Stability and Control of Nonlinear Systems Course-work 2

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1 System state-space representation

First, We can choose the following state variables:

$$x_1 = \theta(t)$$

$$x_2 = r(t)$$

$$x_3 = \dot{\theta}(t)$$

$$x_4 = \dot{r}(t)$$

Then, the state-space equations can be obtained:

$$\dot{x} = f(x) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -\frac{2x_2x_3x_4}{{x_2}^2+1} + \frac{1}{{x_2}^2+1}u_1 \\ {x_2}^2 x_3 + u_2 \end{bmatrix}$$

The output of the system is $[\dot{\theta}(t), \dot{r}(t)]^T = [x_3, x_4]$, then output representation can be obtained:

$$y = h(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The state-space equations can be presented in Matlab as follows.

```matlab
function dx = system_dynamics(t, x, u)
    % define the differential equations
    dx = [
        x(3);
        x(4);
        (u(1) - 2*x(4)*x(2)*x(3)) / (x(2)^2 + 1);
        (u(2) + x(2)*x(3)^2);
    ];
end
```

2 Proof of the passivity of the system

According to the Lyapunov function, the following equation can be obtained:

$$V(x) = \frac{1}{2}(r^2+1)\dot{\theta}^2 + \frac{1}{2}\dot{r}^2$$

Substitute the physical quantities in the equation into the state variables, the Lyapunov function can be obtained as:

$$V(x) = \frac{1}{2}({x_2}^2+1){x_3}^2 + \frac{1}{2}{x_4}^2$$

Then, the following equation can be obtained.

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x,u) = \begin{bmatrix} 0 & x_2 {x_3}^2 & ({x_2}^2+1)x_3 & x_4 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ -\frac{2x_2x_3x_4}{{x_2}^2+1} + \frac{1}{{x_2}^2+1}u_1 \\ {x_2}^2 x_3 + u_2 \end{bmatrix} = x_3u_1 + x_4u_2.$$

Since $y^T u = [x_3, x_4][u_1, u_2]^T = x_2u_1 + x_4u_2$, as a result, $\dot{V} \leq y^T u$ can be proved, indicating that the system is passive.

3 PD Controller Stability Analysis

The PD feedback controller is described by the following equation.

$$\begin{aligned} u_1 &= \quad -\dot{\theta} - (\theta - \theta_d) \\ u_2 &= \quad -\dot{r} - (r - r_d) \end{aligned}$$

Substitute the parameters of the PD controller into the state-space equations, the following equations can be obtained:

$$\dot{x} = f(x) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -\frac{2x_2x_3x_4 + x_1 + x_3}{{x_2}^2+1} \\ {x_2}^2 x_3 - x_2 - x_4 \end{bmatrix}$$

Based on the given hint, the modified Lyapunov function can be defined by incorporating the potential energy term:

$$V(x) = \frac{1}{2}(r^2+1)\dot{\theta}^2 + \frac{1}{2}\dot{r}^2 + \frac{1}{2}(\theta - \theta_d)^2 + \frac{1}{2}(r - r_d)^2$$

By substituting the physical quantities in the equation into the state variables, the Lyapunov function is derived as:

$$V(x) = \frac{1}{2}({x_2}^2+1){x_3}^2 + \frac{1}{2}{x_4}^2 + \frac{1}{2}{x_1}^2 + \frac{1}{2}{x_2}^2$$

The four parts of the equation can only be 0 when $x_1 = x_2 = x_3 = x_4 = 0$, $V(x)$ is positive definite. Besides, When$\|x\| \to \infty$, it is clear that $V(x) \to \infty$.

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x,u) = \begin{bmatrix} x_1 & x_2({x_3}^2+1) & ({x_2}^2+1)x_3 & x_4 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ -\frac{2x_2x_3x_4 + x_1 + x_3}{{x_2}^2+1} \\ {x_2}^2 x_3 - x_2 - x_4 \end{bmatrix} = -{x_3}^2 - {x_4}^2$$

In conclusion, the modified Lyapunov function $V(x)$ is radially infinite, positive definite, and $\dot{V}(x)$ is negative definite, proving that the system is globally asymptotically stable at the origin.

4 System simulation for different constant values of θ_d and r_d

This section explores the performance of the PD controller by varying the values of θ_d and r_d. Four scenarios were selected, involving changes in the two constant inputs and the system's initial state$\mathbf{x}_0$.

The first two simulations set the initial value $\mathbf{x}_0$ as $[0,0,0,0]^T$ and $[1,1,1,1]^T$ respectively, while θ_d, r_d are set to be $[1,1]^T$ and $[1,0]^T$ resp. These simulations aim to evaluate the control performance of the PD controller under conditions where the initial state is relatively small, and the deviation between the constant input and the initial state is minimal.

```matlab
theta_d = 1; r_d = 1;
x0 = [0, 0, 0, 0]; % Initial State
[t, x] = ode45(@(t, x) system_dynamics(t, x, pd_controller(x, theta_d, r_d)), [0 30], x0);
plot(t, x(:, 1), 'r', t, x(:, 2), 'b');  % red line x1,blue line x2
title('The output of this system', 'Interpreter', 'latex');
xlabel('time $t$', 'Interpreter', 'latex');
ylabel('state', 'Interpreter', 'latex');
legend('x1', 'x2');
grid on;

function u = pd_controller(x, theta_d, r_d)
    % define the PD controller
    u1 = -x(3) - (x(1) - theta_d); % input u1
    u2 = -x(4) - (x(2) - r_d);     % input u2
    u = [u1; u2]; % Return control amount
```

First, Let $\theta_d = 1, r_d = 1, \mathbf{x}_0 = [0, 0, 0, 0]^T$. The simulation results are shown as follows:

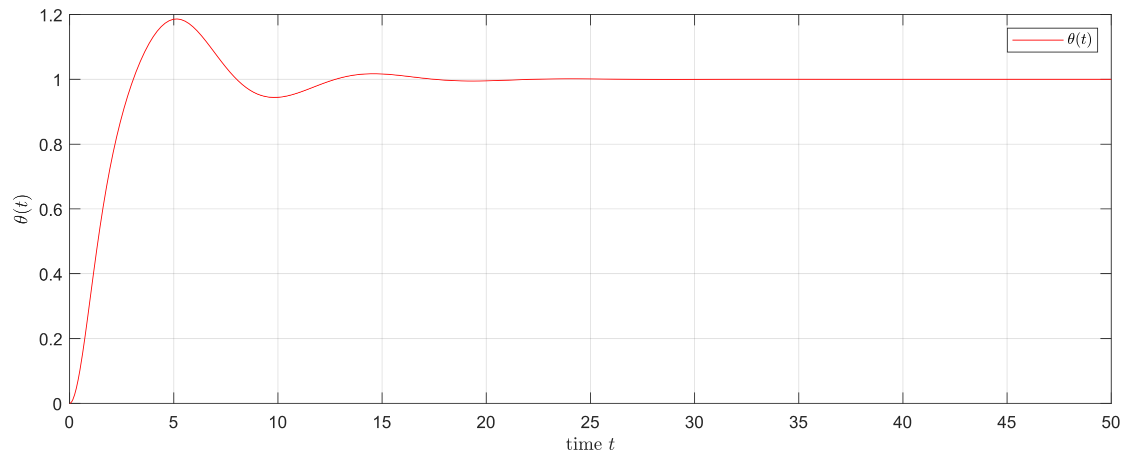

Figure 1: The variation of θ over time

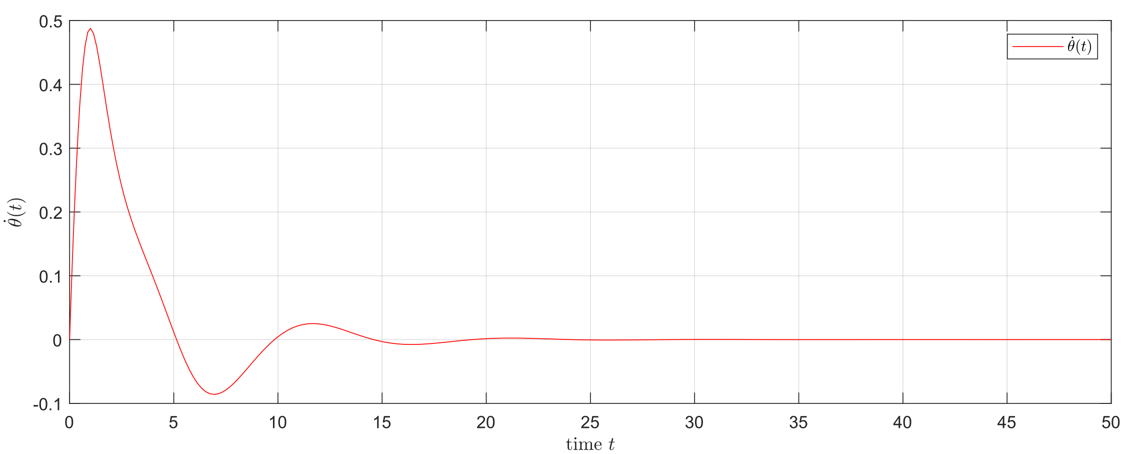

Figure 2: The variation of $\dot{\theta}$ over time

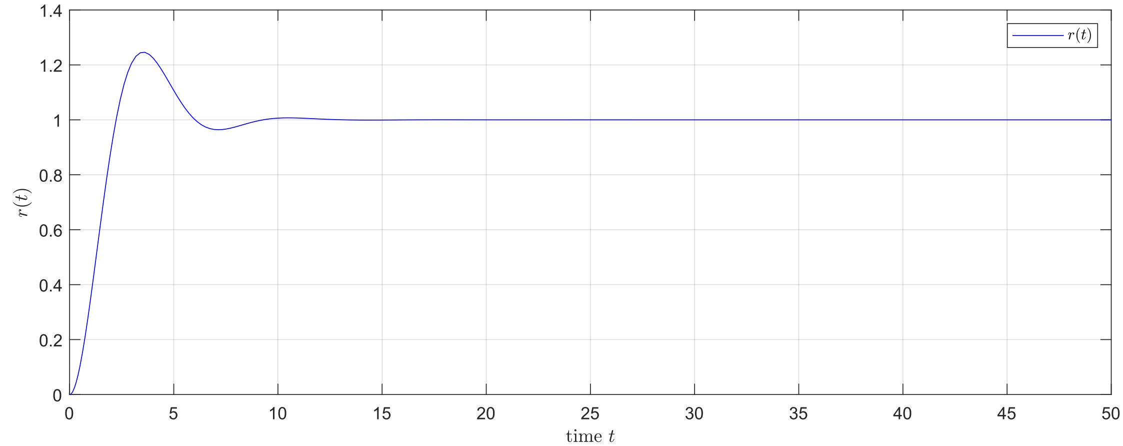

Figure 3: The variation of r over time

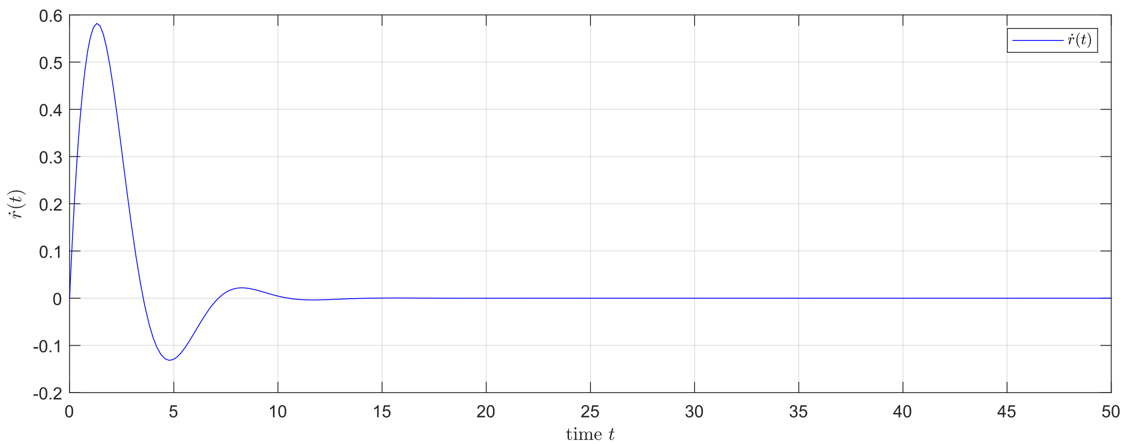

Figure 4: The variation of $\dot{r}$ over time

Then, Let $\theta_d = 2, r_d = 1, \mathbf{x}_0 = [1, 1, 1, 1]^T$. The simulation results are shown as follows:

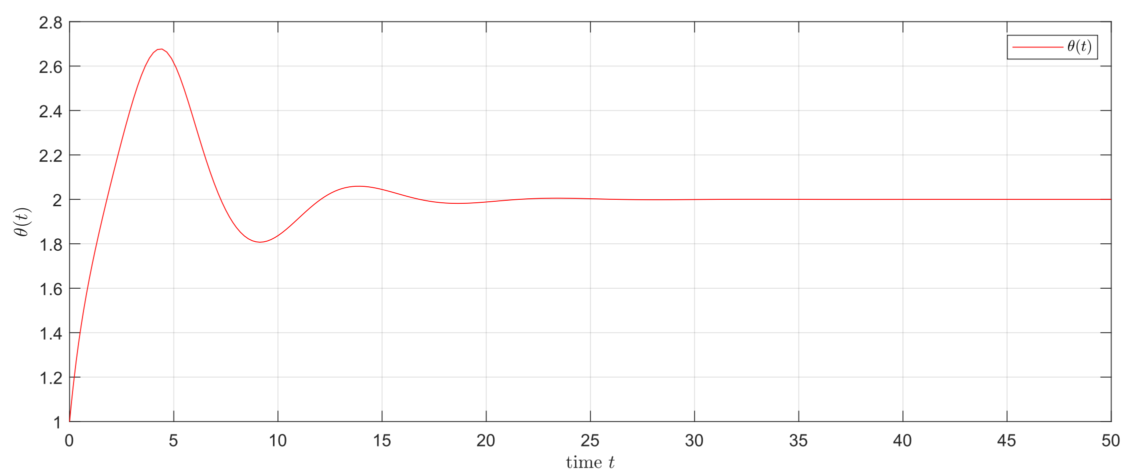

Figure 5: The variation of θ over time

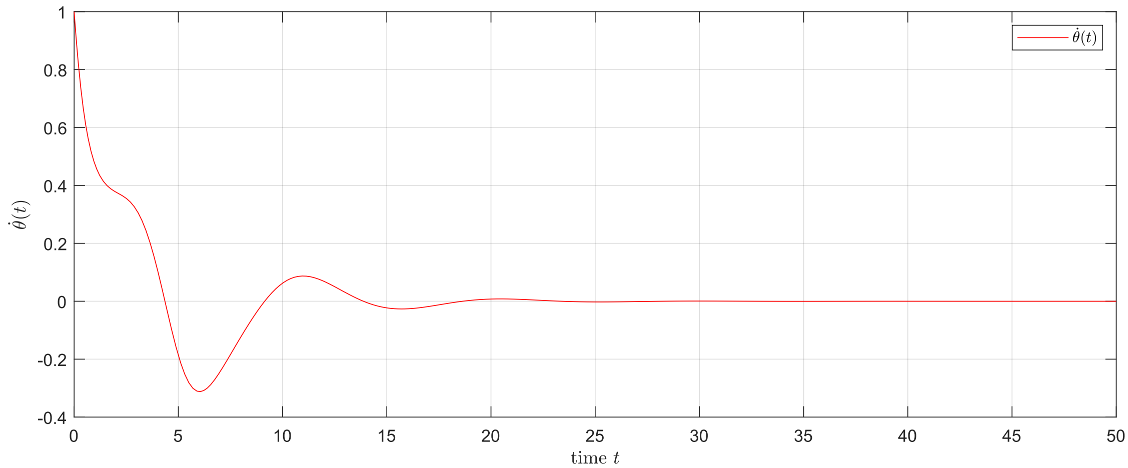

Figure 6: The variation of $\dot{\theta}$ over time

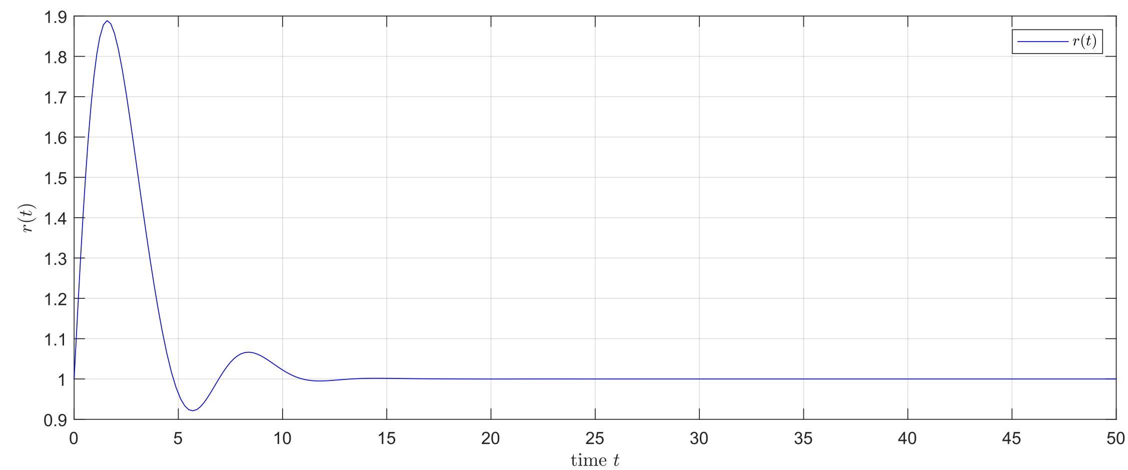

Figure 7: The variation of r over time

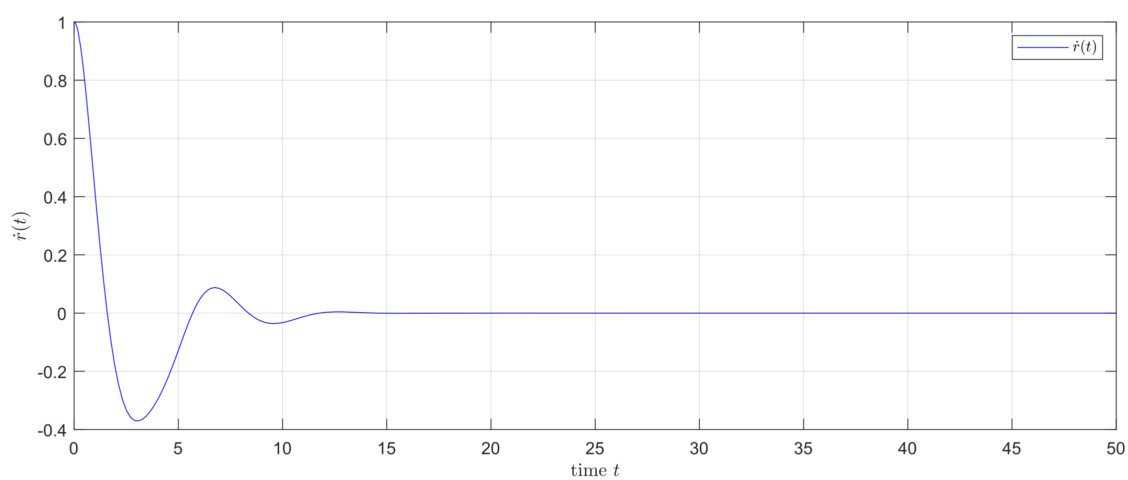

Figure 8: The variation of $\dot{r}$ over time

The simulation results indicate that the system begins adjusting from its initial state x_0 based on the reference values.After a brief period, the states θ and r converge rapidly to their respective constant reference values θ_d and r_d. The convergence process is swift, with the transition completed within 20 seconds. The graphs demonstrate that the system exhibits minimal oscillations during this process. Consequently, the overall control performance is highly satisfactory and aligns well with expectations.

Then, Let $\theta_d = 15, r_d = 10, \mathbf{x}_0 = [10, 10, 0, 0]^T$. The simulation results are shown as follows:

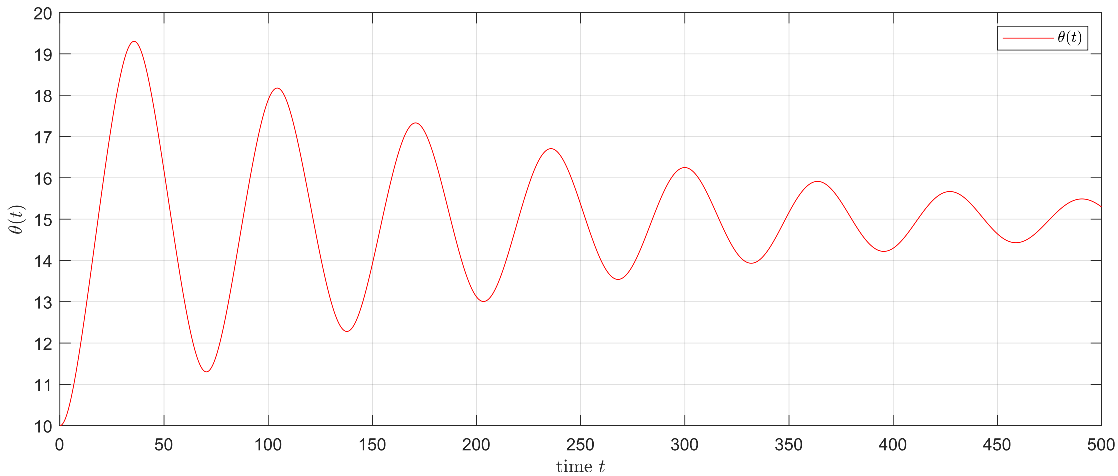

Figure 9: The variation of θ over time

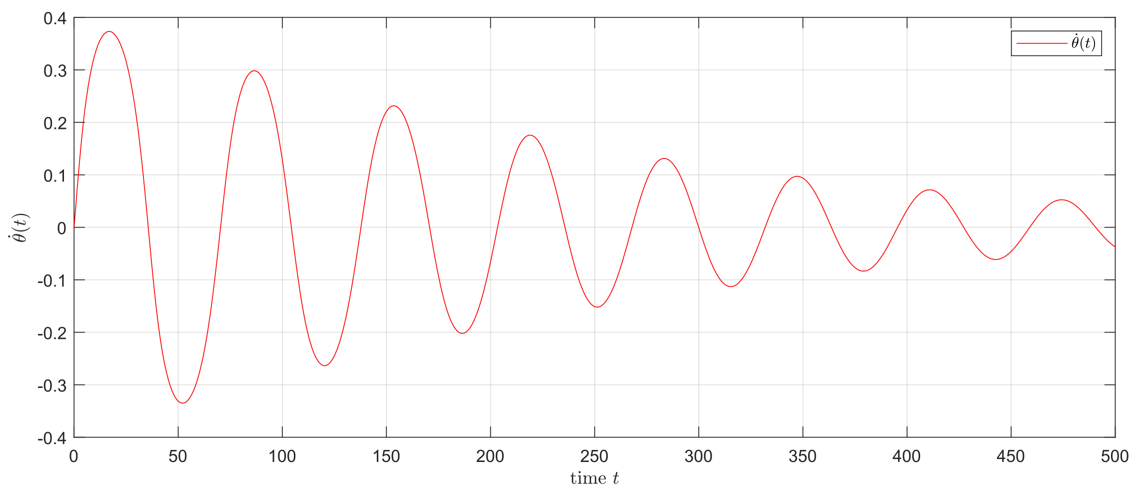

Figure 10: The variation of $\dot{\theta}$ over time

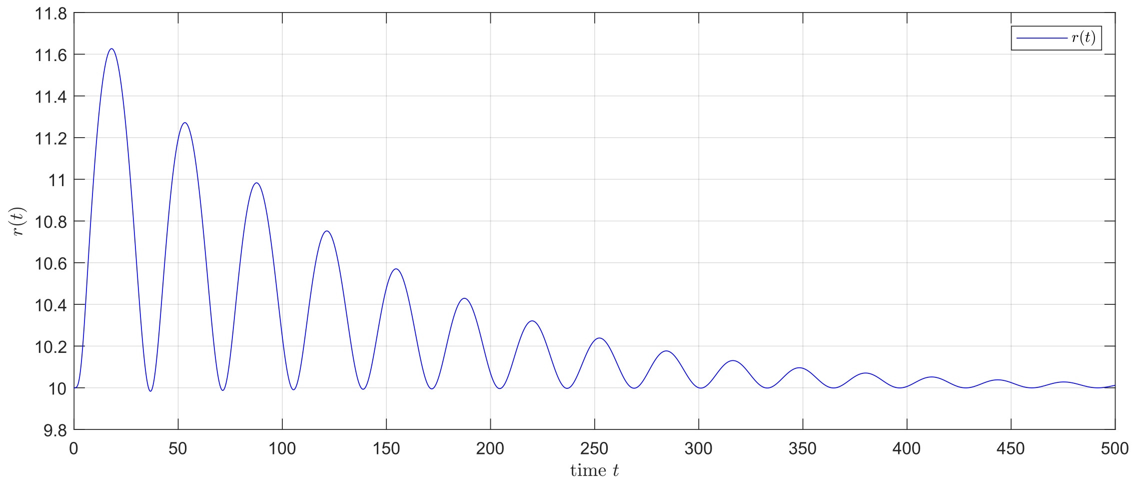

Figure 11: The variation of r over time

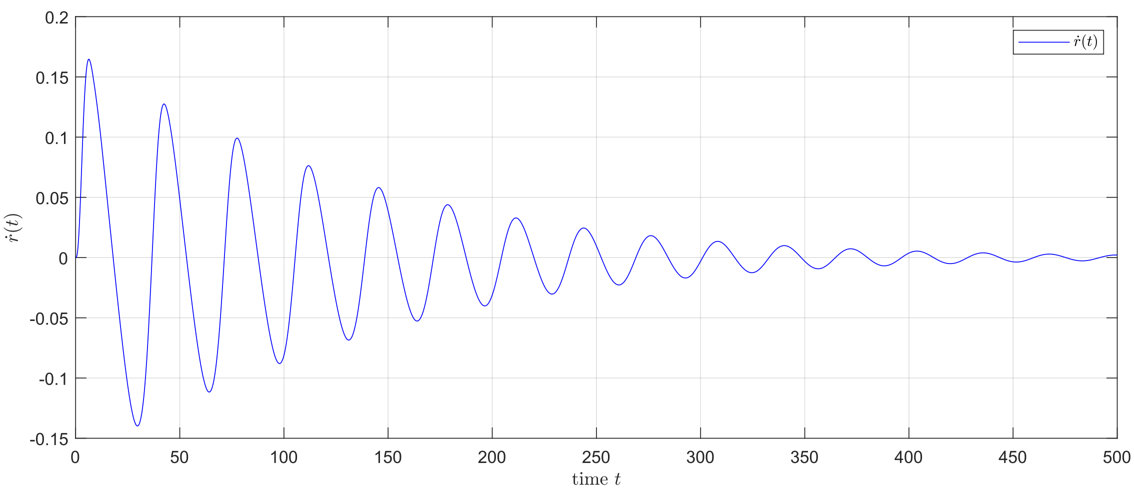

Figure 12: The variation of $\dot{r}$ over time

The simulation results indicate that the system begins to adjust from its initial state x_0 based on the reference values. The states θ and r both converge to the constant reference values θ_d and r_d. However, the convergence rates for both θ and r are very slow, with a convergence time of over 500 seconds. Additionally, the graphs clearly show that the system experiences significant oscillations. In this context, the performance of the PD controller is poor.

Based on all the simulation outcomes, it can be clearly seen that the PD controller functions effectively in driving the controlled system to converge to a fixed input value within a finite time interval. This finding is in line with the previously deduced Lyapunov equation. It shows that the PD controller can be relied upon to ensure the stable and accurate operation of the controlled system.

5 Disproof of the ISS property of the system

The goal of this section is to verify whether the system is not Input-to-State Stable (ISS) by constructing an unstable feedback controller $r_d = 0$, $\theta_d = csign(k(x(t))$, where $k(x) : R^4 \to R$ and c is a constant.

According to the hint, we can use the Lyapunov function again to maximize the amount of power transferred to the system.

$$V(x) = \frac{1}{2}(r^2+1)\dot{\theta}^2 + \frac{1}{2}\dot{r}^2 + \frac{1}{2}(\theta-\theta_d)^2 + \frac{1}{2}(r-r_d)^2$$

By setting the constant C to 100, feedback control system is constructed. The initial state is selected as $[10, 5, 0, 0]^T$. Following the given hint, we can let $\theta_d = 100 \cdot \mathrm{sign}(V(x))$. The corresponding MATLAB code is provided below.

```
%Disprove of ISS property matlab code
x0 = [15, 5, 0, 0]
[t, x] = ode45(@dynamics1, [0 100], x0);  %initial state is set in advance
figure(1);
subplot(4,1,1);
plot(t, x(:, 1), 'r','linewidth',1);
xlabel('time $t$', 'Interpreter', 'latex');
ylabel('$\theta(t)$', 'Interpreter', 'latex');
legend('$\theta(t)$', 'Interpreter', 'latex');
subplot(4,1,2);
plot(t, x(:, 2), 'b','linewidth',1);
xlabel('time $t$', 'Interpreter', 'latex');
ylabel('$r(t)$', 'Interpreter', 'latex');
legend('$r(t)$', 'Interpreter', 'latex');
subplot(4,1,3);
plot(t, x(:, 3), 'r','linewidth',1);
xlabel('time $t$', 'Interpreter', 'latex');
ylabel('$\dot{\theta}(t)$', 'Interpreter', 'latex');
legend('$\dot{\theta}(t)$', 'Interpreter', 'latex');
subplot(4,1,4);
plot(t, x(:, 4), 'b','linewidth',1);
xlabel('time $t$', 'Interpreter', 'latex');
ylabel('$\dot{r}(t)$', 'Interpreter', 'latex');
legend('$\dot{r}(t)$', 'Interpreter', 'latex');
grid on;
% function of the four dimensional system;
function dx = dynamics1(t, x)
    %Calculate the Lyapunov function V(x) for the current state, without
    V = 0.5 * ((x(2)^2 + 1) * (x(3))^2) + 0.5 * (x(4))^2 + 0.5 * (x(1))^2 + 0.5 * (x(2))^2;
    % update thetad
    c = 100;
    theta_d = c * sign(V);
    rd = 0;
    % calculation of state derivatives
    denominator = (x(2))^2 + 1 ;
    dx = [x(3);
          x(4);
          (-2 * x(2) * x(3) * x(4) -x(3) - (x(1) - theta_d)) / denominator;
          (x(3))^2 * x(2) - x(4) - (x(2) - rd)];
end
```

From the simulation results shown below, it can be observed that when we set $\theta_d = 100 \cdot \mathrm{sign}(V(x))$, the states x_1 and x_2 diverge. Although the input remains bounded, the system exhibits instability, contradicting the expected Input-to-State Stability (ISS) property.

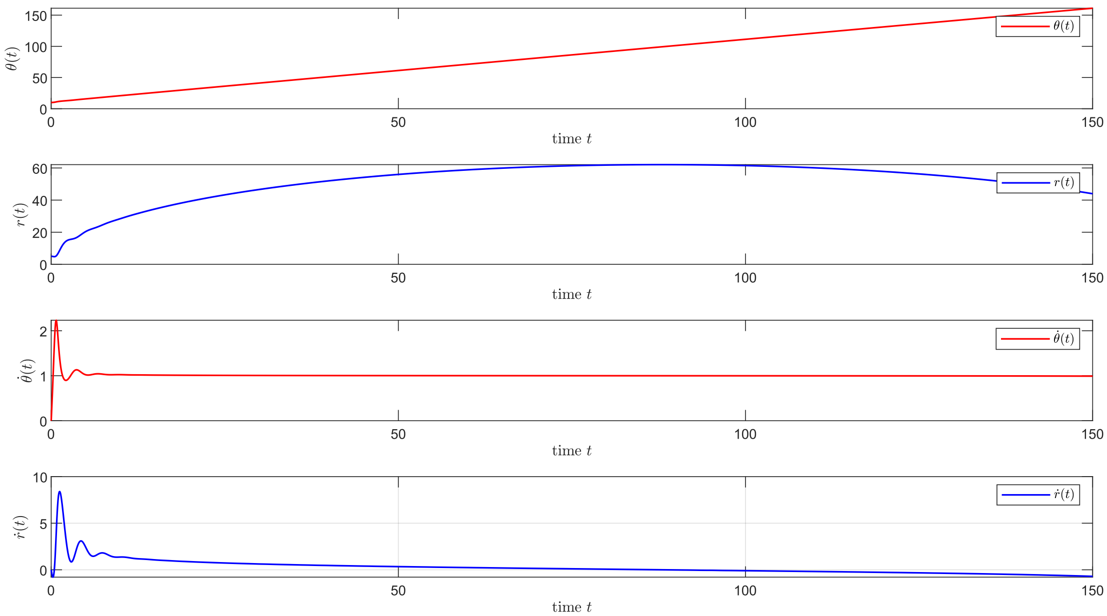

Figure 13: The variation of system state over time

6 Nonlinear PD Controller Stability Analysis

Modify the controller by using a nonlinear PD control, and the u_1 and u_2 are as follows:

$$u_1 = -\dot{\theta} - (\theta - \theta_d)$$
$$u_2 = -\dot{r} - (r - r_d) - (r^3 - r_d^3)$$

Substitute the physical quantities in the equation into the state variables, The following equation can be obtained.

$$u_1 = -x_2 - (x_1 - \theta_d)$$
$$u_2 = -x_4 - (x_2 - r_d) - ({x_2}^3 - {r_d}^3)$$

According to the equation of u_2, we can know that the linear proportional term $-(r - r_d)$ is used to reduce error. The nonlinear term $-(r^3 - r_d^3)$ is used to improve control over large errors. The differential term $-\dot{r}$ is used to increase the damping effect of the system and improve dynamic performance.The introduction of nonlinear terms can increase the adaptability of the control system to nonlinear dynamics.

The Matlab code for constructing the nonlinear controller is shown below.

```
function u = pd_controller2(x, theta_d, r_d)
    % define PD control
    u1 = -x(3) - (x(1) - theta_d);
    u2 = -x(4) - (x(2) - r_d)-(x(2)^2-r_d^2);
    u = [u1; u2];
end
```

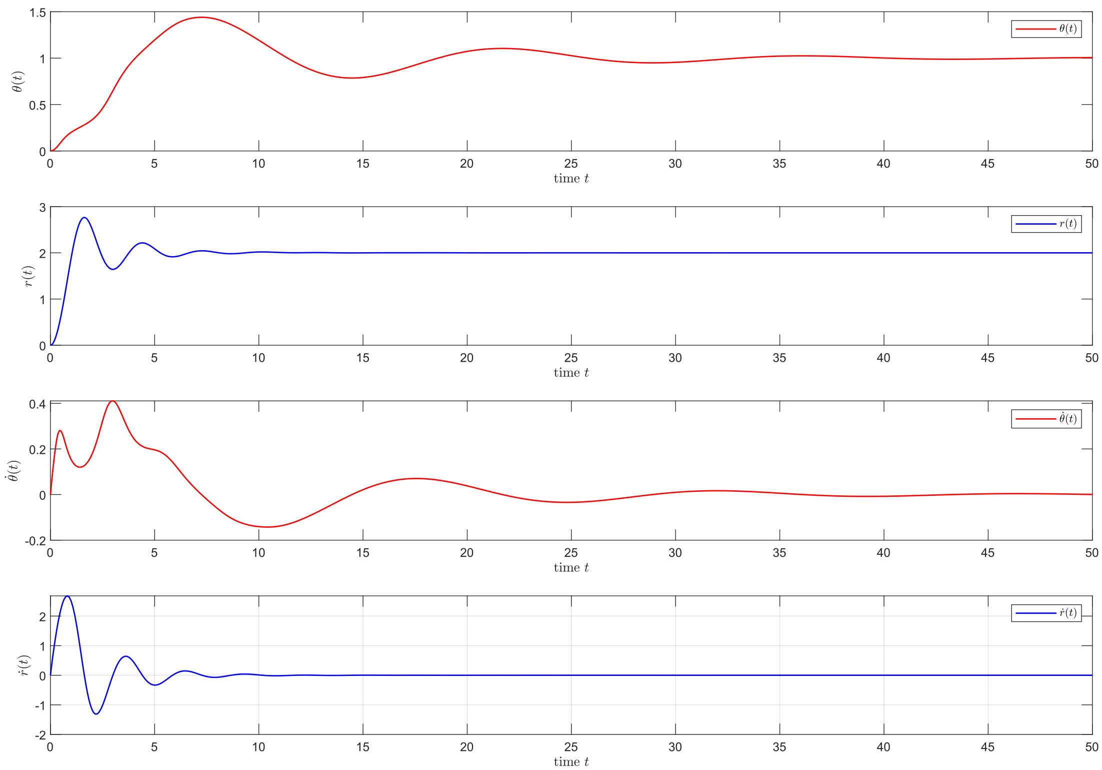

Figure 14: The variation of system state over time

By setting the initial state $x_0 = [0, 0, 0, 0]^T$, $\theta_d = 1$, and $r_d = 2$, the simulation results are as follows. We can clearly see that initially, despite some oscillations, the system states eventually converge to the input value.

The convergence rates for both θ and r are rapid, leading to overall control performance that is highly satisfactory and in line with our expectations.