

Derivations for the CSM model

1 Definition of the model

A gestalt, a perceptual object, is characterised by a covariance component for the joint distribution of visual neural activity.

$$p(v \mid g) = \mathcal{N}(v; 0, C_v) \quad (1)$$

$$C_v = \sum_{k=1}^K g_k C_k \quad (2)$$

where K is the fixed number of possible gestalts in the visual scene and g_k is the strength of the gestalt number k , coming from a K -dimensional Gamma prior distribution with shape and scale parameters α_g and θ_g controlling the sparsity of the prior.

$$p(g) = \text{Gam}(g; \alpha_g, \theta_g) \quad (3)$$

The global contrast of the image patch is encoded by a scalar variable z , also coming from a Gamma prior

$$p(z) = \text{Gam}(z; \alpha_z, \theta_z) \quad (4)$$

The pixel intensities are generated from the neural activity through a set of linear projective field models, possibly Gabor filters, A , scaled by the contrast and adding some independent observational noise.

$$p(x \mid v, z) = \mathcal{N}(x; zAv, \sigma_x I) \quad (5)$$

2 Likelihoods

2.1 Likelihood of g and z

by intuition:

$$p(x \mid z, g) = \mathcal{N}(x; 0, \sigma_x I + z^2 A \left(\sum_{k=1}^K g_k U_k^T U_k \right) A^T) \quad (6)$$

by algebraic derivation:

$$p(x \mid z, g) = \int_{-\infty}^{\infty} p(x, v \mid z, g) dv = \frac{1}{z^{D_v} \sqrt{\det(A^T A)}} \mathcal{N}\left(\frac{1}{z} A^+ x; 0, \frac{\sigma_x}{z^2} (A^T A)^{-1} + \sum_{k=1}^K g_k U_k^T U_k\right) \quad (7)$$

$$f(x, z) \equiv \frac{1}{z} A^+ x, \quad h(z) \equiv \frac{1}{z^{D_v}} \quad (8)$$

$$C(z, g) \equiv \frac{\sigma_x}{z^2} (A^T A)^{-1} + \sum_{k=1}^K g_k U_k^T U_k \quad (9)$$

$$p(x \mid z, g) = \frac{1}{\sqrt{\det(A^T A)}} h(z) \mathcal{N}(f(x, z); 0, C(z, g)) \quad (10)$$

2.2 Log-likelihood of the parameters

$$p(X \mid C_{1..K}) = \prod_{n=1}^N p(x_n \mid C_{1..K}) \quad (11)$$

$$p(X \mid C_{1..K}) = \prod_{n=1}^N \iint_{-\infty}^{\infty} p(x_n \mid z, g) p(g) p(z) dg dz \quad (12)$$

$$(13)$$

approximation of the integrals by samples from the priors $p(g_n)$ and $p(z_n)$

$$p(X \mid C_{1..K}) \approx \prod_{n=1}^N \frac{1}{L} \sum_{l=1}^L p(x_n \mid z^l, g^l) \quad (14)$$

using 6

$$\log p(X \mid C_{1..K}) \approx -N \log L + \sum_{n=1}^N \log \sum_{l=1}^L \mathcal{N}(x_n; 0, \sigma_x I + z^2 A \left(\sum_{k=1}^K g_k U_k^T U_k \right) A^T) \quad (15)$$

$$C_{int} = \sigma_x I + z^2 A \left(\sum_{k=1}^K g_k U_k^T U_k \right) A^T \quad (16)$$

$$\mathcal{L}_n^{int} = \sum_{l=1}^L \mathcal{N}(x_n; 0, C_{int}) \quad (17)$$

$$\log p(X \mid C_{1..K}) \approx -N \log L + \sum_{n=1}^N \log \mathcal{L}_n^{int} \quad (18)$$

using 10

$$p(X | C_{1..K}) \approx \left(L \sqrt{\det(A^T A)} \right)^{-N} \prod_{n=1}^N \sum_{l=1}^L h(z^l) \mathcal{N}(f(x_n, z^l); 0, C(z^l, g^l)) \quad (19)$$

$$\log p(X | C_{1..K}) \approx -N(\log L + \frac{1}{2} \log(\det(A^T A))) + \sum_{n=1}^N \log \left[\sum_{l=1}^L h(z^l) \mathcal{N}(f(x_n, z^l); 0, C(z^l, g^l)) \right] \quad (20)$$

$$h^l \equiv h(z^l), \quad f_n^l \equiv f(x_n, z^l), \quad C^l \equiv C(z^l, g^l) \quad (21)$$

$$\mathcal{L}_n^l \equiv h^l \mathcal{N}(f_n^l; 0, C^l), \quad \mathcal{L}_n \equiv \sum_{l=1}^L \mathcal{L}_n^l \quad (22)$$

$$\log p(X | C_{1..K}) \approx -N(\log L + \frac{1}{2} \log(\det(A^T A))) + \sum_{n=1}^N \log \mathcal{L}_n \quad (23)$$

2.2.1 Derivative w.r.t. $U_{1..K}$ based on 10

$$\begin{aligned} \frac{\partial \log p(X | C_{1..K})}{\partial [U_k]_{i,j}} &\approx \sum_{n=1}^N \frac{\partial \log \mathcal{L}_n}{\partial [U_k]_{i,j}} = \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \frac{\partial \mathcal{L}_n}{\partial [U_k]_{i,j}} = \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \sum_{l=1}^L \frac{\partial \mathcal{L}_n^l}{\partial [U_k]_{i,j}} = \\ &= \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \sum_{l=1}^L \text{Tr} \left[\frac{\partial \mathcal{L}_n^l}{\partial C^l} \frac{\partial C^l}{\partial [U_k]_{i,j}} \right] \end{aligned} \quad (24)$$

the derivatives in this formula are the following

$$\begin{aligned} \frac{\partial \mathcal{L}_n^l}{\partial C^l} &= h^l \frac{\partial}{\partial C^l} \mathcal{N}(f_n^l; 0, C^l) = h^l \mathcal{N}(f_n^l; 0, C^l) \frac{\partial}{\partial C^l} \log \mathcal{N}(f_n^l; 0, C^l) = \\ &= -\frac{h^l}{2} \mathcal{N}(f_n^l; 0, C^l) [(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}] \end{aligned} \quad (25)$$

$$\frac{\partial C(z, g)}{\partial [U_k]_{i,j}} = g_k \frac{\partial (U_k^T U_k)}{\partial [U_k]_{i,j}} = g_k (U_k^T J^{ij} + J^{ji} U_k) \equiv g_k \hat{U}_k^{ij} \quad (26)$$

substituting back to the derivative

$$\begin{aligned} \frac{\partial \log p(X | C_{1..K})}{\partial [U_k]_{i,j}} &\approx \\ &\approx -\frac{1}{2} \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \sum_{l=1}^L h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) \text{Tr} \left[[(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}] \hat{U}_k^{ij} \right] = \\ &= -\frac{1}{2} \text{Tr} \left[\sum_{n=1}^N \left(\frac{1}{\mathcal{L}_n} \sum_{l=1}^L h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) [(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}] \right) \hat{U}_k^{ij} \right] \end{aligned} \quad (27)$$

The regularities of the \hat{U}_k matrices allow us to replace the trace with a much more efficient computation:

$$M_k = -\frac{1}{2} \sum_{n=1}^N \left(\frac{1}{\mathcal{L}_n} \sum_{l=1}^L h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) [(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}] \right) \quad (28)$$

$$\frac{\partial \log p(X | C_{1..K})}{\partial [U_k]_{i,j}} \approx \text{Tr} [M_k \hat{U}_k^{ij}] = \sum_{a=1}^{Dv} [M_k]_{j,a} [U_k]_{i,a} + [M_k]_{a,j} [U_k]_{i,a} \quad (29)$$

2.2.2 Derivative w.r.t. $U_{1..K}$ based on 6

$$\begin{aligned} \frac{\partial \log p(X | C_{1..K})}{\partial [U_k]_{i,j}} &\approx \sum_{n=1}^N \frac{\partial \log \mathcal{L}_n^{int}}{\partial [U_k]_{i,j}} = \sum_{n=1}^N \frac{1}{\mathcal{L}_n^{int}} \frac{\partial \mathcal{L}_n^{int}}{\partial [U_k]_{i,j}} = \sum_{n=1}^N \frac{1}{\mathcal{L}_n^{int}} \sum_{l=1}^L \frac{\partial \mathcal{N}(x_n; 0, C_{int})}{\partial [U_k]_{i,j}} = \\ &= \sum_{n=1}^N \frac{1}{\mathcal{L}_n} \sum_{l=1}^L \text{Tr} \left[\frac{\partial \mathcal{N}(x_n; 0, C_{int})}{\partial C_{int}} \frac{\partial C_{int}}{\partial [U_k]_{i,j}} \right] \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{\partial \mathcal{N}(x_n; 0, C_{int})}{\partial C_{int}} &= \frac{\partial}{\partial C_{int}} \mathcal{N}(f_n^l; 0, C_{int}) = \mathcal{N}(f_n^l; 0, C_{int}) \frac{\partial}{\partial C_{int}} \log \mathcal{N}(f_n^l; 0, C_{int}) = \\ &= -\frac{1}{2} \mathcal{N}(f_n^l; 0, C_{int}) [(C_{int})^{-1} - (C_{int})^{-1} f_n^l (f_n^l)^T (C_{int})^{-1}] \\ \frac{\partial C(z, g)}{\partial [U_k]_{i,j}} &= g_k \frac{\partial (U_k^T U_k)}{\partial [U_k]_{i,j}} = g_k (U_k^T J^{ij} + J^{ji} U_k) \equiv g_k \hat{U}_k^{ij} \end{aligned} \quad (31)$$

2.3 Complete-data log-likelihood

2.3.1 Expectation w.r.t. the posterior

2.3.2 Derivative w.r.t. $C_{1..K}$

3 Posteriors

3.1 Full posterior

$$p(v, g, z | x) = p(x | v, g, z) p(v | g) p(g) p(z) \frac{1}{p(x)} \sim p(x | v, z) p(v | g) p(g) p(z) \quad (33)$$

so the log-posterior will be the following, up to an additive constant, using Gamma priors over g and z defined by shape and scale parameters:

$$\begin{aligned} \log p(v, g, z | x) &\sim \log p(x | v, z) + \log p(v | g) + \log p(g) + \log p(z) = \\ &= \log \mathcal{N}(x; zAv, \sigma_x I) + \log \mathcal{N}(v; 0, C_v) + \log \text{Gam}(g; sh_g, sc_g) + \log \text{Gam}(z; sh_z, sc_z) \end{aligned} \quad (34)$$

logarithms of the used pdfs look as follows:

$$\log \mathcal{N}(y; \mu, C) = -\frac{1}{2} [\log(2\pi) + \log \det(C) + (y - \mu)^T C^{-1} (y - \mu)] \quad (35)$$

$$\log \text{Gam}(y; sh, sc) = \log(1) - \log(\Gamma(sh)) - sh \log(sc) + (sh - 1) \log(y) - \frac{y}{sc} \quad (36)$$

3.1.1 Derivative w.r.t. v

3.1.2 Derivative w.r.t. g

3.1.3 Derivative w.r.t. z

3.2 Marginal posterior of g

3.2.1 MAP estimate of g

$$g_{MAP} = \arg \max_g p(g | x) = \arg \max_g \frac{p(x | g)p(g)}{p(x)} = \arg \max_g p(x | g)p(g) \quad (37)$$

$$p(x | g) = \int_{-\infty}^{\infty} p(x | z, g)p(z)dz \approx \frac{1}{L} \sum_{l=1}^L p(x | g, z^l) \quad (38)$$

Appendix

3.3 Merging two Gaussian distributions

We want to merge two Gaussians over x and v into one over v

$$p(x | v, z)p(v | g) = \mathcal{N}(x; zAv, \sigma_x I) \mathcal{N}(v; 0, C_v) \quad (39)$$

The Gaussian over x spelled out is

$$\mathcal{N}(x; zAv, \sigma_x I) = \sqrt{\frac{1}{(2\pi)^{D_x} \sigma_x^{D_x}}} e^{-\frac{1}{2\sigma_x} (x - zAv)^T (x - zAv)} \quad (40)$$

$$-\frac{1}{2} (x - zAv)^T (x - zAv) = -\frac{1}{2} (x^T x - zv^T A^T x - zx^T Av + z^2 v^T A^T Av) \quad (41)$$

as $v^T A^T x = (x^T Av)^T$, and both are scalars, thus equal to their transposes, it's also true that $v^T A^T x = x^T Av$

$$\begin{aligned} -\frac{1}{2} (x - zAv)^T (x - zAv) &= -\frac{1}{2} (x^T x - 2zx^T Av + z^2 v^T A^T Av) = \\ &= -\frac{x^T x}{2} + zx^T Av - \frac{z^2}{2} v^T A^T Av \end{aligned} \quad (42)$$

we have the identity for any symmetric matrix M and vector b that

$$-\frac{1}{2}v^T M v + b^T v = -\frac{1}{2}(v - M^{-1}b)^T M (v - M^{-1}b) + \frac{1}{2}b^T M^{-1}b \quad (43)$$

making the substitution $M = z^2 A^T A$ and $b = (zx^T A)^T = zA^T x$, yielding $M^{-1} = \frac{1}{z^2}(A^T A)^{-1}$ and $M^{-1}b = \frac{1}{z}(A^T A)^{-1}A^T x = \frac{1}{z}A^+x$, where A^+ is the Moore-Penrose pseudoinverse of A . Thus we get

$$\begin{aligned} & -\frac{1}{2}(x - zAv)^T (x - zAv) = \\ & = -\frac{x^T x}{2} - \frac{1}{2}\left(v - \frac{1}{z}A^+x\right)^T z^2 A^T A \left(v - \frac{1}{z}A^+x\right) + \frac{1}{2}(A^T x)^T (A^T A)^{-1} A^T x = \\ & = -\frac{x^T x}{2} - \frac{1}{2}\left(v - \frac{1}{z}A^+x\right)^T z^2 A^T A \left(v - \frac{1}{z}A^+x\right) + \frac{1}{2}x^T A A^{-1} A^{-T} A^T x = \\ & = -\frac{x^T x}{2} - \frac{1}{2}\left(v - \frac{1}{z}A^+x\right)^T z^2 A^T A \left(v - \frac{1}{z}A^+x\right) + \frac{x^T x}{2} = \\ & = -\frac{1}{2}\left(v - \frac{1}{z}A^+x\right)^T z^2 A^T A \left(v - \frac{1}{z}A^+x\right) \end{aligned} \quad (44)$$

which implies

$$e^{-\frac{1}{2\sigma_x}(x - zAv)^T (x - zAv)} = e^{-\frac{1}{2}\left(v - \frac{1}{z}A^+x\right)^T \frac{z^2}{\sigma_x} A^T A \left(v - \frac{1}{z}A^+x\right)} \quad (45)$$

meaning that

$$\mathcal{N}(x; zAv, \sigma_x I) = \alpha \mathcal{N}\left(v; \frac{1}{z}A^+x, \frac{\sigma_x}{z^2}(A^T A)^{-1}\right) \quad (46)$$

and as the formulas in the exponents are equal, the constant α is given by the ratio of the normalisation terms

$$\sqrt{\frac{1}{(2\pi)^{D_x} \sigma_x^{D_x}}} = \alpha \sqrt{\frac{1}{(2\pi)^{D_v} \det(\frac{\sigma_x}{z^2}(A^T A)^{-1})}} \quad (47)$$

$$\alpha = \sqrt{\frac{(2\pi)^{D_v} \frac{\sigma_x^{D_v}}{z^{2D_v}} \det((A^T A)^{-1})}{(2\pi)^{D_x} \sigma_x^{D_x}}} \quad (48)$$

$$\alpha = \sqrt{\frac{(2\pi)^{D_v} \sigma_x^{D_v}}{(2\pi)^{D_x} \sigma_x^{D_x} z^{2D_v} \det(A^T A)}} \quad (49)$$

making the simplifying assumption $D_x = D_v$ we arrive to

$$\mathcal{N}(x; zAv, \sigma_x I) = \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}\left(v; \frac{1}{z}A^+x, \frac{\sigma_x}{z^2}(A^T A)^{-1}\right) \quad (50)$$

we can merge two Gaussian distributions over v into one by using the following formula

$$\mathcal{N}(v; \mu_1, C_1) \mathcal{N}(v; \mu_2, C_2) = \mathcal{N}(\mu_1; \mu_2, C_1 + C_2) \mathcal{N}(v; \mu_c, C_c) \quad (51)$$

where $C_c = (C_1^{-1} + C_2^{-1})^{-1}$ and $\mu_c = C_c(C_1^{-1}\mu_1 + C_2^{-1}\mu_2)$. Substitution to these formulas yields

$$\begin{aligned} & \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}) \mathcal{N}(v; 0, C_v) = \\ & \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}(\frac{1}{z} A^+ x; 0, \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v) \mathcal{N}(v; \mu_c, C_c) \end{aligned} \quad (52)$$

$$C_c = (\frac{z^2}{\sigma_x} (A^T A) + C_v^{-1})^{-1} \quad (53)$$

$$\mu_c = C_c \frac{z}{\sigma_x} (A^T A) A^+ x = \frac{z}{\sigma_x} C_c A^T x \quad (54)$$

3.4 Rules of differentiation

Assuming that y and a are vectors and M is a symmetric matrix of appropriate dimension, and f is a scalar function, and s is a scalar variable.

$$\frac{\partial}{\partial y} y^T M y = 2M y \quad (55)$$

$$\frac{\partial}{\partial y} a^T y = a \quad (56)$$

$$\frac{\partial}{\partial M} y^T M^{-1} y = -M^{-1} y y^T M^{-1} \quad (57)$$

$$\frac{\partial}{\partial M} \log \det M = M^{-1} \quad (58)$$

$$\frac{\partial}{\partial s} f(M(s)) = \text{Tr} \left[\frac{\partial f}{\partial M} \frac{\partial M}{\partial s} \right] \quad (59)$$

$$\frac{\partial}{\partial s} s M = M \quad (60)$$

$$\frac{\partial}{\partial s} f(s) = f(s) \frac{\partial}{\partial s} \log f(s) \quad (61)$$

$$\frac{\partial}{\partial M} \log \mathcal{N}(y; a, M) = M^{-1} - M^{-1} (y - a)(y - a)^T M^{-1} \quad (62)$$

3.5 Equivalence of the likelihood formulas

Expanding [10](#) yields

$$\begin{aligned}
p(x \mid z, g) &= \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \frac{1}{\sqrt{(2\pi)^{D_v} \det C(z, g)}} e^{-\frac{1}{2} \left(\frac{1}{z} A^+ x \right)^T C^{-1}(z, g) \frac{1}{z} A^+ x} = \\
&= \frac{1}{\sqrt{(2\pi z^2)^{D_v} \det(A^T A C(z, g))}} e^{-\frac{1}{2z^2} x^T A^{+T} C^{-1}(z, g) A^+ x} \quad (63) \\
C(z, g) &\equiv \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \quad (64)
\end{aligned}$$

so in the exponent, in the place of the covariance matrix, we have

$$\begin{aligned}
&\frac{1}{z^2} ((A^T A)^{-1} A^T)^T \left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right]^{-1} (A^T A)^{-1} A^T = \\
&= \frac{1}{z^2} A (A^T A)^{-1} \left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right]^{-1} (A^T A)^{-1} A^T = \\
&= \frac{1}{z^2} A \left[\left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right] (A^T A) \right]^{-1} (A^T A)^{-1} A^T = \\
&= \frac{1}{z^2} A \left[\frac{\sigma_x}{z^2} I + C_v A^T A \right]^{-1} (A^T A)^{-1} A^T = \\
&= \frac{1}{z^2} A \left[(A^T A) \left[\frac{\sigma_x}{z^2} I + C_v A^T A \right] \right]^{-1} A^T = \\
&= \frac{1}{z^2} A \left[\frac{\sigma_x}{z^2} A^T A + A^T A C_v A^T A \right]^{-1} A^T \quad (65)
\end{aligned}$$

assuming that A is invertible this is equal to

$$\begin{aligned}
&\frac{1}{z^2} [A^{-1}]^{-1} \left[\frac{\sigma_x}{z^2} A^T A + A^T A C_v A^T A \right]^{-1} [A^{-T}]^{-1} = \\
&\frac{1}{z^2} [A^{-1}]^{-1} \left[A^{-T} \left[\frac{\sigma_x}{z^2} A^T A + A^T A C_v A^T A \right] \right]^{-1} = \\
&\frac{1}{z^2} [A^{-1}]^{-1} \left[\frac{\sigma_x}{z^2} A + A C_v A^T A \right]^{-1} = \\
&\frac{1}{z^2} \left[\left[\frac{\sigma_x}{z^2} A + A C_v A^T A \right] A^{-1} \right]^{-1} = \\
&\frac{1}{z^2} \left[\frac{\sigma_x}{z^2} I + A C_v A^T \right]^{-1} = \\
&= [\sigma_x I + z^2 A C_v A^T]^{-1} \quad (66)
\end{aligned}$$

under the square root we have

$$\begin{aligned}
z^{2D_v} \det(A^T A C(z, g)) &= \det \left(z^2 A^T A \left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right] \right) = \\
&= \det([\sigma_x I + z^2 A^T A C_v]) = \det([\sigma_x I + z^2 A^T C_v A]) \quad (67)
\end{aligned}$$