# Chapter 9

One- and Two-Sample Estimation Problems

Confidence Intervals for the Mean When σ Is Known

Confidence Intervals for the Mean When σ Is Unknown

Statistical inference

consists of those methods by which one makes inferences or generalizations about a population.

Statistical inference may be divided into two major areas:

estimation and   tests of hypotheses.

Chap 9   Chap 10
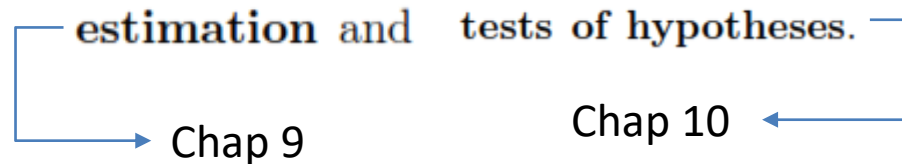
Statistical inference is a method of making decisions about the parameters of a population, based on random sampling. It helps to assess the relationship between the dependent and independent variables. The purpose of statistical inference to estimate the uncertainty or sample to sample variation.

Types of Statistical Inference

There are different types of statistical inferences that are extensively used for making conclusions. They are:

➢ One sample hypothesis testing
➢ Confidence Interval
➢ Pearson Correlation
➢ Bi-variate regression
➢ Multi-variate regression
➢ Chi-square statistics and contingency table
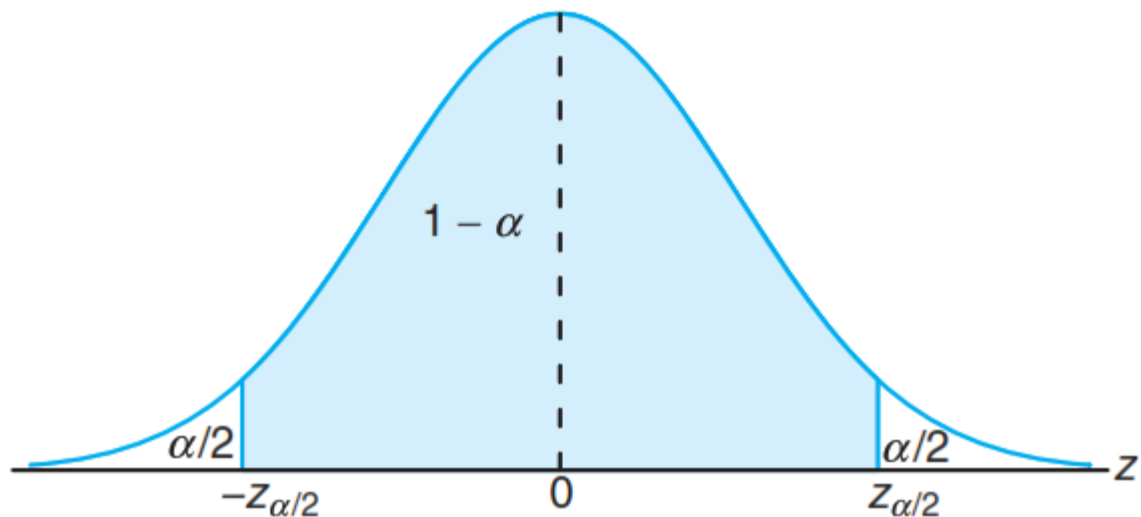➢ ANOVA or T-test

Single Sample: Estimating the Mean

Figure 9.2: $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha.$

confidence coefficient or the degree of confidence,

According to the Central Limit Theorem, we can expect the sampling distribution of $\bar{X}$ to be approximately normally distributed with mean $\mu_{\bar{X}} = \mu$ and

standard deviation $\sigma_{\bar{X}} = \sigma/\sqrt{n}$. Writing $z_{\alpha/2}$ for the z-value above which we find an area of $\alpha/2$ under the normal curve, we can see from Figure 9.2 that

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha,$$

where

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}.$$

Hence,

$$P\left(-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1 - \alpha.$$

Multiplying each term in the inequality by $\sigma/\sqrt{n}$ and then subtracting $\bar{X}$ from each term and multiplying by -1 (reversing the sense of the inequalities), we obtain

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

Proof:

$$-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}$$

$$-z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \bar{X} - \mu < z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

$$-\bar{X} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < -\mu < -\bar{X} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

$$\bar{X} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} > \mu > \bar{X} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

$$\bar{X} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Confidence interval
for the mean;
σ is known

$$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

If $\bar{x}$ is the mean of a random sample of size n from a population with known variance σ^2, a $100(1 - \alpha)\%$ confidence interval for μ is given by

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

$$\hat{\theta}_L = \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \hat{\theta}_U = \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Waiting Times in Emergency Rooms

A survey of 30 emergency room patients found that the average waiting time for treatment was 174.3 minutes. Assuming that the population standard deviation is 46.5 minutes, find the best point estimate of the population mean and the 99% confidence of the population mean.

Solution

The best point estimate is 174.3 minutes. The 99% confidence is interval is

$$\bar{X} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) < \mu < \bar{X} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

$$174.3 - 2.58\left(\frac{46.5}{\sqrt{30}}\right) < \mu < \bar{X} + 2.58\left(\frac{46.5}{\sqrt{30}}\right)$$

$$174.3 - 21.9 < \mu < 174.3 + 21.9$$

$$152.4 < \mu < 196.2$$

Hence, one can be 99% confident that the mean waiting time for emergency room treatment is between 152.4 and 196.2 minutes.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table A.4 (continued) Critical Values of the *t*-Distribution

				α			
v	**0.02**	**0.015**	**0.01**	**0.0075**	**0.005**	**0.0025**	**0.0005**
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725

Table A.4 Critical Values of the t-Distribution

v	α						
	0.40	**0.30**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter.

Solution: The point estimate of μ is $\bar{x} = 2.6$. The z-value leaving an area of 0.025 to the right, and therefore an area of 0.975 to the left, is $z_{0.025} = 1.96$ (Table A.3). Hence, the 95% confidence interval is

$$2.6 - (1.96)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (1.96)\left(\frac{0.3}{\sqrt{36}}\right),$$

which reduces to $2.50 < \mu < 2.70$. To find a 99% confidence interval, we find the z-value leaving an area of 0.005 to the right and 0.995 to the left. From Table A.3 again, $z_{0.005} = 2.575$, and the 99% confidence interval is

$$2.6 - (2.575)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (2.575)\left(\frac{0.3}{\sqrt{36}}\right),$$

or simply

$$2.47 < \mu < 2.73.$$

Example 9.13. Construct a 95% confidence interval for the population mean based on a sample of measurements

$$2.5, \ 7.4, \ 8.0, \ 4.5, \ 7.4, \ 9.2$$

if measurement errors have Normal distribution, and the measurement device guarantees a standard deviation of $\sigma = 2.2$.

Solution. This sample has size $n = 6$ and sample mean $\bar{X} = 6.50$. To attain a confidence level of

$$1 - \alpha = 0.95,$$

$$\alpha = 0.05 \text{ and } \alpha/2 = 0.025.$$

$$\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 6.50 \pm (1.960) \frac{2.2}{\sqrt{6}}$$

$$= 6.50 \pm 1.76 \text{ or } [4.74, \ 8.26].$$

Error in Estimation

The $100(1-\alpha)\%$ confidence interval provides an estimate of the accuracy of our point estimate. If μ is actually the center value of the interval, then $\bar{x}$ estimates μ without error. Most of the time, however, $\bar{x}$ will not be exactly equal to μ and the point estimate will be in error. The size of this error will be the absolute value of the difference between μ and $\bar{x}$, and we can be $100(1-\alpha)\%$ confident that this difference will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. We can readily see this if we draw a diagram of a hypothetical confidence interval, as in Figure 9.4.
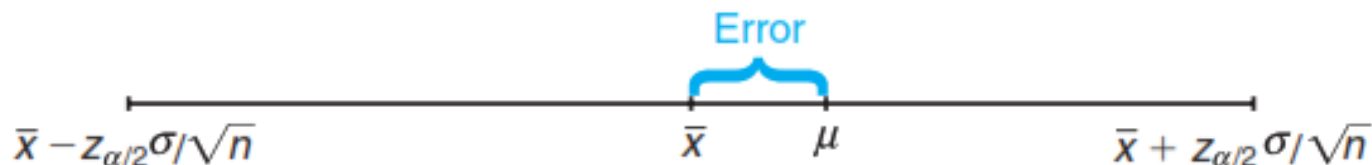

Figure 9.4: Error in estimating μ by $\bar{x}$.

Theorem 9.1: If $\bar{x}$ is used as an estimate of μ, we can be $100(1 - \alpha)\%$ confident that the error will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

Theorem 9.2: If $\bar{x}$ is used as an estimate of μ, we can be $100(1 - \alpha)\%$ confident that the error will not exceed a specified amount e when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{e}\right)^2.$$

Example 9.3:

How large a sample is required if we want to be 95% confident that our estimate of μ in Example 9.2 is off by less than 0.05?

Solution: The population standard deviation is $\sigma = 0.3$. Then, by Theorem 9.2,

$$n = \left[\frac{(1.96)(0.3)}{0.05}\right]^2 = 138.3.$$

Therefore, we can be 95% confident that a random sample of size 139 will provide an estimate $\bar{x}$ differing from μ by an amount less than 0.05.

If $\bar{X}$ is the mean of a random sample of size n from a population with variance σ^2, the one-sided $100(1 - \alpha)\%$ confidence bounds for μ are given by

upper one-sided bound: $\qquad \bar{x} + z_\alpha \sigma / \sqrt{n};$

lower one-sided bound: $\qquad \bar{x} - z_\alpha \sigma / \sqrt{n}.$

In a psychological testing experiment, 25 subjects are selected randomly and their reaction time, in seconds, to a particular stimulus is measured. Past experience suggests that the variance in reaction times to these types of stimuli is 4 sec^2 and that the distribution of reaction times is approximately normal. The average time for the subjects is 6.2 seconds. Give an upper 95% bound for the mean reaction time.

Solution: The upper 95% bound is given by

$$\bar{x} + z_\alpha \sigma/\sqrt{n} = 6.2 + (1.645)\sqrt{4/25} = 6.2 + 0.658$$
$$= 6.858 \text{ seconds.}$$

Hence, we are 95% confident that the mean reaction time is less than 6.858 seconds.

The Case of σ Unknown

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

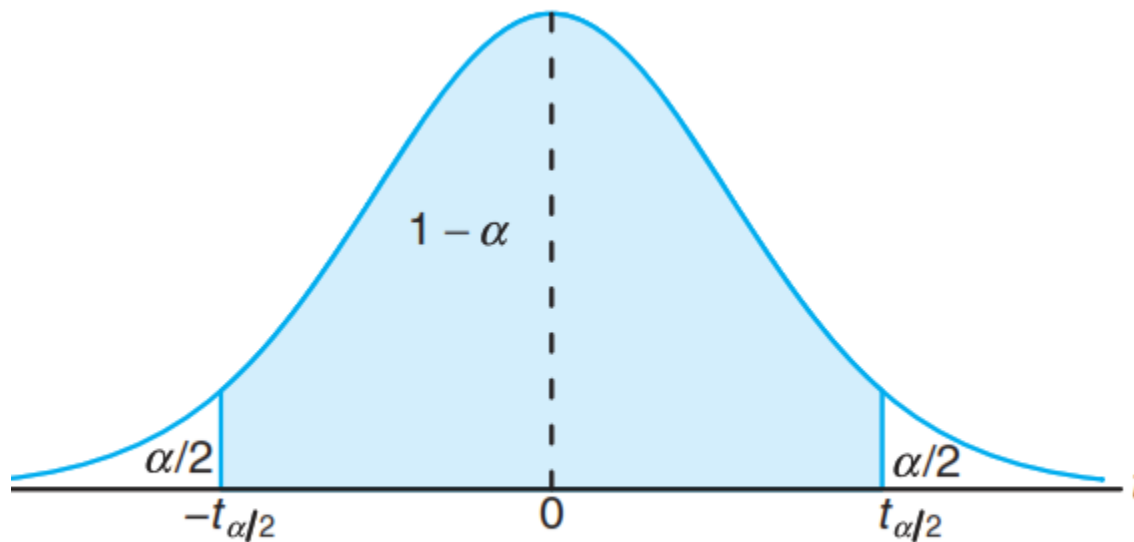has a Student t-distribution with $n - 1$ degrees of freedom.

Figure 9.5: $P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1 - \alpha.$

Confidence interval for the mean; σ is unknown

$$\bar{X} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ is a critical value from T-distribution with $n - 1$ degrees of freedom

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal population with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}},$$

where $t_{\alpha/2}$ is the t-value with $v = n - 1$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

Home Fires Started by Candles

The data represent a sample of the number of home fires started by candles for the past several years. (Data are from the National Fire Protection Association.) Find the 99% confidence interval for the mean number of home fires started by candles each year.

| 5460 | 5900 | 6090 | 6310 | 7160 | 8440 | 9930 |

Solution

$$\bar{X} - t_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right) < \mu < \bar{X} + t_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)$$

$$7041.4 - 3.707\left(\frac{1610.3}{\sqrt{7}}\right) < \mu < 7041.4 + 3.707\left(\frac{1610.3}{\sqrt{7}}\right)$$

$$7041.4 - 2256.2 < \mu < 7041.4 + 2256.2$$

$$4785.2 < \mu < 9297.6$$

Conclusion:

One can be 99% confident that the population mean number of home fires started by candles each year is between 4785.2 and 9297.6, based on a sample of home fires occurring over a period of 7 years.

Example 9.5:

The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.

Solution: The sample mean and standard deviation for the given data are

$$\bar{x} = 10.0 \qquad \text{and} \qquad s = 0.283.$$

Using Table A.4, we find $t_{0.025} = 2.447$ for $v = 6$ degrees of freedom.

95% confidence interval for μ is

$$10.0 - (2.447)\left(\frac{0.283}{\sqrt{7}}\right) < \mu < 10.0 + (2.447)\left(\frac{0.283}{\sqrt{7}}\right),$$

which reduces to $9.74 < \mu < 10.26.$

Concept of a Large-Sample Confidence Interval

Often statisticians recommend that even when normality cannot be assumed, σ is unknown, and $n \geq 30$, s can replace σ and the confidence interval

$$\bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$$

Example 9.6:

Scholastic Aptitude Test (SAT) mathematics scores of a random sample of 500 high school seniors in the state of Texas are collected, and the sample mean and standard deviation are found to be 501 and 112, respectively. Find a 99% confidence interval on the mean SAT mathematics score for seniors in the state of Texas.

Solution:

we find $z_{0.005} = 2.575$. Hence, a 99% confidence interval for μ is

$$501 \pm (2.575)\left(\frac{112}{\sqrt{500}}\right) = 501 \pm 12.9,$$

$$488.1 < \mu < 513.9.$$

Two Samples: Estimating the Difference between Two Means

If $\bar{x}_1$ and $\bar{x}_2$ are means of independent random samples of sizes n_1 and n_2 from populations with known variances σ_1^2 and σ_2^2, respectively, a $100(l - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\bar{x}_1 - \bar{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\bar{x}_1 - \bar{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

A study was conducted in which two types of engines, A and B, were compared. Gas mileage, in miles per gallon, was measured. Fifty experiments were conducted using engine type A and 75 experiments were done with engine type B. The gasoline used and other conditions were held constant. The average gas mileage was 36 miles per gallon for engine A and 42 miles per gallon for engine B. Find a 96% confidence interval on $\mu_B - \mu_A$, where μ_A and μ_B are population mean gas mileages for engines A and B, respectively. Assume that the population standard deviations are 6 and 8 for engines A and B, respectively.

Solution:

The point estimate of $\mu_B - \mu_A$ is $\bar{x}_B - \bar{x}_A = 42 - 36 = 6$. Using $\alpha = 0.04$, we find $z_{0.02} = 2.05$ from Table A.3. Hence, with substitution in the formula above, the 96% confidence interval is

$$6 - 2.05\sqrt{\frac{64}{75} + \frac{36}{50}} < \mu_B - \mu_A < 6 + 2.05\sqrt{\frac{64}{75} + \frac{36}{50}},$$

or simply $3.43 < \mu_B - \mu_A < 8.57$.

9.35 A random sample of size $n_1 = 25$, taken from a normal population with a standard deviation $\sigma_1 = 5$, has a mean $\bar{x}_1 = 80$. A second random sample of size $n_2 = 36$, taken from a different normal population with a standard deviation $\sigma_2 = 3$, has a mean $\bar{x}_2 = 75$. Find a 94% confidence interval for $\mu_1 - \mu_2$.

Solution:

$$n_1 = 25, n_2 = 36, \bar{x}_1 = 80, \bar{x}_2 = 75, \sigma_1 = 5, \sigma_2 = 3,$$

$$(80 - 75) - (1.88)\sqrt{25/25 + 9/36} < \mu_1 - \mu_2 < (80 - 75) + (1.88)\sqrt{25/25 + 9/36},$$

$$\boxed{2.9 < \mu_1 - \mu_2 < 7.1.}$$

9.36 Two kinds of thread are being compared for strength. Fifty pieces of each type of thread are tested under similar conditions. Brand A has an average tensile strength of 78.3 kilograms with a standard deviation of 5.6 kilograms, while brand B has an average tensile strength of 87.2 kilograms with a standard deviation of 6.3 kilograms. Construct a 95% confidence interval for the difference of the population means.

Solution:

$n_A = 50, n_B = 50, \bar{x}_A = 78.3, \bar{x}_B = 87.2, \sigma_A = 5.6,$ and $\sigma_B = 6.3.$

$$(87.2 - 78.3) \pm 1.96\sqrt{5.6^2/50 + 6.3^2/50} = 8.9 \pm 2.34,$$

$$\boxed{\text{or } 6.56 < \mu_A - \mu_B < 11.24.}$$

Confidence interval for the difference between two means

Confidence interval for the difference of means; known standard deviations	$\bar{X} - \bar{Y} \pm z_{\alpha/2}\sqrt{\dfrac{\sigma_X^2}{n} + \dfrac{\sigma_Y^2}{m}}$

Example 9.16 (DELAYS AT NODES). Internet connections are often slowed by delays at nodes. Let us determine if the delay time increases during heavy-volume times.

Five hundred packets are sent through the same network between 5 pm and 6 pm (sample X), and three hundred packets are sent between 10 pm and 11 pm (sample Y). The early sample has a mean delay time of 0.8 sec with a standard deviation of 0.1 sec whereas the second sample has a mean delay time of 0.5 sec with a standard deviation of 0.08 sec. Construct a 99.5% confidence interval for the difference between the mean delay times.

Solution:

We have $n = 500$, $\bar{X} = 0.8$, $s_X = 0.1$; $m = 300$, $\bar{Y} = 0.5$, $s_Y = 0.08$.

$$
\begin{aligned}
\bar{X} - \bar{Y} \;\pm\; & z_{0.0025}\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}} = (0.8 - 0.5) \pm (2.81)\sqrt{\frac{(0.1)^2}{500} + \frac{(0.08)^2}{300}} \\
= \;& \underline{0.3 \pm 0.018} \text{ or } \underline{[0.282, 0.318]}.
\end{aligned}
$$

Variances Unknown but Equal

Consider the case where σ_1^2 and σ_2^2 are unknown. If $\sigma_1^2 = \sigma_2^2 = \sigma^2$, we obtain a standard normal variable of the form

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma^2[(1/n_1) + (1/n_2)]}}.$$

Confidence Interval for $\mu_1 - \mu_2$, $\sigma_1^2 = \sigma_2^2$ but Both Unknown

Pooled Estimate of Variance

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}.$$

If $\bar{x}_1$ and $\bar{x}_2$ are the means of independent random samples of sizes n_1 and n_2, respectively, from approximately normal populations with unknown but equal variances, a $100(1 - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is given by

$$(\bar{x}_1 - \bar{x}_2) - t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

where s_p is the pooled estimate of the population standard deviation and $t_{\alpha/2}$ is the t-value with $v = n_1 + n_2 - 2$ degrees of freedom, leaving an area of $\alpha/2$ to the right.
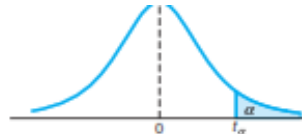
Table A.4 Critical Values of the t-Distribution

v	α						
	0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

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9.38 Two catalysts in a batch chemical process, are being compared for their effect on the output of the process reaction. A sample of 12 batches was prepared using catalyst 1, and a sample of 10 batches was prepared using catalyst 2. The 12 batches for which catalyst 1 was used in the reaction gave an average yield of 85 with a sample standard deviation of 4, and the 10 batches for which catalyst 2 was used gave an average yield of 81 and a sample standard deviation of 5. Find a 90% confidence interval for the difference between the population means, assuming that the populations are approximately normally distributed with equal variances.

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}.$$

Solution:

$$n_1 = 12, n_2 = 10, \bar{x}_1 = 85, \bar{x}_2 = 81, s_1 = 4, s_2 = 5,$$

$$s_p = 4.478 \qquad\qquad t_{0.05} = 1.725$$

$$(85 - 81) \pm (1.725)(4.478)\sqrt{1/12 + 1/10} = 4 \pm 3.31,$$

$$\boxed{0.69 < \mu_1 - \mu_2 < 7.31.}$$

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}.$$

$$(\bar{x}_1 - \bar{x}_2) - t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

9.50 Two levels (low and high) of insulin doses are given to two groups of diabetic rats to check the insulin-binding capacity, yielding the following data:

Low dose: $n_1 = 8$ $\bar{x}_1 = 1.98$ $s_1 = 0.51$
High dose: $n_2 = 13$ $\bar{x}_2 = 1.30$ $s_2 = 0.35$

Assume that the variances are equal. Give a 95% confidence interval for the difference in the true average insulin-binding capacity between the two samples.

Solution:

$$n_1 = 8, n_2 = 13, \bar{x}_1 = 1.98, \bar{x}_2 = 1.30, s_1 = 0.51, s_2 = 0.35,$$

$$s_p = 0.416.$$

$$(1.98 - 1.30) \pm (2.093)(0.416)\sqrt{1/8 + 1/13} = 0.68 \pm 0.39,$$

$$\boxed{0.29 < \mu_1 - \mu_2 < 1.07.}$$

Confidence Interval for $\mu_D = \mu_1 - \mu_2$ for Paired Observations	If $\bar{d}$ and s_d are the mean and standard deviation, respectively, of the normally distributed differences of n random pairs of measurements, a $100(1 - \alpha)\%$ confidence interval for $\mu_D = \mu_1 - \mu_2$ is

$$\bar{d} - t_{\alpha/2}\frac{s_d}{\sqrt{n}} < \mu_D < \bar{d} + t_{\alpha/2}\frac{s_d}{\sqrt{n}},$$

where $t_{\alpha/2}$ is the t-value with $v = n - 1$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

Table A.4 (continued) Critical Values of the t-Distribution

				α			
v	0.02	0.015	0.01	0.0075	0.005	0.0025	0.0005
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725

Table A.4 Critical Values of the t-Distribution

v	α 0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
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13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
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20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

9.44 Referring to Exercise 9.43, find a 99% confidence interval for $\mu_1 - \mu_2$ if tires of the two brands are assigned at random to the left and right rear wheels of 8 taxis and the following distances, in kilometers, are recorded:

Taxi	Brand A	Brand B
1	34,400	36,700
2	45,500	46,800
3	36,700	37,700
4	32,000	31,100
5	48,400	47,800
6	32,800	36,400
7	38,100	38,900
8	30,100	31,500

$$\bar{d} - t_{\alpha/2}\frac{s_d}{\sqrt{n}} < \mu_D < \bar{d} + t_{\alpha/2}\frac{s_d}{\sqrt{n}},$$

Assume that the differences of the distances are approximately normally distributed.

Solution:

$n = 8, \bar{d} = -1112.5, s_d = 1454$, with $t_{0.005} = 3.499$ with 7 degrees of freedom.

$$-1112.5 \pm (3.499)\frac{1454}{\sqrt{8}} = -1112.5 \pm 1798.7,$$

$$\boxed{-2911.2 < \mu_D < 686.2.}$$