatt hours.
3. $\alpha = 0.05$.
4. Critical region: $t < -1.796$, where $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ with 11 degrees of freedom.
5. Computations: $\bar{x} = 42$ kilowatt hours, $s = 11.9$ kilowatt hours, and $n = 12$. Hence,

$$t = \frac{42 - 46}{11.9/\sqrt{12}} = -1.16, \qquad P = P(T < -1.16) \approx 0.135.$$

6. Decision: Do not reject H_0

10.21 An electrical firm manufactures light bulbs that have a lifetime that is approximately normally distributed with a mean of 800 hours and a standard deviation of 40 hours. Test the hypothesis that $\mu = 800$ hours against the alternative, $\mu \neq 800$ hours, if a random sample of 30 bulbs has an average life of 788 hours. Use a P-value in your answer.

Solution:

The hypotheses are

$$H_0 : \mu = 800,$$
$$H_1 : \mu \neq 800.$$

Now, $z = \frac{788-800}{40/\sqrt{30}} = -1.64,$

and P-value$= 2P(Z < -1.64) = (2)(0.0505) = 0.1010.$

$\boxed{\text{Hence, the mean is not significantly different from 800 for } \alpha < 0.101.}$

10.24 The average height of females in the freshman class of a certain college has historically been 162.5 centimeters with a standard deviation of 6.9 centimeters. Is there reason to believe that there has been a change in the average height if a random sample of 50 females in the present freshman class has an average height of 165.2 centimeters? Use a P-value in your conclusion. Assume the standard deviation remains the same.

Solution:

$$H_0 : \mu = 162.5 \text{ centimeters},$$
$$H_1 : \mu \neq 162.5 \text{ centimeters}.$$

Now, $z = \frac{165.2-162.5}{6.9/\sqrt{50}} = 2.77$,

and P-value$= 2P(Z > 2.77) = (2)(0.0028) = 0.0056$.

$\boxed{\text{Decision: reject } H_0 \text{ and conclude that } \mu \neq 162.5.}$

10.25 It is claimed that automobiles are driven on average more than 20,000 kilometers per year. To test this claim, 100 randomly selected automobile owners are asked to keep a record of the kilometers they travel. Would you agree with this claim if the random sample showed an average of 23,500 kilometers and a standard deviation of 3900 kilometers? Use a P-value in your conclusion.

Solution:

$$H_0 : \mu = 20,000 \text{ kilometers},$$
$$H_1 : \mu > 20,000 \text{ kilometers}.$$

Now, $z = \frac{23,500-20,000}{3900/\sqrt{100}} = 8.97$.

P-value$= P(Z > 8.97) \approx 0$.

Decision: reject H_0 and conclude that $\mu \neq 20,000$ kilometers.

Two Samples: Tests on Two Means

Two-Sample Pooled t-Test

For the two-sided hypothesis

$$H_0\colon\ \mu_1 = \mu_2,$$
$$H_1\colon\ \mu_1 \neq \mu_2,$$

we reject H_0 at significance level α when the computed t-statistic

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}},$$

where

$$s_p^2 = \frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

exceeds $t_{\alpha/2, n_1+n_2-2}$ or is less than $-t_{\alpha/2, n_1+n_2-2}$.

Two Samples: Tests on Two Means

Two independent random samples of sizes n_1 and n_2, respectively, are drawn from two populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2. We know that the random variable

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

if we can assume that $\sigma_1 = \sigma_2 = \sigma$, $$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sigma\sqrt{1/n_1 + 1/n_2}}.$$

H_0: $\mu_1 - \mu_2 = d_0$. $$z = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}},$$

with a two-tailed critical region in the case of a two-sided alternative. That is, reject H_0 in favor of H_1: $\mu_1 - \mu_2 \neq d_0$ if $z > z_{\alpha/2}$ or $z < -z_{\alpha/2}$. One-tailed critical regions are used in the case of the one-sided alternatives. The reader should, as before, study the test statistic and be satisfied that for, say, H_1: $\mu_1 - \mu_2 > d_0$, the signal favoring H_1 comes from large values of z. Thus, the upper-tailed critical region applies.

Example 10.6: An experiment was performed to compare the abrasive wear of two different laminated materials. Twelve pieces of material 1 were tested by exposing each piece to a machine measuring wear. Ten pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The samples of material 1 gave an average (coded) wear of 85 units with a sample standard deviation of 4, while the samples of material 2 gave an average of 81 with a sample standard deviation of 5. Can we conclude at the 0.05 level of significance that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume the populations to be approximately normal with equal variances.

Solution: Let μ_1 and μ_2 represent the population means of the abrasive wear for material 1 and material 2, respectively.

1. H_0: $\mu_1 - \mu_2 = 2$.
2. H_1: $\mu_1 - \mu_2 > 2$.
3. $\alpha = 0.05$.
4. Critical region: $t > 1.725$,

   where $t = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}}$ with $v = 20$

5. Computations:

$$\bar{x}_1 = 85, \qquad s_1 = 4, \qquad n_1 = 12,$$
$$\bar{x}_2 = 81, \qquad s_2 = 5, \qquad n_2 = 10.$$

$$s_p = \sqrt{\frac{(11)(16) + (9)(25)}{12 + 10 - 2}} = 4.478,$$

$$t = \frac{(85 - 81) - 2}{4.478\sqrt{1/12 + 1/10}} = 1.04,$$

$$P = P(T > 1.04) \approx 0.16.$$

6. $\boxed{\text{Decision: Do not reject } H_0.}$

Class Activity

10.33 A study was conducted to see if increasing the substrate concentration has an appreciable effect on the velocity of a chemical reaction. With a substrate concentration of 1.5 moles per liter, the reaction was run 15 times, with an average velocity of 7.5 micromoles per 30 minutes and a standard deviation of 1.5. With a substrate concentration of 2.0 moles per liter, 12 runs were made, yielding an average velocity of 8.8 micromoles per 30 minutes and a sample standard deviation of 1.2. Is there any reason to believe that this increase in substrate concentration causes an increase in the mean velocity of the reaction of more than 0.5 micromole per 30 minutes? Use a 0.01 level of significance and assume the populations to be approximately normally distributed with equal variances.

Solution:

$H_0: \mu_1 - \mu_2 = 0.5$ micromoles per 30 minutes,

$H_1: \mu_1 - \mu_2 > 0.5$ micromoles per 30 minutes.

$\alpha = 0.01$.

Critical region: $t > 2.485$ with 25 degrees of freedom.

Computation: $s_p^2 = \frac{(14)(1.5)^2+(11)(1.2)^2}{25} = 1.8936$,

and $t = \frac{(8.8-7.5)-0.5}{\sqrt{1.8936}\sqrt{1/15+1/12}} = 1.50$.

$\boxed{\text{Decision: Do not reject } H_0.}$

10.38 A UCLA researcher claims that the average life span of mice can be extended by as much as 8 months when the calories in their diet are reduced by approximately 40% from the time they are weaned. The restricted diets are enriched to normal levels by vitamins and protein. Suppose that a random sample of 10 mice is fed a normal diet and has an average life span of 32.1 months with a standard deviation of 3.2 months, while a random sample of 15 mice is fed the restricted diet and has an average life span of 37.6 months with a standard deviation of 2.8 months. Test the hypothesis, at the 0.05 level of significance, that the average life span of mice on this restricted diet is increased by 8 months against the alternative that the increase is less than 8 months. Assume the distributions of life spans for the regular and restricted diets are approximately normal with equal variances.

Solution:

The hypotheses are

$$H_0: \mu_1 - \mu_2 = 8,$$
$$H_1: \mu_1 - \mu_2 < 8.$$

$\alpha = 0.05$ and the critical region is $t < -1.714$ with 23 degrees

Computation: $s_p = \sqrt{\frac{(9)(3.2)^2+(14)(2.8)^2}{23}} = 2.963,$

and $t = \frac{5.5-8}{2.963\sqrt{1/10+1/15}} = -2.07.$

$\boxed{\text{Decision: Reject } H_0 \text{ and conclude that } \mu_1 - \mu_2 < 8 \text{ months.}}$

Hypothesis Tests for the Means of Two Populations

Pooled T-Test (Assume Equal Standard Deviations: σ1 = σ2)

Independent Samples Test

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples
4. Equal population standard deviations

Step 1 The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is

Ha: $\mu_1 \neq \mu_2$ or	Ha: $\mu_1 < \mu_2$ or	Ha: $\mu_1 > \mu_2$
(Two tailed)	(Left tailed)	(Right tailed)

Step 2 Decide on the significance level, α.

Step 3 Compute the value of the test statistic

Step 3 Compute the value of the test statistic

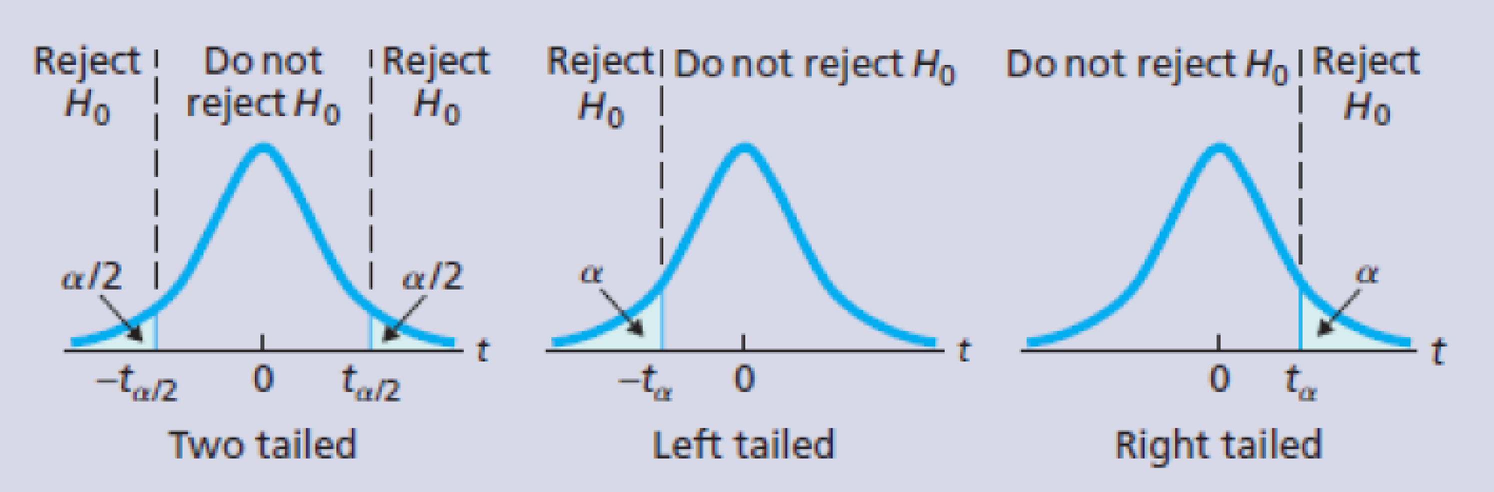
$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1) + (1/n_2)}}$$

Where $$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}.$$

Step 4 The critical value(s) are

$\pm t_{\alpha/2}$	(Two tailed)	or
$-t_{\alpha}$	(Left tailed)	or
$+t_{\alpha}$	(Right tailed)	

with df = $n_1 + n_2 - 2$. Use T Table to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Example:

Annual salaries ($1000s) for 35 faculty members in private institutions and 30 faculty members in public institutions. At the 5% significance level, do the data provide sufficient evidence to conclude that mean salaries for faculty in private and public institutions differ?

Summary statistics for the samples	
Private institutions	**Public institutions**
$\overline{x_1} = 88.19$	$\overline{x_2} = 73.18$
$s_1 = 26.21$	$s_2 = 23.95$
$n_1 = 35$	$n_2 = 30$

Step 1 State the null and alternative hypotheses.
The null and alternative hypotheses are, respectively,
H_0: $\mu_1 = \mu_2$ (mean salaries are the same)
H_a: $\mu_1 \ = \mu_2$ (mean salaries are different),
where μ_1 and μ_2 are the mean salaries of all faculty in private and public institutions, respectively.
Note that the hypothesis test is two tailed.

Step 2 Decide on the significance level, α.
The test is to be performed at the 5% significance level, or $\alpha = 0.05$.

Step 3 Compute the value of the test statistic

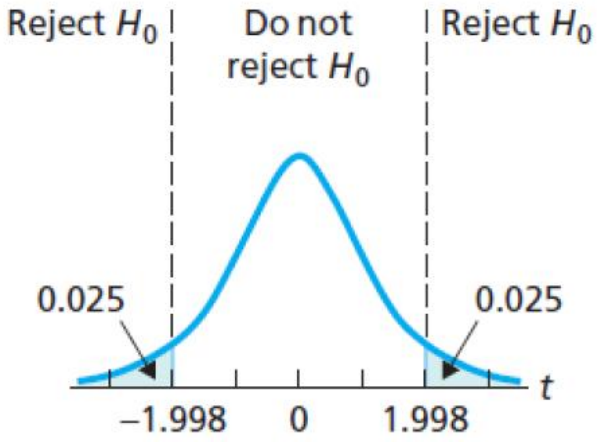
$$s_p = \sqrt{\frac{(35-1)\cdot(26.21)^2 + (30-1)\cdot(23.95)^2}{35+30-2}} = 25.19.$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1)+(1/n_2)}} = \frac{88.19 - 73.18}{25.19\sqrt{(1/35)+(1/30)}} = 2.395.$$

Step 4 The critical values for a two-tailed test are $\pm t_{\alpha/2}$ with df = $n_1 + n_2 - 2$. Use T Table to find the critical values.

Since $n_1 = 35$ and $n_2 = 30$, so df = 35 + 30 − 2 = 63.

Also, from Step 2, we have $\alpha = 0.05$. Using T Table with df = 63, we find that the critical values are $\pm t_{\alpha/2} = \pm t_{0.05/2} = \pm t_{0.025} = \pm 1.998$, as shown in the Figure.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

From Step 3, the value of the test statistic is $t = 2.395$, which falls in the rejection region. Thus we reject H_0. The test results are statistically significant at the 5% level.

Step 6 Interpretation: At the 5% significance level, the data provide sufficient evidence to conclude that a difference exists between the mean salaries of faculty in private and public institutions.

Paired T Test: (Dependent Sample Test)

Inferences for Two Population Means, Using Paired Samples

A paired sample may be appropriate when the members of the two populations have a natural pairing.
Each pair in a **paired sample** consists of a member of one population and that member's corresponding member in the other population.

Paired t-Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions

1. Simple random paired sample
2. Normal differences or large sample

Step 1 **The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is**

$$H_a\colon \mu_1 \neq \mu_2 \quad \text{or} \quad H_a\colon \mu_1 < \mu_2 \quad \text{or} \quad H_a\colon \mu_1 > \mu_2$$

(Two tailed) **or** **(Left tailed)** **or** **(Right tailed)**

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

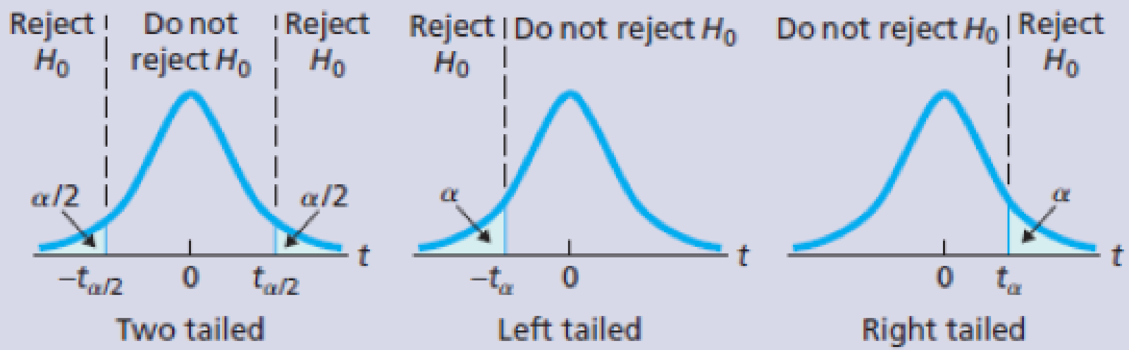
$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

and denote that value t_0.

Step 4 The critical value(s) are

$\pm t_{\alpha/2}$ (Two tailed)	or	$-t_{\alpha}$ (Left tailed)	or	t_{α} (Right tailed)

with df $= n - 1$. Use Table IV to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Table A.4 Critical Values of the t-Distribution

v	α = 0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

Table A.4 (continued) Critical Values of the t-Distribution

v	α = 0.02	0.015	0.01	0.0075	0.005	0.0025	0.0005
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.689
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.660
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.290

Class Activity

10.42 Five samples of a ferrous-type substance were used to determine if there is a difference between a laboratory chemical analysis and an X-ray fluorescence analysis of the iron content. Each sample was split into two subsamples and the two types of analysis were applied. Following are the coded data showing the iron content analysis:

	Sample				
Analysis	**1**	**2**	**3**	**4**	**5**
X-ray	2.0	2.0	2.3	2.1	2.4
Chemical	2.2	1.9	2.5	2.3	2.4

Assuming that the populations are normal, test at the 0.05 level of significance whether the two methods of analysis give, on the average, the same result.

Solution:

The hypotheses are

$$H_0 : \mu_1 = \mu_2,$$
$$H_1 : \mu_1 \neq \mu_2.$$

$\alpha = 0.05$.

Critical regions $t < -2.776$ or $t > 2.776$, with 4 degrees of freedom.

Computation: $\bar{d} = -0.1$, $s_d = 0.1414$,

$$t = \frac{-0.1}{0.1414/\sqrt{5}} = -1.58.$$

Decision: Do not reject H_0

and conclude that the two methods are not significantly different.

Paired T Test: (Dependent Sample Test)

Inferences for Two Population Means, Using Paired Samples

Example

Ages of Married People The U.S. Census Bureau publishes information on the ages of married people in *Current Population Reports*. Suppose that we want to decide whether, in the United States, the mean age of married men differs from the mean age of married women.

Couple	Husband	Wife	Difference, d
1	59	53	6
2	21	22	−1
3	33	36	−3
4	78	74	4
5	70	64	6
6	33	35	−2
7	68	67	1
8	32	28	4
9	54	41	13
10	52	44	8
			36

We want to perform the hypothesis test
H_0: $\mu_1 = \mu_2$ (mean ages of married men and women are the same)
H_a: $\mu_1 \ = \mu_2$ (mean ages of married men and women differ).

Step 1 State the null and alternative hypotheses.
Let μ_1 denote the mean age of all married men, and let μ_2 denote the mean age of all married women. Then the null and alternative hypotheses are, respectively,
$H0$: $\mu_1 = \mu_2$ (mean ages are equal)
Ha: $\mu_1 \ = \mu_2$ (mean ages differ).
Note that the hypothesis test is two tailed.

Step 2 Decide on the significance level, α.
We are to perform the test at the 5% significance level, so $\alpha = 0.05$.

Step 3 Compute the value of the test statistic
The paired differences (d-values) of the sample pairs are shown in the last column of above Table.
We need to determine the sample mean and sample standard deviation of those paired differences.

$$\overline{d} = \frac{\Sigma d_i}{n} = \frac{36}{10} = 3.6,$$

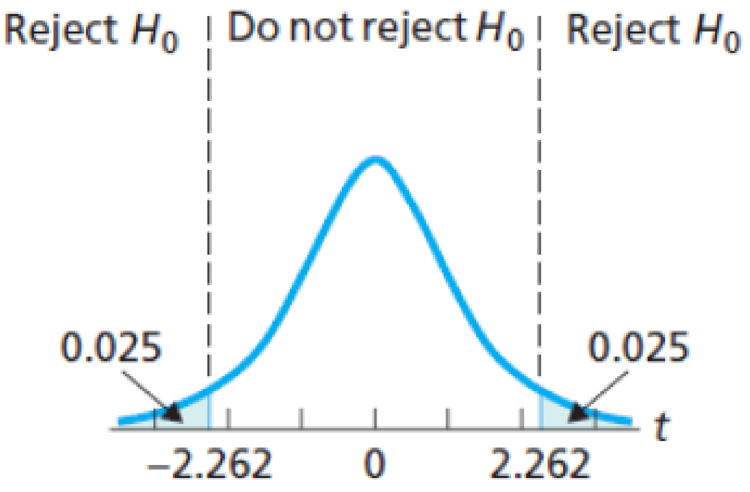
and

$$s_d = \sqrt{\frac{\Sigma d_i^2 - (\Sigma d_i)^2/n}{n-1}} = \sqrt{\frac{352 - (36)^2/10}{10-1}} = 4.97.$$

Consequently, the value of the test statistic is

$$t = \frac{\overline{d}}{s_d/\sqrt{n}} = \frac{3.6}{4.97/\sqrt{10}} = 2.291.$$

Step 4 The critical values for a two-tailed test are $\pm t\alpha/2$ with df = $n - 1$. Use T Table to find the critical values.
We have $n = 10$ and $\alpha = 0.05$. for df = 10 − 1 = 9, $\pm t_{0.05/2} = \pm t_{0.025} = \pm 2.262$, as shown in Fig.



Step 5 If the value of the test statistic falls in the rejection region, reject $H0$; otherwise, do not reject $H0$.
From Step 3, the value of the test statistic is $t = 2.291$, which falls in the rejection region depicted in Fig. Thus we reject $H0$. The test results are statistically significant at the 5% level.

Step 6 Interpret the results of the hypothesis test.
Interpretation At the 5% significance level, the data provide sufficient evidence to conclude that the mean age of married men differs from the mean age of married women.

Choice of Sample Size for Testing Means

Table 10.3: Tests Concerning Means

H_0	Value of Test Statistic	H_1	Critical Region
$\mu = \mu_0$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$; σ known	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu = \mu_0$	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$; $v = n - 1$, σ unknown	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$z = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$; σ_1 and σ_2 known	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}}$; $v = n_1 + n_2 - 2$, $\sigma_1 = \sigma_2$ but unknown, $s_p^2 = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_D = d_0$ paired observations	$t = \dfrac{\bar{d} - d_0}{s_d/\sqrt{n}}$; $v = n - 1$	$\mu_D < d_0$ $\mu_D > d_0$ $\mu_D \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

Example1:

Class Activity

If an unauthorized person accesses a computer account with the correct username and password (stolen or cracked), can this intrusion be detected? Recently, a number of methods have been proposed to detect such unauthorized use. The time between keystrokes, the time a key is depressed, the frequency of various keywords is measured and compared with those of the account owner. If there are significant differences, an intruder is detected.
A longtime authorized user of the account makes an average 0.2 seconds between keystrokes.
One day, as someone typed the correct username and password.
The following times between keystrokes were recorded when a user typed the username and password:
.24, .22, .26, .34, .35, .32, .33, .29, .19, .36, .30, .15, .17, .28, .38, .40, .37, .27 seconds

I. Construct a 99% confidence interval for the mean time between keystrokes assuming normal distribution of these times.
II. Is this evidence of an unauthorized attempt at a 5% level of significance

Example2:

CD writing is energy consuming; therefore, it affects the battery lifetime on laptops. To estimate the effect of CD writing, 30 users are asked to work on their laptops until the "low battery" sign comes on. Eighteen users without a CD writer worked an average of 5.3 hours with a standard deviation of 1.4 hours. The other twelve, who used their CD writer, worked an average of 4.8 hours with a standard deviation of 1.6 hours. Assuming Normal distributions with equal population variances.

I. Construct a 95% confidence interval for the battery life reduction caused by CD writing.
II. Does a CD writer consume extra energy, and therefore, does it reduce the battery life on a laptop at $\alpha = 0.05$ [Critical value = 1.645]