Stability of port-Hamiltonian descriptor systems

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The port-Hamiltonian idea

- unified modeling approach for systems (mostly) based on the energy described by the Hamiltonian
- ports can be introduced to describe the exchange of the system with "outer world"
- power conserving interconnections guarantee that coupled systems are again port-Hamiltonian
- many recent results on well-posedness, stability, structure preserving discretizations, model order reduction
- In this talk we focus on a recent generalizations to descriptor systems or differential-algebraic equations.

Dissipative Hamiltonian DAEs

dissipative Hamiltonian DAEs

- In the following we consider $\frac{d}{dt}Ex(t) = DQx(t)$ where

$$D + D^* \leq 0, \quad Q^*E = E^*Q$$

are called **port-Hamiltonian** DAEs and if $E^*Q \geq 0$ the DAEs are called **dissipative Hamiltonian (dH)**.

- If $E^*Q \geq 0$ one might ask for the stability of solutions and using $V(x) = x^*E^*Qx$ as a **Lyapunov candidate**. This leads to

$$\tfrac{d}{dt}x(t)^*E^*Qx(t) = x(t)^*Q^*D^*Qx(t) + x(t)^*Q^*DQx(t) \leq 0$$

However if Q has a kernel we cannot conclude stability.

- A DAE is called **stable** if for all solutions there exists $M > 0$ such that

$$\sup_{t\geq 0} \|x(t)\| \leq M.$$

dH-DAEs are unstable

- Dissipative Hamiltonian DAEs are **not stable** as the following example from MEHL, MEHRMANN, WOJTYLAK '18 shows:

$$Ex = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = DQx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

  This is an ODE with Jordan block of size two at 0 and hence unstable.

- One can show that the size of the Jordan blocks at 0 is at most two.

Sufficient condition for stability

- If $\ker Q \subseteq \ker E$ and (E, DQ) is regular, i.e. $\lambda E - DQ$ is invertible for some $\lambda \in \mathbb{C}$.

$$\Rightarrow \{0\} = \ker E \cap \ker DQ \supseteq \ker E \cap \ker Q \supseteq \ker Q$$

$$\Rightarrow \text{Q injective} \Rightarrow \textbf{Q invertible}.$$

- Hence the DAE (E, DQ) is equivalent to (Q^*E, Q^*DQ) which is called **positive real** (Berger, Reis '13) or **semi-dissipative Hamiltonian** (Mehl, Mehrmann, Wojtylak '18, Achleitner, Arnold, Mehrmann '21) and these are known to be **stable**.
- Condition not necessary, consider $Q = 0$ and $E = I_n$; Hence not sufficient even in the ODE case if Q has kernel.
- Another sufficient condition is that $P_{\operatorname{ran} Q} D|_{\operatorname{ran} Q}$ is (strictly) dissipative.
- But maybe the converse is true: **Stability $\Rightarrow$ dH-DAE!?**

Stability of DAEs

How to solve DAEs

- If $E, A \in \mathbb{C}^{n\times n}$ then there exists invertible $S, T \in \mathbb{C}^{n\times n}$ such that

$$SET = \begin{bmatrix} I_{n_0} & & & \\ & N_\alpha & & \\ & & K_\beta & \\ & & & K_\gamma^T \end{bmatrix}, \quad SAT = \begin{bmatrix} A_0 & & & \\ & I_\alpha & & \\ & & L_\beta & \\ & & & L_\gamma^T \end{bmatrix}$$

  with block diagonal

$$N_\alpha \text{ with blocks } \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix} \quad sK_\beta - L_\beta \text{ with blocks } \begin{bmatrix} s & 1 & & \\ & \ddots & \ddots & \\ & & s & 1 \end{bmatrix}$$

  which is called the **Kronecker form** or quasi-Kronecker form (Berger, Trenn '12)
- Hence: To **solve** the DAE one has to consider each block **separately**
- Roughly speaking A_0 describes the ODE-part, the α-blocks have only the trivial solution, β is under-determined, γ is over-determined.

Consequence: Stability of DAEs

- Hence we can characterize the stability based on the Kronecker form:
- A DAE with $E, A \in \mathbb{C}^{n\times n}$ is **stable** $\iff$ regular, all eigenvalues of A_0 are in the closed left half plane and semi-simple on the imaginary axis.
- For a Lyapunov-like characterization recall the **system space** (Reis, Rendel, Voigt '16)

$$\mathcal{V}_{\mathrm{sys}} := \{x(0) \mid x \text{ smooth and } \tfrac{d}{dt}Ex(t) = Ax(t)\}.$$

Proposition (G, Haller '21)

The DAE given by $E, A \in \mathbb{C}^{n\times n}$ is stable if and only if there exists $X >_{E\mathcal{V}_{\mathrm{sys}}} 0$ with $X(E\mathcal{V}_{\mathrm{sys}}) = E\mathcal{V}_{\mathrm{sys}}$ solving the generalized Lyapunov inequality

$$E^*XA + A^*XE \leq_{\mathcal{V}_{\mathrm{sys}}} 0.$$

- Related results on Lyapunov inequalities for DAEs by Lewis '86, Stykel '02...

Stable DAEs are also dissipative Hamiltonian

- If (E, A) is stable and $X >_{E\mathcal{V}_{\text{sys}}} 0$ with $X(E\mathcal{V}_{\text{sys}}) = E\mathcal{V}_{\text{sys}}$ is a solution of
$$E^* XA + A^* XE \leq_{\mathcal{V}_{\text{sys}}} 0.$$
- Set $Q := XE$ then for the pseudo-inverse $Q^\dagger$ define $D := AQ^\dagger$ and we have
$$A =_{\mathcal{V}_{\text{sys}}} DQ, \quad Q^T E >_{\mathcal{V}_{\text{sys}}} 0.$$
- If the DAE is stable with **index one**, i.e. $N_\alpha = 0$ in the Kronecker form we have $Q^T E \geq 0$ and $A = DQ$ **(on the whole space)**. In this case $\mathcal{V}_{\text{sys}} = A^{-1}(E\mathbb{C}^n)$.

Further remarks

- If we have a stabilizable system of the form

$$\frac{d}{dt}Ex(t) = Ax(t) + Bu(t)$$

in the sense that for all initial values $x_0 \in \mathcal{V}_{\mathrm{sys}}$ there exists u and $M > 0$ such that

$$\sup_{t\geq 0} \|x(t)\| \leq M.$$

- Furthermore, (E, A) is assumed to be regular with only semi-simple eigenvalues on the imaginary axis then there exists solutions to the **algebraic Bernoulli equation**

$$A^* X_1 E + E^* X_1 A =_{\mathcal{V}_{\mathrm{sys}}} E^T X_1 B B^T X_1 E.$$

This solution can be used to rewrite the stabilizable system as a port-Hamiltonian one on the system space.

Extension to infinite dimensional case

- Let X, Z be Hilbert spaces and $E, Q : X \to Z$ bounded and Q with bounded inverse, D densely defined and closed in Z and strictly dissipative, i.e. $(Dx, x) + (x, Dx) < 0$ and Q^*E has closed range.
- Then the DAE is equivalent to

$$\frac{d}{dt} \underbrace{Q^*E}_{=:\hat{E}\geq 0} x(t) = \underbrace{Q^*DQ}_{=:\hat{D}} x(t)$$

If we use the pseudo-inverse $(\sqrt{\hat{E}})^\dagger$ and $(\cdot,\cdot)_w := (\sqrt{\hat{E}}^\dagger \cdot, \sqrt{\hat{E}}^\dagger \cdot)$ to obtain

$$\begin{aligned}(\hat{E}x, \hat{D}x)_w = (\hat{E}\hat{D}^{-1}y, y)_w = (\sqrt{\hat{E}}^\dagger \hat{E}\hat{D}^{-1}y, \sqrt{\hat{E}}y) = (P_{\operatorname{ran}\hat{E}}\sqrt{\hat{E}}\hat{D}^{-1}y, \sqrt{\hat{E}}^\dagger y)\\ = (\sqrt{\hat{E}}^\dagger \sqrt{\hat{E}}\hat{D}^{-1}y, y) = (\hat{D}^{-1}y, y) \leq 0 \quad \forall x = \hat{D}^{-1}y \in \operatorname{ran} Q^*E.\end{aligned}$$

- If $\ker Q^*E \cap \ker Q^*DQ = \{0\}$ then we have regularity and we can show that there exists an exponentially stable semigroup on a certain subspace

Geometric pH-DAE framework

The geometric pH-DAE framework

- Let $\mathcal{D}$ be a subspace of $\mathbb{C}^n \times \mathbb{C}^n$ and $\mathcal{D} = \operatorname{im} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$
- Then a DAE $\frac{d}{dt}Ex(t) = Ax(t)$ is called pH in G, Haller, Reis based on Maschke, van der Schaft, Jeltsema if

$$\operatorname{im} \begin{bmatrix} E \\ A \end{bmatrix} = \mathcal{D}\mathcal{L} := \{(x,z) \mid (x,y) \in \mathcal{L},\ (y,z) \in \mathcal{D}\}$$

for some $\mathcal{D} = \operatorname{im} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$, $D_1, D_2 \in \mathbb{C}^{n\times n}$ and $\mathcal{L} = \operatorname{im} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, $L_1, L_2 \in \mathbb{C}^{n\times n}$ which are assumed to fulfill

$$D_2^* D_1 + D_1^* D_2 \leq 0 \ \text{ (dissipative)}, \quad L_2^* L_1 = L_1^* L_2 \ \text{(symmetric)}$$

- The simplest examples are DAEs where $E = E^*$ and $A + A^* \leq 0$ then we can simply define $\mathcal{L} = \operatorname{graph} E$ and $\mathcal{D} = \operatorname{graph} A$.

Connection to dH-DAEs

- It **contains** the **previous setting** by choosing $\mathcal{D} = \operatorname{graph} D$ and $\mathcal{L} = \operatorname{im} \begin{bmatrix} E \\ Q \end{bmatrix}$ with $Q^*E \geq 0$ or $Q^*E = E^*Q$.
- However Maschke, van der Schaft '18 assume that $\dim \mathcal{L} = n$ which is not assumed in Mehl, Mehrmann, Wojtylak '18 but is implies by the regularity of $[E, A]$!
- Let $E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and consider $\mathcal{D} = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{L} = \left(\operatorname{graph} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1}$.
- Then $\mathcal{D}$ is dissipative, $\mathcal{L}$ is nonnegative, and $\operatorname{im} \begin{bmatrix} E \\ A \end{bmatrix} = \mathcal{D}\mathcal{L}$.
- One can show that it is **not possible** to rewrite $\mathcal{D}\mathcal{L} = (\operatorname{graph} D)\hat{\mathcal{L}}$ for some dissipative matrix $D \in \mathbb{C}^{2\times 2}$ and a nonnegative relation $\hat{\mathcal{L}} \subset \mathbb{C}^4$.

Regularity, spectrum and stability in the geometric setting

- What is known about the Kronecker form when the DAE is given by a product $\mathcal{D}\mathcal{L}$ of a dissipative and a nonnegative linear relation?

Proposition (G, Haller, Reis '21)

Let $\mathcal{D} = \operatorname{ran}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ be dissipative and $\mathcal{L} = \operatorname{ran}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ nonnegative Lagrangian with $L_1, L_1^T L_2 \geq 0$ and $D_2 + D_2^T \leq 0$. Further, let $\operatorname{im}\begin{bmatrix} E \\ A \end{bmatrix} := \mathcal{D}\mathcal{L}$. Then with $\mathcal{X} := \operatorname{ran} D_1 \cap \operatorname{ran} L_2$ the following holds.

(a) $[E, A]$ is regular if and only if $\ker P_{\mathcal{X}} L_1|_{\mathcal{X}} \cap \ker P_{\mathcal{X}} D_2|_{\mathcal{X}} = \{0\}$ and $D_2(\ker D_1) \cap L_1(\ker L_2) = \{0\}$.

(b) If $[E, A]$ is regular the eigenvalues are in the closed left half plane and the index is at most three.

(c) $[E, A]$ is stable if additionally $L_1(\ker L_2) = \{0\}$ holds.

- More results on the Kronecker form if $\mathcal{D}$ and $\mathcal{L}$ are in addition maximal, i.e. $\dim \mathcal{D} = \dim \mathcal{L} = n$.

Final remarks

- For the previous setting $A = DQ$ and $Q^*E = E^*Q \geq 0$ we have $D_1 = I$, $D_2 = D$, $L_1 = E, L_2 = Q$ this implies $\mathcal{X} = \operatorname{ran} D_1 \cap \operatorname{ran} L_2 = \operatorname{ran} Q$
- Hence $[E, A]$ is **regular** if and only if

$$
\begin{aligned}
&\ker P_{\mathcal{X}} L_1|_{\mathcal{X}} \cap \ker P_{\mathcal{X}} D_2|_{\mathcal{X}} = \{0\}, \quad D_2(\ker D_1) \cap L_1(\ker L_2) = \{0\} \\
\Leftrightarrow\ &\ker P_{\operatorname{ran} Q} E|_{\operatorname{ran} Q} \cap \ker P_{\operatorname{ran} Q} D|_{\operatorname{ran} Q} = \{0\} \\
\Leftrightarrow\ &\ker E \cap \ker Q^*RQ \cap \ker Q^*JQ = \{0\}, D = J - R, J = -J^*, R = R^* \geq 0
\end{aligned}
$$

  which was also obtained recently by Faulwasser, Maschke, Philipp, Schaller, Worthmann
- The stability condition $L_1(\ker L_2) = \{0\}$ is equivalent to $\ker Q \subseteq \ker E$.

Final remarks

- The infinite dimensional pH-ODE setting of Jacob Zwart '12 we have

$$\frac{\partial}{\partial t}x(\xi,t) = P_1\frac{\partial}{\partial \xi}(\mathcal{H}(\xi)x(\xi,t)) + P_0(\mathcal{H}(\xi)x(\xi,t)) := A(\mathcal{H}x)$$

  where $P_1 = P_1^* \in \mathbb{C}^{n\times n}$ is invertible, $P_0 = -P_0^*$ and $\mathcal{H} \in L^\infty((a,b),\mathbb{C}^{n\times n})$ satisfies $mI_n \leq \mathcal{H}(\xi) \leq MI_n$ for some constants $0 < m < M$ and A maximally dissipative

- this can be rewritten in our setting using the subspaces

$$\mathcal{D} = \operatorname{graph} A, \quad \mathcal{L} = (\operatorname{graph}\mathcal{H})^{-1} = \operatorname{graph}\mathcal{H}^{-1} \quad \Longrightarrow \quad \left(\begin{smallmatrix} x \\ \dot{x}\end{smallmatrix}\right) \in \operatorname{ran}\left[\begin{smallmatrix}\mathcal{H}^{-1} \\ A\end{smallmatrix}\right] = \mathcal{D}\mathcal{L}$$

Final remarks

- A natural generalization would be

$$\frac{\partial}{\partial t}E(\xi)x(\xi,t) = P_1\frac{\partial}{\partial \xi}(\mathcal{H}(\xi)x(\xi,t)) + P_0(\mathcal{H}(\xi)x(\xi,t)) := A(\mathcal{H}x)$$

  where H is uniformly positive multiplication operator and A dissipative and E is positive.
- This leads to the subspaces

$$\mathcal{D} = \operatorname{graph} A, \quad \mathcal{L} = \operatorname{ran}\begin{bmatrix} E \\ \mathcal{H} \end{bmatrix} = \operatorname{ran}\begin{bmatrix} E\mathcal{H}^{-1} \\ I \end{bmatrix} \quad \Longrightarrow \quad \begin{pmatrix} Ex \\ \frac{d}{dt}Ex \end{pmatrix} \in \mathcal{D}\mathcal{L}$$

- Hence we are in the previous setting with $\hat{E} = E\mathcal{H}^{-1} \geq 0$
- **Next step:** Generalize the finite dimensional results to multi-valued $\mathcal{D}$ and $\mathcal{L}$ with nontrivial kernel.

List of some references used during the talk

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