First-Order Logic

Russell and Norvig:
Chapter 8, Sections 8.1-8.3

In which we notice that the world is blessed with many objects, some of which are related to other objects, and in which we endeavor to reason about them

Outline

- Why FOL?
- Syntax and semantics of FOL
- Using FOL
- Wumpus world in FOL

Propositional logic, pros and cons

- ☺ Propositional logic is declarative
- ☺ Propositional logic allows partial (disjunctive/negated) information
  - (unlike most data structures and databases)
- ☺ Propositional logic is compositional:
  - meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

Propositional logic, pros and cons

- ☺ Meaning in propositional logic is context-independent
  - (unlike natural language, where meaning depends on context)
- ☹ Propositional logic has very limited expressive power
  - (unlike natural language)
  - E.g., cannot say "pits cause breezes in adjacent squares"

    except by writing one sentence for each square

Why not use Natural Language?

- It serves a different purpose:
  - Communication
    rather than representation
- It is not compositional
  - Context matters
- It can be ambiguous
  - Again, context matters

Create a new language

- Builds on propositinal logic
- But is inspired by natural language!

Language	Ontological Commitment (What exists in the world)	Epistemological Commitment (What an agent believes about facts)
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief $\in [0, 1]$
Fuzzy logic	facts with degree of truth $\in [0, 1]$	known interval value

Figure 8.1 Formal languages and their ontological and epistemological commitments.

First-order logic

- Whereas propositional logic assumes the world contains facts,
- first-order logic (like natural language) assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, …
  - Relations: red, round, prime, brother of, bigger than, part of, comes between, …
  - Functions: father of, best friend, one more than, plus, …

Relations do or do not hold between objects

$$
\begin{array}{rcl}
\mathit{Sentence} & \rightarrow & \mathit{AtomicSentence} \mid \mathit{ComplexSentence} \\
\mathit{AtomicSentence} & \rightarrow & \mathit{Predicate} \mid \mathit{Predicate}(\mathit{Term}, \ldots) \mid \mathit{Term} = \mathit{Term} \\
\mathit{ComplexSentence} & \rightarrow & (\,\mathit{Sentence}\,) \\
& \mid & \neg\, \mathit{Sentence} \\
& \mid & \mathit{Sentence} \land \mathit{Sentence} \\
& \mid & \mathit{Sentence} \lor \mathit{Sentence} \\
& \mid & \mathit{Sentence} \;\Rightarrow\; \mathit{Sentence} \\
& \mid & \mathit{Sentence} \;\Leftrightarrow\; \mathit{Sentence} \\
& \mid & \mathit{Quantifier}\ \mathit{Variable}, \ldots\ \mathit{Sentence} \\
\\
\mathit{Term} & \rightarrow & \mathit{Function}(\mathit{Term}, \ldots) \\
& \mid & \mathit{Constant} \\
& \mid & \mathit{Variable} \\
\\
\mathit{Quantifier} & \rightarrow & \forall \mid \exists \\
\mathit{Constant} & \rightarrow & A \mid X_1 \mid \mathit{John} \mid \cdots \\
\mathit{Variable} & \rightarrow & a \mid x \mid s \mid \cdots \\
\mathit{Predicate} & \rightarrow & \mathit{True} \mid \mathit{False} \mid \mathit{After} \mid \mathit{Loves} \mid \mathit{Raining} \mid \cdots \\
\mathit{Function} & \rightarrow & \mathit{Mother} \mid \mathit{LeftLeg} \mid \cdots \\
\textsc{Operator Precedence} & : & \neg, =, \land, \lor, \Rightarrow, \Leftrightarrow
\end{array}
$$

Syntax of FOL: Basic elements

Constants	KingJohn, 2, Richard,...
Predicates	Brother, >, Crown, ...
Functions	Sqrt, LeftLegOf,...
Variables	x, y, a, b,...
Connectives	$\neg$, $\Rightarrow$, $\wedge$, $\vee$, $\Leftrightarrow$
Equality	=
Quantifiers	$\forall$, $\exists$

Atomic sentences

Atomic sentences state facts

Term = *function* ($term_1$,...,$term_n$)
or *constant* or *variable*

Atomic sentence = *predicate* ($term_1$,...,$term_n$)
or $term_1 = term_2$

Examples:

Brother(KingJohn,RichardTheLionheart)

>(Length(LeftLegOf(Richard)),Length(LeftLegOf(KingJohn)))

Complex sentences

Complex sentences = Made from atomic sentences using connectives

$\neg S$, $S_1 \wedge S_2$, $S_1 \vee S_2$, $S_1 \Rightarrow S_2$, $S_1 \Leftrightarrow S_2$,

Examples:

$Sibling(KingJohn,Richard) \Rightarrow Sibling(Richard,KingJohn)$

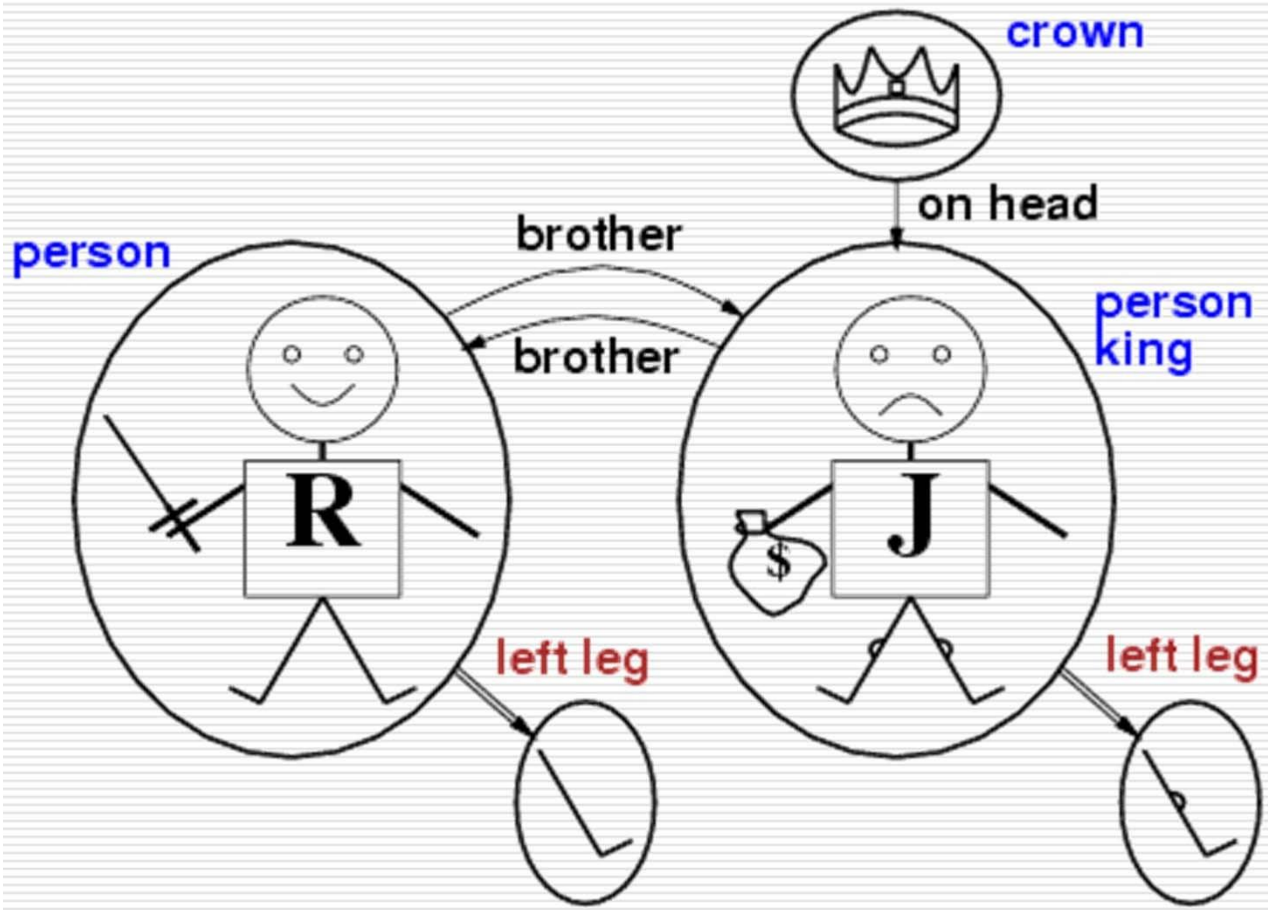
$>(1,2) \vee \leq (1,2)$

$<(1,2) \wedge \neg >(1,2)$

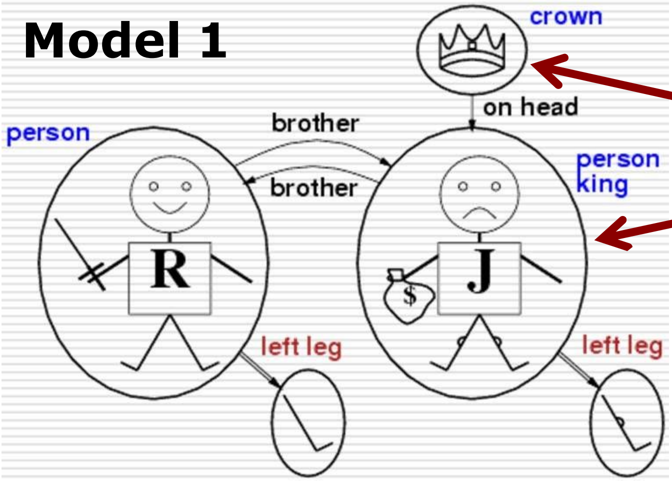
Truth in first-order logic

- Sentences are true with respect to a model
- A model contains objects (domain elements) and relations among them
- A model specifies an interpretation (referents) for
  constant symbols → objects
  predicate symbols → relations
  function symbols → functional relations
- An atomic sentence $predicate(term_1,\ldots,term_n)$ is true iff the objects referred to by $term_1,\ldots,term_n$ are in the relation referred to by *predicate*

Models and Interpretations



Models and Interpretations



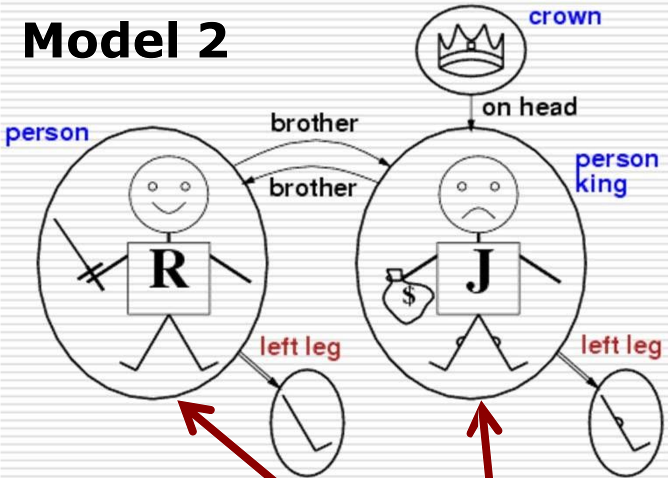
Interpretation I.

Constant: A

Constant: B

Atomic sentence: Brother(A, B)? → **False**

Models and Interpretations



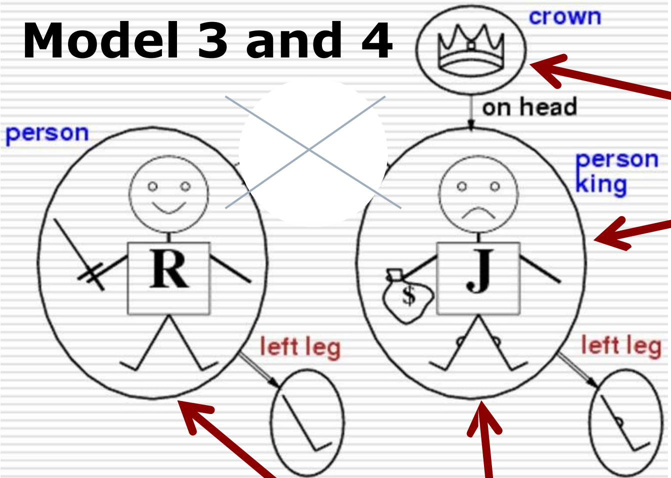
Atomic sentence: Brother(A, B)? → **True**

Interpretation II.

Constant: A

Constant: B

Models and Interpretations



Interpretation I.

Constant: A

Constant: B

Atomic sentence: Brother(A, B)? → **False**

Interpretation II.

Constant: A

Constant: B

Universal quantification

- ∀ *<variables> <sentence>*

Everyone in HR is smart:
$\forall x\ At(x,HR) \Rightarrow Smart(x)$

- $\forall x$ *P* is true in a model *m* iff *P* is true with *x* being **each possible object** in the model

- Roughly speaking, equivalent to the conjunction of instantiations of *P*

$$
\begin{aligned}
& At(KingJohn,HR) \Rightarrow Smart(KingJohn) \\
\wedge\ & At(Richard,HR) \Rightarrow Smart(Richard) \\
\wedge\ & At(HR,HR) \Rightarrow Smart(HR) \\
\wedge\ & \ldots
\end{aligned}
$$

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- Common mistake: using $\wedge$ as the main connective with $\forall$:

  $\forall x\ At(x,HR) \wedge Smart(x)$

  means "Everyone is at HR and everyone is smart"

Existential quantification

- $\exists$<*variables*> <*sentence*>

  Someone at HR is smart:
  $\exists x$ At(x,HR) $\wedge$ Smart(x)

- $\exists x$ *P* is true in a model *m* iff *P* is true with *x* being **some possible object** in the model

- Roughly speaking, equivalent to the disjunction of instantiations of *P*

  At(KingJohn,HR) $\wedge$Smart(KingJohn)
  $\vee$ At(Richard,HR) $\wedge$Smart(Richard)
  $\vee$ At(HR,HR) $\wedge$Smart(HR)
  $\vee$...

Another mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake:
  using $\Rightarrow$ as the main connective with $\exists$:

  $$\exists x \; At(x,HR) \Rightarrow Smart(x)$$

  true if there is anyone who is not at HR!

Properties of quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$

- $\exists x\ \forall y$ is not the same as $\forall y\ \exists x$
- $\exists x\ \forall y$ Loves(x,y)
  - "There is a person who loves everyone in the world"
- $\forall y\ \exists x$ Loves(x,y)
  - "Everyone in the world is loved by at least one person"

- Quantifier duality: each can be expressed using the other

  $\forall x$ Likes(x,IceCream) $\neg\exists x\ \neg$Likes(x,IceCream)

  $\exists x$ Likes(x,Broccoli) $\neg\forall x\ \neg$Likes(x,Broccoli)

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

- E.g., definition of *Sibling* in terms of *Parent*:

  $\forall x,y \; Sibling(x,y) \Leftrightarrow$
  $[\neg(x = y) \wedge \exists m,f \; \neg(m = f) \wedge Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y)]$

Assertions and Queries in FOL

$TELL\big(KB, King(John)\big)$

$TELL(KB, \forall x\ King(x) \rightarrow Person(x)\ )$

$ASK(KB, King(John))$ return True

$ASK(KB, \exists x\ Person(x))$ return True

$ASKVARS(KB, Person(x))$ yeilds $\{x/John, x/Richard\}$, binding list

Using FOL – Kinship Domain

"The son of my father is my brother"
"One's grandmother is the mother of one's parent" etc.

- **Domain:** People
- **Unary predicates:** Male, Female
- **Relations (binary predicates):** Parent, Sibling, Brother, Sister, Child, Daughter, Son, Spouse, Wife, Husband, Grandparent, Grandchild, Cousin, Aunt, Uncle
- **Functions:** Mother, Father

Using FOL

The kinship domain:

- Brothers are siblings

  $\forall$x,y *Brother(x,y)* $\Leftrightarrow$ *Sibling(x,y)*
- One's mother is one's female parent

  $\forall$m,c *Mother(c)* = m $\Leftrightarrow$ *(Female(m)* $\wedge$*Parent(m,c))*
- "Sibling" is symmetric

  $\forall$x,y *Sibling(x,y)* $\Leftrightarrow$ *Sibling(y,x)*
- "Jen is female"

  Female(Jen)

*Some sentences are **Axioms** (i.e. definitions, facts) while others are **Theorems** derived from those.*

Wumpus World

- Perceives STENCH adjacent to WUMPUS
- Perceives BREEZE adjacent to PIT
- Perceives GLITTER in GOLD room
- Perceives BUMP when hitting wall
- Can move forwards, turn left, turn right or shoot an arrow. Arrow flies in facing direction until hitting a wall or killing a WUMPUS
- Perceives SCREAM if WUMPUS gets killed
- Can pick up GOLD if in same room

Interacting with FOL KBs

- Suppose a wumpus-world agent is using an FOL KB and perceives STENCH and BREEZE (but no GLITTER) at *t=5*:

  `TELL`(KB,Percept([STENCH,BREEZE,None],5))
  `ASK`(KB,∃a BestAction(a,5))

  i.e., does the KB entail some best action at *t=5*?

- Answer: *Yes*, {*a/Shoot*} ← substitution (binding list)

- Given a sentence *S* and a substitution q,
- *S*q denotes the result of plugging q into *S*; e.g.,
  S = Smarter(x,y)
  q = {x/Hillary,y/Bill}
  *S*q = Smarter(Hillary,Bill)

- `Ask`(KB,S) returns some/all q such that KB ⊨ Sq

KB for the wumpus world

- Perception
  - $\forall t,s,b$ Percept([s,b,GLITTER],t) $\Rightarrow$ Glitter(t)

- Reflex
  - $\forall t$ Glitter(t) $\Rightarrow$ BestAction(Grab,t)

Deducing hidden properties

Instead of:

- $Adjacent(Square_{1,2}, Square_{1,1})$
- $Adjacent(Square_{3,4}, Square_{4,4})$

Do:

- $\forall x,y,a,b\ Adjacent([x,y],[a,b]) \Leftrightarrow$
  $[a,b] \in \{[x+1,y], [x-1,y],[x,y+1],[x,y-1]\}$

Properties of squares:

- $\forall s,t\ At(\text{Agent},s,t) \wedge \text{Breeze}(t) \Rightarrow \text{Breezy}(s)$

Squares are breezy near a pit:

- Diagnostic rule---infer cause from effect
  $\forall s\ \text{Breezy}(s) \Rightarrow \exists r\ \text{Adjacent}(r,s) \wedge \text{Pit}(r)$
- Causal rule---infer effect from cause
  $\forall r\ \text{Pit}(r) \Rightarrow [\forall s\ \text{Adjacent}(r,s) \Rightarrow \text{Breezy}(s)\ ]$

Inference Example

“The laws says that it’s a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of American, has some missiles, and all of its missiles were sold to it by Colonel West, who is American”

Prove that Colonel West is a criminal.

Inference Example

“The laws says that it’s a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of American, has some missiles, and all of its missiles were sold to it by Colonel West, who is American”

KB:

R1: $American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \rightarrow Criminal(x)$

R2: $Owns(Nono, M_1)$

R3: $Missle(M_1)$

R4: $Missle(x) \rightarrow Weapon(x)$ A missile is a wepon

R5: $Missle(x) \land Owns(Nono, x) \rightarrow Sells(West, x, Nono)$ All missiles sold by West

R6: $Enemy(x, America) \rightarrow Hostile(x)$ Enemies of America are hostile

R7: $American(West)$ West is an American

R8: $Enemy(Nono, America)$

Fransisco Iacobelli https://youtu.be/I0QAdG68K0o

Inference Example

Iteration 1:

R5 satisfied with {*x/M_1*} and R9: *Sells(West, M_1, Nono)* is added

R4 satisfied with {*x/M_1*} and R10: *Weapon(M_1)* is added

R6 satisfied with {*x/Nono*} and R11: *Hostile(Nono)* is added

Iteration 2:

R1 satisfied with {*x/West, y/M_1, z/Nono*} and *Criminal(West)* is added

Summary

- First-order logic:
  - objects and relations are semantic primitives
  - syntax: constants, functions, predicates, equality, quantifiers
- Increased expressive power: sufficient to define Wumpus world
  - We did not have to write sentence for every square!