

Differential-algebraic equations in Hilbert spaces

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Motivation

Navier-Stokes equations

- Consider the following example of incompressible Navier-Stokes equations at low Reynolds numbers:

$$\begin{aligned}\rho(\xi) \frac{\partial v}{\partial t}(t, \xi) &= -\nabla p(t, \xi) + \mu \Delta v(t, \xi) + \hat{f}(t, \xi), \\ \operatorname{div} v(t, \xi) &= 0,\end{aligned}$$

- It consists of a coupled differential equation (1st line) and an algebraic equation (2nd line). This can be formalized as a **differential-algebraic equation (DAE)**

$$\frac{d}{dt} Ex(t) = \frac{d}{dt} \begin{bmatrix} \rho(\cdot) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v_\xi(t) \\ p_\xi(t) \end{pmatrix} = \begin{bmatrix} \mu \Delta & -\nabla \\ \operatorname{div} & 0 \end{bmatrix} \begin{pmatrix} v_\xi(t) \\ p_\xi(t) \end{pmatrix} + \begin{pmatrix} \hat{f}_\xi(t) \\ 0 \end{pmatrix} = Ax(t) + f(t)$$

on suitable Hilbert spaces encoding possible boundary conditions

DAE vs. ODE Toy examples

Consider a simple example in \mathbb{R}^2

$$\begin{aligned}\dot{x}_2(t) &= x_1(t) + f_1(t) \\ 0 &= x_2(t)\end{aligned}$$

Hence $x_2 = 0$ which implies $x_1 = -f_1$. Consequently, not for all initial values a solution exists. Furthermore, x_1 has the same regularity as f_1 and might be non-smooth.

$$\begin{aligned}\dot{x}_2(t) &= x_1(t) \\ 0 &= x_2(t) + f_2(t)\end{aligned}$$

Then $x_2 = -f_2$ and thus, f_2 has to be weakly differentiable. Hence not for all initial values and for all right hand sides a solution exists.

Reminder: How to solve finite dimensional DAEs?

Given $E, A \in \mathbb{C}^{n \times n}$ and $f \in L^1([0, t_f])$, $t_f > 0$ the equation

$$\frac{d}{dt}Ex(t) = Ax(t) + f(t), \quad (Ex)(0) = Ex_0.$$

If $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$, the DAE is called **regular** and there exists invertible $W, T \in \mathbb{C}^{n \times n}$ such that

$$\frac{d}{dt} \begin{pmatrix} x_{\mathfrak{J}} \\ x_N \end{pmatrix} = \frac{d}{dt} WET = WAT + Wf = \begin{pmatrix} \mathfrak{J} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{\mathfrak{J}} \\ x_N \end{pmatrix} + \begin{pmatrix} f_{\mathfrak{J}} \\ f_N \end{pmatrix}$$

where N is nilpotent and \mathfrak{J} in Jordan normal form. This is a decoupling into an **ODE-part** and an **algebraic** part. The above form is called **Weierstraß-Kronecker form**.

GANTMACHER or KUNKEL, MEHRMANN

How to solve a finite dimensional DAE?

- The solution of

$$\frac{d}{dt} \begin{pmatrix} x_{\mathfrak{J}} \\ N x_N \end{pmatrix} = \frac{d}{dt} WET = WAT + Wf = \begin{pmatrix} \mathfrak{J} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{\mathfrak{J}} \\ x_N \end{pmatrix} + \begin{pmatrix} f_{\mathfrak{J}} \\ f_N \end{pmatrix}$$

is then given in BERGER, ILCHMANN, TRENN '12 by

$$x_{\mathfrak{J}}(t) = e^{\mathfrak{J}t} x_0 + \int_0^t e^{\mathfrak{J}(t-\tau)} f_{\mathfrak{J}}(t-\tau) d\tau, \quad x_N(t) = \sum_{i=0}^{k-1} N^i f_N^{(i)}(t).$$

- The smallest number $k \in \mathbb{N}$ such that $N^k = 0$ is the **index** of the DAE.
- **Difficulty:** Solution exists only for a sufficiently smooth f and not for every initial value x_0 .

Extension to Hilbert spaces

How can we decouple a DAE in finite dimensions?

- Let $E, A \in \mathbb{C}^{n \times n}$. Consider the Wong sequence

$$\mathcal{V}_0 := \mathbb{C}^n, \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), i \geq 0.$$

where A^{-1} is the set theoretic inverse.

- Idea: If $x(t) \in \mathcal{V}_i$ then $\dot{x}(t) \in \mathcal{V}_i$ (\mathcal{V}_i is closed) and $Ax(t) \in E\mathcal{V}_i \Rightarrow x(t) \in \mathcal{V}_{i+1}$
- One can show $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$, $i \in \mathbb{N}$, and since \mathbb{C}^n is finite dimensional the sequence $(\mathcal{V}_i)_i$ terminates.
- Does not work without further assumptions in Hilbert spaces.

How can we decouple a DAE in finite dimensions?

- A second way to obtain such a decoupling for $\frac{d}{dt}Ex(t) = Ax(t)$, $E, A \in \mathbb{C}^{n \times n}$ via resolvents:

$$\begin{aligned}\frac{d}{dt}Ex(t) &= Ax(t) = (A - \lambda E)x(t) + \lambda Ex(t) \\ \iff \frac{d}{dt}(A - \lambda E)^{-1}Ex(t) &= x(t) + \lambda(A - \lambda E)^{-1}Ex(t)\end{aligned}$$

- There exists $T \in \mathbb{C}^{n \times n}$ such that $T^{-1}(A - \lambda E)^{-1}ET$ is in Jordan canonical form. The nilpotent Jordan blocks are precisely N , the algebraic part of the Weierstraß form.
- Problem: no canonical form for operators in Hilbert spaces.

How can we decouple a DAE in infinite dimensions?

Assume that $E, A : X \rightarrow X$ are linear bounded on a Hilbert space X .

- Wong-sequence approach

$$\mathcal{V}_{i+1} = A^{-1}(E\mathcal{V}_i), \quad i \geq 0, \quad \mathcal{V}_0 = X,$$

- **Problem:** $\{\mathcal{V}_i\}_{i \geq 0}$ must not terminate and \mathcal{V}_i are not closed

- Example 1: $X = \ell_2(\mathbb{N})$, $A = Id$, and let E be the right shift

$$(x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots).$$

- Example 2: $A : X \supset D \rightarrow X$ closed densely defined with bounded inverse A^{-1} and $E = I_X$ then $\mathcal{V}_i = A^{-i}X$ is not closed.

How can we decouple a DAE in infinite dimensions?

- TROSTORFF, WAURICK '18 considered Hilbert spaces X and used an additional resolvent growth condition:

$$\exists \omega > 0 : \quad \|(\lambda E - A)^{-1}\| \leq M|\lambda|^{k-1}, \quad \text{for all } \lambda \in \mathbb{C}_\omega^+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega\}, \quad (1)$$

- In the finite dimensional case and if $E = N$ for some nilpotent N and $A = I_n$ we obtain

$$(sN - I_n)^{-1} = - \sum_{i=0}^{\infty} (-sN)^i = - \sum_{i=0}^{k-1} (-sN)^i$$

and hence the smallest k such that (1) holds is the index of the DAE.

- If additionally $E\mathcal{V}_k$ is closed TROSTORFF, WAURICK '18 showed that $\mathcal{V}_k = \mathcal{V}_{k+1}$ and

$$W(sE - A)T = \begin{bmatrix} sI - \mathfrak{J} & 0 \\ 0 & sN - I \end{bmatrix}, \quad \mathfrak{J} \text{ bounded, } N \text{ nilpotent}$$

What is known for unbounded A ?

- Tractability approach from REIS, TISCHENDORF '05, REIS '06 for Hilbert spaces X . Construction of a sequence of projectors with quite restrictive closedness assumptions. (very technical)
- A DAE with tractability index k can be transformed with W injective, dense range and T invertible to

$$WET = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}, \quad WAT = \begin{bmatrix} I & K \\ 0 & \mathfrak{J} \end{bmatrix}$$

here N is nilpotent with index k and $K \neq 0$.

- A solution can be obtained on the subspace via \mathfrak{J} , but the finite tractability index **does not imply** that \mathfrak{J} generates a C_0 -semigroup.
- We have **no explicit** expression in terms of E and A for the **initial values** for which a solution exists.

New decoupling approach via pseudo-resolvents

- Assume that $\lambda, \mu \in \rho(E, A) := \{\lambda \in \mathbb{C} \mid \lambda E - A \text{ bijective}\}$ then

$$\begin{aligned} A - \lambda E - (A - \mu E) &= (\mu - \lambda)E \\ \iff (A - \mu E)^{-1} - (A - \lambda E)^{-1} &= (\mu - \lambda)(A - \lambda E)^{-1}E(A - \mu E)^{-1} \\ \implies E(A - \mu E)^{-1} - E(A - \lambda E)^{-1} &= (\mu - \lambda)E(A - \lambda E)^{-1}E(A - \mu E)^{-1}. \end{aligned}$$

- The expression $\lambda \rightarrow R(\lambda) = E(\lambda E - A)^{-1}$ is a **pseudo-resolvent**¹, i.e. it satisfies

$$\frac{R(\lambda) - R(\mu)}{\mu - \lambda} = R(\lambda)R(\mu), \quad \text{for all } \lambda, \mu \in \rho(E, A), \lambda \neq \mu.$$

- Properties: R is holomorphic with $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ and $R(\mu)Z = R(\lambda)Z$ for all $\lambda, \mu \in \rho(E, A)$

- pseudo-resolvents are the natural resolvents for multi-valued linear operators

¹HILLE'48

Space decoupling

- Let X be a Banach space. Assume that the pseudo-resolvent $R : \Omega \rightarrow L(X)$ satisfies

$$\exists M > 0 \quad \exists (\lambda_n)_n, \lambda_n \nearrow \infty : \|\lambda_n R(\lambda_n)\| \leq M. \quad (2)$$

Then $\ker R(\lambda) \cap \overline{\operatorname{ran} R(\lambda)} = \{0\}$ for some $\lambda \in \Omega$.

- If in X every bounded sequence has a weakly convergent subsequence, then (2) implies²

$$X = \ker R(\lambda) \dot{+} \underbrace{\overline{R(\lambda)X}}_{=\mathcal{V}_1}.$$

This property holds if X is either reflexive or a Hilbert space.

- However we need to allow for a polynomial growth of the pseudo-resolvent to include higher index examples (as in TROSTORFF, WAURICK)

²KATO '59, FOIAS '57, YOSIDA '61, FAVINI, YAGI '99

Main result: Decoupling of infinite dimensional DAEs

Consider the DAE with $E : X \rightarrow Z$ bounded and $A : X \supset D(A) \rightarrow Z$ with Hilbert spaces X and Z . Assume that $\rho(E, A) \supseteq (\omega, \infty)$, let $R(\lambda) := E(A - \lambda E)^{-1}$ and assume

$$\|R(\lambda)x\| \leq \frac{M\|x\|}{\lambda - \omega}, \quad \forall x \in R(\mu)^{k-1}X. \quad (D_k)$$

Theorem 1 (G., Reis)

Assume that the DAE satisfies the above condition and let $\mu \in (\omega, \infty)$ then³

$$\text{graph } \mathfrak{J} = \{(R(\mu)z, z + \mu R(\mu)z), z \in \overline{R(\mu)^k Z}\}.$$

defines an closed densely defined operator on $\overline{R(\mu)^k Z}$ with $(\mathfrak{J} - \lambda)^{-1} = R(\lambda)|_{\overline{R(\mu)^k Z}}$.

Furthermore \mathfrak{J} generates a C_0 -semigroup, e.g. if $M = 1$ in $(D_k) \rightarrow$ solution theory

³extends previous results by BASKAKOV, CHERNISHOV and TROSTORFF, WAURICK

Discussion

- For A unbounded the condition (D_k) is stronger than the resolvent growth condition used in TROSTORFF, WAURICK but equivalent to it for bounded A
- There are examples where the tractability index of REIS, TISCHENDORF does not exist but (D_k) holds (heat-wave coupling).
- To solve inhomogeneous problems in Banach spaces one might use the space decomposition (as in finite dimensions)

$$X = \overline{R(\mu)^k X} \dot{+} \ker R(\mu)^k$$

which holds only under additional assumptions (e.g. (D_k) for the adjoint pseudo-resolvent R')

Sufficient conditions for (D_k)

Dissipative-Hamiltonian DAEs

- A typical assumption which appears in many applications (e.g. Navier-Stokes) is

$$E = E^* \geq 0, \quad E \text{ closed range}, \quad A \text{ maximally dissipative}$$

and these DAEs are closely related to generalized port-Hamiltonian systems VAN DER SCHAFT, MASCHKE '18, MEHL, MEHRMANN, WOJTYLAK '18.

- It also contains the infinite dimensional port-Hamiltonian equations from JACOB, ZWART '12

$$\frac{d}{dt}x(t) = \mathcal{J}Hx(t)$$

where $\mathcal{J} = -\mathcal{J}^*$ and H uniformly bounded from below.

- Show that the above DAEs fulfill (D_2) , i.e.

$$(D_2) \quad \|E(\lambda E - A)^{-1}x\| \leq \frac{\|x\|}{\lambda} \quad \forall \lambda > 0 \quad \forall x \in \text{ran } E(\lambda E - A)^{-1}$$

Dissipativity condition

Given $E : X \rightarrow Z$ and $A : X \supset D(A) \rightarrow Z$ densely defined and closed then we will use in the following the generalized dissipativity condition

$$\|(\lambda E - A)x\| \geq \lambda \|Ex\|, \quad \forall x \in D(A). \quad (3)$$

(equivalent condition obtained in FAVINI, YAGI '99)

Theorem 2 (G., Reis)

- (a) If DAE satisfies (3) and $\lambda E - A$ is surjective, $\ker E \cap \ker A = \{0\}$ then (D_1) holds.
- (b) If $E = E^* \geq 0$ with closed range and A maximal dissipative then (D_2) holds. If in addition $y \in \ker E \cap D(A) \wedge Ay \in \text{ran } E \Rightarrow y = 0$ then (D_1) holds.
- (c) If $Z = X \times Y$ and $E = \begin{bmatrix} E_0 \\ 0 \end{bmatrix}$, $E_0 \geq 0$ with closed range and

$$(Ax, \begin{bmatrix} x \\ 0 \end{bmatrix}) + (\begin{bmatrix} x \\ 0 \end{bmatrix}, Ax) \leq 0, \quad \forall x \in D(A)$$

and regular then (D_2) holds.

A lot of examples fit into our setting

- (i) PDE-ODE coupling on Hilbert spaces: Let A be maximally dissipative, B bounded, $E_0, A_0 \in \mathbb{C}^{n \times n}$ with $E_0 \geq 0$ and A_0 dissipative, $\ker A_0 \cap \ker E_0 = \{0\}$ then

$$\frac{d}{dt} \begin{bmatrix} I_H & 0 \\ 0 & E_0 \end{bmatrix} x(t) = \begin{bmatrix} A & B \\ -B^* & A_0 \end{bmatrix} x(t)$$

is regular, i.e. $\rho(E, A) \neq \emptyset$ and fulfill (D_2) due to (b).

- (ii) Generalized boundary control problems of the form

$$\frac{d}{dt} \begin{bmatrix} E \\ 0 \end{bmatrix} x(t) = \begin{bmatrix} A \\ \Gamma \end{bmatrix} x(t) + \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$$

with A dissipative, $\rho(E|_{\ker \Gamma}, A|_{\ker \Gamma}) \neq \emptyset$, Γ surjective, $E \geq 0$ with closed range fulfill (D_2) due to (c).

- (iii) Coupled PDEs such as heat-wave coupling⁴ satisfy (D_1) due to (b).

⁴SCHWENNINGER, ZWART '14, BATTY, PAUNONEN, SEIFERT '15

List of some references used during the talk

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