國立陽明交通大學 期貨與選擇權

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1. Basics

(a) Derivation of (Eq.13.2 and Eq.13.3)

To construct a riskless portfolio in a one-step binomial model and derive the option price formula $f = e^{-rT}[pf_u + (1-p)f_d]$ where $p = \frac{e^{rT}-d}{u-d}$, follow these steps:

Step 1: Set Up the Binomial Model

- **Initial Stock Price:** S_0
- **Up Move:** Stock price moves up to $S_u = S_0 u$
- **Down Move:** Stock price moves down to $S_d = S_0 d$
- **Option Payoffs:**
  - **Up State:** f_u
  - **Down State:** f_d
- **Risk-Free Rate:** r
- **Time Period:** T

Step 2: Construct a Riskless Portfolio

We aim to find Δ (the number of shares to hold) and B (the amount to borrow or lend at the risk-free rate) such that the portfolio replicates the option's payoffs in both states.

Set up the following equations based on the portfolio's payoffs:

1. **Up State Payoff:**

$$\Delta S_u + Be^{rT} = f_u(1)$$

2. **Down State Payoff:**

$$\Delta S_d + Be^{rT} = f_d(2)$$

Subtract the second equation from the first to eliminate Be^{rT}:

$$\Delta(S_u - S_d) = f_u - f_d$$

Solve for Δ:

$$\Delta = \frac{f_u - f_d}{S_u - S_d} = \frac{f_u - f_d}{S_0(u - d)}$$

Now, solve for B using one of the payoff equations (we'll use the up state):

$$B = \frac{f_u - \Delta S_u}{e^{rT}}$$

Substitute Δ into B:

$$B = \frac{f_u - \left(\frac{f_u - f_d}{S_0(u-d)}\right) S_0 u}{e^{rT}}$$

Simplify B:

$$B = \frac{f_u - \left(\frac{f_u - f_d}{u-d}\right) u}{e^{rT}}$$

$$B = \frac{(u-d)f_u - u(f_u - f_d)}{(u-d)e^{rT}}$$

$$B = \frac{uf_d - df_u}{(u-d)e^{rT}}$$

Step 3: Determine the Option Price

The cost of setting up the portfolio today is the option price f:

$$f = \Delta S_0 + B$$

$$f = \left(\frac{f_u - f_d}{u - d}\right) + \frac{uf_d - df_u}{(u-d)e^{rT}}$$

Step 4: Express the Option Price Using Risk-Neutral Probabilities

Introduce the risk-neutral probability p:

$$p = \frac{e^{rT} - d}{u - d}$$

Note that

$$1 - p = \frac{u - e^{rT}}{u - d}$$

.

Now, rewrite f in terms of p:

1. Multiply numerator and denominator in f to combine terms:

$$f = \frac{(f_u - f_d)e^{rT} + uf_d - df_u}{(u-d)e^{rT}}$$

2. Rearrange the numerator:

$$f = \frac{e^{rT}(f_u - f_d) + uf_d - df_u}{(u-d)e^{rT}}$$

3. Group like terms:

$$f = \frac{f_u(e^{rT} - d) + f_d(u - e^{rT})}{(u-d)e^{rT}}$$

4. Recognize that $e^{rT} - d = p(u-d)$ and $u - e^{rT} = (1-p)(u-d)$:

$$f = \frac{f_u p(u-d) + f_d(1-p)(u-d)}{(u-d)e^{rT}}$$

5. Simplify the expression:

$$f = \frac{(u-d)[pf_u + (1-p)f_d]}{(u-d)e^{rT}}$$

$$f = \frac{pf_u + (1-p)f_d}{e^{rT}}$$

6. Finally, write the option price formula:

$$f = e^{-rT}[pf_u + (1-p)f_d]$$

Conclusion

By constructing a riskless portfolio and using the properties of the binomial model, we have shown in detail that the option price is:

$$f = e^{-rT}[pf_u + (1-p)f_d]$$

where:

$$p = \frac{e^{rT} - d}{u - d}$$

This formula represents the discounted expected payoff of the option under the risk-neutral probability measure.

Answer:

By constructing a riskless portfolio and using the binomial model, we find that:

$$f \;=\; e^{-rT}[\,p\,f_u + (1–p)\,f_d\,], \quad where \quad p = [\,e^(rT)–d\,]/(u–d)$$

(b) End-of-Chapter exercise 21.7.

With the usual notation,

$$p = \frac{a - d}{u - d}$$

$$1 - p = \frac{u - a}{u - d}$$

if $a < d$ or $a > u$, one of the two probabilities is negative. This happens when

$$e^{(r-q)\Delta t} < e^{-\sigma\sqrt{\Delta t}}$$

or

$$e^{(r-q)\Delta t} > e^{\sigma\sqrt{\Delta t}}$$

This in turn happens when $(q-r)\sqrt{\Delta t} > \sigma$ or $(r-q)\sqrt{\Delta t} < \sigma$. Hence, negative probabilities
occur when

$$\sigma < \left|(r-q)\sqrt{\Delta t}\right|$$

This is the condition in footnote 8.

2.Computing Option Prices Using Binomial Model

a. Compute u, d, p for each setting

- $\Delta t = 12 * T$ (one month)
  – For 24 time steps (Delta t = 0.0833): u = 1.0905, d = 0.9170, p = 0.5024

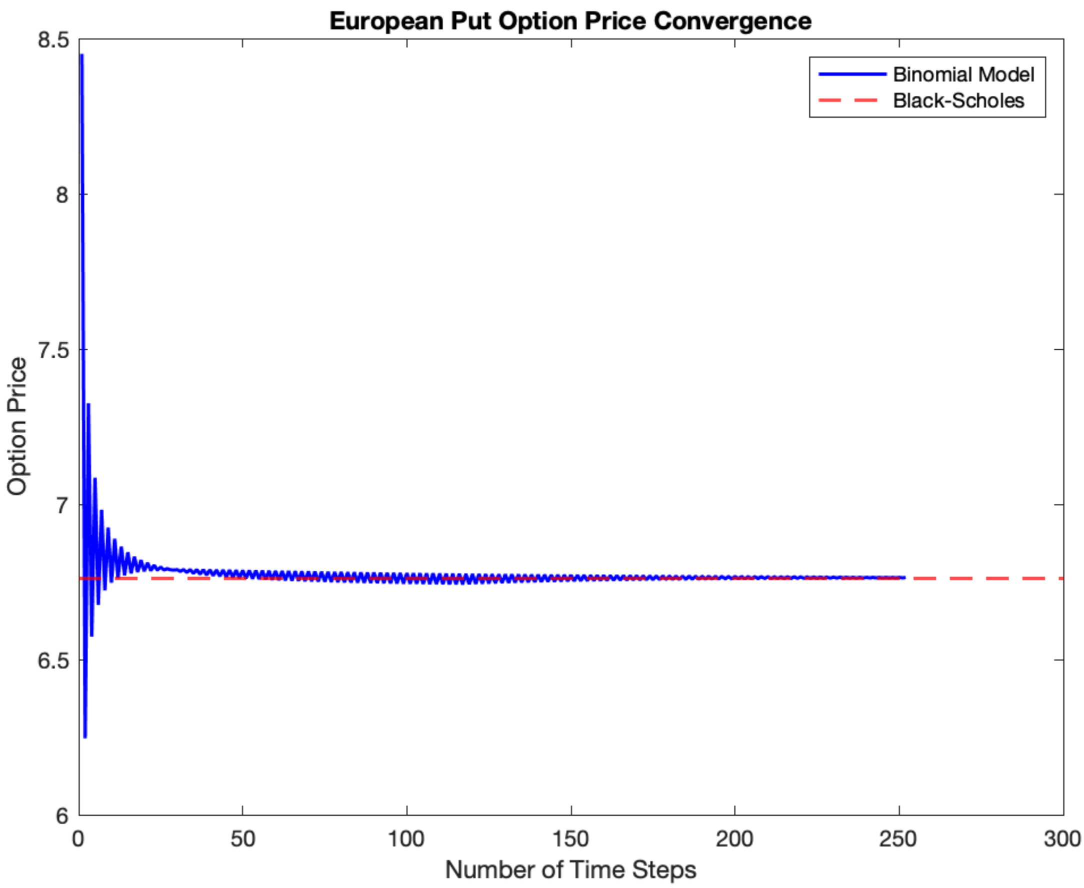
- $\Delta t = 52 * T$ (one week)
  – For 104 time steps (Delta t = 0.0192): u = 1.0425, d = 0.9593, p = 0.5012

- $\Delta t = 252 * T$ (one day)
  – For 504 time steps (Delta t = 0.0040): u = 1.0191, d = 0.9813, p = 0.5005

b. European Put Option Prices

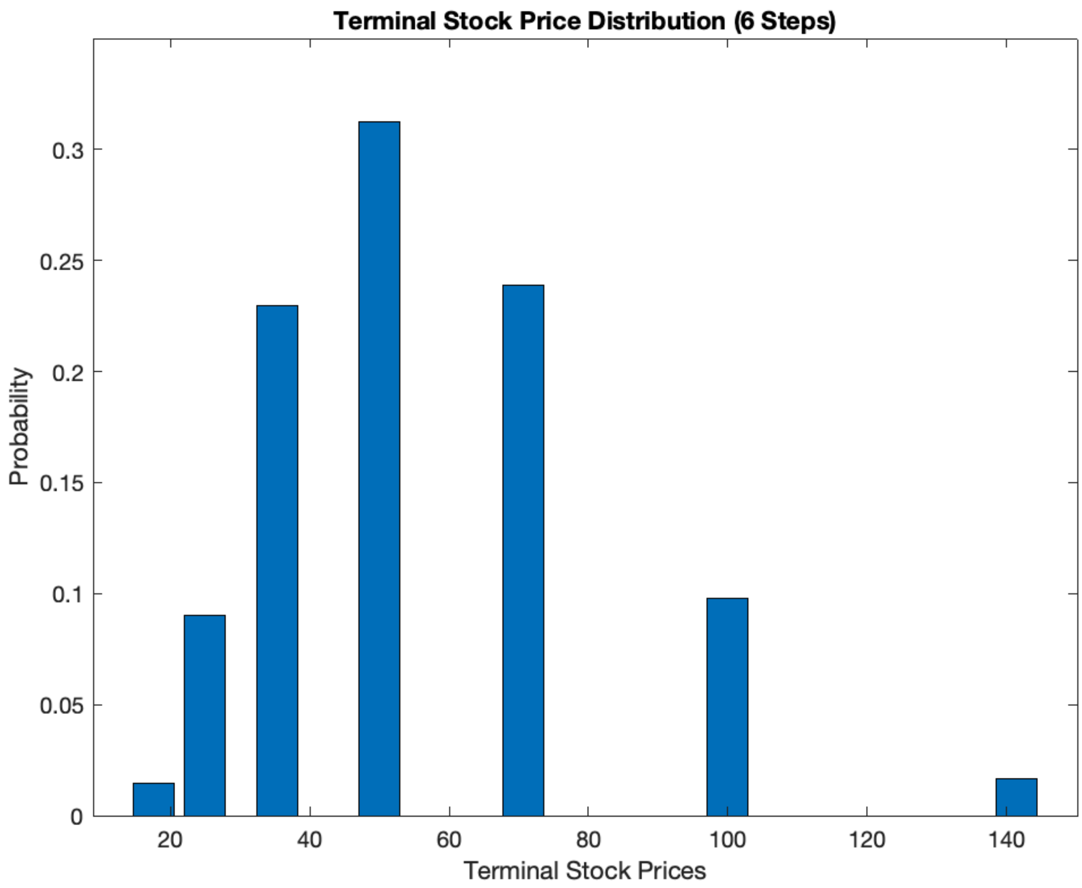
- European Put Option Price with 24 steps: 6.7874
- European Put Option Price with 104 steps: 6.7775
- European Put Option Price with 504 steps: 6.7570
- Black-Scholes Put Option Price: 6.7601
  The larger steps is, the closer it is to the theoretical value

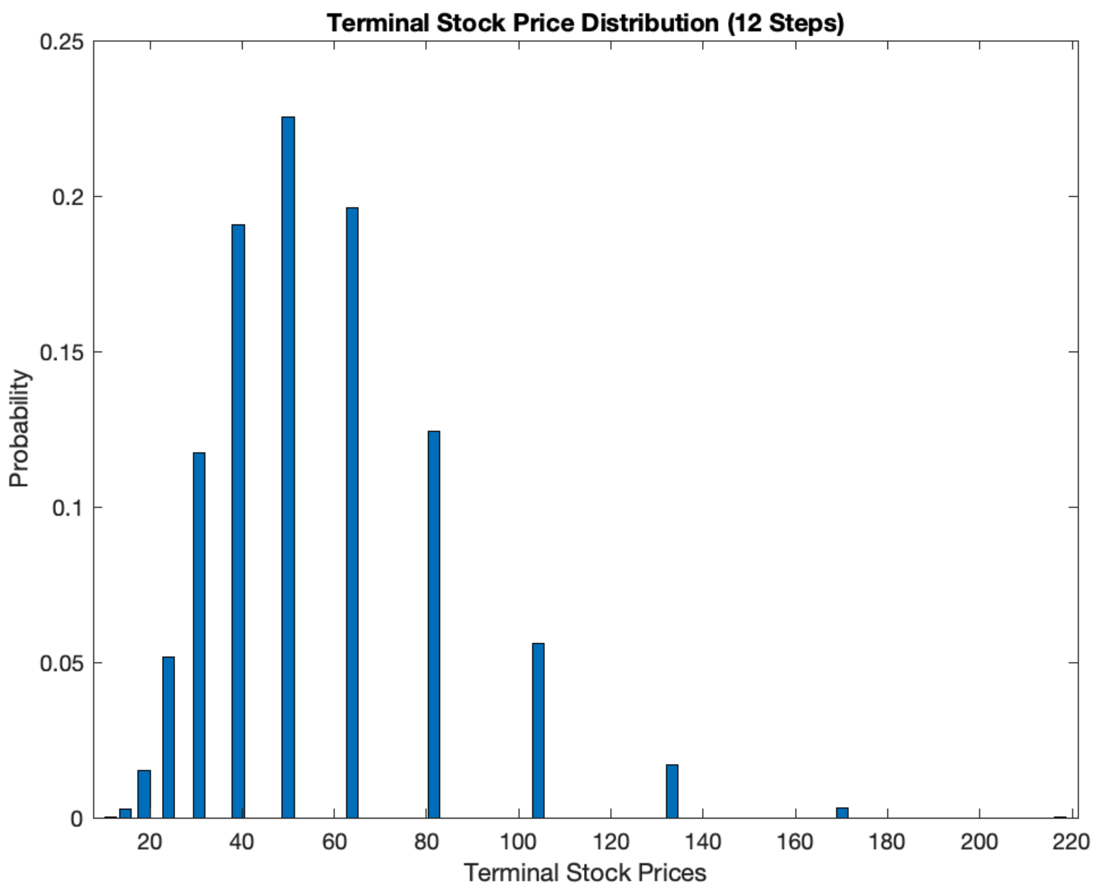
c. Plot Results for Varying Time Steps

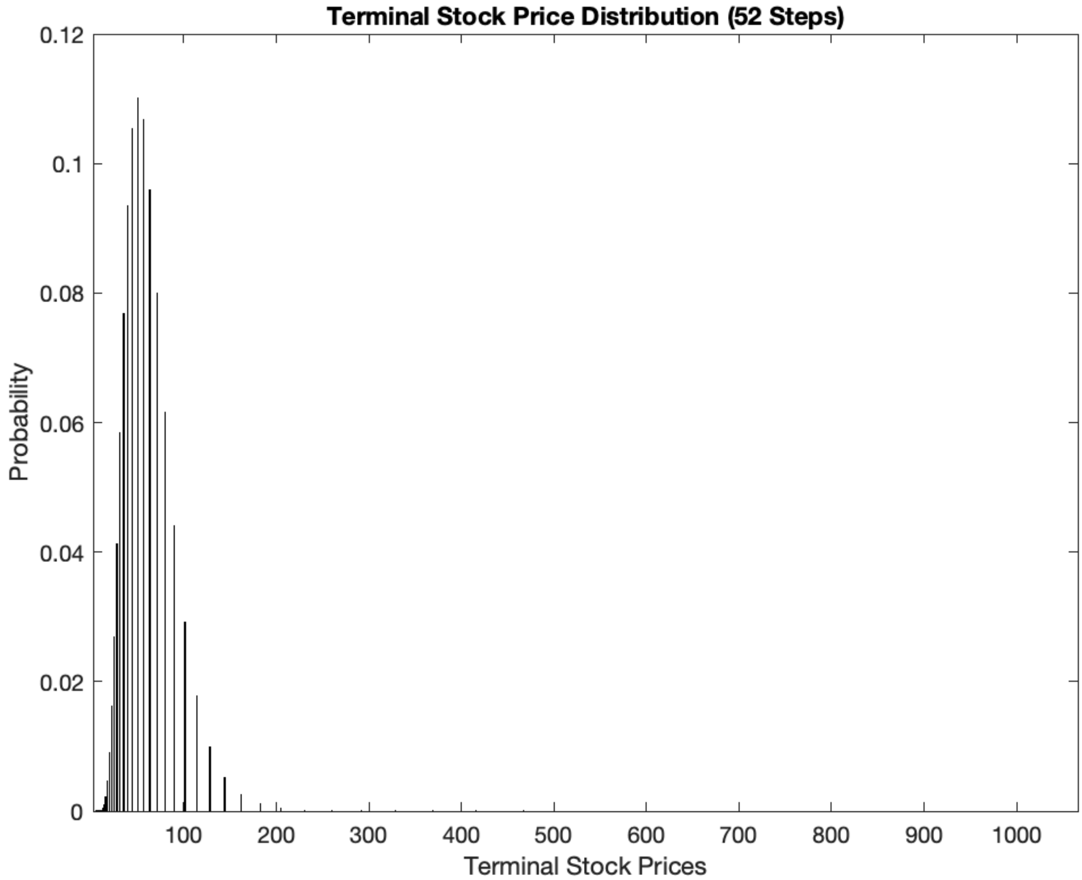


When the step value is very low, it can be significantly closer to the theoretical value when the step value is increased. However, when the step value is greater than 50, the approaching value is not obvious enough.

d. Terminal Stock Price Distributions







These results are close to the log normal distribution, which is similar to the Black-Scholes results.

e. American Put Option Prices

- American Put Option Price with 24 steps: 7.5075
- American Put Option Price with 104 steps: 7.4856
- American Put Option Price with 504 steps: 7.4710
  American put option usually early exercise so the price will usually higher than European put option