Paper review: Statistical analysis and simulation of random shocks in stochastic Burgers equation

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Statistical analysis and simulation of random shocks in stochastic Burgers equation

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Motivation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + \sigma f(x,t;\omega), \quad x \in [0,2\pi] \;\; t \geq 0,$$

$$u(x,0;\omega) = u_0(x;\omega)$$

$$u(0,t;\omega) = u(2\pi,t;\omega),$$

- Burger's equation is a very important equation, and is used as a case study for for variety of different stochastic methods.
- It has nonlinear term (advection term)
- It has a laplacian term (diffusion term)
- it has source term
- Depend on which term is dominant this equation could have different behavior.
- It can be considered as simple 1D Navier_Stockes equations

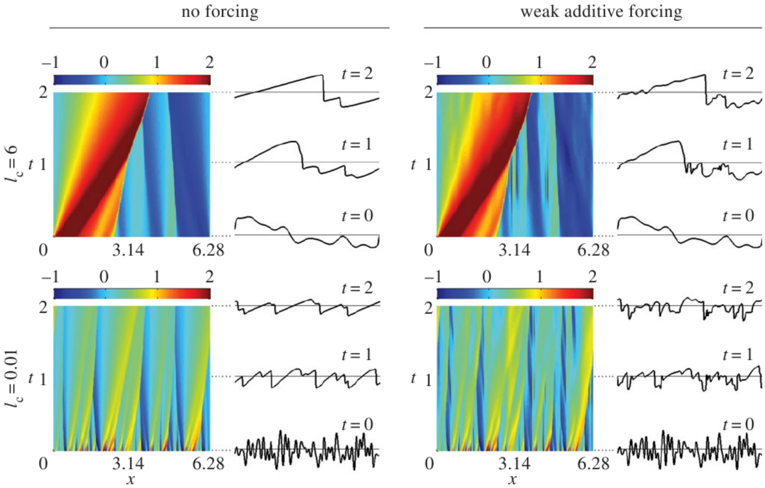

Figure 1. Sample solutions of the Burgers problem (2.1) in the inviscid limit $\nu \to 0$. Here, we consider two initial conditions with different correlation lengths randomly sampled from (3.6) and a realization of the random forcing term (3.5). It is seen that at $t = 2$ the velocity field already developed the triangular-shaped shock structure that is characteristic of the Burgers turbulence regime. Note that even weak additive forcing ($\sigma = 0.05$) can influence the solution, especially for rough initial conditions ($l_c = 0.01$). (Online version in colour.)

Assumptions

Assumptions:

- For sake of simplicity, they considered just 1D Burger's equation.
- To be able to find an analytical answer, they neglect the diffusion effect, so it can be thought of a formulation for inviscid flow.
- They assumed that the initial condition and source term are square integrable random fields. (previouse picture shows their effect on the problem).

Assumptions Cont.

- They further consider f as a smooth noise.
- Under these assumptions one can write $f(x, t; \omega)$ and $u_0(x; \omega)$ in terms of series expansions involving proper sets of random variables.

$$u_0(x; \boldsymbol{\eta}) = \sum_{k=1}^{l} \eta_k(\omega)\phi_k(x), \quad f(x, t; \boldsymbol{\xi}) = \sum_{j=1}^{m} \xi_j(\omega)\Psi_j(x, t).$$

And then the solution can be written in form of:

$$u(x, t; \omega) = U(x, t; \boldsymbol{\eta}(\omega), \boldsymbol{\xi}(\omega)).$$

Approach

- They use the modeling of probability density function(pdf) itself in the deterministic sens.
- They use the Mori–Zwanzig (MZ) formalism which relies on deriving reduced-order kinetic equations for the stochastic velocity field in the limits of small viscosity and small perturbations.
- They combine this approach with the adaptive discontinuous Galerkin (DG)

Toward pdf

The authors in their previous work, showed that under these assumptions, Burger's equation admit the following joint pdf:

$$p(x,t;a,\boldsymbol{b}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \delta(a - U(x,t;\boldsymbol{A}_0,\boldsymbol{B}))\delta(\boldsymbol{b}-\boldsymbol{B})q(\boldsymbol{A}_0,\boldsymbol{B})\,\mathrm{d}\boldsymbol{A}_0\,\mathrm{d}\boldsymbol{B},$$

where $a \in \mathbb{R}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{A}_0 \in \mathbb{R}^l$, $\boldsymbol{B} \in \mathbb{R}^m$, $q(\boldsymbol{A}_0,\boldsymbol{B})$ denotes the (possibly compactly supported) joint PDF of the random vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$, and $\delta(\boldsymbol{b}-\boldsymbol{B})$ is a multi-dimensional Dirac delta function, i.e.

$$\delta(\boldsymbol{b}-\boldsymbol{B}) \stackrel{\text{def}}{=} \prod_{k=1}^{m} \delta(b_k - B_k).$$

Toward pdf Cont.

Putting the above relation into the PDE results:

$$\frac{\partial p(t)}{\partial t} + \int_{-\infty}^{a} \frac{\partial p(t)}{\partial x}\, \mathrm{d}a' + a\frac{\partial p(t)}{\partial x} = -\sigma f(x,t;\boldsymbol{b})\frac{\partial p(t)}{\partial a} - \nu\frac{\partial}{\partial a}\left\langle \frac{\partial^2 u}{\partial x^2}\delta(a-u(x,t))\right\rangle,$$

$$\left\langle \frac{\partial^2 u}{\partial x^2}\delta(a-u(x,t))\right\rangle \stackrel{\mathrm{def}}{=} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2}\delta(a-U(x,t;\boldsymbol{A}_0,\boldsymbol{B})) \times \delta(\boldsymbol{b}-\boldsymbol{B})q(\boldsymbol{A}_0,\boldsymbol{B})\,\mathrm{d}\boldsymbol{A}_0\,\mathrm{d}\boldsymbol{B},$$

Solving the above relation requires further assumptions, for more simplicity they just neglect the last term and solve it for invicsid flow.

Reduced-order PDF equations: Mori–Zwanzig (MZ) approach

We can write the pde of pdf inform of:

$$\frac{\partial p(t)}{\partial t} = [\mathcal{L}_0 + \sigma \mathcal{L}_1(t)] p(t),$$

where

$$\mathcal{L}_0 \stackrel{\text{def}}{=} -\int_{-\infty}^{a} \mathrm{d}a' \frac{\partial}{\partial x} - a \frac{\partial}{\partial x} \quad \text{and} \quad \mathcal{L}_1(t) \stackrel{\text{def}}{=} -f(x,t;b) \frac{\partial}{\partial a}.$$

by using the following transformation,

$$w(t) = \mathrm{e}^{-t\mathcal{L}_0} p(t),$$

we can simplify it more in form of:

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \sigma \mathcal{N}(t) w(t), \quad \mathcal{N}(t) \stackrel{\text{def}}{=} \mathrm{e}^{-t\mathcal{L}_0} \mathcal{L}_1(t)\, \mathrm{e}^{t\mathcal{L}_0}.$$

MZ approach Cont.

- $\mathcal{L}_0$ depends only on the phase variable a, representing the velocity field, but not on the phase variables b associated with the random forcing term.
- PDF of $u(x, t; \omega)$ can be, in principle, obtained by inverting equation and then integrating it with respect to b.
- This operation can be conveniently represented in terms of an orthogonal projection operator.
- So we can define the following projection ($q(b)$ denotes the joint PDF of the random vector ξ appearing in the forcing term):

$$\mathcal{P}p(t) \stackrel{\text{def}}{=} q(\boldsymbol{b}) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(t)\, \mathrm{d}\boldsymbol{b},$$

MZ approach Cont.

- With some simplification we can write:

$$\frac{\partial p_u(t)}{\partial t} = \mathcal{L}_0 p_u(t) + \mathrm{e}^{t\mathcal{L}_0} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial \mathcal{P}w(t)}{\partial t} \, \mathrm{d}\boldsymbol{b}.$$

- Or in projected space:

$$\frac{\partial \mathcal{P}w(t)}{\partial t} = \hat{\mathcal{K}}(t)\mathcal{P}w(t) + \hat{\mathcal{H}}(t)\mathcal{Q}w(0),$$

where $\mathcal{Q} \stackrel{\text{def}}{=} \mathcal{I} - \mathcal{P}$,

$$\hat{\mathcal{K}}(t) \stackrel{\text{def}}{=} \sigma \mathcal{P}\mathcal{N}(t)[\mathcal{I} - \sigma \hat{\Sigma}(t)]^{-1},$$

$$\hat{\mathcal{H}}(t) \stackrel{\text{def}}{=} \sigma \mathcal{P}\mathcal{N}(t)[\mathcal{I} - \sigma \hat{\Sigma}(t)]^{-1}\hat{\mathcal{G}}(t,0)$$

$$\hat{\Sigma}(t) \stackrel{\text{def}}{=} \int_0^t \hat{\mathcal{G}}(t,s)\mathcal{Q}\mathcal{N}(s)\mathcal{P}\hat{\mathcal{Z}}(t,s)\,\mathrm{d}s,$$

and $$\hat{\mathcal{G}}(t,s) \stackrel{\text{def}}{=} \overleftarrow{\mathcal{T}} \exp\left[\sigma \int_s^t \mathcal{Q}\mathcal{N}(\tau)\,\mathrm{d}\tau\right], \quad \hat{\mathcal{Z}}(t,s) \stackrel{\text{def}}{=} \overrightarrow{\mathcal{T}} \exp\left[-\sigma \int_s^t \mathcal{N}(\tau)\,\mathrm{d}\tau\right].$$

Kinetic equations for the two-point PDF

Almost same story can be repeated for joint distribution, but clearly more complicated:

$$p_2(x_1, x_2, t; a_1, a_2, \boldsymbol{b}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{2} \delta(a_i - U(x_i, t; \boldsymbol{A}_0, \boldsymbol{B}))$$
$$\times\, \delta(\boldsymbol{b} - \boldsymbol{B}) q(\boldsymbol{A}_0, \boldsymbol{B})\, \mathrm{d}\boldsymbol{A}_0\, \mathrm{d}\boldsymbol{B}.$$

Such PDF satisfies the obvious limiting condition

$$\lim_{x_1 \to x_2} p_2(x_1, x_2, t; a_1, a_2, \boldsymbol{b}) = \delta(a_1 - a_2) p(x_1, t; a_1, \boldsymbol{b}),$$

$$\frac{\partial p_2(t)}{\partial t} = [\mathcal{H}_0 + \sigma \mathcal{H}_1(t)] p_2(t),$$

where

$$\mathcal{H}_0 \stackrel{\text{def}}{=} -\sum_{i=1}^{2} \left(\int_{-\infty}^{a_i} \mathrm{d}a_i' \frac{\partial}{\partial x_i} + a_i \frac{\partial}{\partial x_i} \right), \quad \mathcal{H}_1(t) \stackrel{\text{def}}{=} -\sum_{i=1}^{2} f(x_i, t; \boldsymbol{b}) \frac{\partial}{\partial a_i}.$$

Simulation

$$u_0(x;\eta) = \sin(x) + \eta(\omega)$$

$$f(x,t;\omega) = \xi(\omega)\sin(t),$$

where $\xi(\omega)$ and $\eta(\omega)$ are independent zero-mean Gaussian random variables with standard deviation $\pi/10$ and 1, respectively. In this hypothesis, the joint PDF of u_0 and ξ is

$$p(0) = \frac{5}{\pi^2}\exp\left[-\frac{(a-\sin(x))^2}{2}\right]\exp\left[-50\frac{b^2}{\pi^2}\right].$$

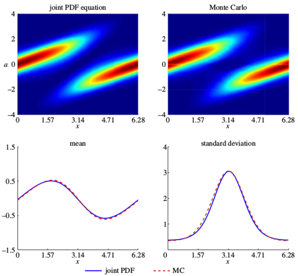

Figure 3. PDF of the velocity field: validation of the joint PDF equation. Shown is a comparison between the marginalized solution to equation (2.8) at $t = 1$ and a kernel density estimation [53] of the PDF of the velocity based on 50 000 MC samples. We also compare the mean and the standard deviation at $t = 1$ as computed from the PDF and the MC approaches. (Online version in colour.)

Effect of approximation

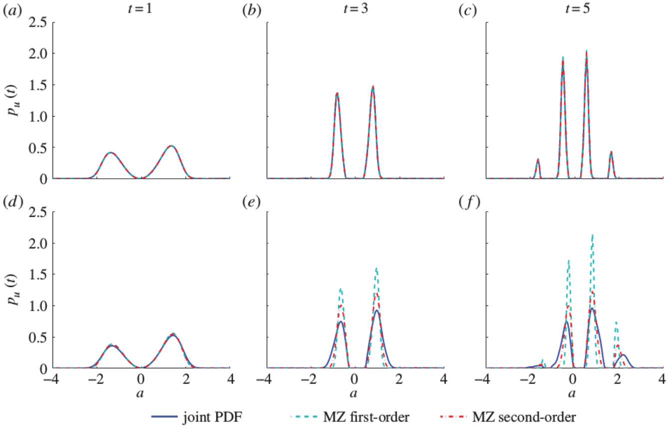

Figure 5. Randomly forced Burgers equation. One-point PDF of the velocity field at $x = \pi$ for exponentially correlated, homogeneous (in space) random forcing processes with correlation time $\tau = 0.01$ and amplitude $\sigma = 0.01$ (*a*–*c*) and $\sigma = 0.1$ (*d*–*f*). Shown are results obtained from the joint PDF equation (2.9), and two different truncations of the MZ-PDF equation (2.28). (Online version in colour.)