For the optimization problem

$$\minrac{1}{2}\paralleleta\parallel^2+C\sum_{i=1}^n\xi_i$$
 subject to: $y_i(eta_0+x^Teta)+\xi_i\geq 1, \xi_i\geq 0, i=1,\dots,n$

derive the dual maximization problem:

$$egin{aligned} F_D(lpha) &= \sum_{i=1}^n lpha_i - rac{1}{2} \sum_{i=1}^n \sum_{j=1}^n lpha_i lpha_j y_i y_j x_i^T x_j \ & ext{subject to: } 0 \leq lpha_i \leq C, \sum_{i=1}^n lpha_i y_i = 0 \end{aligned}$$

解:(以下用中文和书写)

Optimization problem 的拉格朗日函数为:

$$L(eta,eta_0,\xi,lpha,\mu) == rac{1}{2}\,eta^Teta + C\sum_{i=1}^n \xi_i + \sum_{i=1}^n lpha_i[1-y_i(eta_0+x^Teta)-\xi_i] - \sum_{i=1}^n \mu_i \xi_i \ ext{subject to:} \ \lambda_i \geq 0, \mu_i \geq 0$$

其对偶问题为:

$$heta_D(lpha,\mu) = \min_{eta,eta_0,\xi} L(eta,eta_0,\xi,lpha,\mu)$$

求偏导并令其为 0:

$$egin{aligned} rac{\partial L}{\partial eta} &= eta + \sum_{i=1}^n lpha_i (-y_i x_i) = 0, \ rac{\partial L}{\partial eta_0} &= -\sum_{i=1}^n lpha_i y_i = 0, \ rac{\partial L}{\partial ar{\epsilon}} &= C - lpha_i - \mu_i \end{aligned}$$

解得:

$$eta = \sum_{i=1}^n lpha_i y_i x_i, \ \sum_{i=1}^n lpha_i y_i = 0, \ C = lpha_i + \mu_i \ ext{subject to: } lpha_i \geq 0, \ \mu_i \geq 0, i = 1, \dots, n$$

因此:

$$\begin{split} \theta_D(\alpha,\mu) &= \min_{\beta,\beta_0,\xi} L(\beta,\beta_0,\xi,\alpha,\mu) \\ &= \frac{1}{2} \left(\sum_{i=1}^n \alpha_i y_i x_i \right)^2 + \sum_{i=1}^n \alpha_i [1 - y_i \beta_0 - y_i \beta x_i^T] \\ &= \frac{1}{2} \left(\sum_{i=1}^n \alpha_i y_i x_i \right)^2 + \sum_{i=1}^n \alpha_i + \beta_0 \sum_{i=1}^n \alpha_i y_i - \left(\sum_{i=1}^n \alpha_i y_i x_i \right)^2 \\ &= -\frac{1}{2} \left(\sum_{i=1}^n \alpha_i y_i x_i \right)^2 + \sum_{i=1}^n \alpha_i \\ &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j x_i x_j + \sum_{i=1}^n \alpha_i \\ &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j x_i x_j + \sum_{i=1}^n \alpha_i \\ \text{subject to: } \alpha_i \geq 0, \ \mu_i \geq 0, i = 1, \dots, n \\ C &= \alpha_i + \mu_i, \\ \sum_{i=1}^n \alpha_i y_i &= 0, \end{split}$$

而前三项条件可以简化为:

$$0 \leq \alpha_i \leq C, i = 1, \ldots, n$$

从而 dual maximization problem 可以写作:

$$egin{aligned} F_D(lpha) &= \sum_{i=1}^n lpha_i - rac{1}{2} \sum_{i=1}^n \sum_{j=1}^n lpha_i lpha_j y_i y_j x_i^T x_j \ & ext{subject to: } 0 \leq lpha_i \leq C, \sum_{i=1}^n lpha_i y_i = 0 \end{aligned}$$