Model selection for the gestalt model

1 The likelihood function

In order to evaluate the log-likelihood for cross validation steps, we have to use an approximation.

$$\begin{aligned} p(X \mid C_{1..k}) &= \prod_{n=1}^{N} \iiint_{-\infty}^{\infty} p(x_n \mid v_n, z_n) p(v_n \mid g_n) p(g_n) p(z_n) \mathrm{d}v_n \mathrm{d}g_n \mathrm{d}z_n = \\ &= \prod_{n=1}^{N} \iint_{-\infty}^{\infty} p(g_n) p(z_n) \int_{-\infty}^{\infty} p(x_n \mid v_n, z_n) p(v_n \mid g_n) \mathrm{d}v_n \mathrm{d}g_n \mathrm{d}z_n \end{aligned} \tag{1}$$

The Gaussian over x can be rewritten to a Gaussian over v in the following way

$$\mathcal{N}(x; zAv, \sigma_x I) = (2\pi)^{\frac{D_v - D_x}{2}} \sigma_x^{\frac{D_v - D_x}{2}} \frac{1}{z^{D_v}} \sqrt{\det(A^T A)} \mathcal{N}(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}) \tag{2}$$

and with $D_x = D_v$

$$\mathcal{N}(x; zAv, \sigma_x I) = \frac{1}{z^{D_v}} \sqrt{\det(A^T A)} \cdot \mathcal{N}(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}) \tag{3}$$

the product of the two conditionals can be rewritten as a Gaussian over v times another Gaussian

$$\begin{aligned} p(x \mid v, z) p(v \mid g) = \frac{1}{z^{D_v}} \sqrt{\det(A^T A)} \cdot \mathcal{N}(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}) \mathcal{N}(v; 0, C_v) = \\ \frac{1}{z^{D_v}} \sqrt{\det(A^T A)} \cdot \mathcal{N}(\frac{1}{z} A^+ x; 0, \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v) \cdot \mathcal{N}(v; \mu_c, C_c) \end{aligned} \tag{4}$$

where v only appears in the second Gaussian, so integrating this formula will set that term to 1 and leave everything else as it is.

$$p(X \mid C_{1..k}) = \det(A^T A)^{\frac{N}{2}} \prod_{n=1}^{N} \int_{-\infty}^{\infty} p(z_n) \frac{1}{z^{D_v}} \int_{-\infty}^{\infty} p(g_n) \mathcal{N}(\frac{1}{z} A^+ x; 0, \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v) \mathrm{d}g_n \mathrm{d}z_n \tag{5}$$

2 Approximation with sampling

The double integral may be approximated by averaging over samples from $p(g)$ and $p(z)$

$$p(X \mid C_{1..k}) \approx \prod_{n=1}^{N} \frac{1}{L^2} \sum_{l_1=1}^{L} \sum_{l_2=1}^{L} \int_{-\infty}^{\infty} p(x_n \mid v, z^{l_2}) p(v \mid g^{l_1}) \mathrm{d}v \tag{6}$$

$$p(X \mid C_{1..k}) \approx \prod_{n=1}^{N} \frac{1}{L^2} \sum_{l_1=1}^{L} \sum_{l_2=1}^{L} \frac{1}{z^{D_v}} \sqrt{\det(A^T A)} \cdot \mathcal{N}(\frac{1}{z} A^{+} x; 0, \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v) \tag{7}$$