MID2 Syllabus

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Chap-3
Random Variables and Probability Distributions

A random variable is a function that associates a real number with each element in the sample space.

- A random variable is called discrete random variable if its set of possible outcome is countable.

Definition 3.4: The set of ordered pairs $(x, f(x))$ is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_x f(x) = 1$,
3. $P(X = x) = f(x)$.

Example 3.1: Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values y of the random variable Y, where Y is the number of red balls, are

Sample Space	y
RR	2
RB	1
BR	1
BB	0

Example 3.8: A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Thus, the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

Activity

3.26 From a box containing 4 black balls and 2 green balls, 3 balls are drawn in succession, each ball being replaced in the box before the next draw is made. Find the probability **distribution** for the number of green balls.

Solution:

Denote by X the number of green balls in the three draws. Let G and B stand for the colors of green and black, respectively.

Simple Event	x	$P(X = x)$
BBB	0	$(2/3)^3 = 8/27$
GBB	1	$(1/3)(2/3)^2 = 4/27$
BGB	1	$(1/3)(2/3)^2 = 4/27$
BBG	1	$(1/3)(2/3)^2 = 4/27$
BGG	2	$(1/3)^2(2/3) = 2/27$
GBG	2	$(1/3)^2(2/3) = 2/27$
GGB	2	$(1/3)^2(2/3) = 2/27$
GGG	3	$(1/3)^3 = 1/27$

The probability mass function for X is then

x	0	1	2	3
$P(X = x)$	$8/27$	$4/9$	$2/9$	$1/27$

Definition 3.5: The **cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for } -\infty < x < \infty.$$

Example 3.10: Find the cumulative distribution function of the random variable X in Example 3.9. Using $F(x)$, verify that $f(2) = 3/8$.

the probability distribution $f(x) = P(X = x)$ is

$$f(x) = \frac{1}{16}\binom{4}{x}, \quad \text{for } x = 0, 1, 2, 3, 4.$$

$$F(0) = f(0) = \frac{1}{16},$$
$$F(1) = f(0) + f(1) = \frac{5}{16},$$
$$F(2) = f(0) + f(1) + f(2) = \frac{11}{16},$$
$$F(3) = f(0) + f(1) + f(2) + f(3) = \frac{15}{16},$$
$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1.$$

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{1}{16}, & \text{for } 0 \leq x < 1, \\ \frac{5}{16}, & \text{for } 1 \leq x < 2, \\ \frac{11}{16}, & \text{for } 2 \leq x < 3, \\ \frac{15}{16}, & \text{for } 3 \leq x < 4, \\ 1 & \text{for } x \geq 4. \end{cases}$$

$$f(2) = F(2) - F(1) = \frac{11}{16} - \frac{5}{16} = \frac{3}{8}.$$

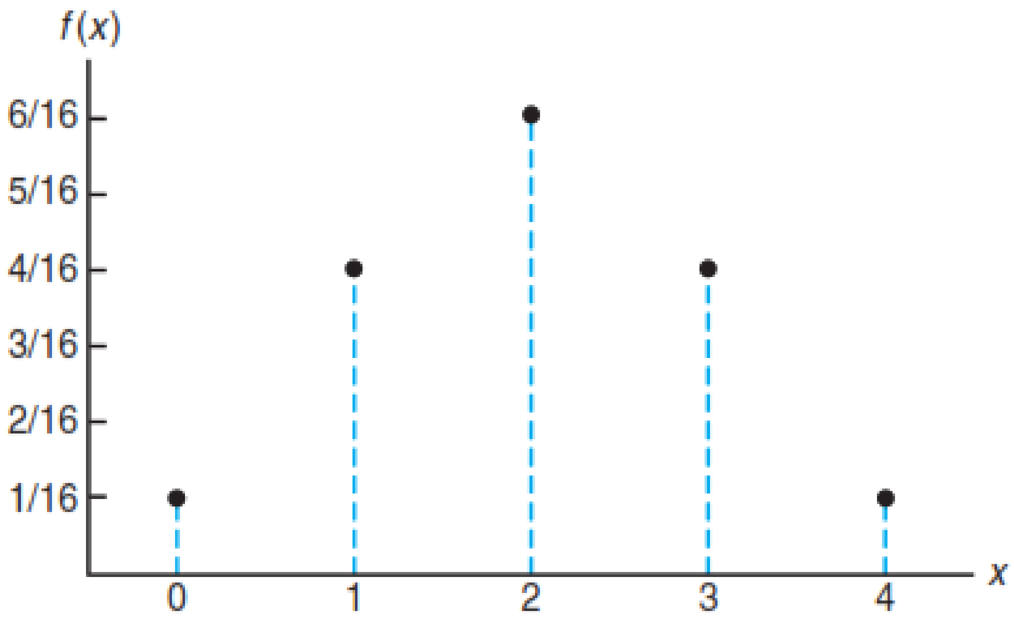

Figure 3.1: Probability mass function plot.

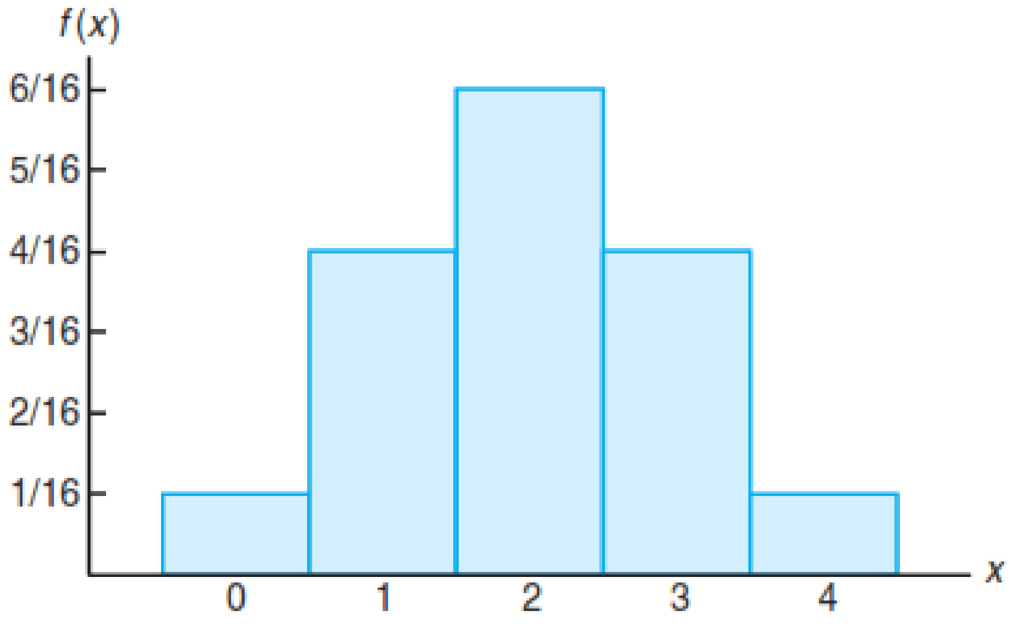

Figure 3.2: Probability histogram.

Continuous Probability Distributions

$$P(a < X \leq b) = P(a < X < b) + P(X = b) = P(a < X < b).$$

$$P(a < X < b) = \int_a^b f(x)\, dx.$$

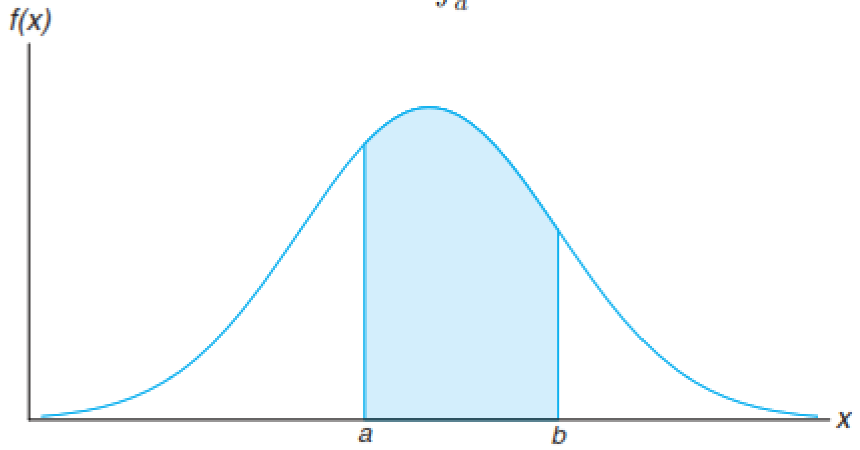

Figure 3.5: $P(a < X < b)$.

Definition 3.6: The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
3. $P(a < X < b) = \int_a^b f(x)\, dx$.

Example 3.11: Suppose that the error in the reaction temperature, in °C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

.

(a) Verify that $f(x)$ is a density function.

(b) Find $P(0 < X \leq 1)$.

Solution:

(a) $$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-1}^{2} \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_{-1}^{2} = \frac{8}{9} + \frac{1}{9} = 1.$$

(b) $$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_0^1 = \frac{1}{9}.$$

Example 3.12: For the density function of Example 3.11, find $F(x)$, and use it to evaluate $P(0 < X \leq 1)$.

Definition 3.7: The **cumulative distribution function** $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \quad \text{for } -\infty < x < \infty.$$

$$P(a < X < b) = F(b) - F(a) \text{ and } f(x) = \frac{dF(x)}{dx},$$

Example 3.12: For the density function of Example 3.11, find $F(x)$, and use it to evaluate $P(0 < X \leq 1)$.

***Solution*:** For $-1 < x < 2$,

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-1}^{x} \frac{t^2}{3} dt = \left.\frac{t^3}{9}\right|_{-1}^{x} = \frac{x^3+1}{9}.$$

Therefore,

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{x^3+1}{9}, & -1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

The cumulative distribution function $F(x)$ is expressed in Figure 3.6. Now

$$P(0 < X \leq 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9},$$

Practice:

3.14 The waiting time, in hours, between successive speeders spotted by a radar unit is a continuous random variable with cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-8x}, & x \geq 0. \end{cases}$$

Find the probability of waiting less than 12 minutes between successive speeders

(a) using the cumulative distribution function of X;

(b) using the probability density function of X.

Solution:

(a) $P(X < 0.2) = F(0.2) = 1 - e^{-1.6} = 0.7981$;

(b) $f(x) = F'(x) = 8e^{-8x}$. Therefore, $P(X < 0.2) = 8\int_0^{0.2} e^{-8x}\, dx = -e^{-8x}\big|_0^{0.2} = 0.7981$.

3.17 A continuous random variable X that can assume values between $x = 1$ and $x = 3$ has a density function given by $f(x) = 1/2$.

(a) Show that the area under the curve is equal to 1.

(b) Find $P(2 < X < 2.5)$.

(c) Find $P(X \leq 1.6)$.

Solution:

(a) Area $= \int_1^3 (1/2)\, dx = \frac{x}{2}\big|_1^3 = 1$.

(b) $P(2 < X < 2.5)\ \int_2^{2.5} (1/2)\, dx = \frac{x}{2}\big|_2^{2.5} = \frac{1}{4}$.

(c) $P(X \leq 1.6) = \int_1^{1.6} (1/2)\, dx = \frac{x}{2}\big|_1^{1.6} = 0.3$.

3.21 Consider the density function

$$f(x) = \begin{cases} k\sqrt{x}, & 0 < x < 1, \\ 0, & \text{elsewhere}. \end{cases}$$

(a) Evaluate k.

(b) Find $F(x)$ and use it to evaluate

$$P(0.3 < X < 0.6).$$

Solution:

(a) $1 = k\int_0^1 \sqrt{x}\, dx = \frac{2k}{3} x^{3/2}\big|_0^1 = \frac{2k}{3}$. Therefore, $k = \frac{3}{2}$.

(b) $F(x) = \frac{3}{2}\int_0^x \sqrt{t}\, dt = t^{3/2}\big|_0^x = x^{3/2}$.
$P(0.3 < X < 0.6) = F(0.6) - F(0.3) = (0.6)^{3/2} - (0.3)^{3/2} = 0.3004$.

Practice:

3.11 A shipment of 7 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If x is the number of defective sets purchased by the hotel, find the probability distribution of X. Express the results graphically as a probability histogram.

$$f(x) = \frac{\binom{2}{x}\binom{5}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2.$$

x	0	1	2
$f(x)$	2/7	4/7	1/7

3.15 Find the cumulative distribution function of the random variable X representing the number of defectives in Exercise 3.11. Then using $F(x)$, find

(a) $P(X = 1)$;

(b) $P(0 < X \leq 2)$.

Solution:

3.15 The c.d.f. of X is

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ 2/7, & \text{for } 0 \leq x < 1, \\ 6/7, & \text{for } 1 \leq x < 2, \\ 1, & \text{for } x \geq 2. \end{cases}$$

(a) $P(X = 1) = P(X \leq 1) - P(X \leq 0) = 6/7 - 2/7 = 4/7$;

(b) $P(0 < X \leq 2) = P(X \leq 2) - P(X \leq 0) = 1 - 2/7 = 5/7$.

Joint Probability Distributions

The function $f(x,y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x,y) \geq 0$ for all (x,y),
2. $\sum_x \sum_y f(x,y) = 1$,
3. $P(X = x, Y = y) = f(x,y)$.

For any region A in the xy plane, $P[(X,Y) \in A] = \sum \sum_A f(x,y)$.

Example 3.14:

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

(a) the joint probability function $f(x,y)$,

(b) $P[(X,Y) \in A]$, where A is the region $\{(x,y)|x+y \leq 1\}$.

Solution possible pairs of values (x,y) are $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(0,2)$, and $(2,0)$

$$\binom{8}{2} = 28. \qquad \binom{2}{1}\binom{3}{1} = 6. \qquad f(0,1) = 6/28$$

Solution:

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},$$

$x = 0, 1, 2;\ y = 0, 1, 2;$ and $0 \le x + y \le 2$.

	$f(x,y)$	x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$P[(X,Y) \in A] = P(X + Y \le 1) = f(0,0) + f(0,1) + f(1,0)$$

$$= \frac{3}{28} + \frac{3}{14} + \frac{9}{28} = \frac{9}{14}.$$

The **marginal distributions** of X alone and of Y alone are

$$g(x) = \sum_{y} f(x,y) \quad \text{and} \quad h(y) = \sum_{x} f(x,y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

for the continuous case.

Example 3.16

Show that the column and row totals of Table 3.1 give the marginal distribution of X alone and of Y alone.

		x			Row
	$f(x,y)$	0	1	2	Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Solution:

$$g(0) = f(0,0) + f(0,1) + f(0,2) = \frac{3}{28} + \frac{3}{14} + \frac{1}{28} = \frac{5}{14},$$

$$g(1) = f(1,0) + f(1,1) + f(1,2) = \frac{9}{28} + \frac{3}{14} + 0 = \frac{15}{28},$$

$$g(2) = f(2,0) + f(2,1) + f(2,2) = \frac{3}{28} + 0 + 0 = \frac{3}{28},$$

x	0	1	2
$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
$h(y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$

Example 4.6

Find the expected value of $g(X, Y) = XY$.

Example 4.13

Find the covariance of X and Y.

	$f(x,y)$	$x=0$	$x=1$	$x=2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Solution:

$$\begin{aligned} E(XY) &= \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y) \\ &= (0)(0)f(0,0) + (0)(1)f(0,1) \\ &\quad + (1)(0)f(1,0) + (1)(1)f(1,1) + (2)(0)f(2,0) \\ &= f(1,1) = \frac{3}{14}. \end{aligned}$$

Example 4.13

Find the covariance of X and Y.

x	0	1	2
$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
$h(y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$

$$\mu_X = \sum_x x f(x,y), \quad \mu_Y = \sum_y y f(x,y),$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y.$$

Solution:

$f(x,y)$		$x=0$	$x=1$	$x=2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$\mu_X = \sum_{x=0}^{2} x g(x) = (0)\left(\frac{5}{14}\right) + (1)\left(\frac{15}{28}\right) + (2)\left(\frac{3}{28}\right) = \frac{3}{4},$$

and

$$\mu_Y = \sum_{y=0}^{2} y h(y) = (0)\left(\frac{15}{28}\right) + (1)\left(\frac{3}{7}\right) + (2)\left(\frac{1}{28}\right) = \frac{1}{2}.$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{9}{56}.$$

Definition 4.5: Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y, respectively. The correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}. \qquad -1 \le \rho_{XY} \le 1.$$

Example 4.15:

Find the correlation coefficient between X and Y in Example 4.13.

$$\sigma^2 = \sum_x x^2 f(x) - \mu^2 = E(X^2) - \mu^2.$$

$f(x,y)$		$x=0$	$x=1$	$x=2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$E(X^2) = (0^2)\left(\frac{5}{14}\right) + (1^2)\left(\frac{15}{28}\right) + (2^2)\left(\frac{3}{28}\right) = \frac{27}{28}$$

$$E(Y^2) = (0^2)\left(\frac{15}{28}\right) + (1^2)\left(\frac{3}{7}\right) + (2^2)\left(\frac{1}{28}\right) = \frac{4}{7},$$

$$\sigma_X^2 = \frac{27}{28} - \left(\frac{3}{4}\right)^2 = \frac{45}{112} \text{ and } \sigma_Y^2 = \frac{4}{7} - \left(\frac{1}{2}\right)^2 = \frac{9}{28}.$$

the correlation coefficient between X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-9/56}{\sqrt{(45/112)(9/28)}} = -\frac{1}{\sqrt{5}}.$$

Class Activity

3.39 From a sack of fruit containing 3 oranges, 2 apples, and 3 bananas, a random sample of 4 pieces of fruit is selected. If X is the number of oranges and Y is the number of apples in the sample, find

(a) the joint probability distribution of X and Y;

(b) $P[(X, Y) \in A]$, where A is the region that is given by $\{(x, y) \mid x + y \leq 2\}$.

Solution:

(a) We can select x oranges from 3, y apples from 2, and $4 - x - y$ bananas from 3 in $\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}$ ways. A random selection of 4 pieces of fruit can be made in $\binom{8}{4}$ ways. Therefore,

$$f(x, y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}}, \qquad x = 0, 1, 2, 3; \quad y = 0, 1, 2; \quad 1 \leq x + y \leq 4.$$

(b) $P[(X, Y) \in A] = P(X + Y \leq 2) = f(1, 0) + f(2, 0) + f(0, 1) + f(1, 1) + f(0, 2)$

$$\boxed{= 3/70 + 9/70 + 2/70 + 18/70 + 3/70 = 1/2.}$$

The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$,
3. $P[(X, Y) \in A] = \int \int_A f(x, y)\, dx\, dy$, for any region A in the xy plane.

Example 3.15:

A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x, y) = \begin{cases} \frac{2}{5}(2x + 3y), & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify condition 2 of Definition 3.9.

(b) Find $P[(X, Y) \in A]$, where $A = \{(x, y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$

Solution:

a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy = 1$$

b) $$P[(X,Y)\in A] = P\left(0 < X < \frac{1}{2}, \frac{1}{4} < Y < \frac{1}{2}\right)$$
$$= \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\,dx\,dy = \frac{13}{160}.$$

Example:

Home work

A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X = the proportion of time that the drive-up facility is in use (at least one customer is being served or waiting to be served) and Y = the proportion of time that the walk-up window is in use. Then the set of possible values for (X, Y) is the rectangle $D = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$. Suppose the joint pdf of (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x + y^2) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate:

(a) $P\left(0 \le X \le \frac{1}{4}, 0 \le Y \le \frac{1}{4}\right)$ (b) $P\left(\frac{1}{4} \le Y \le \frac{3}{4}\right)$ (c) $P(Y \le .5 \mid X = .8)$

Example 3.16

Find $g(x)$ and $h(y)$ for the joint density function of Example 3.15.

$$f(x, y) = \begin{cases} \frac{2}{5}(2x + 3y), & 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_0^1 \frac{2}{5}(2x + 3y)\, dy$$

$$\boxed{= \left(\frac{4xy}{5} + \frac{6y^2}{10} \right) \Bigg|_{y=0}^{y=1} = \frac{4x + 3}{5},}$$

for $0 \le x \le 1$, and $g(x) = 0$ elsewhere.

Similarly,

$$\boxed{h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^1 \frac{2}{5}(2x + 3y)\, dx = \frac{2(1 + 3y)}{5},}$$

for $0 \le y \le 1$, and $h(y) = 0$ elsewhere.

Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)}, \text{ provided } g(x) > 0.$$

Similarly, the conditional distribution of X given that $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \text{ provided } h(y) > 0.$$

$$P(a < X < b \mid Y = y) = \sum_{a<x<b} f(x|y), \qquad \text{(Discrete)}$$

$$P(a < X < b \mid Y = y) = \int_a^b f(x|y)\, dx. \qquad \text{(Continuous)}$$

Example:

The joint density for the random variables (X, Y), where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f(x, y) = \begin{cases} 10xy^2, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal densities $g(x)$, $h(y)$, and the conditional density $f(y|x)$.

(b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased by 0.25 unit.

Solution

(a) By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{x}^{1} 10xy^2\, dy$$

$$= \frac{10}{3}xy^3 \Big|_{y=x}^{y=1} = \frac{10}{3}x(1 - x^3),\ 0 < x < 1,$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_{0}^{y} 10xy^2\, dx = 5x^2y^2\Big|_{x=0}^{x=y} = 5y^4,\ 0 < y < 1.$$

$$f(y|x) = \frac{f(x, y)}{g(x)} = \frac{10xy^2}{\frac{10}{3}x(1 - x^3)} = \frac{3y^2}{1 - x^3},\ 0 < x < y < 1.$$

$$P(a < X < b \mid Y = y) = \int_a^b f(x|y)\,dx.. \quad \text{(formula)}$$

(b) Therefore,

$$P\left(Y > \frac{1}{2} \,\middle|\, X = 0.25\right) = \int_{1/2}^{1} f(y \mid x = 0.25)\,dy$$

$$= \int_{1/2}^{1} \frac{3y^2}{1 - 0.25^3}\,dy = \frac{8}{9}.$$

Class Activity

Given the joint density function

$$f(x, y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find $g(x)$, $h(y)$, $f(x|y)$, and evaluate $P(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3})$.

Solution:

By definition of the marginal density. for $0 < x < 2$,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{x(1+3y^2)}{4}dy$$
$$= \left(\frac{xy}{4} + \frac{xy^3}{4}\right)\Bigg|_{y=0}^{y=1} = \frac{x}{2},$$

and for $0 < y < 1$,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{x(1+3y^2)}{4}dx$$
$$= \left(\frac{x^2}{8} + \frac{3x^2y^2}{8}\right)\Bigg|_{x=0}^{x=2} = \frac{1+3y^2}{2}.$$

using the conditional density definition, for $0 < x < 2$,

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{x(1+3y^2)/4}{(1+3y^2)/2} = \frac{x}{2},$$

$$P\left(\frac{1}{4} < X < \frac{1}{2} \;\middle|\; Y = \frac{1}{3}\right) = \int_{1/4}^{1/2} \frac{x}{2}dx = \frac{3}{64}.$$

Practice:

3.38 If the joint probability distribution of X and Y is given by

$$f(x, y) = \frac{x+y}{30}, \quad \text{for } x = 0, 1, 2, 3;\ y = 0, 1, 2,$$

find

(a) $P(X \leq 2, Y = 1)$;
(b) $P(X > 2, Y \leq 1)$;
(c) $P(X > Y)$;
(d) $P(X + Y = 4)$.

3.46 Referring to Exercise 3.38, find
(a) the marginal distribution of X;
(b) the marginal distribution of Y.

3.40 A fast-food restaurant operates both a drive-through facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-through and walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y), & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density of X.
(b) Find the marginal density of Y.
(c) Find the probability that the drive-through facility is busy less than one-half of the time.

Practice:

3.42 Let X and Y denote the lengths of life, in years, of two components in an electronic system. If the joint density function of these variables is

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x > 0,\ y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

find $P(0 < X < 1 \mid Y = 2)$

3.43 Let X denote the reaction time, in seconds, to a certain stimulus and Y denote the temperature (°F) at which a certain reaction starts to take place. Suppose that two random variables X and Y have the joint density

$$f(x,y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find

(a) $P(0 \le X \le \frac{1}{2}$ and $\frac{1}{4} \le Y \le \frac{1}{2})$;

(b) $P(X < Y)$.

Example 4.14: The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$f(x,y) = \begin{cases} 8xy, & 0 \le y \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.

Covariance and correlation for Continuous random variable

Example 4.16: Find the correlation coefficient of X and Y in Example 4.14.

***Solution*:** We first compute the marginal density functions.

5 Steps:

1-Marginal density function

$$g(x) = \begin{cases} 4x^3, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases} \quad \text{and} \quad h(y) = \begin{cases} 4y(1-y^2), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

5 Steps:

2-Mean

$$\mu_X = E(X) = \int_0^1 4x^4 \; dx = \frac{4}{5} \text{ and } \mu_Y = \int_0^1 4y^2(1-y^2) \; dy = \frac{8}{15}.$$

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2 \; dx \; dy = \frac{4}{9}.$$

3-covariance

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$

4-variance

$$E(X^2) = \int_0^1 4x^5 \; dx = \frac{2}{3} \text{ and } E(Y^2) = \int_0^1 4y^3(1-y^2) \; dy = 1 - \frac{2}{3} = \frac{1}{3},$$

$$\sigma_X^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75} \text{ and } \sigma_Y^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}.$$

5-Correlarion

$$\boxed{\rho_{XY} = \frac{4/225}{\sqrt{(2/75)(11/225)}} = \frac{4}{\sqrt{66}}.}$$

3.44 Each rear tire on an experimental airplane is supposed to be filled to a pressure of 40 pounds per square inch (psi). Let X denote the actual air pressure for the right tire and Y denote the actual air pressure for the left tire. Suppose that X and Y are random variables with the joint density function

$$f(x,y) = \begin{cases} k(x^2+y^2), & 30 \le x < 50,\ 30 \le y < 50, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find k.

(b) Find $P(30 \le X \le 40 \text{ and } 40 \le Y < 50)$.

(c) Find the probability that both tires are underfilled.

4.46 Find the covariance of the random variables X and Y of Exercise 3.44 on page 105.

Find the correlation coefficient between X and Y

Activity

Joint Probability distribution
in Continuous random variable

Solution 3.44:

(a) $1 = k\int_{30}^{50}\int_{30}^{50}(x^2+y^2)\,dx\,dy = k(50-30)\left(\int_{30}^{50}x^2\,dx + \int_{30}^{50}y^2\,dy\right) = \frac{392k}{3}\cdot 10^4.$

So, $k = \frac{3}{392}\cdot 10^{-4}.$

(b) $P(30 \le X \le 40,\ 40 \le Y \le 50) = \frac{3}{392}\cdot 10^{-4}\int_{30}^{40}\int_{40}^{50}(x^2+y^2)\,dy\,dx$

$= \frac{3}{392}\cdot 10^{-3}(\int_{30}^{40}x^2\,dx + \int_{40}^{50}y^2\,dy) = \frac{3}{392}\cdot 10^{-3}\left(\frac{40^3-30^3}{3} + \frac{50^3-40^3}{3}\right) = \frac{49}{196}.$

(c) $P(30 \le X \le 40,\ 30 \le Y \le 40) = \frac{3}{392}\cdot 10^{-4}\int_{30}^{40}\int_{30}^{40}(x^2+y^2)\,dx\,dy$

$= 2\frac{3}{392}\cdot 10^{-4}(40-30)\int_{30}^{40}x^2\,dx = \frac{3}{196}\cdot 10^{-3}\frac{40^3-30^3}{3} = \frac{37}{196}.$

Solution 3.46:

Covariance

From previous exercise, $k = \left(\frac{3}{392}\right)10^{-4}$, and $g(x) = k\left(20x^2 + \frac{98000}{3}\right)$, with
$\mu_X = E(X) = \int_{30}^{50} xg(x)\,dx = k\int_{30}^{50}\left(20x^3 + \frac{98000}{3}x\right)\,dx = 40.8163.$
Similarly, $\mu_Y = 40.8163$. On the other hand,
$E(XY) = k\int_{30}^{50}\int_{30}^{50} xy(x^2+y^2)\,dy\,dx = 1665.3061.$
Hence, $\sigma_{XY} = E(XY) - \mu_X\mu_Y = 1665.3061 - (40.8163)^2 = -0.6642.$

variance

$$\boxed{\sigma^2 = \sum_x x^2 f(x) - \mu^2 = E(X^2) - \mu^2.}$$

Correlation

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} =$$

Example 4.1: A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Activity

Find
Mean and Variance

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

Example 4.3: Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{20{,}000}{x^3}, & x > 100, \\ 0, & \text{elsewhere}. \end{cases}$$

Find the expected life of this type of device.

formula

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\mu = E(X) = \sum_x x f(x)$$

$$\sigma^2 = E(X^2) - \mu^2.$$

Example 4.4: Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Solution:

$$E[g(X)] = E(2X-1) = \sum_{x=4}^{9}(2x-1)f(x) = \$12.67.$$

Example 4.5: Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

Solution:

$$E(4X+3) = \int_{-1}^{2} \frac{(4x+3)x^2}{3}\,dx = \frac{1}{3}\int_{-1}^{2}(4x^3+3x^2)\,dx = 8.$$

Example 4.20: The weekly demand for a certain drink, in thousands of liters, at a chain of convenience stores is a continuous random variable $g(X) = X^2 + X - 2$, where X has the density function

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of the weekly demand for the drink.

Solution:

$$E(X^2 + X - 2) = E(X^2) + E(X) - E(2).$$

$$E(X) = \int_1^2 2x(x-1)\,dx = \frac{5}{3} \text{ and } E(X^2) = \int_1^2 2x^2(x-1)\,dx = \frac{17}{6}.$$

Now

$$E(X^2 + X - 2) = \frac{17}{6} + \frac{5}{3} - 2 = \frac{5}{2},$$

Example 4.22: If X and Y are random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 4$ and covariance $\sigma_{XY} = -2$, find the variance of the random variable $Z = 3X - 4Y + 8$.

Solution:

$$\begin{aligned} \sigma_Z^2 &= \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 \\ &= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \\ &= (9)(2) + (16)(4) - (24)(-2) = 130. \end{aligned}$$

Practice:

4.43 The length of time, in minutes, for an airplane to obtain clearance for takeoff at a certain airport is a random variable $Y = 3X - 2$, where X has the density function

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of the random variable Y.

4.60 Suppose that X and Y are independent random variables having the joint probability distribution

$f(x, y)$		$x = 2$	$x = 4$
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

Find
(a) $E(2X - 3Y)$;
(b) $E(XY)$.

4.23 Find the expected value of $g(X, Y) = XY^2$.
Find μ_X and μ_Y. In Question **4.60**

4.36 Suppose that the probabilities are 0.4, 0.3, 0.2, and 0.1, respectively, that 0, 1, 2, or 3 power failures will strike a certain subdivision in any given year. Find the mean and variance of the random variable X representing the number of power failures striking this subdivision.

4.50 For a laboratory assignment, if the equipment is working, the density function of the observed outcome X is

$$f(x) = \begin{cases} 2(1 - x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the variance and standard deviation of X.

4.59 If a random variable X is defined such that

$$E[(X - 1)^2] = 10 \text{ and } E[(X - 2)^2] = 6,$$

find μ and σ^2.

ANY Questions?