Stability of port-Hamiltonian descriptor systems

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Outline

- In this talk I consider generalized port-Hamiltonian **differential-algebraic equations** (DAEs) which can be described by

$$\tfrac{d}{dt}Ex(t) = Ax(t), \quad Ex(0) = Ex_0.$$

- These equations also incorporate algebraic constraints which makes E in general not invertible.

- Unique solvability is guaranteed if (E, A) is regular, i.e. $\lambda E - A$ is invertible for some $\lambda \in \mathbb{C}$. (but not for all initial values)

- Study **regularity** and **stability** of DAEs under more structural assumptions, namely the port-Hamiltonian or dissipative Hamiltonian structure.

Dissipative Hamiltonian DAEs

dissipative Hamiltonian DAEs

- In the following we consider $\frac{d}{dt} Ex(t) = DQx(t)$ where

$$D + D^* \leq 0, \quad Q^*E = E^*Q$$

are called **port-Hamiltonian** DAEs and if $E^*Q \geq 0$ the DAEs are called **dissipative Hamiltonian (dH)**.

- If $E^*Q \geq 0$ one might ask for the stability of solutions and using $V(x) = x^*E^*Qx$ as a **Lyapunov candidate**. This leads to

$$\frac{d}{dt}x(t)^*E^*Qx(t) = x(t)^*Q^*D^*Qx(t) + x(t)^*Q^*DQx(t) \leq 0$$

However if Q has a kernel we cannot conclude stability.

- A DAE is called **stable** if for all solutions there exists $M > 0$ such that

$$\sup_{t \geq 0} \|x(t)\| \leq M.$$

dH-DAEs are unstable

- Dissipative Hamiltonian DAEs are **not stable** as the following example from MEHL, MEHRMANN, WOJTYLAK '18 shows:

$$Ex = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = DQx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

This is an ODE with Jordan block of size two at 0 and hence unstable.

- One can show that the size of the Jordan blocks at 0 is **at most two**, see e.g. MEHL, MEHRMANN, WOJTYLAK '20 or G, HALLER, REIS '21 or FAULWASSER, MASCHKE, PHILIPP, SCHALLER, WORTHMANN '21

Sufficient condition for stability

- If $\ker Q \subseteq \ker E$ and (E, DQ) is regular, i.e. $\lambda E - DQ$ is invertible for some $\lambda \in \mathbb{C}$.

$$\Rightarrow \{0\} = \ker E \cap \ker DQ \supseteq \ker E \cap \ker Q \supseteq \ker Q$$

$$\Rightarrow Q \text{ injective} \Rightarrow \mathbf{Q \text{ invertible}}.$$

- Hence the DAE (E, DQ) is equivalent to (Q^*E, Q^*DQ) which is called **positive real** (BERGER, REIS '13) or **semi-dissipative Hamiltonian** (MEHL, MEHRMANN, WOJTYLAK '18,'20, ACHLEITNER, ARNOLD, MEHRMANN '21) and these are known to be **stable**.

- Condition **not necessary**, consider $Q = 0$ and $E = I_n$ and in particular even for ODEs if Q has kernel.

- Also sufficient for stability is that $P_{\mathrm{im}\, Q} D|_{\mathrm{im}\, Q}$ is (strictly) dissipative.

- What about the converse **Stability $\Rightarrow$ dH-DAE!?**

Stability of DAEs

How to solve DAEs

- If $E, A \in \mathbb{C}^{n \times n}$ then there exists invertible $S, T \in \mathbb{C}^{n \times n}$ such that

$$SET = \begin{bmatrix} I_{n_0} & & & \\ & N_\alpha & & \\ & & K_\beta & \\ & & & K_\gamma^T \end{bmatrix}, \quad SAT = \begin{bmatrix} A_0 & & & \\ & I_\alpha & & \\ & & L_\beta & \\ & & & L_\gamma^T \end{bmatrix}$$

with block diagonal

$$N_\alpha \text{ with blocks } \begin{bmatrix} 0 & 1 & \\ & \ddots & \ddots \\ & & 1 \\ & & 0 \end{bmatrix} \quad sK_\beta - L_\beta \text{ with blocks } \begin{bmatrix} s & 1 & \\ & \ddots & \ddots \\ & & s & 1 \end{bmatrix}$$

which is called the **Kronecker form** or quasi-Kronecker form (BERGER, TRENN '12)

- Hence: To **solve** the DAE one has to consider each block **separately**

- Roughly speaking A_0 describes the ODE-part and the α-blocks the algebraic constrains. Moreover, β is under-determined, γ is over-determined.

Let's have a closer look

- the ODE part $\dot{x} = A_0 x, x(0) = x_0$ can be solved via $x(t) = e^{A_0 t} x_0$. Hence for stability of the DAE this ODE this must be stable.

- the algebraic part consists of blocks which is basically of the form

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$$

Hence the only solution is $x = 0$.

- the under-determined blocks can be solved by integrating

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1(t) = x_2(0) + \int_0^t x_2(\tau) d\tau.$$

- Similarly one can show that the overdetermined block has only the trivial solution.

Consequence: Stability of DAEs

- Hence we can characterize the stability based on the Kronecker form:
- A DAE with $E, A \in \mathbb{C}^{n \times n}$ is **stable** $\iff$ regular, all eigenvalues of A_0 are in the closed left half plane and semi-simple on the imaginary axis.
- For a Lyapunov-like characterization recall the **system space** (REIS, RENDEL, VOIGT '16)

$$\mathcal{V}_{\mathrm{sys}} := \{x(0) \mid x \text{ smooth and } \tfrac{d}{dt} Ex(t) = Ax(t)\}.$$

Proposition (G, Haller '21)

The DAE given by $E, A \in \mathbb{C}^{n \times n}$ is stable if and only if there exists $X >_{E\mathcal{V}_{\mathrm{sys}}} 0$ with $X(E\mathcal{V}_{\mathrm{sys}}) = E\mathcal{V}_{\mathrm{sys}}$ solving the generalized Lyapunov inequality

$$E^* XA + A^* XE \leq_{\mathcal{V}_{\mathrm{sys}}} 0.$$

- Related results on Lyapunov inequalities for DAEs by LEWIS '86, STYKEL '02...

Stable DAEs are also dissipative Hamiltonian

- If (E, A) is stable and $X >_{E\mathcal{V}_{\text{sys}}} 0$ with $X(E\mathcal{V}_{\text{sys}}) = E\mathcal{V}_{\text{sys}}$ is a solution of

$$E^* X A + A^* X E \leq_{\mathcal{V}_{\text{sys}}} 0.$$

- Set $Q := XE$ then for the pseudo-inverse $Q^\dagger$ define $D := AQ^\dagger$ and we have

$$A =_{\mathcal{V}_{\text{sys}}} DQ, \quad Q^T E >_{\mathcal{V}_{\text{sys}}} 0.$$

- If the DAE is stable with **index one**, i.e. $N_\alpha = 0$ in the Kronecker form we have $Q^T E \geq 0$ and $A = DQ$ **(on the whole space)**. In this case $\mathcal{V}_{\text{sys}} = A^{-1}(E\mathbb{C}^n)$.

Further remarks

- If we have a stabilizable system of the form

$$\frac{d}{dt}Ex(t) = Ax(t) + Bu(t)$$

  in the sense that for all initial values $x_0 \in \mathcal{V}_{\mathrm{sys}}$ there exists u and $M > 0$ such that

$$\sup_{t \geq 0} \|x(t)\| \leq M.$$

- Furthermore, (E, A) is assumed to be regular with only semi-simple eigenvalues on the imaginary axis then there exists solutions to the **algebraic Bernoulli equation**

$$A^* X_1 E + E^* X_1 A =_{\mathcal{V}_{\mathrm{sys}}} E^T X_1 B B^T X_1 E.$$

  This solution can be used to rewrite the stabilizable system as a port-Hamiltonian one on the system space.

Geometric pH-DAE framework

The geometric pH-DAE framework

- Let $\mathcal{D}$ be a subspace of $\mathbb{C}^n \times \mathbb{C}^n$ and $\mathcal{D} = \mathrm{im} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$
- these subspaces have certain (skew)-symmetry properties
- Then a DAE $\frac{d}{dt}Ex(t) = Ax(t)$ is called pH in MASCHKE, VAN DER SCHAFT '18 if

$$(\dot{x}(t), e(t)) \in \mathcal{D}, \quad (x(t), e(t)) \in \mathcal{L} \text{ for some function } e$$

- can be reinterpreted as

$$\mathrm{im} \begin{bmatrix} E \\ A \end{bmatrix} = \mathcal{DL} := \{(x, z) \mid (x, y) \in \mathcal{L}, \ (y, z) \in \mathcal{D}\}$$

for some $\mathcal{D} = \mathrm{im} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$, $D_1, D_2 \in \mathbb{C}^{n \times n}$ and $\mathcal{L} = \mathrm{im} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, $L_1, L_2 \in \mathbb{C}^{n \times n}$ which are assumed to fulfill

$$D_2^* D_1 + D_1^* D_2 \leq 0 \text{ (dissipative)}, \quad L_2^* L_1 = L_1^* L_2 \text{ (symmetric)}$$

Connection to dH-DAEs

- It **contains** the **previous setting** by choosing $\mathcal{D} = \text{graph } D$ and $\mathcal{L} = \text{im } \begin{bmatrix} E \\ Q \end{bmatrix}$ with $Q^*E \geq 0$ or $Q^*E = E^*Q$.

- However $\dim \mathcal{L} = \text{im } \begin{bmatrix} E \\ Q \end{bmatrix} < n$ is possible, but only if the DAE (E, A) is singular.

- Let $E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and consider $\mathcal{D} = \text{im } \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{L} = \text{im } \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

- Then $\mathcal{D}$ is dissipative, $\mathcal{L}$ is nonnegative, and $\text{im } \begin{bmatrix} E \\ A \end{bmatrix} = \mathcal{D}\mathcal{L}$.

- One can show that it is **not possible** to rewrite $\mathcal{D}\mathcal{L} = (\text{graph } D)\hat{\mathcal{L}}$ for some dissipative matrix $D \in \mathbb{C}^{2\times 2}$ and a nonnegative relation $\hat{\mathcal{L}} \subset \mathbb{C}^4$.

Regularity, spectrum and stability in the geometric setting

- What is known about the Kronecker form when the DAE is given by a product $\mathcal{D}\mathcal{L}$ of a dissipative and a nonnegative linear relation?

Proposition (G, Haller, Reis '21)

Let $\mathcal{D} = \operatorname{im} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ be dissipative and $\mathcal{L} = \operatorname{im} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ nonnegative Lagrangian with $L_1, L_1^T L_2 \geq 0$ and $D_2 + D_2^T \leq 0$. Further, let $\operatorname{im} \begin{bmatrix} E \\ A \end{bmatrix} := \mathcal{D}\mathcal{L}$. Then with $\mathcal{X} := \operatorname{im} D_1 \cap \operatorname{im} L_2$ the following holds.

(a) (E, A) is regular if and only if $\ker P_{\mathcal{X}} L_1|_{\mathcal{X}} \cap \ker P_{\mathcal{X}} D_2|_{\mathcal{X}} = \{0\}$ and $D_2(\ker D_1) \cap L_1(\ker L_2) = \{0\}$.

(b) If (E, A) is regular the eigenvalues are in the closed left half plane and the index is at most three.

(c) (E, A) is stable if additionally $L_1(\ker L_2) = \{0\}$ holds.

Final remarks

- For the previous setting $A = DQ$ and $Q^*E = E^*Q \geq 0$ we have $D_1 = I$, $D_2 = D$, $L_1 = E$, $L_2 = Q$ this implies $\mathcal{X} = \operatorname{im} D_1 \cap \operatorname{im} L_2 = \operatorname{im} Q$

- Hence $[E, A]$ is **regular** if and only if

$$\ker P_{\mathcal{X}} L_1|_{\mathcal{X}} \cap \ker P_{\mathcal{X}} D_2|_{\mathcal{X}} = \{0\}, \quad D_2(\ker D_1) \cap L_1(\ker L_2) = \{0\}$$

$$\Leftrightarrow \ker P_{\operatorname{im} Q} E|_{\operatorname{im} Q} \cap \ker P_{\operatorname{im} Q} D|_{\operatorname{im} Q} = \{0\}$$

$$\Leftrightarrow \ker E \cap \ker Q^*RQ \cap \ker Q^*JQ = \{0\},$$

$$D = J - R, \ J = -J^*, \ R = R^* \geq 0$$

which was also obtained recently by FAULWASSER, MASCHKE, PHILIPP, SCHALLER, WORTHMANN '21

- The stability condition $L_1(\ker L_2) = \{0\}$ is equivalent to $\ker Q \subseteq \ker E$.

Outlook

- The infinite dimensional pH-ODE setting of Jacob Zwart '12 we have

$$\frac{\partial}{\partial t}x(\xi, t) = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}(\xi)x(\xi, t)) + P_0(\mathcal{H}(\xi)x(\xi, t)) := A(\mathcal{H}x)$$

where $P_1 = P_1^* \in \mathbb{C}^{n \times n}$ is invertible, $P_0 = -P_0^*$ and $\mathcal{H} \in L^\infty((a, b), \mathbb{C}^{n \times n})$ satisfies $mI_n \leq \mathcal{H}(\xi) \leq MI_n$ for some constants $0 < m < M$ and A maximally dissipative

- this can be rewritten in our setting using the subspaces

$$\mathcal{D} = \operatorname{graph} A, \quad \mathcal{L} = \operatorname{graph} \mathcal{H}^{-1} \quad \implies \quad \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \in \operatorname{im} \begin{bmatrix} \mathcal{H}^{-1} \\ A \end{bmatrix} = \mathcal{D}\mathcal{L}$$

Outlook

- A natural generalization would be

$$\frac{\partial}{\partial t} E(\xi) x(\xi, t) = P_1 \frac{\partial}{\partial \xi} (\mathcal{H}(\xi) x(\xi, t)) + P_0 (\mathcal{H}(\xi) x(\xi, t)) := A(\mathcal{H} x)$$

  where H is uniformly positive multiplication operator and A dissipative and E is positive.

- This leads to the subspaces

$$\mathcal{D} = \operatorname{graph} A, \quad \mathcal{L} = \operatorname{im} \left[\begin{smallmatrix} E \\ \mathcal{H} \end{smallmatrix} \right] = \operatorname{im} \left[\begin{smallmatrix} E\mathcal{H}^{-1} \\ I \end{smallmatrix} \right] \quad \Longrightarrow \quad \left(\begin{smallmatrix} Ex \\ \frac{d}{dt} Ex \end{smallmatrix} \right) \in \mathcal{DL}$$

- **Next step:** Generalize the finite dimensional stability and regularity results to Hilbert or Banach space setting.

List of some references used during the talk

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