

Derivations for the CSM model

1 Definition of the model

A gestalt, a perceptual object, is characterised by a covariance component for the joint distribution of visual neural activity.

$$p(v \mid g) = \mathcal{N}(v; 0, C_v) \quad (1)$$

$$C_v = \sum_{k=1}^K g_k U_k^T U_k \quad (2)$$

where K is the fixed number of possible gestalts in the visual scene and g_k is the strength of the gestalt number k , coming from a K -dimensional Gamma prior distribution with shape and scale parameters α_g and θ_g controlling the sparsity of the prior.

$$p(g) = \text{Gam}(g; \alpha_g, \theta_g) \quad (3)$$

The global contrast of the image patch is encoded by a scalar variable z , also coming from a Gamma prior

$$p(z) = \text{Gam}(z; \alpha_z, \theta_z) \quad (4)$$

The pixel intensities are generated from the neural activity through a set of linear projective field models, possibly Gabor filters, A , scaled by the contrast and adding some independent observational noise.

$$p(x \mid v, z) = \mathcal{N}(x; zAv, \sigma_x I) \quad (5)$$

2 Likelihoods

2.1 Likelihood of g and z

by intuition:

$$p(x \mid z, g) = \mathcal{N}(x; 0, \sigma_x I + z^2 A \left(\sum_{k=1}^K g_k U_k^T U_k \right) A^T) \quad (6)$$

by algebraic derivation (see Seq. 4.4):

$$p(x | z, g) = \int_{-\infty}^{\infty} p(x, v | z, g) dv = \frac{1}{z^{D_v} \sqrt{\det(A^T A)}} \mathcal{N}\left(\frac{1}{z} A^+ x; 0, \frac{\sigma_x}{z^2} (A^T A)^{-1} + \sum_{k=1}^K g_k U_k^T U_k\right) \quad (7)$$

$$f(x, z) \equiv \frac{1}{z} A^+ x, \quad h(z) \equiv \frac{1}{z^{D_v}} \quad (8)$$

$$C(z, g) \equiv \frac{\sigma_x}{z^2} (A^T A)^{-1} + \sum_{k=1}^K g_k U_k^T U_k \quad (9)$$

$$p(x | z, g) = \frac{1}{\sqrt{\det(A^T A)}} h(z) \mathcal{N}(f(x, z); 0, C(z, g)) \quad (10)$$

2.2 Log-likelihood of the parameters

All parameters of the model consist of

$$\zeta = \{\sigma_x, A, U_{1..K}, \alpha_g, \theta_g, \alpha_z, \theta_z\} \quad (11)$$

for n exchangeable observations of x , the likelihood looks like this

$$p(X | \zeta) = \prod_{n=1}^N p(x_n | \zeta) = \prod_{n=1}^N \iint_{-\infty}^{\infty} p(x_n | z, g) p(g) p(z) dg dz \quad (12)$$

approximation of the integrals by samples from the priors $p(g)$ and $p(z)$

$$p(X | \zeta) \approx \prod_{n=1}^N \frac{1}{L} \sum_{l=1}^L p(x_n | z^l, g^l) \quad (13)$$

using Eq. 10

$$p(X | \zeta) \approx \left(L \sqrt{\det(A^T A)} \right)^{-N} \prod_{n=1}^N \sum_{l=1}^L h(z^l) \mathcal{N}(f(x_n, z^l); 0, C(z^l, g^l)) \quad (14)$$

$$\log p(X | \zeta) \approx -N(\log L + \frac{1}{2} \log(\det(A^T A))) + \sum_{n=1}^N \log \left[\sum_{l=1}^L h(z^l) \mathcal{N}(f(x_n, z^l); 0, C(z^l, g^l)) \right] \quad (15)$$

$$h^l \equiv h(z^l), \quad f_n^l \equiv f(x_n, z^l), \quad C^l \equiv C(z^l, g^l) \quad (16)$$

$$\mathcal{L}_n^l \equiv h^l \mathcal{N}(f_n^l; 0, C^l), \quad \mathcal{L}_n \equiv \sum_{l=1}^L \mathcal{L}_n^l \quad (17)$$

$$\log p(X | \zeta) \approx -N(\log L + \frac{1}{2} \log(\det(A^T A))) + \sum_{n=1}^N \log \mathcal{L}_n \quad (18)$$

An equivalent way to write this based on Eq. 6 is the following

$$p(X | \zeta) \approx L^{-N} \prod_{n=1}^N \sum_{l=1}^L \mathcal{N}(x_n; 0, \sigma_x I + z^{l2} A C_v^l A^T) \quad (19)$$

$$C_x^l \equiv \sigma_x I + z^{l2} A C_v^l A^T \quad (20)$$

$$\log p(X | \zeta) \approx -N \log L + \sum_{n=1}^N \log \left[\sum_{l=1}^L \mathcal{N}(x_n; 0, C_x^l) \right] \quad (21)$$

2.2.1 Derivative w.r.t. $U_{1...K}$

Using Eq. 18

$$\begin{aligned} \frac{\partial \log p(X | \zeta)}{\partial [U_k]_{i,j}} &\approx \sum_{n=1}^N \frac{\partial \log \mathcal{L}_n}{\partial [U_k]_{i,j}} = \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \frac{\partial \mathcal{L}_n}{\partial [U_k]_{i,j}} = \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \sum_{l=1}^L \frac{\partial \mathcal{L}_n^l}{\partial [U_k]_{i,j}} = \\ &= \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \sum_{l=1}^L \text{Tr} \left[\frac{\partial \mathcal{L}_n^l}{\partial C^l} \frac{\partial C^l}{\partial [U_k]_{i,j}} \right] \end{aligned} \quad (22)$$

the derivatives in this formula are the following

$$\begin{aligned} \frac{\partial \mathcal{L}_n^l}{\partial C^l} &= h^l \frac{\partial}{\partial C^l} \mathcal{N}(f_n^l; 0, C^l) = h^l \mathcal{N}(f_n^l; 0, C^l) \frac{\partial}{\partial C^l} \log \mathcal{N}(f_n^l; 0, C^l) = \\ &= -\frac{h^l}{2} \mathcal{N}(f_n^l; 0, C^l) [(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}] \end{aligned} \quad (23)$$

$$\frac{\partial C(z, g)}{\partial [U_k]_{i,j}} = g_k \frac{\partial (U_k^T U_k)}{\partial [U_k]_{i,j}} = g_k (U_k^T J^{ij} + J^{ji} U_k) \equiv g_k \hat{U}_k^{ij} \quad (24)$$

substituting back to the derivative

$$\begin{aligned} \frac{\partial \log p(X | \zeta)}{\partial [U_k]_{i,j}} &\approx \\ &\approx -\frac{1}{2} \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \sum_{l=1}^L h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) \text{Tr} \left[[(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}] \hat{U}_k^{ij} \right] = \\ &= -\frac{1}{2} \text{Tr} \left[\sum_{n=1}^N \left(\frac{1}{\mathcal{L}_n} \sum_{l=1}^L h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) [(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}] \right) \hat{U}_k^{ij} \right] \end{aligned} \quad (25)$$

The regularities of the \hat{U}_k matrices allow us to replace the trace with a much more efficient computation:

$$M_k = -\frac{1}{2} \sum_{n=1}^N \left(\frac{1}{\mathcal{L}_n} \sum_{l=1}^L h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) [(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}] \right) \quad (26)$$

$$\frac{\partial \log p(X | \zeta)}{\partial [U_k]_{i,j}} \approx \text{Tr} [M_k \hat{U}_k^{ij}] = \sum_{a=1}^{Dv} [M_k]_{j,a} [U_k]_{i,a} + [M_k]_{a,j} [U_k]_{i,a} \quad (27)$$

2.2.2 Derivative w.r.t. σ_x

Using Eq. 21

$$\begin{aligned} \frac{\partial \log p(X | \zeta)}{\partial \sigma_x} &\approx \sum_{n=1}^N \frac{\partial}{\partial \sigma_x} \log \left[\sum_{l=1}^L \mathcal{N}(x_n; 0, C_x^l) \right] = \\ &= \sum_{n=1}^N \frac{1}{\sum_{l=1}^L \mathcal{N}(x_n; 0, C_x^l)} \sum_{l=1}^L \frac{\partial}{\partial \sigma_x} \mathcal{N}(x_n; 0, C_x^l) = \\ &= \sum_{n=1}^N \frac{1}{\sum_{l=1}^L \mathcal{N}(x_n; 0, C_x^l)} \sum_{l=1}^L \text{Tr} \left[\frac{\partial}{\partial C_x^l} \mathcal{N}(x_n; 0, C_x^l) \frac{\partial C_x^l}{\partial \sigma_x} \right] \end{aligned} \quad (28)$$

the derivatives in this formula are the following

$$\begin{aligned} \frac{\partial}{\partial C_x^l} \mathcal{N}(x_n; 0, C_x^l) &= \mathcal{N}(x_n; 0, C_x^l) \frac{\partial}{\partial C_x^l} \log \mathcal{N}(x_n; 0, C_x^l) = \\ &= -\frac{1}{2} \mathcal{N}(x_n; 0, C_x^l) [(C_x^l)^{-1} - (C_x^l)^{-1} x_n x_n^T (C_x^l)^{-1}] \end{aligned} \quad (29)$$

$$\frac{\partial C_x^l}{\partial \sigma_x} = I \quad (30)$$

substituting back to the derivative

$$\begin{aligned} &\frac{\partial \log p(X | \zeta)}{\partial \sigma_x} \approx \\ &\approx -\frac{1}{2} \sum_{n=1}^N \frac{1}{\sum_{l=1}^L \mathcal{N}(x_n; 0, C_x^l)} \sum_{l=1}^L \mathcal{N}(x_n; 0, C_x^l) \text{Tr} [(C_x^l)^{-1} - (C_x^l)^{-1} x_n x_n^T (C_x^l)^{-1}] \end{aligned} \quad (31)$$

2.3 Complete-data log-likelihood

$$p(V, G, Z, X | \zeta) = \prod_{n=1}^N p(x_n | v_n, z_n) p(v_n | g_n) p(g_n) p(z_n) \quad (32)$$

the logarithm of this will be

$$\begin{aligned}
\log p(V, G, Z, X \mid \zeta) &= \\
&= \sum_{n=1}^N [\log p(g_n) + \log p(z_n) + \log p(x_n \mid v_n) + \log p(v_n \mid g_n)] = \\
&= N(\log p(g) + \log p(z)) + \sum_{n=1}^N \log p(x_n \mid v_n) + \log p(v_n \mid g_n) = \\
&= c + \sum_{n=1}^N \log p(v_n \mid g_n)
\end{aligned} \tag{33}$$

where c is constant with respect to the parameters $U_{1...K}$.

2.3.1 Expectation w.r.t. the posterior

$$\mathcal{L} = \iiint_{-\infty}^{\infty} p(V, G, Z \mid X) \log p(V, G, Z, X \mid \zeta) dV dG dZ. \tag{34}$$

We can approximate this integral by averaging over L samples from the full posterior, separately for each observation x_n , as defined in Seq. 3.1. As we will seek the values of the Cholesky components $U_{1...K}$ that maximise this integral, we can discard each term not depending on these parameters, only leaving the term of the form $p(v \mid g)$. This way we arrive to the following expression

$$\begin{aligned}
\mathcal{L} &\sim \sum_{n=1}^N \frac{1}{L} \sum_{l=1}^L -\frac{1}{2} \left[\log \left(\det \left(C_v^{(l,n)} \right) \right) + v^{(l,n)T} \left(C_v^{(l,n)} \right)^{-1} v^{l,n} \right] = \\
&= -\frac{1}{2L} \sum_{m=1}^{NL} \left[\log \left(\det \left(C_v^m \right) \right) + v^{mT} \left(C_v^m \right)^{-1} v^m \right]
\end{aligned} \tag{35}$$

noting that the double summation over L samples over all N observations always happens on the same terms, so we can substitute it with a single sum that iterates over the full sample set.

2.3.2 Derivative w.r.t. $U_{1...K}$

Using the derivative of C_v with respect to U_k^{ij} as defined in Eq. 24, by the chain rule, the derivative of \mathcal{L} according to an element of U_k looks like this

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial [U_k]_{i,j}} &= -\frac{1}{2L} \sum_{m=1}^{NL} \text{Tr} \left[\frac{\partial \mathcal{L}^m}{\partial C_v^m} \frac{\partial C_v^m}{\partial [U_k]_{i,j}} \right] = \\
&- \frac{1}{2L} \sum_{m=1}^{LN} \text{Tr} \left[\left[(C_v^m)^{-1} - (C_v^m)^{-1} v^m v^{mT} (C_v^m)^{-1} \right] g_k^m \hat{U}_k^{ij} \right] = \\
&- \frac{1}{2L} \text{Tr} \left[\sum_{m=1}^{LN} g_k^m \left[(C_v^m)^{-1} - (C_v^m)^{-1} (v^m v^{mT}) (C_v^m)^{-1} \right] \hat{U}_k^{ij} \right]
\end{aligned} \tag{36}$$

The regularities of the \hat{U}_k matrices allow us to replace the trace with a much more efficient computation:

$$M_k = -\frac{1}{2L} \left[\sum_{m=1}^{LN} g_k^m \left[(C_v^m)^{-1} - (C_v^m)^{-1} (v^m v^{mT}) (C_v^m)^{-1} \right] \right] \tag{37}$$

$$\frac{\partial \mathcal{L}}{\partial [U_k]_{i,j}} = \sum_{a=1}^{Dv} [M_K]_{j,a} [U_k]_{i,a} + [M_k]_{a,j} [U_k]_{i,a} \tag{38}$$

3 Posteriors

3.1 Full posterior

$$p(v, g, z | x) = p(x | v, g, z) p(v | g) p(g) p(z) \frac{1}{p(x)} \sim p(x | v, z) p(v | g) p(g) p(z) \tag{39}$$

so the log-posterior will be the following, up to an additive constant, using Gamma priors over g and z defined by shape and scale parameters:

$$\begin{aligned}
\log p(v, g, z | x) &\sim \log p(x | v, z) + \log p(v | g) + \log p(g) + \log p(z) = \\
&= \log \mathcal{N}(x; zAv, \sigma_x I) + \log \mathcal{N}(v; 0, C_v) + \log \text{Gam}(g; \alpha_g, \theta_g) + \log \text{Gam}(z; \alpha_z, \theta_z)
\end{aligned} \tag{40}$$

using the logarithms of the used pdfs from Sec. 4.2 and discarding all terms not dependent on any of the three variables we get

$$\begin{aligned}
\log p(v, g, z | x) &\sim -\frac{1}{2\sigma_x} (x - zAv)^T (x - zAv) - \\
&- \frac{1}{2} \left[\log(\det(C_v)) + v^T C_v^{-1} v \right] + \\
&+ \sum_{j=1}^K \left[(\alpha_g - 1) \log(g_j) - \frac{g_j}{\theta_g} \right] + (\alpha_z - 1) \log(z) - \frac{z}{\theta_z}
\end{aligned} \tag{41}$$

rearranging the first quadratic term according to Eq. 84 and discarding the term not dependent on v yields

$$\begin{aligned} \log p(v, g, z | x) \sim & -\frac{z}{2\sigma_x} (zv^T A^T A v - 2x^T A v) - \\ & -\frac{1}{2} [\log(\det(C_v)) + v^T C_v^{-1} v] + \\ & + \sum_{j=1}^K \left[(\alpha_g - 1) \log(g_j) - \frac{g_j}{\theta_g} \right] + \\ & + (\alpha_z - 1) \log(z) - \frac{z}{\theta_z} \end{aligned} \quad (42)$$

3.1.1 Derivative w.r.t. v

$$\log p(v, g, z | x) \sim -\frac{1}{2} \left[\frac{z^2}{\sigma_x} v^T A^T A v - \frac{2z}{\sigma_x} x^T A v + v^T C_v^{-1} v \right] + f_1(g, z) \quad (43)$$

lumping the two quadratic forms together

$$\log p(v, g, z | x) \sim \frac{z}{\sigma_x} x^T A v - \frac{1}{2} v^T \left[\frac{z^2}{\sigma_x} A^T A + C_v^{-1} \right] v + f_1(g, z) \quad (44)$$

Taking the derivative using Eq. 72 and 73 we get

$$\frac{\partial}{\partial v} \log p(v, g, z | x) = \frac{z}{\sigma_x} A^T x - \left[\frac{z^2}{\sigma_x} A^T A + C_v^{-1} \right] v \quad (45)$$

3.1.2 Derivative w.r.t. g

$$\begin{aligned} \log p(v, g, z | x) \sim & -\frac{1}{2} [\log \det(C_v) + v^T C_v^{-1} v] + \\ & + \sum_{j=1}^K \left[(\alpha_g - 1) \log(g_j) - \frac{g_j}{\theta_g} \right] + f_2(v, z) \end{aligned} \quad (46)$$

Taking the derivative w.r.t. a single g_i using Eq. 76 we get

$$\begin{aligned} \frac{\partial}{\partial g_i} \log p(v, g, z | x) = & -\frac{1}{2} \text{Tr} \left[\frac{\partial}{\partial C_v} [\log \det(C_v) + v^T C_v^{-1} v] \frac{\partial C_v}{\partial g_i} \right] + \\ & + \frac{\partial}{\partial g_i} \left[(\alpha_g - 1) \log(g_i) - \frac{g_i}{\theta_g} \right] \end{aligned} \quad (47)$$

using Eq. 75, 74 and 77 we arrive to

$$\frac{\partial}{\partial g_i} \log p(v, g, z | x) = -\frac{1}{2} \text{Tr} \left[[C_v^{-1} - C_v^{-1} v v^T C_v^{-1}] C_i \right] + \frac{\alpha_g - 1}{g_i} - \frac{1}{\theta_g} \quad (48)$$

3.1.3 Derivative w.r.t. z

$$\log p(v, g, z | x) \sim -\frac{z}{2\sigma_x} (zv^T A^T Av - 2x^T Av) + (\alpha_z - 1) \log(z) - \frac{z}{\theta_z} + f_3(g, v) \quad (49)$$

$$\frac{\partial}{\partial z} \log p(v, g, z | x) = \frac{1}{\sigma_x} [x^T Av - zv^T A^T Av] + \frac{\alpha_z - 1}{z} - \frac{1}{\theta_z} \quad (50)$$

3.2 Conditional posteriors

3.2.1 Conditional posterior of v

$$p(v | x, g, z) = \frac{p(x | v, z, g)p(v | z, g)}{p(x | z, g)} = \frac{\mathcal{N}(x; zAv, \sigma_x I) \mathcal{N}(v; 0, C_v)}{\int_{-\infty}^{\infty} \mathcal{N}(x; zAv, \sigma_x I) \mathcal{N}(v; 0, C_v) dv} \quad (51)$$

the product of two Gaussians in the numerator of Eq. 51 can also be written as a Gaussian over v as in Seq. 4.3:

$$\mathcal{N}(x; zAv, \sigma_x I) \mathcal{N}(v; 0, C_v) = c \mathcal{N}(v; \mu_{post}, C_{post}) \quad (52)$$

The denominator of Eq. 51 is the integral of this formula, which evaluates to c , as the Gaussian integrates to one. This cancels the constant in the numerator, making the conditional posterior equal to the combined Gaussian over v , which, after expanding μ_{post} and C_{post} , is

$$p(v | x, g, z) = \mathcal{N} \left(v; \frac{z}{\sigma_x} \left(\frac{z^2}{\sigma_x} A^T A + C_v^{-1} \right)^{-1} A^T x, \left(\frac{z^2}{\sigma_x} A^T A + C_v^{-1} \right)^{-1} \right) \quad (53)$$

3.2.2 Conditional posterior of g

$$p(g | X, V, z) = \frac{p(X | g, V, z)p(g | V, z)}{p(X | V, z)} = \frac{p(V | g)p(g)}{p(V)} \quad (54)$$

taking the logarithm and discarding constant terms

$$\log p(g | X, V) \sim -\frac{1}{2} [\log(\det(C_v)) + v^T C_v^{-1} v] + \log p(g) \quad (55)$$

The unnormalised conditionals of single elements of g , assuming an independent prior look as follows

$$\begin{aligned} \log p(g_j | g_{\neg j}, X, V) &= \frac{p(V | g_j, g_{\neg j}, X)p(g_j | g_{\neg j}, X)}{p(V | g_{\neg j}, X)} = \\ &= \frac{p(V | g)p(g_j)}{p(V | g_{\neg j})} \sim p(V | g)p(g_j) \end{aligned} \quad (56)$$

3.2.3 Conditional posterior of z

$$p(z \mid X, V, g) = \frac{p(X \mid g, z, V)p(z \mid V, g)}{p(X \mid V, g)} \sim p(X \mid z, V)p(z) \quad (57)$$

the log-posterior being

$$\log p(z \mid X, V) \sim -\frac{1}{2} \left[D_x \log(\sigma_x) + \frac{1}{\sigma_x} (x - zAv)^T (x - zAv) \right] + \log p(z) \quad (58)$$

3.3 Marginal posteriors

3.3.1 Marginal posterior of g and z

$$p(g, z \mid x) \sim p(x \mid g, z)p(g)p(z) \quad (59)$$

from Eq. 6

$$\begin{aligned} \log p(g, z \mid x) \sim & -\frac{1}{2} \left[\log \det(C_x) + x^T C_x^{-1} x \right] + (\alpha_z - 1) \log(z) - \frac{z}{\theta_z} + \\ & + (\alpha_g - 1) \sum_{k=1}^K \log(g_k) - \frac{1}{\theta_g} \sum_{k=1}^K g_k \end{aligned} \quad (60)$$

$$C_x = \sigma_x I + z^2 A \left(\sum_{k=1}^K g_k U_k^T U_k \right) A^T \quad (61)$$

the derivative w.r.t g_k

$$\begin{aligned} \frac{\partial}{\partial g_k} \log p(g, z \mid x) &= \text{Tr} \left[\frac{\partial}{\partial C_x} \log p(g, z \mid x) \frac{\partial C_x}{\partial g_k} \right] + \frac{\alpha_g - 1}{g_k} - \frac{1}{\theta_g} = \\ &= -\frac{z^2}{2} \text{Tr} \left[[C_x^{-1} - C_x^{-1} x x^T C_x^{-1}] A U_k^T U_k A^T \right] + \frac{\alpha_g - 1}{g_k} - \frac{1}{\theta_g} \end{aligned} \quad (62)$$

the derivative w.r.t z

$$\begin{aligned} \frac{\partial}{\partial z} \log p(g, z \mid x) &= \text{Tr} \left[\frac{\partial}{\partial C_x} \log p(g, z \mid x) \frac{\partial C_x}{\partial z} \right] + \frac{\alpha_z - 1}{z} - \frac{1}{\theta_z} = \\ &= -z \text{Tr} \left[[C_x^{-1} - C_x^{-1} x x^T C_x^{-1}] A C_v A^T \right] + \frac{\alpha_z - 1}{z} - \frac{1}{\theta_z} \end{aligned} \quad (63)$$

3.3.2 Marginal posterior of g

A maximum a posterior estimate of g can be given as follows

$$g_{MAP} = \arg \max_g p(g | x) = \arg \max_g \frac{p(x | g)p(g)}{p(x)} = \arg \max_g p(x | g)p(g) \quad (64)$$

$$p(x | g) = \int_{-\infty}^{\infty} p(x | z, g)p(z)dz \approx \frac{1}{L} \sum_{l=1}^L p(x | g, z^l) \quad (65)$$

3.3.3 Marginal posterior of v

$$p(v | x) = \iint_{-\infty}^{\infty} p(v | x, g, z)p(g, z | x)dgdz \quad (66)$$

$$p(v | x) \approx \frac{1}{L} \sum_{l=1}^L p(v | x, g^l, z^l), \quad g^l, z^l \sim p(g, z | x) \quad (67)$$

where $p(v | x, g, z)$ is given by Eq. 53, so we approximate the marginal posterior with a finite mixture of Gaussians, for which the covariance is given in the following form

$$C_{v|x} \approx \frac{1}{L} \sum_{l=1}^L C_{v|xgz}^l + (\mu_{v|xgz}^l - \frac{1}{L} \sum_{m=1}^L \mu_{v|xgz}^m)(\mu_{v|xgz}^l - \frac{1}{L} \sum_{m=1}^L \mu_{v|xgz}^m)^T \quad (68)$$

$$C_{v|x} \approx E[C_{v|xgz}^l]_l + \text{Cov}[\mu_{v|xgz}^l]_l \quad (69)$$

$$C_{v|xgz} = \left(\frac{z^2}{\sigma_x} A^T A + \left[\sum_{k=1}^K g_k C_k \right]^{-1} \right)^{-1} \quad (70)$$

$$\mu_{v|xgz} = \frac{z}{\sigma_x} C_{v|xgz} A^T x \quad (71)$$

4 Appendix

4.1 Rules of differentiation

Assuming that y and a are vectors and M is a symmetric matrix of appropriate dimension, and f is a scalar function, and s is a scalar variable.

$$\frac{\partial}{\partial y} y^T M y = 2 M y \quad (72)$$

$$\frac{\partial}{\partial y} a^T y = a \quad (73)$$

$$\frac{\partial}{\partial M} y^T M^{-1} y = -M^{-1} y y^T M^{-1} \quad (74)$$

$$\frac{\partial}{\partial M} \log \det M = M^{-1} \quad (75)$$

$$\frac{\partial}{\partial s} f(M(s)) = \text{Tr} \left[\frac{\partial f}{\partial M} \frac{\partial M}{\partial s} \right] \quad (76)$$

$$\frac{\partial}{\partial s} s M = M \quad (77)$$

$$\frac{\partial}{\partial s} f(s) = f(s) \frac{\partial}{\partial s} \log f(s) \quad (78)$$

$$\frac{\partial}{\partial M} \log \mathcal{N}(y; a, M) = -\frac{1}{2} [M^{-1} - M^{-1}(y - a)(y - a)^T M^{-1}] \quad (79)$$

4.2 Logarithms of used PDFs

$$\log \mathcal{N}(y; \mu, C) = -\frac{1}{2} [D \log(2\pi) + \log \det(C) + (y - \mu)^T C^{-1} (y - \mu)] \quad (80)$$

$$\log \text{Gam}(y; \alpha, \zeta) = \log(1) - \log(\Gamma(\alpha)) - \alpha \log(\zeta) + (\alpha - 1) \log(y) - \frac{y}{\zeta} \quad (81)$$

4.3 Merging two Gaussian distributions

We want to merge two Gaussians over x and v into one over v

$$p(x | v, z) p(v | g) = \mathcal{N}(x; z A v, \sigma_x I) \mathcal{N}(v; 0, C_v) \quad (82)$$

The Gaussian over x spelled out is

$$\mathcal{N}(x; z A v, \sigma_x I) = \sqrt{\frac{1}{(2\pi)^{D_x} \sigma_x^{D_x}}} e^{-\frac{1}{2\sigma_x} (x - z A v)^T (x - z A v)} \quad (83)$$

rearranging the quadratic term:

$$\begin{aligned} -\frac{1}{2} (x - z A v)^T (x - z A v) &= -\frac{1}{2} (x^T x - z v^T A^T x - z x^T A v + z^2 v^T A^T A v) = \\ &= -\frac{1}{2} (x - z A v)^T (x - z A v) = -\frac{1}{2} (x^T x - 2 z x^T A v + z^2 v^T A^T A v) = \\ &= -\frac{x^T x}{2} + z x^T A v - \frac{z^2}{2} v^T A^T A v \end{aligned} \quad (84)$$

as $v^T A^T x = (x^T A v)^T$, and both are scalars, thus equal to their transposes, it's also true that $v^T A^T x = x^T A v$. We have the identity for any symmetric matrix M and vector b that

$$-\frac{1}{2}v^T M v + b^T v = -\frac{1}{2}(v - M^{-1}b)^T M (v - M^{-1}b) + \frac{1}{2}b^T M^{-1}b \quad (85)$$

making the substitution $M = z^2 A^T A$ and $b = (z x^T A)^T = z A^T x$, yielding $M^{-1} = \frac{1}{z^2} (A^T A)^{-1}$ and $M^{-1}b = \frac{1}{z} (A^T A)^{-1} A^T x = \frac{1}{z} A^+ x$, where A^+ is the Moore-Penrose pseudoinverse of A . Thus we get

$$\begin{aligned} & -\frac{1}{2}(x - z A v)^T (x - z A v) = \\ & = -\frac{x^T x}{2} - \frac{1}{2}\left(v - \frac{1}{z} A^+ x\right)^T z^2 A^T A \left(v - \frac{1}{z} A^+ x\right) + \frac{1}{2}(A^T x)^T (A^T A)^{-1} A^T x = \\ & = -\frac{x^T x}{2} - \frac{1}{2}\left(v - \frac{1}{z} A^+ x\right)^T z^2 A^T A \left(v - \frac{1}{z} A^+ x\right) + \frac{1}{2}x^T A A^{-1} A^{-T} A^T x = \\ & = -\frac{x^T x}{2} - \frac{1}{2}\left(v - \frac{1}{z} A^+ x\right)^T z^2 A^T A \left(v - \frac{1}{z} A^+ x\right) + \frac{x^T x}{2} = \\ & = -\frac{1}{2}\left(v - \frac{1}{z} A^+ x\right)^T z^2 A^T A \left(v - \frac{1}{z} A^+ x\right) \end{aligned} \quad (86)$$

which implies

$$e^{-\frac{1}{2\sigma_x}(x - z A v)^T (x - z A v)} = e^{-\frac{1}{2}\left(v - \frac{1}{z} A^+ x\right)^T \frac{z^2}{\sigma_x} A^T A \left(v - \frac{1}{z} A^+ x\right)} \quad (87)$$

meaning that

$$\mathcal{N}(x; z A v, \sigma_x I) = \alpha \mathcal{N}\left(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}\right) \quad (88)$$

and as the formulas in the exponents are equal, the constant α is given by the ratio of the normalisation terms

$$\sqrt{\frac{1}{(2\pi)^{D_x} \sigma_x^{D_x}}} = \alpha \sqrt{\frac{1}{(2\pi)^{D_v} \det(\frac{\sigma_x}{z^2} (A^T A)^{-1})}} \quad (89)$$

$$\alpha = \sqrt{\frac{(2\pi)^{D_v} \frac{\sigma_x^{D_v}}{z^{2D_v}} \det((A^T A)^{-1})}{(2\pi)^{D_x} \sigma_x^{D_x}}} \quad (90)$$

$$\alpha = \sqrt{\frac{(2\pi)^{D_v} \sigma_x^{D_v}}{(2\pi)^{D_x} \sigma_x^{D_x} z^{2D_v} \det(A^T A)}} \quad (91)$$

making the simplifying assumption $D_x = D_v$ we arrive to

$$\mathcal{N}(x; zAv, \sigma_x I) = \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}) \quad (92)$$

we can merge two Gaussian distributions over v into one by using the following formula

$$\mathcal{N}(v; \mu_1, C_1) \mathcal{N}(v; \mu_2, C_2) = \mathcal{N}(\mu_1; \mu_2, C_1 + C_2) \mathcal{N}(v; \mu_c, C_c) \quad (93)$$

where $C_c = (C_1^{-1} + C_2^{-1})^{-1}$ and $\mu_c = C_c(C_1^{-1}\mu_1 + C_2^{-1}\mu_2)$. Substitution to these formulas yields

$$\begin{aligned} & \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}) \mathcal{N}(v; 0, C_v) = \\ & \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}(\frac{1}{z} A^+ x; 0, \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v) \mathcal{N}(v; \mu_c, C_c) \end{aligned} \quad (94)$$

$$C_c = (\frac{z^2}{\sigma_x} (A^T A) + C_v^{-1})^{-1} \quad (95)$$

$$\mu_c = C_c \frac{z}{\sigma_x} (A^T A) A^+ x = \frac{z}{\sigma_x} C_c A^T x \quad (96)$$

4.4 Equivalence of the two likelihood formulas of CSM

Expanding [10](#) yields

$$\begin{aligned} p(x | z, g) &= \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \frac{1}{\sqrt{(2\pi)^{D_v} \det C(z, g)}} e^{-\frac{1}{2} (\frac{1}{z} A^+ x)^T C^{-1}(z, g) \frac{1}{z} A^+ x} = \\ &= \frac{1}{\sqrt{(2\pi z^2)^{D_v} \det(A^T A C(z, g))}} e^{-\frac{1}{2z^2} x^T A^{+T} C^{-1}(z, g) A^+ x} \\ & \quad C(z, g) \equiv \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \end{aligned} \quad (97)$$

so in the exponent, in the place of the covariance matrix, we have

$$\begin{aligned} & \frac{1}{z^2} ((A^T A)^{-1} A^T)^T \left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right]^{-1} (A^T A)^{-1} A^T = \\ &= \frac{1}{z^2} A (A^T A)^{-1} \left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right]^{-1} (A^T A)^{-1} A^T = \\ &= \frac{1}{z^2} A \left[\left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right] (A^T A) \right]^{-1} (A^T A)^{-1} A^T = \\ &= \frac{1}{z^2} A \left[\frac{\sigma_x}{z^2} I + C_v A^T A \right]^{-1} (A^T A)^{-1} A^T = \\ &= \frac{1}{z^2} A \left[(A^T A) \left[\frac{\sigma_x}{z^2} I + C_v A^T A \right] \right]^{-1} A^T = \\ &= \frac{1}{z^2} A \left[\frac{\sigma_x}{z^2} A^T A + A^T A C_v A^T A \right]^{-1} A^T \end{aligned} \quad (99)$$

assuming that A is invertible this is equal to

$$\begin{aligned}
& \frac{1}{z^2} [A^{-1}]^{-1} \left[\frac{\sigma_x}{z^2} A^T A + A^T A C_v A^T A \right]^{-1} [A^{-T}]^{-1} = \\
& \frac{1}{z^2} [A^{-1}]^{-1} \left[A^{-T} \left[\frac{\sigma_x}{z^2} A^T A + A^T A C_v A^T A \right] \right]^{-1} = \\
& \frac{1}{z^2} [A^{-1}]^{-1} \left[\frac{\sigma_x}{z^2} A + A C_v A^T A \right]^{-1} = \\
& \frac{1}{z^2} \left[\left[\frac{\sigma_x}{z^2} A + A C_v A^T A \right] A^{-1} \right]^{-1} = \\
& \frac{1}{z^2} \left[\frac{\sigma_x}{z^2} I + A C_v A^T \right]^{-1} = \\
& = [\sigma_x I + z^2 A C_v A^T]^{-1}
\end{aligned} \tag{100}$$

under the square root we have

$$\begin{aligned}
& z^{2D_v} \det(A^T A C(z, g)) = z^{2D_v} \det(A^T) \det(A) \det(C(z, g)) = z^{2D_v} \det(A) \det(C(z, g)) \det(A^T) \\
& = z^{2D_v} \det(A) \det\left(\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v\right) \det(A^T) = z^{2D_v} \det(A) \det\left(\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v\right) \det(A^T) = \\
& = z^{2D_v} \det(A) \det\left(\frac{\sigma_x}{z^2} A^{-1} A^{-T} + C_v\right) \det(A^T) = z^{2D_v} \det \left[A \left[\frac{\sigma_x}{z^2} A^{-1} A^{-T} + C_v \right] A^T \right] = \\
& = z^{2D_v} \det \left[\frac{\sigma_x}{z^2} I + A C_v A^T \right] = \det [\sigma_x I + z^2 A C_v A^T]
\end{aligned} \tag{101}$$

4.5 Precision component formulation of the CSM model

The model can be equally well parametrised by precision components

$$p(v | g) = \mathcal{N}(v; 0, \Lambda_v^{-1}) \tag{102}$$

$$\Lambda_v = \sum_{k=1}^K g_k \Lambda_k \tag{103}$$

in this case the conditional posterior over v takes the form

$$p(v | x, g) = \mathcal{N} \left(v; \frac{1}{\sigma_x} \left(\frac{1}{\sigma_x} A^T A + \Lambda_v \right)^{-1} A^T x, \left(\frac{1}{\sigma_x} A^T A + \Lambda_v \right)^{-1} \right) \tag{104}$$

and the conditional posterior of g will look as follows

$$\log p(g | X, V) \sim -\frac{1}{2} [\log(\det(\Lambda_v^{-1})) + v^T \Lambda_v v] + \log p(g) \tag{105}$$

The gradient of the expectation of the complete-data log-likelihood with respect to the joint posterior will look like this

$$\Lambda_k = U_k^T U_k \quad (106)$$

$$\frac{\partial \mathcal{L}}{\partial [U_k]_{i,j}} = \frac{1}{L} \sum_{m=1}^{LN} g_k^m \text{Tr} \left[\left[(\Lambda_v^m)^{-1} - v^m v^{mT} \right] \hat{U}_k^{ij} \right] \quad (107)$$

4.6 Batches of observations

For a single set of component activations g and contrast z , we might have a batch of v and x values of size B . This modifies expressions as follows.

Conditional posterior of g (Eq. 55)

$$\log p(g \mid X, V) \sim -\frac{1}{2} \left[B \log(\det(C_v)) + \sum_{b=1}^B v_b^T C_v^{-1} v_b \right] + \log p(g) \quad (108)$$

Conditional posterior of z (Eq. 58)

$$\log p(z \mid X, V) \sim -\frac{1}{2} \left[B D_x \log(\sigma_x) + \frac{1}{\sigma_x} \sum_{b=1}^B (x_b - z A v_b)^T (x_b - z A v_b) \right] + \log p(z) \quad (109)$$

Matrix formula in the derivative of the expectation of the complete-data log-likelihood (Eq. 37)

$$M_k = -\frac{1}{2L} \left[\sum_{m=1}^{LN} g_k^m \left[B (C_v^m)^{-1} - (C_v^m)^{-1} \left(\sum_{b=1}^B v^{m,b} v^{(m,b)T} \right) (C_v^m)^{-1} \right] \right] \quad (110)$$

4.7 Latents affecting the mean

of v , instead of the covariance

$$p_m(v \mid g) = \mathcal{N}(v; Bg, \sigma_v I) \quad (111)$$

$$p_m(v \mid x, g, z) = \mathcal{N}(v; \mu_m, C_m) \quad (112)$$

$$C_m = \left(\frac{z^2}{\sigma_x} A^T A + \frac{1}{\sigma_v} I \right)^{-1} \quad (113)$$

$$\mu_m = C_m \left(\frac{z}{\sigma_x} A^T x + \frac{1}{\sigma_v} Bg \right) \quad (114)$$