Homework 10.10
By Gaozhe

The prime problem

$$\min \ \frac{1}{2}\|\beta\|^2+C\sum_{i=1}^{n}\varepsilon_i$$

subject to $y_i(\beta_0+x_i^T\beta)+\varepsilon_i \geq 1$

The Lagrangian

$$L(\beta_0,\beta,\varepsilon,\alpha)=\frac{1}{2}\|\beta\|^2+C\sum_{i=1}^{n}\varepsilon_i-\sum_{i=1}^{n}\alpha_i(y_i(\beta_0+x_i^T\beta)+\varepsilon_i-1)$$

The dual problem

$$\max_{\alpha}\ \min_{\beta_0,\beta,\varepsilon} L(\beta_0,\beta,\varepsilon,\alpha)$$

We can find the patical derivative of the Lagrangian with respect to $\beta_0,\beta,\varepsilon$

$$\frac{\partial L(\beta_0,\beta,\varepsilon,\alpha)}{\partial\beta}=\beta-\sum_{i=1}^{n}\alpha_i y_i x_i=0$$

$$\frac{\partial L(\beta_0,\beta,\varepsilon,\alpha)}{\partial\beta_0}=-\sum_{i=1}^{n}\alpha_i y_i=0$$

$$\frac{\partial L(\beta_0,\beta,\varepsilon,\alpha)}{\partial\varepsilon}=C-\alpha=0$$

Then, we have

$$\beta=\sum_{i=1}^{n}\alpha_i y_i x_i \quad (1)$$

$$\sum_{i=1}^{n}\alpha_i y_i=0 \quad (2)$$

$$\sum_{i=1}^{n}(C-\alpha_i)=0 \quad (3)$$

$$\begin{aligned}
L(\beta_0,\beta,\varepsilon,\alpha)&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j x_i^T x_j+C\sum_{i=1}^{n}\varepsilon_i-\sum_{i=1}^{n}\alpha_i y_i\beta_0-\sum_{i=1}^{n}\alpha_i y_i x_i^T\sum_{j=1}^{n}\alpha_j y_j x_j))-\sum_{i=1}^{n}\alpha_i\varepsilon_i+\sum_{i=1}^{n}\alpha_{ii}\\
&=\sum_{i=1}^{n}\alpha_{ii}-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j x_i^T x_j+\sum_{i=1}^{n}(C-\alpha_i)\varepsilon_i-\sum_{i=1}^{n}\alpha_i y_i\beta_0\\
&=\sum_{i=1}^{n}\alpha_{ii}-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j x_i^T x_j
\end{aligned}$$