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For the optimization problem

$$\min \frac{1}{2} \parallel \beta \parallel^2 + C\sum_{i=1}^{n} \xi_i$$
$$\text{subject to: } y_i(\beta_0 + x^T\beta) + \xi_i \ge 1, \xi_i \ge 0, i = 1, \ldots, n$$

derive the dual maximization problem:

$$F_D(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j y_i y_j x_i^T x_j$$
$$\text{subject to: } 0 \le \alpha_i \le C, \sum_{i=1}^{n} \alpha_i y_i = 0$$

解: (以下用中文和书写)

Optimization problem 的拉格朗日函数为:

$$L(\beta, \beta_0, \xi, \alpha, \mu) == \frac{1}{2}\beta^T\beta + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i[1 - y_i(\beta_0 + x^T\beta) - \xi_i] - \sum_{i=1}^{n}\mu_i\xi_i$$
$$\text{subject to: } \lambda_i \ge 0, \mu_i \ge 0$$

其对偶问题为:

$$\theta_D(\alpha, \mu) = \min_{\beta, \beta_0, \xi} L(\beta, \beta_0, \xi, \alpha, \mu)$$

求偏导并令其为 0:

$$\frac{\partial L}{\partial \beta} = \beta + \sum_{i=1}^{n}\alpha_i(-y_i x_i) = 0,$$
$$\frac{\partial L}{\partial \beta_0} = -\sum_{i=1}^{n}\alpha_i y_i = 0,$$
$$\frac{\partial L}{\partial \xi} = C - \alpha_i - \mu_i$$

解得:

$$\beta = \sum_{i=1}^{n}\alpha_i y_i x_i,$$
$$\sum_{i=1}^{n}\alpha_i y_i = 0,$$
$$C = \alpha_i + \mu_i$$
$$\text{subject to: } \alpha_i \ge 0,\ \mu_i \ge 0, i = 1, \ldots, n$$

因此:

$$
\begin{aligned}
\theta_D(\alpha,\mu) &= \min_{\beta,\beta_0,\xi} L(\beta,\beta_0,\xi,\alpha,\mu) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\alpha_i y_i x_i\right)^2 + \sum_{i=1}^{n}\alpha_i[1-y_i\beta_0-y_i\beta x_i^T] \\
&= \frac{1}{2}\left(\sum_{i=1}^{n}\alpha_i y_i x_i\right)^2 + \sum_{i=1}^{n}\alpha_i + \beta_0\sum_{i=1}^{n}\alpha_i y_i - \left(\sum_{i=1}^{n}\alpha_i y_i x_i\right)^2 \\
&= -\frac{1}{2}\left(\sum_{i=1}^{n}\alpha_i y_i x_i\right)^2 + \sum_{i=1}^{n}\alpha_i \\
&= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j x_i x_j + \sum_{i=1}^{n}\alpha_i \\
\text{subject to: } & \alpha_i \geq 0,\ \mu_i \geq 0, i=1,\ldots,n \\
& C = \alpha_i + \mu_i, \\
& \sum_{i=1}^{n}\alpha_i y_i = 0,
\end{aligned}
$$

而前三项条件可以简化为:

$$0 \leq \alpha_i \leq C, i=1,\ldots,n$$

从而 dual maximization problem 可以写作:

$$F_D(\alpha) = \sum_{i=1}^{n}\alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j x_i^T x_j$$

$$\text{subject to: } 0 \leq \alpha_i \leq C, \sum_{i=1}^{n}\alpha_i y_i = 0$$