Eigenspaces

1. (a) *Find all eigenvalues and eigenvectors of* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   (b) *Find the corresponding eigenspaces.*

   > ***Definition.*** *If A is an $n \times n$ matrix and λ is a scalar, the λ-eigenspace of A (usually denoted E_λ) is the set of all vectors $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$. So, the nonzero vectors in E_λ are exactly the eigenvectors of A with eigenvalue λ.*

   (c) *Find the algebraic multiplicity and the geometric multiplicity for the eigenvaules of A.*

   > ***Definition.*** *The dimension of the eigenspace is called the* **geometric multiplicity** *of λ. The* ***algebraic multiplicity*** *of an eigenvalue is the multiplicity of the root.*

   > *The* ***algebraic multiplicity*** *of an eigenvalue is the multiplicity of the root. For example, the characteristic polynomial of* $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ *is* $(1-\lambda)^2(2-\lambda)$. *The algebraic multiplicity of the eigenvalue $\lambda = 1$ is 2 and the algebraic multiplicity of the eigenvalue $\lambda = 2$ is 1.*

   (d) *Repeat (a)(b) and (c) for the matrix* $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

> ***Definition.*** *A basis of $\mathbb{R}^n$ consisting of eigenvectors of $A_{n\times n}$ is called an* ***eigenbasis***

   (e) *Is there an eigenbasis for A? How about for B?*

   Solution. *(a-f) In class.*

   (f) *Can you generalize the result of (e)?*

   Solution. *If an $n \times n$ matrix A has n distinct eigenvalues, then each has algebraic multiplicity 1 and therefore geometric multiplicity 1. In this case, the geometric multiplicities of the eigenvalues add up to n, so there will be an eigenbasis for A. So, to summarize: if an $n \times n$ matrix A has n distinct eigenvalues, then there is an eigenbasis for A.*

Note:

(a) $1 \leq$ *the geometric multiplicity of* $\lambda \leq$ *the algebraic multiplicity of* λ.

(b) *Sum of the algebraic multiplicities for all the eigenvaules equals n*

2. Let $A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 3 & -1 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

(a) *Find all real eigenvalues of A, and give their algebraic multiplicities.*

Solution. As usual, we'll start by finding the characteristic polynomial of A:

$$\begin{aligned} f_A(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} 1-\lambda & 1 & 3 & 4 \\ 0 & 3-\lambda & -1 & 4 \\ 0 & 0 & 2-\lambda & 2 \\ 0 & 0 & 1 & 3-\lambda \end{bmatrix} \\ &= (1-\lambda)(3-\lambda)[(2-\lambda)(3-\lambda) - 2] \\ &= (1-\lambda)(3-\lambda)(\lambda^2 - 5\lambda + 4) \\ &= (1-\lambda)(3-\lambda)(\lambda - 1)(\lambda - 4) \end{aligned}$$

So, the eigenvalues are

$\boxed{\text{1, with algebraic multiplicity 2; 3, with algebraic multiplicity 1; and 4, with algebraic multiplicity 1.}}$

(b) *Find the geometric multiplicity of each eigenvalue. (If you are able to determine this without computing the corresponding eigenspace, please do, and explain your reasoning.)*

Solution. We know that any eigenvalue has geometric multiplicity ≥ 1 (if the geometric multiplicity was 0, it wouldn't be an eigenvalue at all) and $\leq$ the algebraic multiplicity. So, if the algebraic multiplicity of a particular eigenvalue is 1, then that eigenvalue also has geometric multiplicity 1. Therefore, we know that $\boxed{\text{the eigenvalues 3 and 4 both have geometric multiplicity 1}}$.

However, we can't determine the geometric multiplicity of 1 in such a straightforward manner; all we know is that the geometric multiplicity is either 1 or 2. To determine it, let's look at the 1-eigenspace:

$$E_1 = \ker(A - I) = \ker \begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

We can find this by row-reducing $A - I$:

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ -2(\mathrm{I}) \\ \\ \\ \end{matrix} \to \begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ \\ \text{swap with (II)} \\ \\ \end{matrix}$$

$$\to \begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} -3(\mathrm{II}) \\ \\ +7(\mathrm{II}) \\ -(\mathrm{II}) \end{matrix}$$

$$\to \begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Although we are not done row-reducing, we can already see that $\operatorname{rank}(A - I) = 3$, so $\dim(\ker(A - I)) = 1$ by the rank-nullity theorem. Therefore, $\boxed{\text{the eigenvalue 1 has geometric multiplicity 1}}$.

(c) *Is there an eigenbasis for A?*

Solution. The geometric multiplicities of the eigenvalues add up to 3, not 4, so $\boxed{\text{there is no eigenbasis}}$.

3. *Find the characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What can you say about the eigenvaules of A?*

Solution. By definition, the characteristic polynomial is

$$\begin{aligned} f_A(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \lambda^2 - (\operatorname{tr} A)\lambda + \det A \end{aligned}$$

In fact, for any n by n matrix A one can show that

- trace A = sum of the eigenvaules of A = $\lambda_1 + \lambda_2 + \cdots + \lambda_n$
- $\det A$ = product of eigenvaules of A = $\lambda_1 \lambda_2 \cdots \lambda_n$

4. *True or false: if $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis for A, then $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis for A^2.*

   Solution. True. Since $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis for A, $(\vec{v}_1, \ldots, \vec{v}_n)$ is a basis of $\mathbb{R}^n$ and each $\vec{v}_i$ is an eigenvector of A. In particular, for each i, there is some scalar λ_i such that $A\vec{v}_i = \lambda_i \vec{v}_i$. Then, $A^2\vec{v}_i = A(A\vec{v}_i) = A(\lambda_i \vec{v}_i) = \lambda_i(A\vec{v}_i) = \lambda_i^2 \vec{v}_i$, so $\vec{v}_i$ is an eigenvector of A^2. Thus, we have shown that $(\vec{v}_1, \ldots, \vec{v}_n)$ is a basis for $\mathbb{R}^n$ and that each $\vec{v}_i$ is an eigenvector of A^2. Therefore, $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis of A^2.

5. *True or false: for any square matrix, A and A^T have the same eigenvaules.*

6. *What is the relation between the eigenvaules of A and A^{-1}? (This is your homework problem!)*

7. *Let A be a noninvertible $n \times n$ matrix. Explain why 0 must be an eigenvalue of A. Find the geometric multiplicity of the eigenvalue 0 in terms of* $\operatorname{rank}(A)$.

   Solution. If A is not invertible, then $\det(A - 0I_n) = \det A = 0$, so 0 is an eigenvalue of A. The geometric multiplicity of 0 is $\dim[\ker(A - 0I_n)] = \dim(\ker A) = n - \operatorname{rank}(A)$.

8. *(Practice on modeling discrete dynamical system)*

   Suppose we'd like to model the population of fish in a lake. We'd like to incorporate the age structure of the population into our model, so let's divide the fish into two groups: young fish and mature adult fish. Let $y(t)$ be the number of young fish and $m(t)$ be the number of mature adult fish in year t.

   Consider the following simplified model. Each year:

   - *20% of the young fish grow into mature adult fish.*
   - *10% of the mature adult fish die due to old age.*
   - *30% of the mature adult fish have one offspring each.*
   - *There is no other birth or death.*

   Express $y(t+1)$ and $m(t+1)$ in terms of $y(t)$ and $m(t)$.

   Solution. Let's first think about $y(t+1)$. Each year, 20% of the young fish mature into adult fish, so 80% of the young fish stay young. In addition, 30% of the adult fish have one offspring each, and these offspring are young. So, $y(t+1) = 0.8y(t) + 0.3m(t)$.

   Next, we'll think about $m(t+1)$. Each year, 20% of the young fish become mature fish. 10% of the mature fish die, leaving 90% of them. So, $m(t+1) = 0.2y(t) + 0.9m(t)$.

   So, we have a system

$$\boxed{\begin{cases} y(t+1) = 0.8y(t) + 0.3m(t) \\ m(t+1) = 0.2y(t) + 0.9m(t) \end{cases}}$$

9. *Getting efficient with 2×2 matrices.*

   Let $A = \begin{bmatrix} 11 & -6 \\ 15 & -8 \end{bmatrix}$.

   (a) *Find the eigenvalues of A with their algebraic multiplicities.*

   Solution. The trace of A is $11 - 8 = 3$, and the determinant is $11(-8) - (-6)15 = 2$, so the characteristic polynomial of A is $f_A(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$. Therefore, the eigenvalues of A are $\boxed{\text{1 and 2, each with algebraic multiplicity 1}}$.

(b) *What does (a) tell you about the geometric multiplicities of the eigenvalues of A?*

Solution. As explained in #2(b), an eigenvalue with algebraic multiplicity 1 must also have geometric multiplicity 1. So, in this case, the eigenvalues 1 and 2 $\boxed{\text{both have geometric multiplicity 1}}$.

(c) *Find a basis of each eigenspace. (Do this by inspection.)*

Solution. Let's start with the 1-eigenspace, which is $\ker(A - 1I) = \ker \begin{bmatrix} 10 & -6 \\ 15 & -9 \end{bmatrix}$. We already know that this subspace is 1-dimensional (that's exactly what it means to say that the corresponding eigenvalue has geometric multiplicity 1), so we're simply looking for a single nonzero vector in this eigenspace. By inspection, $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is one such vector.

Similarly, the 2-eigenspace is $\ker(A - 2I) = \ker \begin{bmatrix} 9 & -6 \\ 15 & -10 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

So, $\boxed{\text{a basis of } E_1 \text{ is } \begin{bmatrix} 3 \\ 5 \end{bmatrix} \text{ and a basis of } E_2 \text{ is } \begin{bmatrix} 2 \\ 3 \end{bmatrix}.}$

(d) *True or false: If B is a 2×2 matrix with eigenvalues 3 and 5, then the matrix $B - 3I_2$ must have rank 1. Explain your reasoning.*

Solution. Since B has two distinct eigenvalues, each must have algebraic multiplicity 1. An eigenvalue with algebraic multiplicity 1 also has geometric multiplicity 1, so both eigenvalues have geometric multiplicity 1. Saying that the eigenvalue 3 has geometric multiplicity 1 is exactly saying that $\dim(\ker(B - 3I_2)) = 1$, so $\operatorname{rank}(B - 3I_2) = 1$ by the rank-nullity theorem.

10. Find all (real) eigenvalues of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ -25 & -3 & 10 \\ 0 & 0 & 2 \end{bmatrix}$. Then find a basis of each eigenspace, and find an eigenbasis if there is one. Please also state the algebraic and geometric multiplicity of each eigenvalue. Do all of your calculations by hand, but try to be efficient! (you should be able to find most of the eigenvectors using inspection.)

Solution. The characteristic polynomial of this matrix is

$$\det \begin{bmatrix} 2-\lambda & 0 & 0 \\ -25 & -3-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(-3-\lambda)(2-\lambda).$$

So, the eigenvalues are 2 (with algebraic multiplicity 2) and -3 (with algebraic multiplicity 1). The 2-eigenspace is

$$E_2 = \ker \begin{bmatrix} 0 & 0 & 0 \\ -25 & -5 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

Although you can find the kernel by row-reducing, it's a little faster to use inspection. Right away, we can see that this matrix has rank 1, so its kernel should be 2-dimensional. A vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in this kernel exactly if $-25x - 5y + 10z = 0$, so we are really looking for a basis of the plane $-25x - 5y + 10z = 0$. Any two non-parallel vectors in this plane form a basis, so one basis is $\left(\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right)$.

The -3-eigenspace is

$$E_{-3} = \ker \begin{bmatrix} 5 & 0 & 0 \\ -25 & 0 & 10 \\ 0 & 0 & 5 \end{bmatrix}$$

Again, let's use inspection to find a basis. It's clear that the matrix we're looking at has rank 2, so we're simply looking for one non-zero vector in the kernel, and you can probably see that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is such a vector.

So:

The matrix has eigenvalues 2 (with algebraic and geometric multiplicity 2) and -3 (with algebraic and geometric multiplicity 1). A basis of the 2-eigenspace is $\left(\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right)$, a basis of the -3-eigenspace is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and an eigenbasis is $\left(\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$.

11. *In exactly one of the two parts in # 10, you should have found an eigenbasis for the given matrix. Let A be the matrix and $\mathfrak{B}$ be the basis of $\mathbb{R}^3$ you found in that part. Find the $\mathfrak{B}$-matrix of the linear transformation $T(\vec{x}) = A\vec{x}$. (How does your answer relate to the eigenvalues of A?)*

Solution. In **??**, you were given the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ -25 & -3 & 10 \\ 0 & 0 & 2 \end{bmatrix}$. Let's use the eigenbasis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$ (eigenvalue 2), $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ (eigenvalue 2), and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (eigenvalue -3). Then, $T(\vec{v}_1) = 2\vec{v}_1$, $T(\vec{v}_2) = 2\vec{v}_2$, and $T(\vec{v}_3) = -3\vec{v}_3$, so we can easily construct the $\mathfrak{B}$-matrix column-by-column; it is $\boxed{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}}$. (Note that this is a diagonal matrix with the eigenvalues on the diagonal.)

- You should understand the terminology *characteristic polynomial*, *eigenspace*, and *eigenbasis* (in particular, an eigenbasis for a matrix A is a basis of what?).
- You should be able to find the eigenvalues of a given matrix A, as well as the *algebraic multiplicity* and *geometric multiplicity* of each eigenvalue. You should be able to use this information to determine whether there is an eigenbasis for the matrix A.
- You should understand the relationship between the algebraic and geometric multiplicity of an eigenvalue.
- (Discrete Dynamical Systems) If A is an $n \times n$ matrix and you are given a basis of $\mathbb{R}^n$ consisting of eigenvectors of A, you should be able to solve the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with any given initial condition $\vec{x}(0)$.