Derivations for the CSM model

1 Definition of the model

A gestalt, a perceptual object, is characterised by a covariance component for the joint distribution of visual neural activity.

$$p(v \mid g) = \mathcal{N}(v; 0, C_v) \tag{1}$$

$$C_v = \sum_{k=1}^{K} g_k U_k^T U_k \tag{2}$$

where K is the fixed number of possible gestalts in the visual scene and g_k is the strength of the gestalt number k, coming from a K-dimensional Gamma prior distribution with shape and scale parameters α_g and θ_g controlling the sparsity of the prior.

$$p(g) = \mathrm{Gam}(g; \alpha_g, \theta_g) \tag{3}$$

The global contrast of the image patch is encoded by a scalar variable z, also coming from a Gamma prior

$$p(z) = \mathrm{Gam}(z; \alpha_z, \theta_z) \tag{4}$$

The pixel intensities are generated from the neural activity through a set of linear projective field models, possibly Gabor filters, A, scaled by the contrast and adding some independent observational noise.

$$p(x \mid v, z) = \mathcal{N}(x; zAv, \sigma_x I) \tag{5}$$

2 Likelihoods

2.1 Likelihood of g and z

by intuition:

$$p(x \mid z, g) = \mathcal{N}(x; 0, \sigma_x I + z^2 A \left(\sum_{k=1}^{K} g_k U_k^T U_k \right) A^T) \tag{6}$$

by algebraic derivation (see Seq. 4.4):

$$p(x \mid z, g) = \int_{-\infty}^{\infty} p(x, v \mid z, g) \mathrm{d}v = \frac{1}{z^{D_v}\sqrt{\det(A^T A)}} \mathcal{N}(\frac{1}{z} A^+ x; 0, \frac{\sigma_x}{z^2}(A^T A)^{-1} + \sum_{k=1}^{K} g_k U_k^T U_k) \quad (7)$$

$$f(x, z) \equiv \frac{1}{z} A^+ x, \;\; h(z) \equiv \frac{1}{z^{D_v}} \quad (8)$$

$$C(z, g) \equiv \frac{\sigma_x}{z^2}(A^T A)^{-1} + \sum_{k=1}^{K} g_k U_k^T U_k \quad (9)$$

$$p(x \mid z, g) = \frac{1}{\sqrt{\det(A^T A)}} h(z) \mathcal{N}(f(x, z); 0, C(z, g)) \quad (10)$$

2.2 Log-likelihood of the parameters

$$p(X \mid U_{1..K}) = \prod_{n=1}^{N} p(x_n \mid U_{1..K}) \quad (11)$$

$$p(X \mid U_{1..K}) = \prod_{n=1}^{N} \iint_{-\infty}^{\infty} p(x_n \mid z, g) p(g) p(z) \mathrm{d}g \mathrm{d}z \quad (12)$$

$$(13)$$

approximation of the integrals by samples from the priors $p(g_n)$ and $p(z_n)$

$$p(X \mid U_{1..K}) \approx \prod_{n=1}^{N} \frac{1}{L} \sum_{l=1}^{L} p(x_n \mid z^l, g^l) \quad (14)$$

using 10

$$p(X \mid U_{1..K}) \approx \left(L\sqrt{\det(A^T A)}\right)^{-N} \prod_{n=1}^{N} \sum_{l=1}^{L} h(z^l) \mathcal{N}(f(x_n, z^l); 0, C(z^l, g^l)) \quad (15)$$

$$\log p(X \mid U_{1..K}) \approx -N(\log L + \frac{1}{2} \log(\det(A^T A))) + \sum_{n=1}^{N} \log \left[\sum_{l=1}^{L} h(z^l) \mathcal{N}(f(x_n, z^l); 0, C(z^l, g^l)) \right] \quad (16)$$

$$h^l \equiv h(z^l), \;\; f_n^l \equiv f(x_n, z^l), \;\; C^l \equiv C(z^l, g^l) \quad (17)$$

$$\mathcal{L}_n^l \equiv h^l \mathcal{N}(f_n^l; 0, C^l), \;\; \mathcal{L}_n \equiv \sum_{l=1}^{L} \mathcal{L}_n^l \quad (18)$$

$$\log p(X \mid U_{1..K}) \approx -N(\log L + \frac{1}{2} \log(\det(A^T A))) + \sum_{n=1}^{N} \log \mathcal{L}_n \quad (19)$$

2.2.1 Derivative w.r.t. $U_{1\dots K}$

$$\frac{\partial \log p(X \mid U_{1..K})}{\partial \left[U_k\right]_{i,j}} \approx \sum_{n=1}^{N} \frac{\partial \log \mathcal{L}_n}{\partial \left[U_k\right]_{i,j}} = \sum_{n=1}^{N} \frac{1}{\mathcal{L}_n} \frac{\partial \mathcal{L}_n}{\partial \left[U_k\right]_{i,j}} = \sum_{n=1}^{N} \frac{1}{\mathcal{L}_n} \sum_{l=1}^{L} \frac{\partial \mathcal{L}_n^l}{\partial \left[U_k\right]_{i,j}} = \\ = \sum_{n=1}^{N} \frac{1}{\mathcal{L}_n} \sum_{l=1}^{L} \mathrm{Tr}\left[\frac{\partial \mathcal{L}_n^l}{\partial C^l} \frac{\partial C^l}{\partial \left[U_k\right]_{i,j}}\right] \tag{20}$$

the derivatives in this formula are the following

$$\frac{\partial \mathcal{L}_n^l}{\partial C^l} = h^l \frac{\partial}{\partial C^l} \mathcal{N}(f_n^l; 0, C^l) = h^l \mathcal{N}(f_n^l; 0, C^l) \frac{\partial}{\partial C^l} \log \mathcal{N}(f_n^l; 0, C^l) = \\ = -\frac{h^l}{2} \mathcal{N}(f_n^l; 0, C^l) \left[(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}\right] \tag{21}$$

$$\frac{\partial C(z,g)}{\partial \left[U_k\right]_{i,j}} = g_k \frac{\partial \left(U_k^T U_k\right)}{\partial \left[U_k\right]_{i,j}} = g_k \left(U_k^T J^{ij} + J^{ji} U_k\right) \equiv g_k \hat{U}_k^{ij} \tag{22}$$

substituting back to the derivative

$$\frac{\partial \log p(X \mid U_{1..K})}{\partial \left[U_k\right]_{i,j}} \approx \\ \approx -\frac{1}{2} \sum_{n=1}^{N} \frac{1}{\mathcal{L}_n} \sum_{l=1}^{L} h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) \mathrm{Tr}\left[\left[(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}\right] \hat{U}_k^{ij}\right] = \\ = -\frac{1}{2} \mathrm{Tr}\left[\sum_{n=1}^{N} \left(\frac{1}{\mathcal{L}_n} \sum_{l=1}^{L} h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) \left[(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}\right]\right) \hat{U}_k^{ij}\right] \tag{23}$$

The regularities of the $\hat{U}_k$ matrices allow us to replace the trace with a much more efficient computation:

$$M_k = -\frac{1}{2} \sum_{n=1}^{N} \left(\frac{1}{\mathcal{L}_n} \sum_{l=1}^{L} h^l g_k^l \mathcal{N}(f_n^l; 0, C^l) \left[(C^l)^{-1} - (C^l)^{-1} f_n^l (f_n^l)^T (C^l)^{-1}\right]\right) \tag{24}$$

$$\frac{\partial \log p(X \mid U_{1..K})}{\partial \left[U_k\right]_{i,j}} \approx \mathrm{Tr}\left[M_k \hat{U}_k^{ij}\right] = \sum_{a=1}^{Dv} \left[M_k\right]_{j,a} \left[U_k\right]_{i,a} + \left[M_k\right]_{a,j} \left[U_k\right]_{i,a} \tag{25}$$

2.3 Complete-data log-likelihood

$$p(V, G, Z, X \mid U_{1..K}) = \prod_{n=1}^{N} p(x_n \mid v_n, z_n) p(v_n \mid g_n) p(g_n) p(z_n) \tag{26}$$

the logarithm of this will be

$$\begin{aligned}
&\log p(V, G, Z, X \mid U_{1..K}) = \\
&= \sum_{n=1}^{N} [\log p(g_n) + \log p(z_n) + \log p(x_n \mid v_n) + \log p(v_n \mid g_n)] = \\
&= N(\log p(g) + \log p(z)) + \sum_{n=1}^{N} \log p(x_n \mid v_n) + \log p(v_n \mid g_n) = \\
&= c + \sum_{n=1}^{N} \log p(v_n \mid g_n)
\end{aligned} \tag{27}$$

where c is constant with respect to the parameters $U_{1...K}$.

2.3.1 Expectation w.r.t. the posterior

$$\mathcal{L} = \iiint_{-\infty}^{\infty} p(V, G, Z \mid X) \log p(V, G, Z, X \mid U_{1..K}) \mathrm{d}V \mathrm{d}G \mathrm{d}Z. \tag{28}$$

We can approximate this integral by averaging over L samples from the full posterior, separately for each observation x_n, as defined in Seq. 3.1. As we will seek the values of the Cholesky components $U_{1...K}$ that maximise this integral, we can discard each term not depending on these parameters, only leaving the term of the form $p(v \mid g)$. This way we arrive to the following expression

$$\begin{aligned}
\mathcal{L} \sim \sum_{n=1}^{N} \frac{1}{L} \sum_{l=1}^{L} -\frac{1}{2} \left[\log \left(\det \left(C_v^{(l,n)} \right) \right) + v^{(l,n)T} \left(C_v^{l,n} \right)^{-1} v^{l,n} \right] = \\
= -\frac{1}{2L} \sum_{m=1}^{NL} \left[\log \left(\det \left(C_v^m \right) \right) + v^{mT} \left(C_v^m \right)^{-1} v^m \right]
\end{aligned} \tag{29}$$

noting that the double summation over L samples over all N observations always happens on the same terms, so we can substitute it with a single sum that iterates over the full sample set.

2.3.2 Derivative w.r.t. $U_{1...K}$

Using the derivative of C_v with respect to U_k^{ij} as defined in Eq. 22, by the chain rule, the derivative of $\mathcal{L}$ according to an element of U_k looks like this

$$\frac{\partial \mathcal{L}}{\partial \left[U_k\right]_{i,j}} = -\frac{1}{2L}\sum_{m=1}^{NL} \mathrm{Tr}\left[\frac{\partial \mathcal{L}^m}{\partial C_v^m}\frac{\partial C_v^m}{\partial \left[U_k\right]_{i,j}}\right] =$$
$$-\frac{1}{2L}\sum_{m=1}^{LN} \mathrm{Tr}\left[\left[\left(C_v^m\right)^{-1} - \left(C_v^m\right)^{-1} v^m v^{mT} \left(C_v^m\right)^{-1}\right] g_k^m \hat{U}_k^{ij}\right] = \tag{30}$$
$$-\frac{1}{2L}\mathrm{Tr}\left[\sum_{m=1}^{LN} g_k^m \left[\left(C_v^m\right)^{-1} - \left(C_v^m\right)^{-1}\left(v^m v^{mT}\right)\left(C_v^m\right)^{-1}\right]\hat{U}_k^{ij}\right]$$

The regularities of the $\hat{U}_k$ matrices allow us to replace the trace with a much more efficient computation:

$$M_k = -\frac{1}{2L}\left[\sum_{m=1}^{LN} g_k^m \left[\left(C_v^m\right)^{-1} - \left(C_v^m\right)^{-1}\left(v^m v^{mT}\right)\left(C_v^m\right)^{-1}\right]\right] \tag{31}$$

$$\frac{\partial \mathcal{L}}{\partial \left[U_k\right]_{i,j}} = \sum_{a=1}^{Dv} \left[M_K\right]_{j,a}\left[U_k\right]_{i,a} + \left[M_k\right]_{a,j}\left[U_k\right]_{i,a} \tag{32}$$

3 Posteriors

3.1 Full posterior

$$p(v,g,z \mid x) = p(x \mid v,g,z)p(v \mid g)p(g)p(z)\frac{1}{p(x)} \sim p(x \mid v,z)p(v \mid g)p(g)p(z) \tag{33}$$

so the log-posterior will be the following, up to an additive constant, using Gamma priors over g and z defined by shape and scale parameters:

$$\log p(v,g,z \mid x) \sim \log p(x \mid v,z) + \log p(v \mid g) + \log p(g) + \log p(z) =$$
$$= \log \mathcal{N}(x; zAv, \sigma_x I) + \log \mathcal{N}(v; 0, C_v) + \log \mathrm{Gam}(g; \alpha_g, \theta_g) + \log \mathrm{Gam}(z; \alpha_z, \theta_z) \tag{34}$$

using the logarithms of the used pdfs from Sec. 4.2 and discarding all terms not dependent on any of the three variables we get

$$\log p(v,g,z \mid x) \sim -\frac{1}{2\sigma_x}(x - zAv)^T(x - zAv) -$$
$$-\frac{1}{2}\left[\log\left(\det\left(C_v\right)\right) + v^T C_v^{-1} v\right] + \tag{35}$$
$$+\sum_{j=1}^{K}\left[(\alpha_g - 1)\log(g_j) - \frac{g_j}{\theta_g}\right] + (\alpha_z - 1)\log(z) - \frac{z}{\theta_z}$$

rearranging the first quadratic term according to Eq. 69 and discarding the term not dependent on v yields

$$\begin{aligned} \log p(v, g, z \mid x) \sim & -\frac{z}{2\sigma_x}\left(zv^T A^T Av - 2x^T Av\right) - \\ & -\frac{1}{2}\left[\log\left(\det\left(C_v\right)\right) + v^T C_v^{-1} v\right] + \\ & + \sum_{j=1}^{K}\left[(\alpha_g - 1)\log(g_j) - \frac{g_j}{\theta_g}\right] + \\ & +(\alpha_z - 1)\log(z) - \frac{z}{\theta_z} \end{aligned} \tag{36}$$

3.1.1 Derivative w.r.t. v

$$\log p(v, g, z \mid x) \sim -\frac{1}{2}\left[\frac{z^2}{\sigma_x} v^T A^T Av - \frac{2z}{\sigma_x} x^T Av + v^T C_v^{-1} v\right] + f_1(g, z) \tag{37}$$

lumping the two quadratic forms together

$$\log p(v, g, z \mid x) \sim \frac{z}{\sigma_x} x^T Av - \frac{1}{2} v^T \left[\frac{z^2}{\sigma_x} A^T A + C_v^{-1}\right] v + f_1(g, z) \tag{38}$$

Taking the derivative using Eq. 57 and 58 we get

$$\frac{\partial}{\partial v}\log p(v, g, z \mid x) = \frac{z}{\sigma_x} A^T x - \left[\frac{z^2}{\sigma_x} A^T A + C_v^{-1}\right] v \tag{39}$$

3.1.2 Derivative w.r.t. g

$$\begin{aligned} \log p(v, g, z \mid x) \sim & -\frac{1}{2}\left[\log\det\left(C_v\right) + v^T C_v^{-1} v\right] + \\ & + \sum_{j=1}^{K}\left[(\alpha_g - 1)\log(g_j) - \frac{g_j}{\theta_g}\right] + f_2(v, z) \end{aligned} \tag{40}$$

Taking the derivative w.r.t. a single g_i using Eq. 61 we get

$$\begin{aligned} \frac{\partial}{\partial g_i}\log p(v, g, z \mid x) = & -\frac{1}{2}\mathrm{Tr}\left[\frac{\partial}{\partial C_v}\left[\log\det\left(C_v\right) + v^T C_v^{-1} v\right]\frac{\partial C_v}{\partial g_i}\right] + \\ & + \frac{\partial}{\partial g_i}\left[(\alpha_g - 1)\log(g_i) - \frac{g_i}{\theta_g}\right] \end{aligned} \tag{41}$$

using Eq. 60, 59 and 62 we arrive to

$$\frac{\partial}{\partial g_i}\log p(v, g, z \mid x) = -\frac{1}{2}\mathrm{Tr}\left[\left[C_v^{-1} - C_v^{-1} vv^T C_v^{-1}\right] C_i\right] + \frac{\alpha_g - 1}{g_i} - \frac{1}{\theta_g} \tag{42}$$

3.1.3 Derivative w.r.t. z

$$\log p(v, g, z \mid x) \sim -\frac{z}{2\sigma_x}\left(zv^T A^T Av - 2x^T Av\right) + (\alpha_z - 1)\log(z) - \frac{z}{\theta_z} + f_3(g, v) \tag{43}$$

$$\frac{\partial}{\partial z}\log p(v, g, z \mid x) = \frac{1}{\sigma_x}\left[x^T Av - zv^T A^T Av\right] + \frac{\alpha_z - 1}{z} - \frac{1}{\theta_z} \tag{44}$$

3.2 Conditional posteriors

3.2.1 Conditional posterior of v

$$p(v \mid x, g, z) = \frac{p(x \mid v, z, g)p(v \mid z, g)}{p(x \mid z, g)} = \frac{\mathcal{N}(x; zAv, \sigma_x I)\mathcal{N}(v; 0, C_v)}{\int_{-\infty}^{\infty}\mathcal{N}(x; zAv, \sigma_x I)\mathcal{N}(v; 0, C_v)\mathrm{d}v} \tag{45}$$

the product of two Gaussians in the numerator of Eq. 45 can also be written as a Gaussian over v as in Seq. 4.3:

$$\mathcal{N}(x; zAv, \sigma_x I)\mathcal{N}(v; 0, C_v) = c\mathcal{N}(v; \mu_{post}, C_{post}) \tag{46}$$

The denominator of Eq. 45 is the integral of this formula, which evaluates to c, as the Gaussian integrates to one. This cancels the constant in the numerator, making the conditional posterior equal to the combined Gaussian over v, which, after expanding μ_{post} and C_{post}, is

$$p(v \mid x, g, z) = \mathcal{N}\left(v; \frac{z}{\sigma_x}\left(\frac{z^2}{\sigma_x}A^T A + C_v^{-1}\right)^{-1} A^T x, \left(\frac{z^2}{\sigma_x}A^T A + C_v^{-1}\right)^{-1}\right) \tag{47}$$

3.2.2 Conditional posterior of g

$$p(g \mid X, V, z) = \frac{p(X \mid g, V, z)p(g \mid V, z)}{p(X \mid V, z)} = \frac{p(V \mid g)p(g)}{p(V)} \tag{48}$$

taking the logarithm and discarding constant terms

$$\log p(g \mid X, V) \sim -\frac{1}{2}\left[\log(\det(C_v)) + v^T C_v^{-1} v\right] + \log p(g) \tag{49}$$

The unnormalised conditionals of single elements of g, assuming an independent prior look as follows

$$\begin{aligned}\log p(g_j \mid g_{\neg j}, X, V) &= \frac{p(V \mid g_j, g_{\neg j}, X)p(g_j \mid g_{\neg j}, X)}{p(V \mid g_{\neg_j}, X)} = \\ &= \frac{p(V \mid g)p(g_j)}{p(V \mid g_{\neg_j})} \sim p(V \mid g)p(g_j)\end{aligned} \tag{50}$$

3.2.3 Conditional posterior of z

$$p(z \mid X, V, g) = \frac{p(X \mid g, z, V)p(z \mid V, g)}{p(X \mid V, g)} \sim p(X \mid z, V)p(z) \tag{51}$$

the log-posterior being

$$\log p(z \mid X, V) \sim -\frac{1}{2}\left[D_x \log(\sigma_x) + \frac{1}{\sigma_x}(x - zAv)^T(x - zAv)\right] + \log p(z) \tag{52}$$

3.3 Marginal posteriors

3.3.1 Marginal posterior of g and z

$$p(g, z \mid x) \sim p(x \mid g, z)p(g)p(z) \tag{53}$$

$$\log p(g, z \mid x) \sim -\frac{1}{2}\left[\log\det(C_x) + x^T C_x^{-1} x\right] + (\alpha_z - 1)\log(z) - \frac{z}{\theta_z} + (\alpha_g - 1)\sum_{k=1}^{K}\log(g_k) - \frac{1}{\theta_g}\sum_{k=1}^{K} g_k \tag{54}$$

3.3.2 Marginal posterior and MAP of g

$$g_{MAP} = \arg\max_g p(g \mid x) = \arg\max_g \frac{p(x \mid g)p(g)}{p(x)} = \arg\max_g p(x \mid g)p(g) \tag{55}$$

$$p(x \mid g) = \int_{-\infty}^{\infty} p(x \mid z, g)p(z)\mathrm{d}z \approx \frac{1}{L}\sum_{l=1}^{L} p(x \mid g, z^l) \tag{56}$$

4 Appendix

4.1 Rules of differentiation

Assuming that y and a are vectors and M is a symmetric matrix of appropriate dimension, and f is a scalar function, and s is a scalar variable.

$$\frac{\partial}{\partial y} y^T M y = 2My \tag{57}$$

$$\frac{\partial}{\partial y} a^T y = a \tag{58}$$

$$\frac{\partial}{\partial M} y^T M^{-1} y = -M^{-1} y y^T M^{-1} \tag{59}$$

$$\frac{\partial}{\partial M} \log\det M = M^{-1} \tag{60}$$

$$\frac{\partial}{\partial s} f(M(s)) = \mathrm{Tr}\left[\frac{\partial f}{\partial M}\frac{\partial M}{\partial s}\right] \tag{61}$$

$$\frac{\partial}{\partial s} sM = M \tag{62}$$

$$\frac{\partial}{\partial s} f(s) = f(s)\frac{\partial}{\partial s}\log f(s) \tag{63}$$

$$\frac{\partial}{\partial M} \log\mathcal{N}(y; a, M) = M^{-1} - M^{-1}(y-a)(y-a)^T M^{-1} \tag{64}$$

4.2 Logarithms of used PDFs

$$\log\mathcal{N}(y; \mu, C) = -\frac{1}{2}\left[D\log(2\pi) + \log\det(C) + (y-\mu)^T C^{-1}(y-\mu)\right] \tag{65}$$

$$\log\mathrm{Gam}(y; \alpha, \theta) = \log(1) - \log(\Gamma(\alpha)) - \alpha\log(\theta) + (\alpha-1)\log(y) - \frac{y}{\theta} \tag{66}$$

4.3 Merging two Gaussian distributions

We want to merge two Gaussians over x and v into one over v

$$p(x \mid v, z)p(v \mid g) = \mathcal{N}(x; zAv, \sigma_x I)\mathcal{N}(v; 0, C_v) \tag{67}$$

The Gaussian over x spelled out is

$$\mathcal{N}(x; zAv, \sigma_x I) = \sqrt{\frac{1}{(2\pi)^{Dx}\sigma_x^{Dx}}} e^{-\frac{1}{2\sigma_x}(x-zAv)^T(x-zAv)} \tag{68}$$

rearranging the quadratic term:

$$\begin{aligned}
-\frac{1}{2}(x-zAv)^T(x-zAv) &= -\frac{1}{2}(x^T x - zv^T A^T x - zx^T Av + z^2 v^T A^T Av) = \\
= -\frac{1}{2}(x-zAv)^T(x-zAv) &= -\frac{1}{2}(x^T x - 2zx^T Av + z^2 v^T A^T Av) = \\
&= -\frac{x^T x}{2} + zx^T Av - \frac{z^2}{2} v^T A^T Av
\end{aligned} \tag{69}$$

as $v^TA^Tx = (x^TAv)^T$, and both are scalars, thus equal to their transposes, it's also true that $v^TA^Tx = x^TAv$. We have the identity for any symmetric matrix M and vector b that

$$-\frac{1}{2}v^TMv + b^Tv = -\frac{1}{2}(v - M^{-1}b)^TM(v - M^{-1}b) + \frac{1}{2}b^TM^{-1}b \tag{70}$$

making the substitution $M = z^2A^TA$ and $b = (zx^TA)^T = zA^Tx$, yielding $M^{-1} = \frac{1}{z^2}(A^TA)^{-1}$ and $M^{-1}b = \frac{1}{z}(A^TA)^{-1}A^Tx = \frac{1}{z}A^+x$, where A^+ is the Moore-Penrose pseudoinverse of A. Thus we get

$$\begin{aligned}
&-\frac{1}{2}(x - zAv)^T(x - zAv) = \\
&= -\frac{x^Tx}{2} - \frac{1}{2}(v - \frac{1}{z}A^+x)^Tz^2A^TA(v - \frac{1}{z}A^+x) + \frac{1}{2}(A^Tx)^T(A^TA)^{-1}A^Tx = \\
&= -\frac{x^Tx}{2} - \frac{1}{2}(v - \frac{1}{z}A^+x)^Tz^2A^TA(v - \frac{1}{z}A^+x) + \frac{1}{2}x^TAA^{-1}A^{-T}A^Tx = \\
&= -\frac{x^Tx}{2} - \frac{1}{2}(v - \frac{1}{z}A^+x)^Tz^2A^TA(v - \frac{1}{z}A^+x) + \frac{x^Tx}{2} = \\
&= -\frac{1}{2}(v - \frac{1}{z}A^+x)^Tz^2A^TA(v - \frac{1}{z}A^+x)
\end{aligned} \tag{71}$$

which implies

$$e^{-\frac{1}{2\sigma_x}(x-zAv)^T(x-zAv)} = e^{-\frac{1}{2}(v-\frac{1}{z}A^+x^T)^T\frac{z^2}{\sigma_x}A^TA(v-\frac{1}{z}A^+x)} \tag{72}$$

meaning that

$$\mathcal{N}(x; zAv, \sigma_xI) = \alpha\mathcal{N}(v; \frac{1}{z}A^+x, \frac{\sigma_x}{z^2}(A^TA)^{-1}) \tag{73}$$

and as the formulas in the exponents are equal, the constant α is given by the ratio of the normalisation terms

$$\sqrt{\frac{1}{(2\pi)^{Dx}\sigma_x^{Dx}}} = \alpha\sqrt{\frac{1}{(2\pi)^{Dv}\det(\frac{\sigma_x}{z^2}(A^TA)^{-1})}} \tag{74}$$

$$\alpha = \sqrt{\frac{(2\pi)^{Dv}\frac{\sigma_x^{D_v}}{z^{2D_v}}\det((A^TA)^{-1})}{(2\pi)^{Dx}\sigma_x^{Dx}}} \tag{75}$$

$$\alpha = \sqrt{\frac{(2\pi)^{Dv}\sigma_x^{D_v}}{(2\pi)^{Dx}\sigma_x^{Dx}z^{2D_v}\det(A^TA)}} \tag{76}$$

making the simplifying assumption $D_x = D_v$ we arrive to

$$\mathcal{N}(x; zAv, \sigma_x I) = \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}) \tag{77}$$

we can merge two Gaussian distributions over v into one by using the following formula

$$\mathcal{N}(v; \mu_1, C_1) \mathcal{N}(v; \mu_2, C_2) = \mathcal{N}(\mu_1; \mu_2, C_1 + C_2) \mathcal{N}(v; \mu_c, C_c) \tag{78}$$

where $C_c = (C_1^{-1} + C_2^{-1})^{-1}$ and $\mu_c = C_c(C_1^{-1}\mu_1 + C_2^{-1}\mu_2)$. Substitution to these formulas yields

$$\begin{aligned} &\frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}(v; \frac{1}{z} A^+ x, \frac{\sigma_x}{z^2} (A^T A)^{-1}) \mathcal{N}(v; 0, C_v) = \\ &\frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \mathcal{N}(\frac{1}{z} A^+ x; 0, \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v) \mathcal{N}(v; \mu_c, C_c) \end{aligned} \tag{79}$$

$$C_c = (\frac{z^2}{\sigma_x} (A^T A) + C_v^{-1})^{-1} \tag{80}$$

$$\mu_c = C_c \frac{z}{\sigma_x} (A^T A) A^+ x = \frac{z}{\sigma_x} C_c A^T x \tag{81}$$

4.4 Equivalence of the two likelihood formulas of CSM

Expanding 10 yields

$$\begin{aligned} p(x \mid z, g) &= \frac{1}{\sqrt{\det(A^T A)}} \frac{1}{z^{D_v}} \frac{1}{\sqrt{(2\pi)^{D_v} \det C(z, g)}} e^{-\frac{1}{2} \left(\frac{1}{z} A^+ x \right)^T C^{-1}(z,g) \frac{1}{z} A^+ x} = \\ &= \frac{1}{\sqrt{(2\pi z^2)^{D_v} \det(A^T A C(z, g))}} e^{-\frac{1}{2z^2} x^T A^{+T} C^{-1}(z,g) A^+ x} \end{aligned} \tag{82}$$

$$C(z, g) \equiv \frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \tag{83}$$

so in the exponent, in the place of the covariance matrix, we have

$$\begin{aligned} &\frac{1}{z^2} \left((A^T A)^{-1} A^T \right)^T \left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right]^{-1} (A^T A)^{-1} A^T = \\ &= \frac{1}{z^2} A (A^T A)^{-1} \left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right]^{-1} (A^T A)^{-1} A^T = \\ &= \frac{1}{z^2} A \left[\left[\frac{\sigma_x}{z^2} (A^T A)^{-1} + C_v \right] (A^T A) \right]^{-1} (A^T A)^{-1} A^T = \\ &= \frac{1}{z^2} A \left[\frac{\sigma_x}{z^2} I + C_v A^T A \right]^{-1} (A^T A)^{-1} A^T = \\ &= \frac{1}{z^2} A \left[(A^T A) \left[\frac{\sigma_x}{z^2} I + C_v A^T A \right] \right]^{-1} A^T = \\ &= \frac{1}{z^2} A \left[\frac{\sigma_x}{z^2} A^T A + A^T A C_v A^T A \right]^{-1} A^T \end{aligned} \tag{84}$$

assuming that A is invertible this is equal to

$$\begin{aligned}
\frac{1}{z^2}\left[A^{-1}\right]^{-1}\left[\frac{\sigma_x}{z^2}A^TA + A^TAC_vA^TA\right]^{-1}\left[A^{-T}\right]^{-1} &= \\
\frac{1}{z^2}\left[A^{-1}\right]^{-1}\left[A^{-T}\left[\frac{\sigma_x}{z^2}A^TA + A^TAC_vA^TA\right]\right]^{-1} &= \\
\frac{1}{z^2}\left[A^{-1}\right]^{-1}\left[\frac{\sigma_x}{z^2}A + AC_vA^TA\right]^{-1} &= \\
\frac{1}{z^2}\left[\left[\frac{\sigma_x}{z^2}A + AC_vA^TA\right]A^{-1}\right]^{-1} &= \\
\frac{1}{z^2}\left[\frac{\sigma_x}{z^2}I + AC_vA^T\right]^{-1} &= \\
= \left[\sigma_xI + z^2AC_vA^T\right]^{-1}
\end{aligned} \tag{85}$$

under the square root we have

$$\begin{aligned}
z^{2D_v}\det(A^TAC(z,g)) = z^{2D_v}\det(A^T)\det(A)\det(C(z,g)) = z^{2D_v}\det(A)\det(C(z,g))\det(A^T) \\
= z^{2D_v}\det(A)\det(\frac{\sigma_x}{z^2}(A^TA)^{-1} + C_v)\det(A^T) = z^{2D_v}\det(A)\det(\frac{\sigma_x}{z^2}(A^TA)^{-1} + C_v)\det(A^T) = \\
= z^{2D_v}\det(A)\det(\frac{\sigma_x}{z^2}A^{-1}A^{-T} + C_v)\det(A^T) = z^{2D_v}\det\left[A\left[\frac{\sigma_x}{z^2}A^{-1}A^{-T} + C_v\right]A^T\right] = \\
= z^{2D_v}\det\left[\frac{\sigma_x}{z^2}I + AC_vA^T\right] = \det\left[\sigma_xI + z^2AC_vA^T\right]
\end{aligned} \tag{86}$$

4.5 Precision component formulation of the CSM model

The model can be equally well parametrised by precision components

$$p(v \mid g) = \mathcal{N}(v; 0, \Lambda_v^{-1}) \tag{87}$$

$$\Lambda_v = \sum_{k=1}^{K} g_k\Lambda_k \tag{88}$$

in this case the conditional posterior over v takes the form

$$p(v \mid x, g) = \mathcal{N}\left(v; \frac{1}{\sigma_x}\left(\frac{1}{\sigma_x}A^TA + \Lambda_v\right)^{-1}A^Tx, \left(\frac{1}{\sigma_x}A^TA + \Lambda_v\right)^{-1}\right) \tag{89}$$

and the conditional posterior of g will look as follows

$$\log p(g \mid X, V) \sim -\frac{1}{2}\left[\log(\det(\Lambda_v^{-1})) + v^T\Lambda_vv\right] + \log p(g) \tag{90}$$

The gradient of the expectation of the complete-data log-likelihod with respect to the joint posterior will look like this

$$\Lambda_k = U_k^T U_k \tag{91}$$

$$\frac{\partial \mathcal{L}}{\partial \left[U_k\right]_{i,j}} = \frac{1}{L} \sum_{m=1}^{LN} g_k^m \mathrm{Tr}\left[\left[(\Lambda_v^m)^{-1} - v^m v^{mT}\right] \hat{U}_k^{ij}\right] \tag{92}$$

4.6 Batches of observations

For a single set of component activations g and contrast z, we might have a batch of v and x values of size B. This modifies expressions as follows.

Conditional posterior of g (Eq. 49)

$$\log p(g \mid X, V) \sim -\frac{1}{2}\left[B \log(\det(C_v)) + \sum_{b=1}^{B} v_b^T C_v^{-1} v_b\right] + \log p(g) \tag{93}$$

Conditional posterior of z (Eq. 52)

$$\log p(z \mid X, V) \sim -\frac{1}{2}\left[BD_x \log(\sigma_x) + \frac{1}{\sigma_x}\sum_{b=1}^{B}(x_b - zAv_b)^T(x_b - zAv_b)\right] + \log p(z) \tag{94}$$

Matrix formula in the derivative of the expectation of the complete-data log-likelihood (Eq. 31)

$$M_k = -\frac{1}{2L}\left[\sum_{m=1}^{LN} g_k^m \left[B\left(C_v^m\right)^{-1} - \left(C_v^m\right)^{-1}\left(\sum_{b=1}^{B} v^{m,b} v^{(m,b)T}\right)\left(C_v^m\right)^{-1}\right]\right] \tag{95}$$