

PORT-HAMILTONIAN DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Introduction

- On this poster we compare port-Hamiltonian formulations of differential-algebraic equations in finite and infinite dimensional spaces. Furthermore, we discuss structural properties of the DAEs such as their index and stability.

- A differential-algebraic equation (DAE) is of the form

$$\frac{d}{dt}Ex(t) = Ax(t), \quad Ex(0) = Ex_0$$

where $E, A \in \mathbb{C}^{n \times n}$ and E is typically not invertible.

- Typically one is interested in the **spectrum**

$$\sigma(E, A) := \{\lambda \in \mathbb{C} \mid \lambda E - A \text{ not invertible}\}$$

and the **index** which is the smallest k for which there exists $M > 0$ such that

$$\|(\lambda E - A)^{-1}\| \leq M\lambda^{k-1}$$

for all $\lambda > 0$ sufficiently large.

The unique solvability of DAEs is guaranteed by the **regularity**, i.e. $\sigma(E, A) \neq \mathbb{C}$.

PH-DAEs in finite dimensions

Energy-based modeling of physical systems reveals an additional structure of the DAEs which is typically referred to as *port-Hamiltonian (pH)* or *dissipative Hamiltonian (dH)*

- MEHL, MEHRMANN, WOJTYLAK '18:

$$\exists Q \in \mathbb{C}^{n \times n} : A = DQ, \quad D + D^* \leq 0, \quad Q^*E = E^*Q$$

- MASCHKE, VAN DER SCHAFT '18:

$$\begin{pmatrix} x(t) \\ e(t) \end{pmatrix} \in \mathcal{L} = \text{im} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad \begin{pmatrix} e(t) \\ \dot{x}(t) \end{pmatrix} \in \mathcal{D} = \text{im} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where $t \geq 0$ and $D_1, D_2, L_1, L_2 \in \mathbb{C}^{n \times n}$ fulfill

$$D_2^*D_1 = -D_1^*D_2, \quad L_2^*L_1 = L_1^*L_2, \quad \dim \mathcal{D} = \dim \mathcal{L} = n.$$

- G, HALLER, REIS '21:

$$\text{im} \begin{bmatrix} E \\ A \end{bmatrix} = \mathcal{DL} = \{(x, z) \mid (x, y) \in \mathcal{L}, (y, z) \in \mathcal{D}\},$$

$$D_2^*D_1 + D_1^*D_2 \leq 0, \quad L_2^*L_1 = L_1^*L_2$$

Properties of pH-DAEs

- All of the above pH-formulations do **not imply** the **regularity, stability** or a **small index**!
- If $Q^*E \geq 0$ then it was shown in [1] that the index of the DAE is **at most two**, the size of Jordan blocks at 0 is **at most two** and the spectrum is contained in the closed left half-plane.

$$sE - DQ = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}$$

$$s\hat{E} - \hat{D}\hat{Q} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 1 \\ -1 & 0 \end{bmatrix}$$

- The approach [1] **can be embedded** as follows

$$\mathcal{D} = \text{graph } D, \quad \mathcal{L} = \text{im} \begin{bmatrix} E \\ Q \end{bmatrix}$$

- However the subspace \mathcal{L} is in [1] **might not fulfill** $\dim \mathcal{L} = n$ as required in [2]. If (E, DQ) is regular then (E, Q) is regular, hence $\dim \mathcal{L} = n$.

- We showed in [3] that if $\mathcal{D}, \mathcal{L} \subset \mathbb{C}^n \times \mathbb{C}^n$ are maximal, i.e. $\dim \mathcal{D} = \dim \mathcal{L} = n$ and $L_2^*L_1 \geq 0$ then the index is at most two, the size of the Jordan blocks at 0 is at most two

Infinite dimensional pH-DAEs

- Let X and Z be Hilbert spaces then we consider

$$\frac{d}{dt}Ex(t) = Ax(t), \quad Ex(0) = Ex_0$$

where $E : X \rightarrow Z$ is bounded and

$A : X \supset \text{dom } A \rightarrow Z$ is closed and densely defined.

- Regularity assumption:** there exists $\lambda \in \mathbb{C}$ for which $\lambda E - A$ has a bounded inverse

- Unclear:** What is the space of consistent initial values for which a unique solution exists and what is the right definition of index.

- Previous results mostly in the "index one" case by BARBU, FAVINI, YAGI, SHOWALTHER, THALLER, TROSTORFF, WAURICK, ... (see right column)

- Previous results on **infinite dimensional pH-systems:**

– JACOB, ZWART '12 considered in [4]

$$\frac{\partial x}{\partial t}(t, \xi) = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}(\xi)x(t, \xi)) + P_0(\mathcal{H}(\xi)x(t, \xi)),$$

where $P_1 = P_1^* \in \mathbb{C}^{n \times n}$ is invertible, $P_0 = -P_0^*$ and $\mathcal{H} \in L^\infty((a, b), \mathbb{C}^{n \times n})$ satisfies $mI_n \leq \mathcal{H}(\xi) \leq MI_n$ for some constants $0 < m < M$

– FAULWASSER, MASCHKE, PHILIPP, SCHALLER '20 considered $\dot{x} = (J - R)x$ where $J = -J^*$ and $-R$ dissipative possibly unbounded and relatively bounded perturbations

- Assumption:** $(Ex, x)_X \geq 0$ for all $x \in X$, E closed range and A is ω -dissipative, i.e. for some $\omega > 0$

$$(Ax, x)_X + (x, Ax)_X \leq -\omega \|x\|^2, \quad x \in \text{dom } A$$

- Then using the inverse of A and using

$X = \ker E \oplus \text{im } E$ and $x = x_K \oplus x_R$ we find

$$\begin{aligned} \frac{d}{dt}A^{-1}Ex(t) &= x(t) \\ \iff x_K &= 0, \quad \frac{d}{dt}P_{\text{im } E}A^{-1}Ex_R(t) = x_R(t) \end{aligned}$$

The resulting DAE on $\text{im } E$ fulfills the resolvent growth assumption (D_1) from the right column.

- Hence there exists an generator for the underlying exponentially stable semigroup on $P_{\text{im } E}A^{-1}E$.

- Next step:** Extension to boundary control systems!

Solutions of DAEs

- Let X, Z be Hilbert spaces, $E : X \rightarrow Z$ bounded and $A : X \supset \text{dom } A \rightarrow Z$ for simplicity bijective and $T > 0$.

- A **classical solution** $x : [0, T] \rightarrow X$ fulfills $x(t) \in \text{dom } A$ and $t \mapsto Ex(t)$ is continuously differentiable.

- An **X -mild solution** fulfills $\int_0^t x(\tau) d\tau \in \text{dom } A$ for all $t \in [0, T]$ and

$$Ex(t) - Ex_0 = A \int_0^t x(\tau) d\tau.$$

- If A is invertible then a **Z -mild solution**

$z : [0, T] \rightarrow Z$ fulfills for all $t \in [0, T]$

$$EA^{-1}z(t) - EA^{-1}z_0 = \int_0^t z(\tau) d\tau.$$

Connection between these solutions:

- If x is X -mild then $z = Ex$ is a Z -mild solution.
- If z is Z -mild then $x = A^{-1}z$ is X -mild.

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Index of DAEs in Hilbert spaces

- Let X be a Hilbert space. If $\dim X < \infty$ there are several equivalent index notions: The index is the smallest k such that

$$(a) \|(\lambda E - A)^{-1}\| \leq M\lambda^{k-1};$$

$$(b) \text{termination of Wong sequences } \mathcal{V}_k = \mathcal{V}_{k+1}$$

$$\mathcal{V}_n := A^{-1}(Ex), \quad \mathcal{V}_0 := X \quad (\Rightarrow \mathcal{V}_{n+1} \subseteq \mathcal{V}_n).$$

- This is **no longer true** in infinite dimensions: Termination is not clear and also closedness of subspaces \mathcal{V}_k is an issue

- TROSTORFF, WAURICK '18: $E, A : X \rightarrow X$ **bounded**, resolvent growth and $E(\mathcal{V}_k)$ closed $\implies \mathcal{V}_k = \mathcal{V}_{k+1}$

- REIS, TISCHENDORF '05: $E : X \rightarrow Z$ bounded with closed range, A **unbounded**, with finite tractability index $k \implies \mathcal{V}_k = \mathcal{V}_{k+1}$

- Pseudo-resolvent (HILLE '49): Let $\Omega \subset \mathbb{C}$ be open then $R : \Omega \rightarrow L(X)$ is called **pseudo-resolvent** iff

$$(\lambda - \mu)R(\mu)R(\lambda) = R(\mu) - R(\lambda), \quad \lambda, \mu \in \Omega.$$

- KATO '59: R pseudo-resolvent, $(\lambda_n)_n$ with $\lambda_n \nearrow \infty$, and $\limsup_{n \rightarrow \infty} \|\lambda_n R(\lambda_n)\| < \infty \implies X = \overline{R(\lambda)X} \dot{+} \ker R(\lambda)$ for all $\lambda \in \Omega$ and independent of the choice of λ

- G, REIS '21: Assume that $\rho(E, A) \supseteq [0, \infty)$, let $R(\lambda) := E(A - \lambda E)^{-1}$ and assume

$$\|R(\lambda)z\| \leq \frac{M\|z\|}{\lambda}, \quad \forall z \in R(0)^{k-1}Z. \quad (D_k)$$

- Main result:**

$$\text{graph } \mathfrak{J} = \{(R(0)z, z), z \in \overline{R(0)^k Z}\}.$$

defines an closed densely defined operator on $\overline{R(0)^k Z}$ with $(\mathfrak{J} - \lambda)^{-1} = R(\lambda)|_{\overline{R(0)^k Z}}$.

PDAE models of power grids

- Power grids** consist of generators supplying power, loads consuming this power and transmission lines which interconnect these.
- This is modeled as a DAE consisting of:
 - **non-linear** ODEs describing the generators (e.g. swing equation);
 - **linear equations** for loads and coupling;
 - lumped parameter ODEs for transmission lines.
- Within the DFG Priority Program 1984 we model the transmission lines more accurately based on the telegraph equation

$$\begin{aligned} C(\xi) \frac{\partial v}{\partial t}(t, \xi) &= -\frac{\partial i}{\partial \xi}(t, \xi) - G(\xi)v(t, \xi), \\ L(\xi) \frac{\partial i}{\partial t}(t, \xi) &= -\frac{\partial v}{\partial \xi}(t, \xi) - R(\xi)i(t, \xi), \\ v(t, 0) &= v_0(t), \quad v(t, 1) = v_1(t), \\ v(0, \xi) &= v^0(\xi), \quad i(0, \xi) = i^0(\xi). \end{aligned}$$

where $\xi \in [a, b]$ is the spatial variable, v is the voltage, i is the current through the transmission line and spatially distributed $C, L, G, R > 0$.

- The above equation can be modeled as a pH boundary control system in the sense of JACOB, ZWART '12!

- Next step:** Extension to nonlinear DAEs and their interconnection.

References

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