Paper review: Statistical analysis and simulation of random shocks in stochastic Burgers equation

H. Aghakhani

MTH 837

April 21, 2015

Table of contents

Motivation

Assumptions

Approach

Solution

Motivation

Motivation

Title and Authors

PROCEEDINGS OF THE ROYAL SOCIETY A

rspa.royalsocietypublishing.org

Research

CrossMark
click for updates

Statistical analysis and simulation of random shocks in stochastic Burgers equation

Heyrim Cho, Daniele Venturi and George E. Karniadakis

Division of Applied Mathematics, Brown University, Providence, RI 02912, USA

Motivation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + \sigma f(x,t;\omega), \quad x \in [0,2\pi] \;\; t \geq 0,$$

$$u(x,0;\omega) = u_0(x;\omega)$$

$$u(0,t;\omega) = u(2\pi,t;\omega),$$

- Burger's equation is a very important equation, and is used as a case study for for variety of different stochastic methods.
- It has nonlinear term (advection term)
- It has a laplacian term (diffusion term)
- it has source term
- Depend on which term is dominant this equation could have different behavior.
- It can be considered as simple 1D Navier_Stockes equations

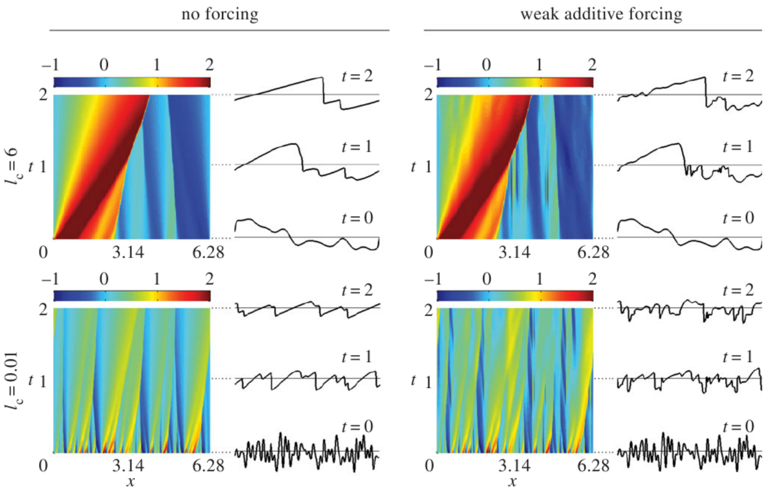

Figure 1. Sample solutions of the Burgers problem (2.1) in the inviscid limit $\nu \rightarrow 0$. Here, we consider two initial conditions with different correlation lengths randomly sampled from (3.6) and a realization of the random forcing term (3.5). It is seen that at $t = 2$ the velocity field already developed the triangular-shaped shock structure that is characteristic of the Burgers turbulence regime. Note that even weak additive forcing ($\sigma = 0.05$) can influence the solution, especially for rough initial conditions ($l_c = 0.01$). (Online version in colour.)

Assumptions

Assumptions:

- For sake of simplicity, they considered just 1D Burger's equation.
- To be able to find an analytical answer, they neglect the diffusion effect, so it can be thought of a formulation for inviscid flow.
- They assumed that the initial condition and source term are square integrable random fields. (previouse picture shows their effect on the problem).

Assumptions Cont.

- They further consider f as a smooth noise.
- Under these assumptions one can write $f(x, t; \omega)$ and $u_0(x; \omega)$ in terms of series expansions involving proper sets of random variables.

$$u_0(x;\boldsymbol{\eta}) = \sum_{k=1}^{l} \eta_k(\omega)\phi_k(x), \quad f(x,t;\boldsymbol{\xi}) = \sum_{j=1}^{m} \xi_j(\omega)\Psi_j(x,t).$$

And then the solution can be written in form of:

$$u(x,t;\omega) = U(x,t;\boldsymbol{\eta}(\omega),\boldsymbol{\xi}(\omega)).$$

Approach

- They use the modeling of probability density function(pdf) itself in the deterministic sens.
- They use the Mori–Zwanzig (MZ) formalism which relies on deriving reduced-order kinetic equations for the stochastic velocity field in the limits of small viscosity and small perturbations.
- They combine this approach with the adaptive discontinuous Galerkin (DG)

Toward pdf

The authors in their previous work, showed that under these assumptions, Burger's equation admit the following joint pdf:

$$p(x,t;a,\boldsymbol{b}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \delta(a - U(x,t;\boldsymbol{A}_0,\boldsymbol{B}))\delta(\boldsymbol{b}-\boldsymbol{B})q(\boldsymbol{A}_0,\boldsymbol{B})\,\mathrm{d}\boldsymbol{A}_0\,\mathrm{d}\boldsymbol{B},$$

where $a \in \mathbb{R}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{A}_0 \in \mathbb{R}^l$, $\boldsymbol{B} \in \mathbb{R}^m$, $q(\boldsymbol{A}_0,\boldsymbol{B})$ denotes the (possibly compactly supported) joint PDF of the random vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$, and $\delta(\boldsymbol{b}-\boldsymbol{B})$ is a multi-dimensional Dirac delta function, i.e.

$$\delta(\boldsymbol{b}-\boldsymbol{B}) \overset{\text{def}}{=} \prod_{k=1}^{m} \delta(b_k - B_k).$$

Toward pdf Cont.

Putting the above relation into the PDE results:

$$\frac{\partial p(t)}{\partial t} + \int_{-\infty}^{a} \frac{\partial p(t)}{\partial x}\,\mathrm{d}a' + a\frac{\partial p(t)}{\partial x} = -\sigma f(x,t;\boldsymbol{b})\frac{\partial p(t)}{\partial a} - \nu\frac{\partial}{\partial a}\left\langle \frac{\partial^2 u}{\partial x^2}\delta(a-u(x,t))\right\rangle,$$

$$\left\langle \frac{\partial^2 u}{\partial x^2}\delta(a-u(x,t))\right\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\partial^2 U}{\partial x^2}\delta(a-U(x,t;\boldsymbol{A}_0,\boldsymbol{B}))$$
$$\times\,\delta(\boldsymbol{b}-\boldsymbol{B})q(\boldsymbol{A}_0,\boldsymbol{B})\,\mathrm{d}\boldsymbol{A}_0\,\mathrm{d}\boldsymbol{B},$$

Solving the above relation requires further assumptions, for more simplicity they just neglect the last term and solve it for invicsid flow.

Reduced-order PDF equations: MZ approach

We can write the pde of pdf inform of:

$$\frac{\partial p(t)}{\partial t} = [\mathcal{L}_0 + \sigma \mathcal{L}_1(t)] p(t),$$

where

$$\mathcal{L}_0 \stackrel{\text{def}}{=} -\int_{-\infty}^{a} \mathrm{d}a' \frac{\partial}{\partial x} - a \frac{\partial}{\partial x} \quad \text{and} \quad \mathcal{L}_1(t) \stackrel{\text{def}}{=} -f(x, t; \boldsymbol{b}) \frac{\partial}{\partial a}.$$

by using the following transformation,

$$w(t) = \mathrm{e}^{-t\mathcal{L}_0} p(t),$$

we can simplify it more in form of:

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \sigma \mathcal{N}(t) w(t), \quad \mathcal{N}(t) \stackrel{\text{def}}{=} \mathrm{e}^{-t\mathcal{L}_0} \mathcal{L}_1(t)\, \mathrm{e}^{t\mathcal{L}_0}.$$

Complex System Modeling and UQ

Complex System Modeling and UQ