Merging two Gaussian distributions

We want to merge two Gaussians over x and v into one over v

$$p(x \mid v, z)p(v \mid g) = \mathcal{N}(x; zAv, \sigma_x I)\mathcal{N}(v; 0, C_v) \tag{1}$$

The Gaussian over x spelled out is

$$\mathcal{N}(x; zAv, \sigma_x I) = \sqrt{\frac{1}{(2\pi)^{Dx}\sigma_x^{Dx}}} e^{-\frac{1}{2\sigma_x}(x-zAv)^T(x-zAv)} \tag{2}$$

$$-\frac{1}{2}(x - zAv)^T(x - zAv) = -\frac{1}{2}(x^Tx - zv^TA^Tx - zxAv + z^2v^TA^TAv) \tag{3}$$

as $v^TA^Tx = (xAv)^T$, and both are scalars, thus equal to their transposes, it's also true that $v^TA^Tx = xAv$

$$\begin{aligned} -\frac{1}{2}(x - zAv)^T(x - zAv) = -\frac{1}{2}(x^Tx - 2zxAv + z^2v^TA^TAv) = \\ = -\frac{x^Tx}{2} + zxAv - \frac{z^2}{2}v^TA^TAv \end{aligned} \tag{4}$$

we have the identity for any symmetric matrix M and vector b that

$$-\frac{1}{2}v^TMv + b^Tv = -\frac{1}{2}(v - M^{-1}b)^TM(v - M^{-1}b) + \frac{1}{2}b^TM^{-1}b \tag{5}$$

making the substitution $M = z^2A^TA$ and $b = (zxA)^T = zA^Tx^T$, yielding $M^{-1} = \frac{1}{z^2}(A^TA)^{-1}$ and $M^{-1}b = \frac{1}{z}(A^TA)^{-1}A^Tx^T = \frac{1}{z}A^+x^T$, where A^+ is the Moore-Penrose pseudoinverse of A. Thus we get

$$
\begin{aligned}
-\frac{1}{2}(x-zAv)^T(x-zAv) = \\
= -\frac{x^Tx}{2} - \frac{1}{2}(v-\frac{1}{z}A^+x^T)^T z^2 A^T A(v-\frac{1}{z}A^+x^T) + \frac{1}{2}(A^Tx^T)^T(A^TA)^{-1}A^Tx^T = \\
= -\frac{x^Tx}{2} - \frac{1}{2}(v-\frac{1}{z}A^+x^T)^T z^2 A^T A(v-\frac{1}{z}A^+x^T) + \frac{1}{2}xAA^{-1}A^{-T}A^Tx^T = \\
-\frac{x^Tx}{2} - \frac{1}{2}(v-\frac{1}{z}A^+x^T)^T z^2 A^T A(v-\frac{1}{z}A^+x^T) + \frac{1}{2}(xAA^{-1}A^{-T}A^Tx^T)^T = \\
-\frac{x^Tx}{2} - \frac{1}{2}(v-\frac{1}{z}A^+x^T)^T z^2 A^T A(v-\frac{1}{z}A^+x^T) + \frac{x^Tx}{2} = \\
-\frac{1}{2}(v-\frac{1}{z}A^+x^T)^T z^2 A^T A(v-\frac{1}{z}A^+x^T)
\end{aligned} \tag{6}
$$

which implies

$$
e^{-\frac{1}{2\sigma_x}(x-zAv)^T(x-zAv)} = e^{-\frac{1}{2}(v-\frac{1}{z}A^+x^T)^T \frac{z^2}{\sigma_x} A^T A(v-\frac{1}{z}A^+x^T)} \tag{7}
$$

meaning that

$$
\mathcal{N}(x; zAv, \sigma_x I) = \alpha \mathcal{N}(v; \frac{1}{z}A^+x^T, \frac{\sigma_x}{z^2}(A^TA)^{-1}) \tag{8}
$$

and as the formulas in the exponents are equal, the constant α is given by the ratio of the normalisation terms

$$
\sqrt{\frac{1}{(2\pi)^{Dx}\sigma_x^{Dx}}} = \alpha \sqrt{\frac{1}{(2\pi)^{Dv}\det(\frac{\sigma_x}{z^2}(A^TA)^{-1})}} \tag{9}
$$

$$
\alpha = \sqrt{\frac{(2\pi)^{Dv}\frac{\sigma_x^{Dv}}{z^{2D_v}}\det((A^TA)^{-1})}{(2\pi)^{Dx}\sigma_x^{Dx}}} \tag{10}
$$

$$
\alpha = \sqrt{\frac{(2\pi)^{Dv}\sigma_x^{D_v}}{(2\pi)^{Dx}\sigma_x^{Dx}z^{2D_v}\det(A^TA)}} \tag{11}
$$

making the simplifying assumption $D_x = D_v$ we arrive to

$$
\mathcal{N}(x; zAv, \sigma_x I) = \frac{1}{\sqrt{\det(A^TA)}}\frac{1}{z^{D_v}}\mathcal{N}(v; \frac{1}{z}A^+x^T, \frac{\sigma_x}{z^2}(A^TA)^{-1}) \tag{12}
$$

we can merge two Gaussian distributions over v into one by using the following formula

$$
\mathcal{N}(v; \mu_1, C_1)\mathcal{N}(v; \mu_2, C_2) = \mathcal{N}(\mu_1; \mu_2, C_1 + C_2)\mathcal{N}(v; \mu_c, C_c) \tag{13}
$$

where $C_c = (C_1^{-1} + C_2^{-1})^{-1}$ and $\mu_c = C_c(C_1^{-1}\mu_1 + C_2^{-1}\mu_2)$