Port-Hamiltonian differential-algebraic equations – H. Gernandt jointly with F.E. Haller (Hamburg), T. Reis (Ilmenau) and V. Mehrmann (Berlin)

What are DAEs?

- A differential-algebraic equation **(DAE)** is of the form
$$\tfrac{d}{dt}Ex(t) = Ax(t), \quad Ex(0) = Ex_0$$
where $E, A \in \mathbb{C}^{n\times n}$ and E is usually **not invertible**.
- Typically one is interested in the **spectrum**
$$\sigma(E,A) := \{\lambda \in \mathbb{C} \mid \lambda E - A \text{ not invertible}\}$$
and the **index** which is the smallest k for which there exists $M > 0$ such that for all $\lambda > 0$ sufficiently large
$$\|(\lambda E - A)^{-1}\| \le M\lambda^{k-1}.$$
The unique solvability of DAEs is guaranteed by the **regularity**, i.e. $\sigma(E,A) \neq \mathbb{C}$.

PH-DAEs in finite dimensions

Energy-based modeling of physical systems reveals an additional structure of the DAEs which is referred to as *port-Hamiltonian* **(pH)**

- Mehl, Mehrmann, Wojtylak '18:
$$\exists Q \in \mathbb{C}^{n\times n}: \ A = DQ, \quad D + D^* \le 0, \quad Q^*E = E^*Q$$
- Maschke, van der Schaft '18:
$$\begin{pmatrix} x(t)\\ e(t)\end{pmatrix} \in \mathcal{L} = \operatorname{im}\begin{bmatrix} L_1\\ L_2\end{bmatrix}, \quad \begin{pmatrix} e(t)\\ \dot{x}(t)\end{pmatrix} \in \mathcal{D} = \operatorname{im}\begin{bmatrix} D_1\\ D_2\end{bmatrix},$$
where $t \ge 0$ and $D_1, D_2, L_1, L_2 \in \mathbb{C}^{n\times n}$ fulfill
$$D_2^*D_1 = -D_1^*D_2, \quad L_2^*L_1 = L_1^*L_2, \quad \dim\mathcal{D} = \dim\mathcal{L} = n.$$
- G, Haller, Reis '21: combines the previous two
$$\operatorname{im}\begin{bmatrix} E\\ A\end{bmatrix} = \mathcal{DL} = \{(x,z) \mid (x,y) \in \mathcal{L}, (y,z) \in \mathcal{D}\},$$
$$D_2^*D_1 + D_1^*D_2 \le 0, \quad L_2^*L_1 = L_1^*L_2$$

Properties of pH-DAEs

- All of the above pH-formulations do **not imply** the **regularity, stability** or a **small index**!
- If $Q^*E \ge 0$ then it was shown in [1] that the index of the DAE is **at most two** the spectrum is contained in the closed left half-plane, but the size of Jordan blocks at 0 **is possibly two**:
$$sE - DQ = s\begin{bmatrix}1&0\\0&1\end{bmatrix} - \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}s&1\\0&s\end{bmatrix} \implies \textbf{unstable}$$
- The approach [1] **can be embedded** via $\mathcal{D} = \operatorname{graph} D$, $\mathcal{L} = \operatorname{im}\begin{bmatrix}E\\Q\end{bmatrix}$
- However the subspace $\mathcal{L}$ is in [1] **might not fulfill** $\dim\mathcal{L} = n$ as required in [2]. If (E, DQ) is regular then (E,Q) is regular, hence $\dim\mathcal{L} = n$.
- We showed in [3] for our more general setting ($\mathcal{D}$ dissipative and $\mathcal{L}$ with $L_2^*L_1 \ge 0$) that the index is at most three, the size of the Jordan blocks at 0 is at most two and that all eigenvalues $\lambda \in \sigma(E,A)$ satisfy $\operatorname{Re}\lambda \le 0$.

Infinite dimensional pH-DAEs

- Let X and Z be Hilbert spaces then we consider
$$\tfrac{d}{dt}Ex(t) = Ax(t), \quad Ex(0) = Ex_0$$
where $E: X \to Z$ is bounded and $A: X \supset \operatorname{dom} A \to Z$ is closed and densely defined.
- **Regularity assumption:** there exists $\lambda \in \mathbb{C}$ for which $\lambda E - A$ has a bounded inverse
- **Open:** What is the space of consistent initial values for which a unique solution exists and what is the right definition of index.
- Previous results mostly in the "index one" case by Barbu, Favini, Yagi, Showalther, Thaller, Trostorff, Waurick,... (see right column)
- Previous results on **infinite dimensional pH-systems**:
  - Jacob, Zwart '12 considered in [4]
$$\tfrac{\partial x}{\partial t}(t,\xi) = P_1\tfrac{\partial}{\partial \xi}(\mathcal{H}(\xi)x(t,\xi)) + P_0(\mathcal{H}(\xi)x(t,\xi)),$$
where $P_1 = P_1^* \in \mathbb{C}^{n\times n}$ is invertible, $P_0 = -P_0^*$ and $\mathcal{H} \in L^\infty((a,b), \mathbb{C}^{n\times n})$ satisfies $mI_n \le \mathcal{H}(\xi) \le MI_n$ for some constants $0 < m < M$
- **Possible pH-setting:** $(Ex,x)_X \ge 0$ for all $x \in X$, E closed range and A is ω-dissipative, i.e. for some $\omega > 0$
$$(Ax,x)_X + (x,Ax)_X \le -\omega\|x\|^2, \quad x \in \operatorname{dom} A$$
- Then using the inverse of A and using $X = \ker E \oplus \operatorname{im} E$ and $x = x_K \oplus x_R$ we find
$$\tfrac{d}{dt}A^{-1}Ex(t) = x(t) \iff x_K = 0, \quad \tfrac{d}{dt}P_{\operatorname{im} E}A^{-1}Ex_R(t) = x_R(t)$$
The resulting DAE on $\operatorname{im} E$ fulfills the resolvent growth assumption (D_1) from the right column.
- Hence there exists an underlying **exponentially stable semigroup** on $P_{\operatorname{im} E}A^{-1}E$.

Solutions of DAEs

- Let X, Z be Hilbert spaces, $E: X \to Z$ bounded and $A: X \supset \operatorname{dom} A \to Z$ for simplicity bijective and $T > 0$.
- A **classical solution** $x: [0,T] \to X$ fulfills $x(t) \in \operatorname{dom} A$ and $t \mapsto Ex(t)$ is continuously differentiable.
- An X-**mild solution** fulfills $\int_0^t x(\tau)d\tau \in \operatorname{dom} A$ for all $t \in [0,T]$ and
$$Ex(t) - Ex_0 = A\int_0^t x(\tau)d\tau.$$
- If A is invertible then a Z-**mild solution** $z: [0,T] \to Z$ fulfills for all $t \in [0,T]$
$$EA^{-1}z(t) - EA^{-1}z_0 = \int_0^t z(\tau)d\tau.$$

Connection between these solutions:

- If x is X-mild then $z = Ex$ is a Z-mild solution.
- If z is Z-mild then $x = A^{-1}z$ is X-mild.

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Index of DAEs in Hilbert spaces

- Let X be a Hilbert space. If $\dim X < \infty$ there are several equivalent index notions: The index is the smallest k such that
(a) $\|(\lambda E - A)^{-1}\| \le M\lambda^{k-1}$;
(b) termination of Wong sequences $\mathcal{V}_k = \mathcal{V}_{k+1}$
$$\mathcal{V}_n := A^{-1}(Ex), \quad \mathcal{V}_0 := X \quad (\Rightarrow \mathcal{V}_{n+1} \subseteq \mathcal{V}_n).$$
- This is **no longer true** in infinite dimensions: Termination is not clear and also closedness of subspaces $\mathcal{V}_k$ is an issue
- Trostorff, Waurick '18: $E, A: X \to X$ **bounded**, resolvent growth and $E(\mathcal{V}_k)$ closed $\implies \mathcal{V}_k = \mathcal{V}_{k+1}$
- Reis, Tischendorf '05: $E: X \to Z$ bounded with closed range, A **unbounded**, with finite tractability index $k \implies \mathcal{V}_k = \mathcal{V}_{k+1}$
- Pseudo-resolvent (Hille '49): Let $\Omega \subset \mathbb{C}$ be open then $R: \Omega \to L(X)$ is called **pseudo-resolvent** iff
$$(\lambda - \mu)R(\mu)R(\lambda) = R(\mu) - R(\lambda), \quad \lambda, \mu \in \Omega.$$
- Kato '59: R pseudo-resolvent, $(\lambda_n)_n$ with $\lambda_n \nearrow \infty$, and $\limsup_{n\to\infty}\|\lambda_n R(\lambda_n)\| < \infty \implies X = \overline{R(\lambda)X} \dotplus \ker R(\lambda)$ for all $\lambda \in \Omega$ and independent of the choice of λ
- G, Reis '21: Assume that $\rho(E,A) \supseteq [0,\infty)$, let $R(\lambda) := E(A - \lambda E)^{-1}$ and assume
$$\|R(\lambda)z\| \le \tfrac{M\|z\|}{\lambda}, \quad \forall \ z \in R(0)^{k-1}Z. \quad \mathbf{(D_k)}$$
- **Main result:** $\operatorname{graph}\mathfrak{J} = \{(R(0)z, z), z \in \overline{R(0)^kZ}\}$ defines an closed densely defined operator on $\overline{R(0)^kZ}$ with $(\mathfrak{J} - \lambda)^{-1} = R(\lambda)|_{\overline{R(0)^kZ}}$.

PDAE models of power grids

- **Power grids** consist of generators supplying power, loads consuming this power and transmission lines which interconnect these.
- This is modeled as a DAE consisting of:
  - **non-linear** ODEs describing the generators (e.g. swing equation);
  - **linear equations** for loads and coupling;
  - lumped parameter ODEs for transmission lines.
- Within the **DFG Priority Program 1984** we plan to model the transmission lines more accurately based on the telegraph equation
$$C(\xi)\tfrac{\partial v}{\partial t}(t,\xi) = -\tfrac{\partial i}{\partial \xi}(t,\xi) - G(\xi)v(t,\xi), \quad v(t,0) = v_0(t), \quad v(t,1) = v_1(t)$$
$$L(\xi)\tfrac{\partial}{\partial t}i(t,\xi) = -\tfrac{\partial}{\partial \xi}v(t,\xi) - R(\xi)i(t,\xi), \quad v(0,\xi) = v^0(\xi), \quad i(0,\xi) = i^0(\xi)$$
where $\xi \in [a,b]$ is the spatial variable, v is the voltage, i is the current through the transmission line and spatially distributed $C, L, G, R > 0$.
- The above equation can be modeled as a pH boundary control system in the sense of Jacob, Zwart '12!
- **Next step:** Extension to nonlinear DAEs and their interconnection.

References

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