deform nuclei π -meson derivation

2019 年 7 月 3 日

The hamiltonian of π -meson nuclear couplings can be express as:

$$H_\pi = -\frac{1}{2}\frac{1}{m_\pi^2}\sum_{\alpha\beta;\alpha'\beta'}\langle\tau_\alpha|\vec{\tau}|\tau_{\alpha'}\rangle\cdot\langle\tau_\beta|\vec{\tau}|\tau_{\beta'}\rangle c_\alpha^\dagger c_\beta^\dagger c_{\beta'}c_{\alpha'}\int d\boldsymbol{r} f_\pi(\xi)\int d\boldsymbol{r'} f_\pi(\xi') \\ \times \bar{f}_\alpha(\boldsymbol{r})\bar{f}_\beta(\boldsymbol{r'})\left(\gamma_5\gamma^k\right)_1\left(\gamma_5\gamma^l\right)_2\partial_k(1)\partial_l(2)v\left(m_\pi;1,2\right)f_{\beta'}(\boldsymbol{r'})f_{\alpha'}(\boldsymbol{r}) \tag{1}$$

where the Yukawa expanded form of propagator $v(m_\pi;1,2)$ is:

$$v(m_\pi;r,r') = \frac{1}{4\pi}\frac{e^{-m_\pi|\boldsymbol{r}-\boldsymbol{r'}|}}{|\boldsymbol{r}-\boldsymbol{r'}|} \tag{2}$$

There have no time term. So the differential ∂_k and ∂_l only include the space differential, which mean that k and l equal 1,2,3.

For the π meson-nuclear coupling, there is the exchange term in energy functional:

$$E_\pi = \frac{1}{2}\frac{1}{m_\pi^2}\sum_{\alpha\beta}\left(2-\delta_{q_\alpha q_\beta}\right)\int d\boldsymbol{r}d\boldsymbol{r'}\bar{f}_\alpha(\boldsymbol{r})\bar{f}_\beta(\boldsymbol{r'})\left(f_\pi\gamma_5\gamma^k\right)_1 \\ \left(f_\pi\gamma_5\gamma^l\right)_2\partial_k(1)\partial_l(2)v(m_\pi;1,2)f_\alpha(\boldsymbol{r'})f_\beta(\boldsymbol{r}) \tag{3}$$

And the space differential of Yukawa propagator can be write as:

$$\boldsymbol{\nabla}_{r'}\boldsymbol{\nabla}_r v(m_\pi;\boldsymbol{r},\boldsymbol{r'}) = m_\pi^2\sum_L\sum_{L_1L_2}^{L\pm1}C_{L010}^{L_10}C_{L010}^{L_20}\mathcal{V}_L^{L_1L_2}(m_\pi;r,r')\boldsymbol{Y}_{LL_1}(\hat{\boldsymbol{r}})\cdot\boldsymbol{Y}_{LL_2}(\hat{\boldsymbol{r}}_2) \tag{4}$$

1 WAV FUNCTION

The wav function in deform nuclear have to express at the Dirac-Woods-Saxon base. So, we use the following marking method.

$$\bar{f}_\alpha(\boldsymbol{r}) = \bar{\psi}_{\nu\pi m}(\boldsymbol{r}_a) \qquad f_\beta(\boldsymbol{r}) = \psi_{\nu'\pi'm'}(\boldsymbol{r}_a) \tag{5}$$

$$\bar{f}_\beta(\boldsymbol{r'}) = \bar{\psi}_{\nu'\pi'm'}(\boldsymbol{r}_b) \qquad f_\alpha(\boldsymbol{r'}) = \psi_{\nu\pi m}(\boldsymbol{r}_b) \tag{6}$$

The $\bar{f}_\alpha, \bar{f}_\beta, f_\beta$ and f_α are wav function's mark in Eq.(3), and in deform nuclei, the good quantum is π, parity and m, the third component of angular momentum. Now we express the wav function at the Dirac-Woods-Saxon base.

$$\bar{\psi}_{\nu\pi m}(\boldsymbol{r}_a) = \sum_a C^\dagger_{a,\nu\pi m}\bar{\psi}_{a\pi m} \qquad \psi_{\nu'\pi' m'}(\boldsymbol{r}_a) = \sum_{a'} C_{a',\nu'\pi' m'}\psi_{a'\pi' m'} \tag{7}$$

$$\bar{\psi}_{\nu'\pi' m'}(\boldsymbol{r}_b) = \sum_{b'} C^\dagger_{b',\nu'\pi' m'}\bar{\psi}_{b'\pi' m'} \qquad \psi_{\nu\pi m}(\boldsymbol{r}_b) = \sum_b C_{b,\nu\pi m}\psi_{b\pi m} \tag{8}$$

Among them, a and b mean $\{n, \kappa\}$, and n is the principal quantum number which dependent on the node of big component of spinor ψ, and κ reflect the angular momentum and spin, in other word, the total angular momentum. C is nothing but the expansion coefficient.

Use this marking method, the π -meson nuclear coupling energy functional can rewrite as:

$$\begin{aligned}
E^E_\pi =& \frac{1}{2}\sum_{\alpha\beta}(2-\delta_{q_\alpha q_\beta})\int d\boldsymbol{r}_a d\boldsymbol{r}_b \sum_L \sum_{L_1L_2}^{L\pm1} C^{L_10}_{L010}C^{L_20}_{L010}\mathcal{V}^{L_1L_2}_L(m_\pi; r_a, r_b)\boldsymbol{Y}_{LL_1}\cdot\boldsymbol{Y}_{LL_2} \\
& \left[f_\pi\bar{\psi}_{\nu\pi m}\gamma_5\boldsymbol{\gamma}\psi_{\nu'\pi' m'}\right]_{\boldsymbol{r}_a}\cdot\left[f_\pi\bar{\psi}_{\nu'\pi' m'}\gamma_5\boldsymbol{\gamma}\psi_{\nu\pi m}\right]_{\boldsymbol{r}_b} \\
=& \frac{1}{2}\sum_{\alpha\beta}\left(2-\delta_{q_\alpha q_\beta}\right)\int d\boldsymbol{r}_a d\boldsymbol{r}_b \sum_{LM} \sum_{L_1L_2}^{L\pm1} C^{L_10}_{L010}C^{L_20}_{L010}\mathcal{V}^{L_1L_2}_L(m_\pi; r_a, r_b)\boldsymbol{Y}^{L_1}_{LM}\boldsymbol{Y}^{L_2*}_{LM} \\
& \left[f_\pi\bar{\psi}_{\nu\pi m}\gamma_5\boldsymbol{\gamma}\psi_{\nu'\pi' m'}\right]_{\boldsymbol{r}_a}\cdot\left[f_\pi\bar{\psi}_{\nu'\pi' m'}\gamma_5\boldsymbol{\gamma}\psi_{\nu\pi m}\right]_{\boldsymbol{r}_b} \\
=& \frac{1}{2}\sum_{\alpha\beta}\left(2-\delta_{q_\alpha q_\beta}\right)\int d\boldsymbol{r}_a d\boldsymbol{r}_b \sum_{LM} \sum_{L_1L_2}^{L\pm1} C^{L_10}_{L010}C^{L_20}_{L010}\mathcal{V}^{L_1L_2}_L(m_\pi; r_a, r_b) \\
& \left[f_\pi\bar{\psi}_{\nu\pi m}\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{Y}^{L_1}_{LM}\psi_{\nu'\pi' m'}\right]_{\boldsymbol{r}_a}\cdot\left[f_\pi\bar{\psi}_{\nu'\pi' m'}\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{Y}^{L_2*}_{LM}\psi_{\nu\pi m}\right]_{\boldsymbol{r}_b}
\end{aligned}$$

According to 《Quantum theory of angular momentum》P.64

$$(\mathscr{M}_J\cdot\mathscr{N}_J) = \sum_M \mathscr{M}_{JM}\mathscr{N}^*_{JM} \tag{9}$$

for the irreducible tensor.

And the term which correspond to the wav function:

$$\bar{\psi}_{\nu\pi m}\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{Y}^{L_1}_{LM}\psi_{\nu'\pi' m'} = \sum_{aa'} C^\dagger_{a,\nu\pi m}C_{a',\nu'\pi' m'}\bar{\psi}_{a\pi m}\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{Y}^{L_1}_{LM}\psi_{a'\pi' m'} \tag{10}$$

$$\bar{\psi}_{\nu'\pi' m'}\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{Y}^{L_2*}_{LM}\psi_{\nu\pi m} = \sum_{bb'} C^\dagger_{b',\nu'\pi' m'}C_{b,\nu\pi m}\bar{\psi}_{b'\pi' m'}\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{Y}^{L_2*}_{LM}\psi_{b\pi m} \tag{11}$$

Now we will write the concrete form of this term. For the spherical nuclei, the spinor can be express as:

$$\psi_{a\pi m}(r,\vartheta,\varphi) = \frac{1}{r}\begin{pmatrix} G(r)\Omega^{l_a}_{j_a m}(\vartheta,\varphi) \\ iF(r)\Omega^{l'_a}_{j_a m}(\vartheta,\varphi)\end{pmatrix} \tag{12}$$

$$\bar{\psi}_{a\pi m}(r,\vartheta,\varphi)=\frac{1}{r}\begin{pmatrix} G(r)\Omega^{l_a\dagger}_{j_am}(\vartheta,\varphi) & iF(r)\Omega^{l'_a\dagger}_{j_am}(\vartheta,\varphi)\end{pmatrix} \tag{13}$$

where Ω is the spinor spherical harmonics, and it can be expressed in terms of scalar spherical harmonic $Y_{LM}(\vartheta,\varphi)$ and spin functions $\chi_{\frac{1}{2}\sigma}$ as:

$$\Omega^{L}_{JM}(\vartheta,\varphi)=\sum_{m\sigma}C^{JM}_{LM\frac{1}{2}\sigma}Y_{LM}(\vartheta,\varphi)\chi_{\frac{1}{2}\sigma} \tag{14}$$

And we have to mentioned that $l_a+l'_a=2j_a$, which mean that $l_a-l'_a=\pm1$, it's depend on the sign of κ.

Then, the wav function (12) can be write as:

$$\begin{aligned}
&\sum_{aa'}C^{\dagger}_{a,\nu\pi m}C_{a',\nu'\pi'm'}\bar{\psi}_{a\pi m}\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{Y}^{L_1}_{LM}\psi_{a'\pi'm'}\\
=&\sum_{aa'}C^{\dagger}_{a,\nu\pi m}C_{a',\nu'\pi'm'}\frac{1}{r_a}\begin{pmatrix} G_a\Omega^{l_a\dagger}_{j_am} & iF_a\Omega^{l'_a\dagger}_{j_am}\end{pmatrix}\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{Y}^{L_1}_{LM}\frac{1}{r_a}\begin{pmatrix} G_{a'}\Omega^{l_{a'}}_{j_{a'}m'}\\ iF_{a'}\Omega^{l'_{a'}}_{j_{a'}m'}\end{pmatrix}\\
=&\sum_{aa'}C^{\dagger}_{a,\nu\pi m}C_{a',\nu'\pi'm'}\frac{1}{r_a^2}\begin{pmatrix} G_a\Omega^{l_a\dagger}_{j_am} & iF_a\Omega^{l'_a\dagger}_{j_am}\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix}\cdot\boldsymbol{Y}^{L_1}_{LM}\begin{pmatrix} G_{a'}\Omega^{l_{a'}}_{j_{a'}m'}\\ iF_{a'}\Omega^{l'_{a'}}_{j_{a'}m'}\end{pmatrix}\\
=&\sum_{aa'}C^{\dagger}_{a,\nu\pi m}C_{a',\nu'\pi'm'}\frac{1}{r_a^2}\begin{pmatrix} iF_a\Omega^{l'_a\dagger}_{j_am} & G_a\Omega^{l_a\dagger}_{j_am}\end{pmatrix}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix}\cdot\boldsymbol{Y}^{L_1}_{LM}\begin{pmatrix} G_{a'}\Omega^{l_{a'}}_{j_{a'}m'}\\ iF_{a'}\Omega^{l'_{a'}}_{j_{a'}m'}\end{pmatrix}\\
=&\sum_{aa'}C^{\dagger}_{a,\nu\pi m}C_{a',\nu'\pi'm'}\frac{1}{r_a^2}\begin{pmatrix} -G_a\Omega^{l_a\dagger}_{j_am}\boldsymbol{\sigma} & iF_a\Omega^{l'_a\dagger}_{j_am}\boldsymbol{\sigma}\end{pmatrix}\cdot\boldsymbol{Y}^{L_1}_{LM}\begin{pmatrix} G_{a'}\Omega^{l_{a'}}_{j_{a'}m'}\\ iF_{a'}\Omega^{l'_{a'}}_{j_{a'}m'}\end{pmatrix}\\
=&\sum_{aa'}C^{\dagger}_{a,\nu\pi m}C_{a',\nu'\pi'm'}\frac{1}{r_a^2}\left[-G_aG_{a'}\Omega^{l_a\dagger}_{j_am}\boldsymbol{\sigma}\cdot\boldsymbol{Y}^{L_1}_{LM}\Omega^{l_{a'}}_{j_{a'}m'}-F_aF_{a'}\Omega^{l'_a\dagger}_{j_am}\boldsymbol{\sigma}\cdot\boldsymbol{Y}^{L_1}_{LM}\Omega^{l'_{a'}}_{j_{a'}m'}\right]
\end{aligned} \tag{15}$$

Use the relations:

$$\boldsymbol{\sigma}\cdot\boldsymbol{Y}^{L_1}_{LM}(\hat{\boldsymbol{r}})=\sum_{k'}\sigma_{k'}\boldsymbol{e}^{k'}\sum_{\mu k}C^{LM}_{L_1\mu 1k}Y_{L_1\mu}\boldsymbol{e}_k \tag{16}$$

and

$$\boldsymbol{e}^{\mu}\boldsymbol{e}_{\nu}=\boldsymbol{e}_{\mu}\boldsymbol{e}^{*}_{\nu}=\delta_{\mu\nu},\qquad(\mu\nu=\pm1,0) \tag{17}$$

So

$$\boldsymbol{\sigma}\cdot\boldsymbol{Y}^{L_1}_{LM}=\sum_{\mu k}C^{LM}_{L_1\mu 1k}Y_{L_1\mu}\sigma_{1k} \tag{18}$$

For the wav function, we can see that:

$$\begin{aligned}
&G_aG_{a'}\Omega^{l_a\dagger}_{j_am}\boldsymbol{\sigma}\cdot\boldsymbol{Y}^{L_1}_{LM}\Omega^{l_{a'}}_{j_{a'}m'}\\
=&G_aG_{a'}\Omega^{l_a\dagger}_{j_am}\sum_{\mu k}C^{LM}_{L_1\mu 1k}Y_{L_1\mu}\sigma_{1k}\Omega^{l_{a'}}_{j_{a'}m'}
\end{aligned}$$

$$
\begin{aligned}
=&G_aG_{a'}\sum_{\mu k}C^{LM}_{L_1\mu 1k}\Omega^{l_a\dagger}_{j_am}Y_{L_1\mu}\sigma_{1k}\Omega^{l_{a'}}_{j_{a'}m'}\\
=&G_aG_{a'}\left[\sum_{\mu k}C^{LM}_{L_1\mu 1k}Y_{L_1\mu}\sigma_{1k}\right]\Omega^{l_a\dagger}_{j_am}\Omega^{l_{a'}}_{j_{a'}m'} \qquad (19)
\end{aligned}
$$

2 附录

$$
\begin{aligned}
&\left\langle a \mid \mathcal{T}^{L_1}_{LM} \mid b\right\rangle = C^{j_am_a}_{j_bm_bLM}\left\langle a \mid\mid \mathcal{T}_{LL_1} \mid\mid b\right\rangle\\
=&C^{j_am_a}_{j_bm_bLM}(-1)^{j_a+L-\frac{1}{2}}\hat{j}_b\hat{L}^{-1}\frac{\kappa_{ab}+\beta_{LL_1}}{\sqrt{4\pi\left|\beta_{LL_1}\right|}}C^{L0}_{j_a\frac{1}{2}j_b-\frac{1}{2}} \qquad (20)\\
& \qquad (21)
\end{aligned}
$$

Also we can write as:

$$
\begin{aligned}
&\left\langle a \mid \mathcal{T}^{L_1}_{LM} \mid b\right\rangle = \Omega^{l_a\dagger}_{j_am_a}\mathcal{T}^{L_1}_{LM}\Omega^{l_b}_{j_bm_b}\\
=&\Omega^{l_a\dagger}_{j_am_a}\sum_{\mu k}C^{LM}_{L_1\mu 1k}Y_{L_1\mu}\sigma_{1k}\Omega^{l_b}_{j_bm_b} \qquad (22)\\
=&\sum_{\mu k}C^{LM}_{L_1\mu 1k}Y_{L_1\mu}\sigma_{1k}\sum_{\lambda}(-1)^{j_a+m_a+j_b+\lambda+\frac{1}{2}}\begin{Bmatrix} l_a & l_b & \lambda\\ j_b & j_a & \frac{1}{2}\end{Bmatrix}\frac{\hat{j}_a\hat{j}_b\hat{l}_a\hat{l}_b}{4\pi\hat{\lambda}}C^{\lambda 0}_{l_a0l_b0}C^{\lambda M}_{j_a-m_aj_bm_b}Y_{\lambda M} \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
&\Omega^{L_1\dagger}_{J_1M_1}\Omega^{L_2}_{J_2M_2}\\
=&\sum_{m_1\sigma_1}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma_1}Y^*_{l_1m_1}\chi^*_{\frac{1}{2}\sigma_1}\sum_{m_2\sigma_2}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma_2}Y_{l_2m_2}\chi_{\frac{1}{2}\sigma_2}\\
=&\sum_{m_1\sigma_1}\sum_{m_2\sigma_2}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma_1}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma_2}Y^*_{l_1m_1}Y_{l_2m_2}\chi^*_{\frac{1}{2}\sigma_1}\chi_{\frac{1}{2}\sigma_2}\\
=&\sum_{m_1m_2}\sum_{\sigma}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma}(-1)^{m_1}Y_{l_1-m_1}Y_{l_2m_2}\\
=&\sum_{m_1m_2}\sum_{\sigma}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma}(-1)^{m_1}\sum_{LM}\frac{\hat{l}_1\hat{l}_2}{4\pi\hat{L}}C^{L0}_{l_10l_20}C^{LM}_{l_1-m_1l_2m_2}Y_{LM}\\
=&\sum_{m_1m_2}\sum_{\sigma}\sum_{LM}(-1)^{m_1}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma}\frac{\hat{l}_1\hat{l}_2}{4\pi\hat{L}}C^{L0}_{l_10l_20}(-1)^{l_1+l_2-L}C^{LM}_{l_2m_2l_1-m_1}Y_{LM}\\
=&\sum_{LM}(-1)^{l_1+l_1+l_2-L}\frac{\hat{l}_1\hat{l}_2}{4\pi\hat{L}}C^{L0}_{l_10l_20}\sum_{m_1m_2\sigma}(-1)^{l_1-m_1}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma}C^{LM}_{l_2m_2l_1-m_1}Y_{LM}\\
=&\sum_{LM}(-1)^{2l_1+l_2-L}\frac{\hat{l}_1\hat{l}_2}{4\pi\hat{L}}C^{L0}_{l_10l_20}(-1)^{\frac{1}{2}+J_1+l_2+L}\prod_{J_1L}C^{J_2M_2}_{J_1M_1LM}\begin{Bmatrix} l_1 & \frac{1}{2} & J_1\\ J_2 & L & l_2\end{Bmatrix}Y_{LM}
\end{aligned}
$$

$$= \sum_{LM} (-1)^{\frac{1}{2}+J_1} \frac{\hat{l}_1 \hat{l}_2 \hat{J}_1 \hat{L}}{4\pi \hat{L}} C^{L0}_{l_1 0 l_2 0} (-1)^{J_1 - M_1} \frac{\hat{J}_2}{\hat{L}} C^{L-M}_{J_1 M_1 J_2 - M_2} \begin{Bmatrix} l_1 & l_2 & L \\ J_2 & J_1 & \frac{1}{2} \end{Bmatrix} Y_{LM}$$

$$= \sum_{LM} (-1)^{\frac{1}{2}+2J_1 - M_1 + J_1 + J_2 - L} \begin{Bmatrix} l_1 & l_2 & L \\ J_2 & J_1 & \frac{1}{2} \end{Bmatrix} \frac{\hat{J}_1 \hat{J}_2 \hat{l}_1 \hat{l}_2}{4\pi \hat{L}} C^{L0}_{l_1 0 l_2 0} C^{LM}_{J_1 - M_1 J_2 M_2} Y_{LM}$$

$$= \sum_{LM} (-1)^{J_1 + M_1 + J_2 + L + \frac{1}{2}} \begin{Bmatrix} l_1 & l_2 & L \\ J_2 & J_1 & \frac{1}{2} \end{Bmatrix} \frac{\hat{J}_1 \hat{J}_2 \hat{l}_1 \hat{l}_2}{4\pi \hat{L}} C^{L0}_{l_1 0 l_2 0} C^{LM}_{J_1 - M_1 J_2 M_2} Y_{LM} \tag{24}$$

the sum of M is necessary?

The orthonormality condition for the basis functions is:

$$\chi^{\dagger}_{Sm} \chi_{Sm'} = \delta_{mm'} \tag{25}$$

The complex conjugate function Y^*_{lm}

$$Y^*_{lm}(\vartheta, \varphi) = Y_{lm}(\vartheta, -\varphi) = (-1)^m Y_{l-m}(\vartheta, \varphi) \tag{26}$$

and a direct product of two spherical harmonics of the same arguments can expanded in series as:

$$Y_{l_1 m_1}(\vartheta, \varphi) Y_{l_2 m_2}(\vartheta, \varphi) = \sum_{LM} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}} C^{L0}_{l_1 0 l_2 0} C^{LM}_{l_1 m_1 l_2 m_2} Y_{LM} \tag{27}$$

The symmetry properties of the Clebsch-Gordan Coefficients:

$$C^{c\gamma}_{a\alpha b\beta} = (-1)^{a+b-c} C_{b\beta a-\alpha} C^{c\gamma}_{b\beta a-\alpha} = (-1)^{a-\alpha} \sqrt{\frac{2c+1}{2b+1}} C^{b-\beta}_{a\alpha c-\gamma} \tag{28}$$

The sums involving products of three CG coefficients:

$$\sum_{\alpha\beta\delta} (-1)^{a-\alpha} C^{c\gamma}_{a\alpha b\beta} C^{e\varepsilon}_{d\delta b\beta} C^{f\varphi}_{d\delta a-\alpha} = \kappa_1 \prod_{cf} C^{e\varepsilon}_{c\gamma f\varphi} \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix} \tag{29}$$

and

$$\kappa_1 = (-1)^{b+c+d+f} \qquad\qquad \prod_{abc\cdots} = [(2a+1)(2b+1)(2c+1)\cdots]^{\frac{1}{2}} \tag{30}$$

$$\Omega^{L_1 \dagger}_{J_1 M_1} \hat{\boldsymbol{S}} \Omega^{L_2}_{J_2 M_2}$$

$$= \sum_{m_1 \sigma_1} C^{J_1 M_1}_{l_1 m_1 \frac{1}{2} \sigma_1} Y^*_{l_1 m_1} \chi^{\dagger}_{\frac{1}{2}\sigma_1} \hat{\boldsymbol{S}} \sum_{m_2 \sigma_2} C^{J_2 M_2}_{l_2 m_2 \frac{1}{2} \sigma_2} Y_{l_2 m_2} \chi_{\frac{1}{2}\sigma_2}$$

$$= \sum_{m_1 \sigma_1} \sum_{m_2 \sigma_2} C^{J_1 M_1}_{l_1 m_1 \frac{1}{2} \sigma_1} C^{J_2 M_2}_{l_2 m_2 \frac{1}{2} \sigma_2} Y^*_{l_1 m_1} Y_{l_2 m_2} \chi^{\dagger}_{\frac{1}{2}\sigma_1} \hat{\boldsymbol{S}} \chi_{\frac{1}{2}\sigma_2}$$

$$= \sum_{m_1 \sigma_1} \sum_{m_2 \sigma_2} C^{J_1 M_1}_{l_1 m_1 \frac{1}{2} \sigma_1} C^{J_2 M_2}_{l_2 m_2 \frac{1}{2} \sigma_2} (-1)^{m_1} \sum_{LM} \frac{\hat{l}_1 \hat{l}_2}{\sqrt{4\pi} \hat{L}} C^{L0}_{l_1 0 l_2 0} C^{LM}_{l_1 - m_1 l_2 m_2} Y_{LM} \sum_{\mu} \frac{\sqrt{3}}{2} C^{\frac{1}{2}\sigma_1}_{\frac{1}{2}\sigma_2 1\mu} \boldsymbol{e}_{1\mu}$$

$$
\begin{aligned}
=&\sum_{m_1\sigma_1}\sum_{m_2\sigma_2}\sum_{LM\mu}(-1)^{l_1-m_1}\frac{\hat{J}_1}{\sqrt{2}}C^{\frac{1}{2}\sigma_1}_{J_1M_1l_1-m_1}(-1)^{l_2-m_2}\frac{\hat{J}_2}{\sqrt{2}}C^{\frac{1}{2}\sigma_2}_{J_2M_2l_2-m_2}(-1)^{l_1+l_2-L}C^{LM}_{l_2m_2l_1-m_1}(-1)^{\frac{1}{2}-\sigma_2}\sqrt{\frac{2}{3}}C^{1\mu}_{\frac{1}{2}\sigma_1\frac{1}{2}-\sigma_2}\\
&\times(-1)^{m_1}\frac{\hat{l}_1\hat{l}_2}{\sqrt{4\pi}\hat{L}}C^{L0}_{l_10l_20}Y_{LM}\frac{\sqrt{3}}{2}\boldsymbol{e}_{1\mu}\\
=&\sum_{m_1\sigma_1}\sum_{m_2\sigma_2}\sum_{LM\mu}(-1)^{L-m_2+\frac{1}{2}-\sigma_2}\frac{\hat{J}_1\hat{J}_2}{2}C^{\frac{1}{2}\sigma_1}_{J_1M_1l_1-m_1}(-1)^{J_2-M_2}\frac{\sqrt{2}}{\hat{l}_2}C^{l_2m_2}_{J_2M_2\frac{1}{2}-\sigma_2}C^{LM}_{l_2m_2l_1-m_1}C^{1\mu}_{\frac{1}{2}\sigma_1\frac{1}{2}-\sigma_2}\frac{\hat{l}_1\hat{l}_2}{\sqrt{4\pi}\hat{L}}C^{L0}_{l_10l_20}Y_{LM}\frac{1}{\sqrt{2}}\boldsymbol{e}_{1\mu}\\
=&\sum_{m_1\sigma_1}\sum_{m_2\sigma_2}\sum_{LM\mu}(-1)^{l_1+\mu+J_2-M_2}(-1)^{\frac{1}{2}-\sigma_1+l_2-m_2}\frac{\hat{J}_1\hat{J}_2}{2}C^{\frac{1}{2}\sigma_1}_{J_1M_1l_1-m_1}\frac{\sqrt{2}}{\hat{l}_2}C^{l_2m_2}_{J_2M_2\frac{1}{2}-\sigma_2}C^{LM}_{l_2m_2l_1-m_1}C^{1\mu}_{\frac{1}{2}\sigma_1\frac{1}{2}-\sigma_2}\\
&\times\frac{\hat{l}_1\hat{l}_2}{\sqrt{4\pi}\hat{L}}C^{L0}_{l_10l_20}Y_{LM}\frac{1}{\sqrt{2}}\boldsymbol{e}_{1\mu}\\
=&\sum_{LM\mu}(-1)^{l_1+\mu+J_2-M_2}\frac{\hat{J}_1\hat{J}_2\hat{l}_1}{\sqrt{16\pi}\hat{L}}(-1)^{J_1+J_2-M_1-M_2}\sqrt{2}\hat{l}_2\hat{L}\sqrt{3}\sum_{J\xi}C^{J\xi}_{LM1-\mu}C^{J\xi}_{J_2M_2J_1-M_1}\begin{Bmatrix}\frac{1}{2}&l_1&J_1\\\frac{1}{2}&l_2&J_2\\1&L&J\end{Bmatrix}C^{L0}_{l_10l_20}Y_{LM}\boldsymbol{e}_{1\mu}\\
=&\sum_{L}(-1)^{l_1+J_1-M_1}\frac{\sqrt{3}\hat{j}_1\hat{j}_2\hat{l}_1\hat{l}_2}{\sqrt{8\pi}}\sum_{J}C^{J\xi}_{J_2M_2J_1-M_1}\begin{Bmatrix}\frac{1}{2}&l_1&J_1\\\frac{1}{2}&l_2&J_2\\1&L&J\end{Bmatrix}C^{L0}_{l_10l_20}\sum_{M\mu}(-1)^{\mu}C^{J\xi}_{LM1-\mu}Y_{LM}\boldsymbol{e}_{1\mu}\\
=&\sum_{JL}(-1)^{l_1+J_1-M_1}\frac{\sqrt{3(2j_1+1)(2j_2+1)(2l_1+1)(2l_2+2)}}{\sqrt{8\pi}}\begin{Bmatrix}l_1&J_1&\frac{1}{2}\\l_2&J_2&\frac{1}{2}\\L&J&1\end{Bmatrix}\\
&\times(-1)^{J_1+J_2-J}C^{J\xi}_{J_1-M_1J_2M_2}C^{L0}_{l_10l_20}\boldsymbol{Y}^{L}_{J\xi}\\
=&(-1)^{J_2+l_1-M_1+1}\frac{\sqrt{3(2j_1+1)(2j_2+1)(2l_1+1)(2l_2+2)}}{\sqrt{8\pi}}\sum_{JL}(-1)^{J}C^{L0}_{l_10l_20}\begin{Bmatrix}l_1&J_1&\frac{1}{2}\\l_2&J_2&\frac{1}{2}\\L&J&1\end{Bmatrix}C^{J\xi}_{J_1-M_1J_2M_2}\boldsymbol{Y}^{L}_{J\xi}\\
=&(-1)^{J_2+l_1+M_1}\frac{\sqrt{3(2j_1+1)(2j_2+1)(2l_1+1)(2l_2+2)}}{\sqrt{8\pi}}\sum_{JL}(-1)^{J}C^{L0}_{l_10l_20}\begin{Bmatrix}l_1&J_1&\frac{1}{2}\\l_2&J_2&\frac{1}{2}\\L&J&1\end{Bmatrix}C^{J\xi}_{J_1-M_1J_2M_2}\boldsymbol{Y}^{L}_{J\xi}
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\Omega^{L_1\dagger}_{J_1M_1}\boldsymbol{\sigma}\cdot\boldsymbol{Y}^{L_1}_{LM}\Omega^{L_2}_{J_2M_2}\\
&=\sum_{m_1\sigma_1}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma_1}Y^*_{l_1m_1}\chi^{\dagger}_{\frac{1}{2}\sigma_1}\sum_{\mu k}C^{LM}_{L_1\mu1k}Y_{L_1\mu}\sigma_{1k}\sum_{m_2\sigma_2}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma_2}Y_{l_2m_2}\chi_{\frac{1}{2}\sigma_2}\\
&=\sum_{m_1\sigma_1m_2\sigma_2}\sum_{\mu k}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma_1}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma_2}C^{LM}_{L_1\mu1k}Y^*_{l_1m_1}Y_{L_1\mu}Y_{l_2m_2}\chi^{\dagger}_{\frac{1}{2}\sigma_1}\sigma_{1k}\chi_{\frac{1}{2}\sigma_2}
\end{aligned}
$$

$$= \sum_{m_1\sigma_1 m_2\sigma_2} \sum_{\mu k} C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma_1} C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma_2} C^{LM}_{L_1\mu 1k} (-1)^{m_1} \sum_{\lambda\xi\lambda'\xi'} \frac{\hat{l}_1\hat{L}_1\hat{l}_2}{4\pi\hat{\lambda}} C^{\lambda'0}_{l_10L_10} C^{\lambda 0}_{\lambda'0l_20} C^{\lambda'\xi'}_{l_1-m_1L_1\mu} C^{\lambda\xi}_{\lambda'\xi'l_2m_2} Y_{\lambda\xi} \frac{\sqrt{3}}{2} C^{\frac{1}{2}\sigma_1}_{\frac{1}{2}\sigma_21k}$$

$$C^{q\kappa}_{p\psi a\alpha} C^{r\rho}_{q\kappa u\nu} C^{s\sigma}_{r\rho c\gamma} C^{p\psi}_{s\sigma v\mu} C^{t\tau}_{u-\nu b\beta} C^{v\mu}_{t\tau d\delta} \tag{32}$$

$a\alpha, b\beta, c\gamma, d\delta$ 是外线。

先决定 $C^{t\tau}_{u-\mu b\beta} C^{v\mu}_{t\tau d\delta}$

$$C^{\lambda\xi}_{\lambda'\xi'l_2m_2} = (-1)^{\lambda'-\xi'} \frac{\hat{\lambda}}{\hat{2}} C^{l_2-m_2}_{\lambda'\xi'\lambda-\xi} \to C^{t\tau}_{u-\nu b\beta} \tag{33}$$

即：

$$(-1)^{\lambda'-\xi'} \frac{\hat{\lambda}}{\hat{l}_2} \hat{l}_2 (-1)^{m_2+3\lambda'+\lambda} \begin{pmatrix} \lambda' & \lambda & l_2 \\ \xi' & -\xi & m_2 \end{pmatrix} = (-1)^{m_2+\lambda-\xi'} \hat{\lambda} \begin{pmatrix} \lambda' & \lambda & l_2 \\ \xi' & -\xi & m_2 \end{pmatrix} \tag{34}$$

则 $t\tau$ 对应 $l_2 - m_2$

$$C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma_2} = (-1)^{l_2-m_2} \frac{\hat{J}_2}{\hat{\frac{1}{2}}} C^{\frac{1}{2}-\sigma_2}_{l_2m_2J_2-M_2} = (-1)^{l_2-m_2+l_2+J_2-\frac{1}{2}} \frac{\hat{J}_2}{\hat{\frac{1}{2}}} C^{\frac{1}{2}\sigma_2}_{l_2-m_2J_2M_2} = (-1)^{J_2-m_2-\frac{1}{2}} \frac{\hat{J}_2}{\sqrt{2}} C^{\frac{1}{2}\sigma_2}_{l_2-m_2J_2M_2} \to C^{v\mu}_{t\tau d\delta} \tag{35}$$

即：

$$(-1)^{J_2-m_2-\frac{1}{2}} \frac{\hat{J}_2}{\hat{\frac{1}{2}}} \hat{\frac{1}{2}} (-1)^{-\sigma_2+3l_2+J_2} \begin{pmatrix} l_2 & J_2 & \frac{1}{2} \\ -m_2 & M_2 & -\sigma_2 \end{pmatrix} = (-1)^{l_2-M_2} \hat{J}_2 \begin{pmatrix} l_2 & J_2 & \frac{1}{2} \\ -m_2 & M_2 & -\sigma_2 \end{pmatrix} \tag{36}$$

然后是四个 CG 系数求和和两条外线

$$C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma_1} = (-1)^{\frac{1}{2}+\sigma_1} \frac{\hat{J}_1}{\hat{l}_1} C^{l_1m_1}_{\frac{1}{2}-\sigma_1J_1M_1} = (-1)^{\frac{1}{2}+\sigma_1} \frac{\hat{J}_1}{\hat{l}_1} C^{l_1m_1}_{\frac{1}{2}-\sigma_1J_1M_1} = (-1)^{1+\sigma_1+J_1-l_1} \frac{\hat{J}_1}{\hat{l}_1} C^{l_1-m_1}_{\frac{1}{2}\sigma_1J_1-M_1} \to C^{q\kappa}_{p\psi a\alpha} \tag{37}$$

即：

$$(-1)^{1+\sigma_1+J_1-l_1} \frac{\hat{J}_1}{\hat{l}_1} \hat{l}_1 (-1)^{m_1+\frac{3}{2}+J_1} \begin{pmatrix} \frac{1}{2} & J_1 & l_1 \\ \sigma_1 & -M_1 & m_1 \end{pmatrix} = \tag{38}$$

$$C^{\lambda'\xi'}_{l_1-m_1L_1\mu} = (-1)^{l_1+m_1} \frac{\hat{\lambda}'}{\hat{L}_1} C^{L_1-\mu}_{l_1-m_1\lambda'-\xi'} \to C^{r\rho}_{q\kappa u\nu} \tag{39}$$

即：

$$(-1)^{l_1+m_1} \frac{\hat{\lambda}'}{\hat{L}_1} \hat{L}_1 (-1)^{\mu+3l_1+\lambda'} \begin{pmatrix} l_1 & \lambda' & L_1 \\ -m_1 & -\xi' & \mu \end{pmatrix} \tag{40}$$

$$C^{LM}_{L_1\mu 1k}=(-1)^{L_1-\mu}\frac{\hat{L}}{\hat{1}}C^{1-k}_{L_1\mu L-M}=(-1)^{L_1-\mu+L_1+L-1}\frac{\hat{L}}{\hat{1}}C^{1k}_{L_1-\mu LM}=(-1)^{L-\mu-1}\frac{\hat{L}}{\hat{1}}C^{1k}_{L_1-\mu LM}\to C^{s\sigma}_{r\rho c\gamma} \tag{41}$$

即：

$$(-1)^{L-\mu-1}\frac{\hat{L}}{\hat{1}}\hat{1}(-1)^{k+3L_1+L}\begin{pmatrix} L_1 & L & 1 \\ -\mu & M & -k \end{pmatrix} \tag{42}$$

$$C^{\frac{1}{2}\sigma_1}_{\frac{1}{2}\sigma_2 1k}=(-1)^{\frac{1}{2}+1-\frac{1}{2}}C^{\frac{1}{2}\sigma_1}_{1k\frac{1}{2}\sigma_2}\to C^{p\psi}_{s\sigma v\mu} \tag{43}$$

即：

$$(-1)^{1}\frac{\hat{1}}{2}(-1)^{\sigma_1+3+\frac{1}{2}}\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ k & \sigma_2 & -\sigma_1 \end{pmatrix} \tag{44}$$

用 3J 符号表示出来：

所以：

$$\begin{aligned}
&\sum_{m_1\sigma_1m_2\sigma_2}\sum_{\mu k\xi'}C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma_1}C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma_2}C^{LM}_{L_1\mu 1k}(-1)^{m_1}C^{\lambda'\xi'}_{l_1-m_1L_1\mu}C^{\lambda\xi}_{\lambda'\xi'l_2m_2}C^{\frac{1}{2}\sigma_1}_{\frac{1}{2}\sigma_2 1k}\\
=&(-1)^{m_1}(-1)^{\lambda'-\xi'+J_2-m_2-\frac{1}{2}+1+\sigma_1+J_1-l_1+l_1+m_1+L-\mu-1+1}\frac{\hat{\lambda}\hat{J_2}\hat{J_1}\hat{\lambda}'\hat{L}}{\hat{l_2}\frac{\hat{1}}{2}\hat{l_1}\hat{L_1}\hat{1}}\\
&\times C^{l_1-m_1}_{\frac{1}{2}\sigma_1J_1-M_1}C^{L_1-\mu}_{l_1-m_1\lambda'-\xi'}C^{1k}_{L_1-\mu LM}C^{\frac{1}{2}\sigma_1}_{1k\frac{1}{2}\sigma_2}C^{l_2-m_2}_{\lambda'\xi'\lambda-\xi}C^{\frac{1}{2}\sigma_2}_{l_2-m_2J_2M_2}\\
=&(-1)^{\lambda'\xi+J_2-\frac{1}{2}+\sigma_1+J_1+L-\mu}\frac{\hat{\lambda}\hat{J_2}\hat{J_1}\hat{\lambda}'\hat{L}}{\hat{l_2}\frac{\hat{1}}{2}\hat{l_1}\hat{L_1}\hat{1}}C^{l_1-m_1}_{\frac{1}{2}\sigma_1J_1-M_1}C^{L_1-\mu}_{l_1-m_1\lambda'-\xi'}C^{1k}_{L_1-\mu LM}C^{\frac{1}{2}\sigma_1}_{1k\frac{1}{2}\sigma_2}C^{l_2-m_2}_{\lambda'\xi'\lambda-\xi}C^{\frac{1}{2}\sigma_2}_{l_2-m_2J_2M_2}
\end{aligned} \tag{45}$$

==

$$C^{q\kappa}_{p\psi a\alpha}C^{g_\lambda}_{q\kappa u\nu}C^{s\sigma}_{g\lambda v\mu}C^{p\psi}_{s\sigma d\delta}C^{r-\rho}_{u-\nu b\beta}C^{v\mu}_{r-\rho c\gamma} \tag{46}$$

$$C^{\lambda\xi}_{\lambda'\xi'l_2m_2}\to C^{l_2-m_2}_{\lambda'-\xi'\lambda-\xi}\to C^{r-\rho}_{u-\nu b\beta} \tag{47}$$

$$C^{J_2M_2}_{l_2m_2\frac{1}{2}\sigma_2}\to C^{\frac{1}{2}\sigma_2}_{l_2-m_2J_2M_2}\to C^{v\mu}_{r-\rho c\gamma} \tag{48}$$

$$C^{J_1M_1}_{l_1m_1\frac{1}{2}\sigma_1}\to C^{\frac{1}{2}-\sigma_1}_{l_1m_1J_1-M_1}\to C^{q\kappa}_{p\psi a\alpha} \tag{49}$$

$$C^{\frac{1}{2}\sigma_2}_{\frac{1}{2}\sigma_1 1k}\to C^{1k}_{\frac{1}{2}-\sigma_1\frac{1}{2}\sigma_2}\to C^{g\lambda}_{q\kappa u\nu} \tag{50}$$

$$\to C^{s\sigma}_{g\lambda v\mu} \tag{51}$$

$$C^{LM}_{L_1\mu 1k}\to C^{p\psi}_{s\sigma d\delta} \tag{52}$$