Differential-algebraic equations in Hilbert spaces

IWOTA Chapman University 09.08.-13.08.21

Hannes Gernandt – joint work with Timo Reis (TU Ilmenau)

August 14th, 2021

Motivation

Navier-Stokes equations

- Consider the following example of incompressible Navier-Stokes equations at low Reynolds numbers:

$$\begin{aligned} \rho(\xi)\tfrac{\partial v}{\partial t}(t,\xi) = -\nabla p(t,\xi) + \mu\Delta v(t,\xi) + \hat{f}(t,\xi), \\ \operatorname{div} v(t,\xi) = 0, \end{aligned}$$

- It consists of a coupled differential equation (1st line) and an algebraic equation (2nd line). This can be formalized as a differential-algebraic equation (DAE)

$$\tfrac{d}{dt}Ex(t) = \tfrac{d}{dt}\begin{bmatrix} \rho(\cdot) & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} v_\xi(t) \\ p_\xi(t) \end{pmatrix} = \begin{bmatrix} \mu\Delta & -\nabla \\ \operatorname{div} & 0 \end{bmatrix}\begin{pmatrix} v_\xi(t) \\ p_\xi(t) \end{pmatrix} + \begin{pmatrix} \hat{f}_\xi(t) \\ 0 \end{pmatrix} = Ax(t) + f(t)$$

on suitable Hilbert spaces encoding possible boundary conditions

DAE vs. ODE Toy examples

Consider a simple example in $\mathbb{R}^2$

$$\begin{aligned}\dot{x}_2(t) &= x_1(t) + f_1(t)\\ 0 &= x_2(t)\end{aligned}$$

Hence $x_2 = 0$ which implies $x_1 = -f_1$. Consequently, not for all initial values a solution exists. Furthermore, x_1 has the same regularity as f_1 and might be non-smooth.

$$\begin{aligned}\dot{x}_2(t) &= x_1(t)\\ 0 &= x_2(t) + f_2(t)\end{aligned}$$

Then $x_2 = -f_2$ and thus, f_2 has to be weakly differentiable. Hence not for all initial values and for all right hand sides a solution exists.

Reminder: How to solve finite dimensional DAEs?

Given $E, A \in \mathbb{C}^{n\times n}$ and $f \in L^1([0,t_f])$, $t_f > 0$ the equation

$$\frac{d}{dt}Ex(t) = Ax(t) + f(t), \quad (Ex)(0) = Ex_0.$$

If $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$, the DAE is called regular and there exists invertible $W, T \in \mathbb{C}^{n\times n}$ such that

$$\frac{d}{dt}\begin{pmatrix} x_{\mathfrak{J}} \\ Nx_N \end{pmatrix} = \frac{d}{dt}WET = WAT + Wf = \begin{pmatrix} \mathfrak{J} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} x_{\mathfrak{J}} \\ x_N \end{pmatrix} + \begin{pmatrix} f_{\mathfrak{J}} \\ f_N \end{pmatrix}$$

where N is nilpotent and $\mathfrak{J}$ in Jordan normal form. This is a decoupling into an ODE-part and an algebraic part. The above form is called Weierstraß-Kronecker form.

Gantmacher or Kunkel, Mehrmann

How to solve a finite dimensional DAE?

- The solution of

$$\frac{d}{dt}\begin{pmatrix} x_{\mathfrak{J}} \\ N x_N \end{pmatrix} = \frac{d}{dt} WET = WAT + Wf = \begin{pmatrix} \mathfrak{J} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} x_{\mathfrak{J}} \\ x_N \end{pmatrix} + \begin{pmatrix} f_{\mathfrak{J}} \\ f_N \end{pmatrix}$$

is then given in Berger, Ilchmann, Trenn '12 by

$$x_{\mathfrak{J}}(t) = e^{\mathfrak{J}t}x_0 + \int_0^t e^{\mathfrak{J}(t-\tau)} f_{\mathfrak{J}}(t-\tau)d\tau, \quad x_N(t) = \sum_{i=0}^{k-1} N^i f_N^{(i)}(t).$$

- The smallest number $k \in \mathbb{N}$ such that $N^k = 0$ is the index of the DAE.
- Difficulty: Solution exists only for a sufficiently smooth f and not for every initial value x_0.

Extension to Hilbert spaces

How can we decouple a DAE in finite dimensions?

- Let $E, A \in \mathbb{C}^{n\times n}$. Consider the Wong sequence

$$\mathcal{V}_0 := \mathbb{C}^n, \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), i \geq 0.$$

  where A^{-1} is the set theoretic inverse.

- Idea: If $x(t) \in \mathcal{V}_i$ then $\dot{x}(t) \in \mathcal{V}_i$ ($\mathcal{V}_i$ is closed) and $Ax(t) \in E\mathcal{V}_i \Rightarrow x(t) \in \mathcal{V}_{i+1}$
- One can show $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$, $i \in \mathbb{N}$, and since $\mathbb{C}^n$ is finite dimensional the sequence $(\mathcal{V}_i)_i$ terminates.
- Does not work without further assumptions in Hilbert spaces.

How can we decouple a DAE in finite dimensions?

- A second way to obtain such a decoupling for $\frac{d}{dt}Ex(t) = Ax(t)$, $E, A \in \mathbb{C}^{n\times n}$ via resolvents:

$$\begin{aligned} \tfrac{d}{dt}Ex(t) &= Ax(t) = (A-\lambda E)x(t) + \lambda Ex(t) \\ \iff \tfrac{d}{dt}(A-\lambda E)^{-1}Ex(t) &= x(t) + \lambda(A-\lambda E)^{-1}Ex(t) \end{aligned}$$

- There exists $T \in \mathbb{C}^{n\times n}$ such that $T^{-1}(A-\lambda E)^{-1}ET$ is in Jordan canonical form. The nilpotent Jordan blocks are precisely N, the algebraic part of the Weiertraß form.

- Problem: no canonical form for operators in Hilbert spaces.

How can we decouple a DAE in infinite dimensions?

Assume that $E, A : X \to X$ are linear bounded on a Hilbert space X.

- Wong-sequence approach

$$\mathcal{V}_{i+1} = A^{-1}(E\mathcal{V}_i),\ i \geq 0, \quad \mathcal{V}_0 = X,$$

- Problem: $\{\mathcal{V}_i\}_{i\geq 0}$ must not terminate and $\mathcal{V}_i$ are not closed

- Example 1: $X = \ell_2(\mathbb{N})$, $A = Id$, and let E be the right shift

$$(x_0, x_1, \ldots) \mapsto (0, x_0, x_1, \ldots).$$

- Example 2: $A : X \supset D \to X$ closed densely defined with bounded inverse A^{-1} and $E = I_X$ then $\mathcal{V}_i = A^{-i}X$ is not closed.

How can we decouple a DAE in infinite dimensions?

- Trostorff, Waurick '18 considered Hilbert spaces X and used an additional resolvent growth condition:

$$\exists\, \omega > 0: \quad \|(\lambda E - A)^{-1}\| \leq M|\lambda|^{k-1}, \quad \text{for all } \lambda \in \mathbb{C}^+_\omega := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega\}, \quad (1)$$

- In the finite dimensional case and if $E = N$ for some nilpotent N and $A = I_n$ we obtain

$$(sN - I_n)^{-1} = -\sum_{i=0}^{\infty}(-sN)^i = -\sum_{i=0}^{k-1}(-sN)^i$$

  and hence the smallest k such that (1) holds is the index of the DAE.

- If additionally $E\mathcal{V}_k$ is closed Trostorff, Waurick '18 showed that $\mathcal{V}_k = \mathcal{V}_{k+1}$ and

$$W(sE - A)T = \begin{bmatrix} sI - \mathfrak{J} & 0 \\ 0 & sN - I \end{bmatrix}, \quad \mathfrak{J} \text{ bounded, } N \text{ nilpotent}$$

What is known for unbounded A?

- Tractability approach from Reis, Tischendorf '05, Reis '06 for Hilbert spaces X. Construction of a sequence of projectors with quite restrictive closedness assumptions. (very technical)
- A DAE with tractability index k can be transformed with W injective, dense range and T invertible to

$$WET = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}, \quad WAT = \begin{bmatrix} I & K \\ 0 & \mathfrak{J} \end{bmatrix}$$

here N is nilpotent with index k and $K \neq 0$.
- A solution can be obtained on the subspace via $\mathfrak{J}$, but the finite tractability index does not imply that $\mathfrak{J}$ generates a C_0-semigroup.
- We have no explicit expression in terms of E and A for the initial values for which a solution exists.

New decoupling approach via pseudo-resolvents

- Assume that $\lambda, \mu \in \rho(E, A) := \{\lambda \in \mathbb{C} \mid \lambda E - A \text{ bijective}\}$ then

$$\begin{aligned} & A - \lambda E - (A - \mu E) = (\mu - \lambda)E \\ \iff \quad & (A - \mu E)^{-1} - (A - \lambda E)^{-1} = (\mu - \lambda)(A - \lambda E)^{-1} E (A - \mu E)^{-1} \\ \implies & E(A - \mu E)^{-1} - E(A - \lambda E)^{-1} = (\mu - \lambda) E (A - \lambda E)^{-1} E (A - \mu E)^{-1}. \end{aligned}$$

- The expression $\lambda \to R(\lambda) = E(\lambda E - A)^{-1}$ is a pseudo-resolvent[1], i.e. it satisfies

$$\frac{R(\lambda) - R(\mu)}{\mu - \lambda} = R(\lambda)R(\mu), \quad \text{for all } \lambda, \mu \in \rho(E, A),\ \lambda \neq \mu.$$

- Properties: R is holomorphic with $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ and $R(\mu)Z = R(\lambda)Z$ for all $\lambda, \mu \in \rho(E, A)$

- pseudo-resolvents are the natural resolvents for multi-valued linear operators

[1] HILLE'48

Space decoupling

- Let X be a Banach space. Assume that the pseudo-resolvent $R : \Omega \to L(X)$ satisfies

$$\exists M > 0 \ \exists (\lambda_n)_n, \lambda_n \nearrow \infty : \|\lambda_n R(\lambda_n)\| \leq M. \tag{2}$$

Then $\ker R(\lambda) \cap \overline{\operatorname{ran} R(\lambda)} = \{0\}$ for some $\lambda \in \Omega$.

- If in X every bounded sequence has a weakly convergent subsequence, then (2) implies[2]

$$X = \ker R(\lambda) \dot{+} \underbrace{\overline{R(\lambda)X}}_{=\mathcal{V}_1}.$$

This property holds if X is either reflexive or a Hilbert space.

- However we need to allow for a polynomial growth of the pseudo-resolvent to include higher index examples (as in Trostorff, Waurick)

[2] Kato '59, Foias '57, Yosida '61, Favini, Yagi '99

Main result: Decoupling of infinite dimensional DAEs

Consider the DAE with $E : X \to Z$ bounded and $A : X \supset D(A) \to Z$ with Hilbert spaces X and Z. Assume that $\rho(E, A) \supseteq (\omega, \infty)$, let $R(\lambda) := E(A - \lambda E)^{-1}$ and assume

$$\|R(\lambda)x\| \leq \frac{M\|x\|}{\lambda-\omega}, \quad \forall \ x \in R(\mu)^{k-1}X. \quad \mathbf{(}D_k\mathbf{)}$$

Theorem 1 (G., Reis)

Assume that the DAE satisfies the above condition and let $\mu \in (\omega, \infty)$ then[3]

$$\operatorname{graph} \mathfrak{J} = \{(R(\mu)z, z + \mu R(\mu)z), z \in \overline{R(\mu)^k Z}\}.$$

defines an closed densely defined operator on $\overline{R(\mu)^k Z}$ with $(\mathfrak{J} - \lambda)^{-1} = R(\lambda)|_{\overline{R(\mu)^k Z}}$.

Furthermore $\mathfrak{J}$ generates a C_0-semigroup, e.g. if $M = 1$ in (D_k) $\to$ solution theory

[3] extends previous results by Baskakov, Chernishov and Trostorff, Waurick

Discussion

- For A unbounded the condition (D_k) is stronger then the resolvent growth condition used in Trostorff, Waurick but equivalent to it for bounded A
- There are examples where the tractability index of Reis, Tischendorf does not exist but (D_k) holds (heat-wave coupling).
- To solve inhomogeneous problems in Banach spaces one might use the space decomposition (as in finite dimensions)

$$X = \overline{R(\mu)^k X} \dotplus \ker R(\mu)^k$$

which holds only under additional assumptions (e.g. (D_k) for the adjoint pseudo-resolvent R')

Sufficient conditions for (D_k)

Dissipative-Hamiltonian DAEs

- A typical assumption which appears in many applications (e.g. Navier-Stokes) is

$$E = E^* \geq 0, \quad E \text{ closed range}, \quad A \text{ maximally dissipative}$$

  and these DAEs are closely related to generalized port-Hamiltonian systems van der Schaft, Maschke '18, Mehl, Mehrmann, Wojtylak '18.
- It also contains the infinite dimensional port-Hamitlonian equations from Jacob, Zwart '12

$$\tfrac{d}{dt}x(t) = \mathcal{J}Hx(t)$$

  where $\mathcal{J} = -\mathcal{J}^*$ and H uniformly bounded from below.
- Show that the above DAEs fulfill (D_2), i.e.

$$(D_2) \quad \|E(\lambda E - A)^{-1}x\| \leq \frac{\|x\|}{\lambda} \quad \forall \lambda > 0 \;\; \forall x \in \operatorname{ran} E(\lambda E - A)^{-1}$$

Dissipativity condition

Given $E : X \to Z$ and $A : X \supset D(A) \to Z$ densely defined and closed then we will use in the following the generalized dissipativity condition

$$\|(\lambda E - A)x\| \geq \lambda \|Ex\|, \quad \forall x \in D(A). \tag{3}$$

(equivalent condition obtained in Favini, Yagi '99)

Theorem 2 (G., Reis)

(a) *If DAE satisfies (3) and $\lambda E - A$ is surjective, $\ker E \cap \ker A = \{0\}$ then (D_1) holds.*

(b) *If $E = E^* \geq 0$ with closed range and A maximal dissipative then (D_2) holds. If in addition $y \in \ker E \cap D(A) \wedge Ay \in \operatorname{ran} E \Rightarrow y = 0$ then (D_1) holds.*

(c) *If $Z = X \times Y$ and $E = \left[\begin{smallmatrix} E_0 \\ 0 \end{smallmatrix}\right]$, $E_0 \geq 0$ with closed range and*

$$(Ax, \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]) + (\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], Ax) \leq 0, \quad \forall\, x \in D(A)$$

and regular then (D_2) holds.

A lot of examples fit into our setting

(i) PDE-ODE coupling on Hilbert spaces: Let A be maximally dissipative, B bounded, $E_0, A_0 \in \mathbb{C}^{n\times n}$ with $E_0 \geq 0$ and A_0 dissipative, $\ker A_0 \cap \ker E_0 = \{0\}$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_H & 0 \\ 0 & E_0 \end{bmatrix} x(t) = \begin{bmatrix} A & B \\ -B^* & A_0 \end{bmatrix} x(t)$$

is regular, i.e. $\rho(E, A) \neq \emptyset$ and fulfill (D_2) due to (b).

(ii) Generalized boundary control problems of the form

$$\frac{d}{dt}\begin{bmatrix} E \\ 0 \end{bmatrix} x(t) = \begin{bmatrix} A \\ \Gamma \end{bmatrix} x(t) + \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$$

with A dissipative, $\rho(E|_{\ker \Gamma}, A|_{\ker \Gamma}) \neq \emptyset$, Γ surjective, $E \geq 0$ with closed range fulfill (D_2) due to (c).

(iii) Coupled PDEs such as heat-wave coupling[4] satisfy (D_1) due to (b).

[4] Schwenninger, Zwart '14, Batty, Paunonen, Seifert '15

List of some references used during the talk

- A. Baskakov, K. Chernishov, Spectral analysis of linear relations and degenerate operator semi-groups, Sb. Math. , 193, 1573–1610, 2002
- T. Berger, A. Ilchmann, S. Trenn, The quasi-Weierstraß form for regular matrix pencils, Linear Algebra Appl., 436, 4052–4069, 2012
- A. Favini, A. Yagi, Degenerate differential equations on Banach spaces, Marcel Dekker, New York, 1999
- T. Reis, Systems Theoretic Aspects of PDAEs and Applications to Electrical Circuits, PhD Thesis, Shaker, Aachen, 2006
- T. Reis, C. Tischendorf, Frequency Domain Methods and Decoupling of Linear Infinite Dimensional Differential Algebraic Systems, Journal Evolution Equations, 5, 357–385, 2005
- F.L. Schwenninger, H. Zwart, Generators with a closure relation, Operators and Matrices, 8, 157–165, 2017
- S. Trostorff, M. Waurick, On higher index differential-algebraic equations in infinite dimensions, Operator Theory Advances and Applications, 268, 477–486, 2018